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LOCAL TIME OF A MULTIFRACTIONAL GAUSSIAN PROCESS

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ABSTRACT. In this paper, a new multifractional Gaussian process is defined by a integral representation. We prove an approximation in law of this process and we prove that the covariance function of this process permits us to obtain a generalization of the sub-fractional Brownian motion [5] by a decomposition in law into the sum of our process and the standard multifractional Brownian motion [1]. We prove also the existence and the joint continuity of the local time of our process.

1. Introduction

Let $X^H := \{X_t^H \,;\, t \geq 0\}$ be the Gaussian process defined by:

$$X_t^H = \int_0^{+\infty} (1 - e^{-\theta t}) \theta^{-\frac{H+1}{2}} dW_\theta,$$

where $H \in (0, 2)$ and $W := \{W_t \,;\, t \geq 0\}$ is a standard Brownian motion. This process was introduced by Lei and Nualart [10] in order to obtain a decomposition in law of the bifractional Brownian motion [9] and was used later by Bardina and Bascompte [2] and Ruiz de Chavez and Tudor [11] in order to obtain a decomposition in law of the sub-fractional Brownian motion $S^H := \{S_t^H \,;\, t \geq 0\}$ with parameter $H \in (0, 2)$. This process was introduced by Bojdecki et *al.* [5]. It is a continuous centered Gaussian process, starting from zero, with covariance function:

$$\mathbb{E}(S_t^H S_s^H) = t^H + s^H - \frac{1}{2}[(t+s)^H + |t-s|^H].$$

In this paper, firstly, we introduce a new multifractional Gaussian process which generalize the process X^H by substituting to the parameter H a function $H(.)$ such that $H(t) \in (0, 2)$ and we prove an approximation in law of this process. Secondly, we prove that the covariance function of this process permits us to obtain a generalization of the sub-fractional Brownian motion by a decomposition in law into the sum of our process and the standard multifractional Brownian motion [1]. We prove also under some assumptions on $H(.)$, the existence, the joint continuity and the Hölder regularity of the local time of our process. Our idea is inspired from the work of Boufoussi et *al.* [6] in case of multifractional Brownian motion. We use the concept of local nondeterminism for Gaussian process introduced by

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Berman [4] and the analytic method used by Berman [3] for the calculation of the moments of local time.

2. The Multifractional Gaussian Process

Definition 2.1. Let $H : [0, +\infty[\to (0,2)$ be a function satisfying: there exists finite and positive constants β, C_2 and C_3 such that

$$C_2|t-s|^\beta \leq |H(t) - H(s)| \leq C_3|t-s|^\beta, \ for\ all\ s,t \geq 0. \tag{2.1}$$

The right hand side of (2.1) means that $H(.)$ is *β-Hölder continuous.*

Definition 2.2. For $t \geq 0$, we denote by $X^{H(t)} := \{X_t^{H(t)} \ ; \ t \geq 0\}$ the *multifractional Gaussian process* with the multifractional function $H(t)$ defined by:

$$X_t^{H(t)} = \int_0^{+\infty} (1 - e^{-\theta t})\theta^{-\frac{H(t)+1}{2}} dW_\theta,$$

where the integration is taken in the mean square sense.

An approximation in law of our process can be obtained by the same argument used by Bardina and Bascompte [2] in case of the process X^H.

Proposition 2.3. *Let $\{N_s \ ; s \geq 0 \}$ be a standard Poisson process and let $\theta \in (0,\pi) \cup (\pi, 2\pi)$. Then the processes,*

$$X_\varepsilon^{H(.)} = \left\{ \frac{2}{\varepsilon} \int_0^{+\infty} (1 - e^{-st}) s^{-\frac{H(t)+1}{2}} \cos\left(\theta N_{\frac{2s}{\varepsilon^2}}\right) ds, \ \ t \in [0,T] \right\}$$

and

$$\tilde{X}_\varepsilon^{H(.)} = \left\{ \frac{2}{\varepsilon} \int_0^{+\infty} (1 - e^{-st}) s^{-\frac{H(t)+1}{2}} \sin\left(\theta N_{\frac{2s}{\varepsilon^2}}\right) ds, \ \ t \in [0,T] \right\}$$

converge in law, in the sense of the finite dimensional distributions, toward two independent processes with the same law that $X^{H(\cdot)}$.

