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A HULL AND WHITE FORMULA FOR A STOCHASTIC VOLATILITY LÉVY MODEL WITH INFINITE ACTIVITY

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Abstract. In this short note, by using techniques of Malliavin calculus for Lévy processes, we obtain an anticipating Itô formula for an infinite activity Lévy process. As an application we derive a Hull and White formula for an infinite activity stochastic volatility Lévy model. There are no assumptions on the Lévy measure and only basic Malliavin calculus assumptions are considered on the stochastic volatility process.

1. Introduction

The main problem of the Black-Scholes formula for option pricing is the assumption of constant volatility for the underlying price. The effort for explaining the smile or skew shapped behavior of the implied volatility observed in markets brought first to consider stochastic volatility models and later to add jumps to these models. Examples of these models can be found in [7]. As it is known, and it is shown in the literature, considering jumps is useful to describe better the short time behaviour of the implied volatility with respect to the strike price (the so called smile or skew of the volatility).

Most famous stochastic volatility models with jumps, as for example the model due to Bates [6], assume a concrete dynamic for the volatility, but this dynamic is difficult to model in practice because volatility is an unobservable parameter. On other hand, as it was shown in [8], any generalization of Black-Scholes model, from the case of deterministic volatility to different cases of stochastic volatility, give a pricing formula, also called Hull and White formula, that depends not on the current volatility but on the future average volatility. The fact that the future average volatility is a non adapted process suggests the use of anticipative calculus techniques as the natural tool to deal with anticipative processes as done in [3] and [2]. In these papers, a general jump diffusion model with no precise assumption on the dynamics of the volatility process is analyzed. In the first of them, the volatility is assumed to be correlated only with the continuous part of the price process whereas in the second one, the volatility is also allowed to be correlated with price jumps. This dependence on jumps makes necessary the use of

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Malliavin calculus for Lévy processes to obtain a suitable Hull and White formula that distinguishes clearly the effects of different correlations.

The goal of this kind of work is mainly to model the plain vanilla price surface given by derivatives markets or the corresponding implied volatility surface, in order to obtain a better comprehension of phenomena like the smile or skew behaviour of the implied volatility with respect the strike price or its behaviour with respect the time to expiry. This is the reason that justifies the assumption of a risk neutral model, under a risk neutral probability measure chosen by the derivative market of reference.

In [2] the jump part is modeled by a Compound Poisson process. Now, we extend it to the case that the jump process is a pure jump Lévy process. More technically we change the finite Lévy measure associated to the Compound Poisson process by an infinite one and we obtain an extension of the Hull and White formula for this more general case.

Our general model covers all cases treated in the literature: correlated stochastic volatility models with jumps (as Bates model for example), uncorrelated models with jumps (Heston-Kou model, see [10]), correlated and uncorrelated models without jumps (Heston, Hull and White, Stein and Stein), or in the case σ constant (but non zero), exponential Lévy models. Detailed presentations of all these models can be shown in [7], [9] and [13], and in the references therein.

Section 2 is devoted to present the model and other preliminaries. Section 3 is a fast summary of Malliavin Calculus for Lévy processes. In section 4 we obtain an Itô formula, necessary for our purposes. In section 5 we obtain the Hull and White formula in our case.

2. Description of the Model and Other Preliminaries

We assume the following model for the log-price process, under a risk neutral measure chosen by the market:

$$X_t = x + rt - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s(\rho dW_s + \sqrt{1-\rho^2}dB_s) + J_t^0,\ t \in [0,T], \quad (2.1)$$

where x_0 is the current log price, $r > 0$ is the instantaneous interest rate, W and B are independent standard Brownian motions, $\rho \in (-1,1)$ and J_t^0 is a pure jump Levy process with possibly infinitely many jumps with triplet $(\gamma_0, 0, \nu)$, with $\gamma_0 \in \mathbb{R}$ and independent of W and B. The volatility process σ is assumed to be adapted to the filtration generated by W and J^0 and its trajectories are assumed to be a.s. square integrable, càdlàg and strictly positive a.e.

In order to $e^{-rt}e^{X_t}$ be a martingale (see for example [7], Proposition 3.18) we must assume

$$\int_{|y|\geq 1} e^y \nu(dy) < \infty \quad (2.2)$$

and

$$\gamma_0 = \int_{\mathbb{R}} (e^y - 1 - y1_{|y|<1})\nu(dy). \quad (2.3)$$

Due to the well known Lévy-Itô decomposition we can write

$$J_t^0 = \int_0^t \int_{\{|y|>1\}} yN(ds,dy) + \lim_{\epsilon\downarrow 0} \int_0^t \int_{\{\epsilon<|y|\le 1\}} y\tilde{N}(ds,dy)$$

where N denotes the Poisson measure associated to Lévy process J, $\tilde{N}(ds,dy) := N(ds,dy) - \nu(dy)ds$ is the compensated Poisson measure and the limit is a.s. and uniformly on compacts.

We will consider the following constants, provided it exist:

$$c_i := \sum_{k=i}^{\infty} \int_{\mathbb{R}} \frac{y^k}{k!}\nu(dy).$$

Observe that in particular we have

$$c_0 := \int_{\mathbb{R}} e^y \nu(dy),$$

$$c_1 := \int_{\mathbb{R}} (e^y - 1)\nu(dy)$$

and

$$c_2 := \int_{\mathbb{R}} (e^y - 1 - y)\nu(dy).$$

Condition (2.2) jointly with the fact that ν is a Lévy measure implies that ν has moments of order $k \ge 2$ and so c_i exist for any $i \ge 2$ but not necessarily for $i = 1$ or $i = 0$. Moreover, the fact that $\int_{\{|y|>1\}} |y|\nu(dy) < \infty$ allows us to define

$$c_2 := \gamma_0 - \int_{\{|y|>1\}} |y|\nu(dy)$$

and to write in all cases,

$$J_t^0 - \gamma_0 t = \int_0^t \int_{\mathbb{R}} y\tilde{N}(ds,dy) - c_2 t,$$

of course interpreting the integral as an a.s. limit uniformly on compacts.

In the case $\int_{\mathbb{R}} |y|\nu(dy) = \infty$ the process has infinite activity and infinite variation. In this case c_0 is infinite and c_1 can be not defined or infinite.

If ν has first order moment the model has infinite activity but finite variation and c_1 is finite. In this case we have $c_2 = c_1 - \int_{\mathbb{R}} y\nu(dy)$ and we can rewrite

$$\int_0^t \int_{\mathbb{R}} y\tilde{N}(ds,dy) - c_2 t = \int_0^t \int_{\mathbb{R}} yN(ds,dy) - c_1 t$$

and simplify the model accordingly.

Finally, if ν is finite, the model has finite activity and in fact it is a Compound Poisson process with $\nu = \lambda Q$ where Q is a probability measure and $\lambda := \nu(\mathbb{R}) > 0$. In this case,

$$c_1 = \int_{\mathbb{R}} (e^y - 1)\nu(dy) = \lambda k = c_0 - \lambda,$$

where $k := \mathbb{E}_Q(e^V) - 1$ and

$$\int_0^t \int_{\mathbb{R}} yN(ds,dy) = \sum_{i=1}^{N_t} V_i,$$

where N is a $\lambda-$Poisson process and V_i are independent and identically distributed copies of V, the random variable, with law Q, that produce the jumps.

So, in the following we will assume, without loosing generality, the model

$$X_t = x + (r - c_2)t - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s(\rho dW_s + \sqrt{1-\rho^2}dB_