The following result concerns the regularity property of $X^{H(\cdot)}$.

Lemma 2.4. *Let $T > 0$ fixed. Assume that $H(.)$ is Hölder continuous with exponent $\beta \in (0,1)$. There exists a finite and positive constant C, such that for all $t, s \in [0,T]$, we have*

$$\mathbb{E}|X_t^{H(t)} - X_s^{H(s)}|^2 \leq C|t-s|^{2\beta}.$$

To prove Lemma 2.4, we need the following result.

Lemma 2.5. *Let $T > 0$ fixed and $[\mu,\nu] \subset (0,2)$. There exists two positive and finite constants $C^1_{\mu,\nu}$ and $C^2_{\mu,\nu}$ such that for all $\lambda, \lambda' \in [\mu,\nu]$,*

$$C^1_{\mu,\nu}|\lambda - \lambda'|^2 \leq \sup_{t\in[0,T]} \mathbb{E}[X_t^\lambda - X_t^{\lambda'}]^2 \leq C^2_{\mu,\nu}|\lambda - \lambda'|^2. \tag{2.2}$$

Proof. We have

$$\mathbb{E}[X_t^\lambda - X_t^{\lambda'}]^2 = \int_0^{+\infty} (1 - e^{-\theta t})^2 (\theta^{-\frac{\lambda+1}{2}} - \theta^{-\frac{\lambda'+1}{2}})^2 d\theta.$$

Using the theorem on finite increments, ([7], Corollary 2.6.1), for the function $x \to \theta^{-\frac{x+1}{2}}$; $x \in (\lambda, \lambda')$, there exists $\xi \in (\lambda, \lambda')$ such that

$$\left|\theta^{-\frac{\lambda+1}{2}} - \theta^{-\frac{\lambda'+1}{2}}\right| = \frac{1}{2}|\ln(\theta)|\theta^{-\frac{\xi+1}{2}}|\lambda - \lambda'|,$$

therefore

$$\begin{aligned}
\mathbb{E}[X_t^\lambda - X_t^{\lambda'}]^2 &= \frac{1}{4}|\lambda-\lambda'|^2 \int_0^{+\infty} (1-e^{-\theta t})^2 \ln^2(\theta)\theta^{-(\xi+1)} d\theta \\
&\leq \frac{1}{4}|\lambda-\lambda'|^2 \sup_{t\in[0,T]} \Big\{ \int_0^1 (1-e^{-\theta t})^2 \ln^2(\theta)\theta^{-(\xi+1)} d\theta \\
&\qquad\qquad + \int_1^{+\infty} (1-e^{-\theta t})^2 \ln^2(\theta)\theta^{-(\xi+1)} d\theta \Big\} \\
&\leq \frac{1}{4}|\lambda-\lambda'|^2 \sup_{t\in[0,T]} \Big\{ \int_0^1 (1-e^{-\theta t})^2 \ln^2(\theta)\theta^{-(\nu+1)} d\theta \\
&\qquad\qquad + \int_1^{+\infty} (1-e^{-\theta t})^2 \ln^2(\theta)\theta^{-(\mu+1)} d\theta \Big\}.
\end{aligned}$$

Consequently

$$\sup_{t\in[0,T]} \mathbb{E}[X_t^\lambda - X_t^{\lambda'}]^2 \leq C_{\mu,\nu}^2 |\lambda - \lambda'|^2.$$

Changing the sup by the inf, we complete the proof of Lemma 2.5. □

Remark 2.6. The theorem on finite increments for the function $x \to e^x$, implies that

$$e^y - e^x < e^y(y-x), \qquad \forall y > x,$$

therefore, there exists a finite and positive constant C_1 such that for any $H \in (0,1)$ and all $s, t \in [a,b] \subset (0,+\infty)$, we have

$$\mathbb{E}[X_t^H - X_s^H]^2 \leq C_1 |t-s|^2, \tag{2.3}$$

where

$$C_1 = \int_0^{+\infty} e^{-2a\theta}\theta^{1-H} d\theta.$$

Now we are ready to prove Lemma 2.4.

Proof of Lemma 2.4. Using the elementary inequality $(a+b)^2 \leq 2(a^2+b^2)$, we obtain

$$\mathbb{E}|X_t^{H(t)} - X_s^{H(s)}|^2 \leq 2\left[\mathbb{E}|X_t^{H(t)} - X_s^{H(t)}|^2 + \mathbb{E}|X_s^{H(t)} - X_s^{H(s)}|^2\right].$$

The first term on the right hand side of the previous expression is the variance of the increments of the process X^H of parameter $H(t)$. Therefore (2.3) implies that

$$\mathbb{E}|X_t^{H(t)} - X_s^{H(t)}|^2 \leq C|t-s|^2.$$

Moreover, by virtue of (2.2), we have

$$\begin{aligned}
\mathbb{E}|X_s^{H(t)} - X_s^{H(s)}|^2 &\leq C|H(t) - H(s)|^2 \\
&\leq C|t-s|^{2\beta}.
\end{aligned}$$

Consequently

$$\begin{aligned}\mathbb{E}|X_t^{H(t)} - X_s^{H(s)}|^2 &\leq C|t-s|^{2\beta}[1+|t-s|^{2(1-\beta)}]\\ &\leq C|t-s|^{2\beta}.\end{aligned}$$

The proof of Lemma 2.4 is done. □

In the next result, we give the formula of the covariance function our process. The proof follows the lines of that given by Bardina and Bascompte [2] in case of the process X^H.

Proposition 2.7. *The process $X^{H(.)}$ is Gaussian, centered, and its covariance function is*

$$\mathbb{E}(X_t^{H(t)}X_s^{H(s)}) = \begin{cases} \frac{\Gamma(1-H')}{H'}[t^{H'}+s^{H'}-(t+s)^{H'}], & if\ H(.) \in (0,1),\\ \frac{\Gamma(2-H')}{H'(H'-1)}[(t+s)^{H'}-t^{H'}-s^{H'}], & if\ H(.)\in(1,2).\end{cases}$$

where $H' = \frac{H(t)+H(s)}{2}$.

As application, we obtain a generalization of the sub-fractional Brownian motion S^H by a decomposition in law into the sum of our process and the standard multifractional Brownian motion [1]. The covariance function of this last process was given by Ayyache et *al.* [1] as follows:

$$\mathbb{E}(B_t^{H(t)}B_s^{H(s)}) = D(H(t),H(s))[t^{H'}+s^{H'}-|t-s|^{H'}],$$

where

$$D(x,y) = \frac{\sqrt{\Gamma(x+1)\Gamma(y+1)\sin(\pi x/2)\sin(\pi y/2)}}{2\Gamma((x+y)/2+1)\sin(\pi(x+y)/4)}.$$

Theorem 2.8. *The centered multifractional Gaussian $S^{H(.)}$ process with a multifrcational function $H(t)\in(0,1)$ defined by the decomposition in law:*

$$S_t^{H(t)} = B_t^{H(t)} + C(H(.))X_t^{H(t)},$$

where $C(H(t)) = \sqrt{\frac{H(t)}{2\Gamma(1-H(t))}}$ and $B^{H(.)}$ and $X^{H(.)}$ are independent, is a generalization of the sub-fractional Brownian motion S^H with parameter $H\in(0,1)$.

3. Local Time and Local Nondeterminism

We begin this section by the definition of local time. For a complete survey on local time, we refer to Geman and Horowitz [8] and the references therein.

Let $X := \{X_t\ ;\ t\geq 0\}$ be a real-valued separable random process with Borel sample functions. For any Borel set $B\subset\mathbb{R}^+$, the occupation measure of X on B is defined as

$$\mu_B(A) = \lambda\{s\in B\ ;\ X_s\in A\},\qquad \forall A\in\mathcal{B}(\mathbb{R}),$$

where λ is the one-dimensional Lebesgue measure on $\mathbb{R}^+$. If μ_B is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$, we say that X has a local time on B and define its local time, $L(B,.)$, to be the Radon-Nikodym derivative of μ_B. Here, x is the so-called space variable and B is the time variable. By standard monotone class arguments, one can deduce that the local time have a measurable modification that satisfies the occupation density formula: for every Borel set $B \subset \mathbb{R}^+$ and every measurable function $f : \mathbb{R} \to \mathbb{R}^+$,

$$\int_B f(X_t)dt = \int_{\mathbb{R}} f(x)L(B,x)dx.$$

Sometimes, we write $L(t,x)$ instead of $L([0,t],x)$.

Here is the outline of the analytic method used by Berman [3] for the calculation of the moments of local time.

For fixed sample function at fixed t, the Fourier transform on x of $L(t,x)$ is the function

$$F(u) = \int_{\mathbb{R}} e^{iux} L(t,x)dx.$$

Using the density of occupation formula, we get

$$F(u) = \int_0^t e^{iuX_s} ds.$$

Therefore, we may represent the local time as the inverse Fourier transform of this function, i.e.,

$$L(t,x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\int_0^t e^{iu(X_s - x)} ds \right) du. \tag{3.1}$$

We end this section by the definition of the concept of local nondeterminism, (LND for short). Let J be an open interval on the t axis. Assume that $\{X_t \ ; \ t \geq 0\}$ is a zero mean Gaussian process without singularities in any interval of the length δ, for some $\delta > 0$, and without fixed zeros, i.e., there exists $\delta > 0$, such that

$$(\mathcal{P}) : \begin{cases} \mathbb{E}(X_t)^2 > 0, \ \ for \ t \in J. \\ \mathbb{E}[X_t - X_s]^2 > 0, \ \ whenever \ 0 < |t-s| < \delta, \end{cases}$$

To introduce the concept of LND, Berman [4] defined the relative conditioning error

$$V_p = \frac{Var\{X_{t_p} - X_{t_{p-1}} / X_{t_1}, ... X_{t_{p-1}}\}}{Var\{X_{t_p} - X_{t_{p-1}}\}},$$

where for $p \geq 2$, $t_1 < ... < t_p$ are arbitrary ordered points in J.

We say that the process X is LND on J if for every $p \geq 2$,

$$\lim_{c \to 0^+} \inf_{0 < t_p - t_1 \leq c} V_p > 0. \tag{3.2}$$

This condition means that a small increment of the process is not almost relatively predictable on the basis of a finite number of observations from the immediate past. Berman [4] has proved, for Gaussian process, that the LND is characterized as follows.

Proposition 3.1. *A Gaussian process X is LND if and only if for every integer $p \geq 2$, there exists two positive constants δ and C_p such that*

$$Var\left(\sum_{i=1}^{p} u_j (X_{t_j} - X_{t_{j-1}}) \right) \geq C_p \sum_{i=1}^{m} u_j^2 Var(X_{t_j} - X_{t_{j-1}}),$$

for all orderer points $t_1 < ... < t_p$ are arbitrary points in J with $t_0 = 0$, $t_p - t_1 \leq \delta$ and $(u_1, ..., u_j) \in \mathbb{R}$.

4. Existence and Joint Continuity of Local Time

The purpose of this section is to present sufficient conditions for the existence of the local time of $X^{H(.)}$. Furthermore, using the LND approach, we show that the local time of $X^{H(.)}$ have a jointly continuous version.

Theorem 4.1. *Assume $H(.)$ satisfying (2.1) with $\beta \in (0,1)$. On each time-interval $[a,b] \subset [0,\infty)$, $X^{H(.)}$ admits a local time which satisfies*

$$\int_{\mathbb{R}} L^2([a,b],x)dx < \infty \qquad a.s.$$

For the proof of Theorem 4.1, we need the following lemma. This result on the regularity of the increments of $X^{H(.)}$ will be the key for the existence and the joint continuity of local times.

Lemma 4.2. *Assume $H(.)$ satisfying (2.1) with $\beta \in (0,1)$. There exists two positive and finite constants $\delta > 0$ and C such that*

$$\mathbb{E}[X_t^{H(t)} - X_s^{H(s)}]^2 \geq C|t-s|^{2\beta}, \tag{4.1}$$

for all $s,t \geq 0$ such that $|t-s| < \delta$.

Proof. By virtue of the elementary inequality $(a+b)^2 \geq \frac{1}{2}a^2 - b^2$, we get

$$\mathbb{E}|X_t^{H(t)} - X_s^{H(s)}|^2 \geq \frac{1}{2}\mathbb{E}|X_s^{H(t)} - X_s^{H(s)}|^2 - \mathbb{E}|X_t^{H(t)} - X_s^{H(t)}|^2.$$

Therefore (2.1), (2.2) and (2.3) implies that

$$\mathbb{E}|X_t^{H(t)} - X_s^{H(s)}|^2 \geq |t-s|^{2\beta}\left[\frac{C^1_{\mu,\nu}C_2}{2} - C_1|t-s|^{2(1-\beta)}\right].$$

Since $\beta < 1$, we can choose $\delta > 0$ small enough such that for all $t,s \geq 0$ with $|t-s| < \delta$, we have

$$\left[\frac{C^1_{\mu,\nu}C_2}{2} - C_1|t-s|^{2(1-\beta)}\right] > 0.$$

Indeed, it suffices to choose

$$\delta < \left(\frac{C^1_{\mu,\nu}C_2}{2C_1} \wedge 1\right)^{1/2(1-\beta)}.$$

Finally,

$$\mathbb{E}[X_t^{H(t)} - X_s^{H(s)}]^2 \geq C|t-s|^2,$$

with $|t-s| < \delta$ and

$$C = \left[\frac{C^1_{\mu,\nu}C_2}{2} - C_1\delta^{2(1-\beta)}\right].$$

The proof of Lemma 4.2 is done. □

Proof of Theorem 4.1. It is well known by Berman [3] that, for a jointly measurable zero-mean Gaussian process $X := \{X(t)\ ;\ t \in [0,T]\}$ with bounded variance, the variance condition

$$\int_0^T \int_0^T \left(\mathbb{E}[X(t)-X(t)]^2\right)^{-1/2} dsdt < \infty$$

is sufficient for the local time $L(t,u)$ of X to exist on $[0,T]$ a.s. and to be square integrable as a function of u. For any $[a,b] \subset [0,\infty)$, and for $I=[a',b'] \subset [a,b]$ such that $|b'-a'| < \delta$, according to (4.1), we have,

$$\int_I \int_I \left(\mathbb{E}[X_t^{H(t)} - X_s^{H(s)}]^2\right)^{-1/2} dsdt < C \int_I \int_I |t-s|^{-\beta} dsdt.$$

The last integral is finite, then $X^{H(.)}$ possesses, on any interval $I \subset [a,b]$ with length $|I| < \delta$, a local time which is square integrable as function of u. Finally, since $[a,b]$ is a finite interval, we can obtain the local time on $[a,b]$ by a patch-up procedure, i.e. we partition $[a,b]$ into $\cup_{i=1}^n [a_{i-1},a_i]$, such that $|a_i - a_{i-1}| < \delta$, and define $L([a,b],x) = \sum_{i=1}^n L([a_{i-1},a_i],x)$, where $a_0 = a$ and $a_n = b$. □

Proposition 4.3. *Assume $H(.)$ is derivable and Hölder continuous with $\beta \in (0,1)$. Then for every $\varepsilon > 0$, and any $T > \varepsilon$, $X^{H(.)}$ is locally LND on $[\varepsilon, T]$.*

Proof. Notice that $X^{H(.)}$ is a zero mean Gaussian process, and $\mathbb{E}(X_t^{H(t)})^2 > 0$ for all $t \in [\varepsilon, T]$. Moreover, thanks to (4.1) also the second point in $(\mathcal{P})$ is satisfied on $[\varepsilon, T]$. It remains to show that $X^{H(.)}$ satisfies (3.2).

We have

$$\sigma(X_\theta^{H(\theta)}\ ,\ \theta \le s) \subset \sigma(W_\theta\ ,\ \theta \le s),\ for\ all\ s \ge 0.$$

Then, for any $t > s$,

$$Var\left(X_t^{H(t)} - X_s^{H(s)}/X_\theta^{H(\theta)}\ ,\ \theta \le s\right) \ge Var\left(X_t^{H(t)} - X_s^{H(s)}/W_\theta\ ,\ \theta \le s\right).$$

The measurability of $\int_0^s (1-e^{-\theta t})\theta^{-\frac{H(t)+1}{2}} dW_\theta$ with respect to $\sigma(W_\theta\ ,\ \theta \le s)$ and the fact that $\int_s^{+\infty} (1-e^{-\theta t})\theta^{-\frac{H(t)+1}{2}} dW_\theta$ is independent of $\sigma(W_\theta\ ,\ \theta \le s)$, (by the independence of the increments of the Brownian motion), implies that

$$\begin{aligned} &Var\left(X_t^{H(t)} - X_s^{H(s)}/X_\theta^{H(\theta)}\ ,\ \theta \le s\right) \\ &\ge \int_s^{+\infty} \left[(1-e^{-\theta t})\theta^{-\frac{H(t)+1}{2}} - (1-e^{-\theta s})\theta^{-\frac{H(s)+1}{2}}\right]^2 d\theta. \end{aligned}$$

Making use of the theorem on finite increments for the function: $x \to f_\theta(x) = (1-e^{-\theta x})\theta^{-\frac{H(x)+1}{2}}$ for $x \in (s,t)$, there exists $\xi \in (s,t)$ such that

$$Var\left(X_t^{H(t)} - X_s^{H(s)}/X_u^{H(u)}\ ,\ u \le s\right) \ge |t-s|^2 \inf_{x \in [\varepsilon,T]} \left(\int_T^{+\infty} (f'_\theta(x))^2 d\theta\right).$$

Therefore the relative prediction error V_p given by (3.2) is at least equal to

$$\frac{\inf_{x\in[\varepsilon,T]} \left(\int_T^{+\infty} (f'_\theta(x))^2 d\theta\right)}{\sup_{x\in[\varepsilon,T]} \left(\int_0^{+\infty} (f'_\theta(x))^2 d\theta\right)},$$

which is bounded away from 0, as t tends to s, and the proof is complete. □

Remark 4.4. When applying the LND to the estimation of the moments of local times of $X^{H(.)}$, the condition $t > \varepsilon$ can be circumvented by slightly adjusting the arguments. Indeed, we can consider $X = X^{H(.)} + Y$, where Y is a standard normal random variable independent of $X^{H(.)}$. Since the occupation density of X has the same local properties as that of $X^{H(.)}$ and $\mathbb{E}(X_t)^2 = \mathbb{E}(X_t^{H(t)})^2 + \mathbb{E}(Y)^2 > 0$ for all $t \in [0, T]$, we can use the LND on the whole interval $[0, T]$ for $X^{H(.)}$.

We end this section by the following main result.

Theorem 4.5. *Assume $H(.)$ is derivable and satisfying (2.1) with $\beta \in (0, 1)$ and let δ the constant appearing in Lemma 4.2. For any integer $p \geq 2$ there exists a finite positive constant C_p such that, for any $t \geq 0$, any $h \in (0, \delta)$, all $x, y \in \mathbb{R}$, and any $0 < \xi < \frac{1-\beta}{2\beta}$,*

$$\mathbb{E}[L(t+h,x) - L(t,x)]^p \leq C_p \frac{h^{p(1-\beta)}}{\Gamma(1+p(1-\beta))}, \tag{4.2}$$

$$\begin{aligned}&\mathbb{E}[L(t+h,y) - L(t,y) - L(t+h,x) + L(t,x)]^p \\ &\leq C_p |y-x|^{p\xi} \frac{h^{p(1-\beta(1+\xi))}}{\Gamma(1+p(1-\beta(1+\xi)))}.\end{aligned} \tag{4.3}$$

Proof. We will prove only (4.3), the proof of (4.2) is similar. It follows from (3.1) that for any $x, y \in \mathbb{R}$, $t, t+h \geq 0$ and for any integer $p \geq 2$,

$$\begin{aligned}&\mathbb{E}[L(t+h,y) - L(t,y) - L(t+h,x) + L(t,x)]^p \\ &= \frac{1}{(2\pi)^p} \int_{[t,t+h]^p} \int_{\mathbb{R}^p} \prod_{j=1}^{p} [e^{-iyu_j} - e^{-ixu_j}] \times \mathbb{E}\left(e^{i\sum_{j=1}^p u_j X_{s_j}^{H(s_j)}}\right) \prod_{j=1}^{p} du_j \prod_{j=1}^{p} ds_j.\end{aligned}$$

Using the elementary inequality $|1 - e^{i\theta}| \leq 2^{1-\xi} |\theta|^\xi$ for all $0 < \xi < 1$ and any $\theta \in \mathbb{R}$, we obtain

$$\begin{aligned}&\mathbb{E}[L(t+h,y) - L(t,y) - L(t+h,x) + L(t,x)]^p \leq (2^\xi \pi)^{-p} p! |x-y|^{p\xi} \\ &\int_{t<t_1<\ldots<t_p<t+h} \int_{\mathbb{R}^p} \prod_{j=1}^{p} |u_j|^\xi \mathbb{E}[exp(i \sum_{j=1}^{p} u_j X_{t_j}^{H(t_j)})] \prod_{j=1}^{p} du_j \prod_{j=1}^{p} t_j,\end{aligned} \tag{4.4}$$

where in order to apply the LND property of $X^{H(.)}$, we replaced the integration over the domain $[t, t+h]$ by over the subset $t < t_1 < \ldots < t_p < t+h$. We deal now with the inner multiple integral over the $u's$. Change the variables of integration by mean of the transformation

$$u_j = v_j - v_{j+1}, j = 1, \ldots, p-1; u_p = v_p.$$

Then the linear combination in the exponent in (4.4) is transformed according to

$$\sum_{j=1}^{p} u_j X_{t_j}^{H(t_j)} = \sum_{j=1}^{p} v_j (X_{t_j}^{H(t_j)} - X_{t_j-1}^{H(t_j)}),$$

where $t_0 = 0$. Since $X^{H(.)}$ is a Gaussian process, the characteristic function in (4.4) has the form

$$exp\left(-\frac{1}{2}Var\left[\sum_{j=1}^{p} v_j(X_{t_j}^{H(t_j)} - X_{t_j-1}^{H(t_j)})\right]\right). \tag{4.5}$$

Since $|x-y|^{\xi} \leq |x|^{\xi} + |y|^{\xi}$ for all $0 < \xi < 1$, it follows that

$$\prod_{j=1}^{p} |u_j|^{\xi} \leq \prod_{j=1}^{p-1} (|v_j|^{\xi} + |v_{j+1}|^{\xi})|v_p|^{\xi}. \tag{4.6}$$

Moreover, the last product is at most equal to a finite sum of 2^{p-1} terms of the form $\prod_{j=1}^{p} |x_j|^{\xi\varepsilon_j}$, where $\epsilon_j = 0, 1$ or 2 and $\sum_{j=1}^{p} \epsilon_j = p$.

Let us write for simply $\sigma_j^2 = \mathbb{E}\left(X_{t_j}^{H(t_j)} - X_{t_j-1}^{H(t_j)}\right)^2$. Combining the result of Proposition 3.1, (4.5) and (4.6), we get that the integral in (4.4) is dominated by the sum over all possible choices of $(\epsilon_1, ..., \epsilon_m) \in \{0,1,2\}^m$ of the following terms

$$\int_{t<t_1<...<t_p<t+h} \int_{\mathbb{R}^p} \prod_{j=1}^{p} |v_j|^{\xi\epsilon_j} exp\left(-\frac{C_p}{2}\sum_{j=1}^{p} v_j^2\sigma_j^2\right) \prod_{j=1}^{p} dt_j dv_j,$$

where C_p is the constant given in Proposition 3.1. The change of variable $x_j = \sigma_j v_j$ converts the last integral to

$$\int_{t<t_1<...<t_p<t+h} \prod_{j=1}^{p} \sigma^{-1-\xi\epsilon_j} dt_1....dt_p \times \int_{\mathbb{R}^p} \prod_{j=1}^{p} |x_j|^{\xi\epsilon_j} exp\left(-\frac{C_p}{2}\sum_{j=1}^{p} x_j^2\right) \prod_{j=1}^{p} dx_j.$$

Let us denote

$$J(p,\xi) = \int_{\mathbb{R}^p} \prod_{j=1}^{p} |x_j|^{\xi\epsilon_j} exp\left(-\frac{C_p}{2}\sum_{j=1}^{p} x_j^2\right) \prod_{j=1}^{p} dx_j.$$

Consequently

$$\begin{aligned} &\mathbb{E}[L(t+h,y) - L(t,y) - L(t+h,x) + L(t,x)]^p \\ &\leq J(p,\xi)C_p|y-x|^{p\xi} \int_{t<t_1<...<t_p<t+h} \prod_{j=1}^{p} \sigma_j^{-1-\xi\epsilon_j} dt_1....dt_p. \end{aligned} \tag{4.7}$$

According to (4.1), for h sufficiently small, namely $0 < h < \delta \wedge 1$, we have

$$\mathbb{E}[X_{t_j}^{H(t_j)} - X_{t_j-1}^{H(t_j)}]^2 \geq C|t_i - t_j|^{2\beta}, \quad \forall\, t_i, t_j \in [t, t+h].$$

It follows that the integral on the right hand side of (4.7) is bounded, up to a constant, by

$$\int_{t<t_1<...<t_p<t+h} \prod_{j=1}^{p} (t_j - t_{j-1})^{-\beta(1+\xi\epsilon_j)} dt_1...dt_p. \tag{4.8}$$

Since, $(t_j - t_{j-1}) < 1$, for all $j \in \{2, ..., p\}$, we have

$$(t_j - t_{j-1})^{-\beta(1+\xi\epsilon_j)} \leq (t_j - t_{j-1})^{-\beta(1+2\xi)}, \quad \forall \epsilon_j \in \{0,1,2\}.$$

Since by Hypothesis $0<\xi<\frac{1}{2\beta}-\frac{1}{2}$, the integral in (4.8) is finite. Moreover, by an elementary calculation, for all $p\geq 1$, $h>0$ and $b_j<1$,

$$\int_{t<s_1<...<s_p<t+h}\prod_{j=1}^{p}(s_j-s_{j-1})^{-b_j}ds_1...ds_p=h^{p-\sum_{j=1}^{p}b_j}\frac{\prod_{j=1}^{p}\Gamma(1-b_j)}{\Gamma(1+h-\sum_{j=1}^{p}b_j)},$$

where $s_0=t$. It follows that (4.8) is dominated by

$$C_p\frac{h^{p(1-\beta(1+\xi))}}{\Gamma(1+p(1-\beta(1+\xi))},$$

where $\sum_{j=1}^{p}\epsilon_j=p$. Consequently

$$\begin{aligned}&\mathbb{E}[L(t+h,y)-L(t,y)-L(t+h,x)+L(t,x)]^p\\ &\leq C_p|y-x|^{p\xi}\frac{h^{p(1-\beta(1+\xi))}}{\Gamma(1+p(1-\beta(1+\xi)))}.\end{aligned}$$

□

Remark 4.6. (1) The process X^H has infinitely differentiable trajectories and it is well-known that in this case the local time does not exist because the occupation measure is singular.
(2) We believe that the same arguments used in this paper can be used for the bifractional Brownian motion and the Gaussian process introduced in Sghir [12].

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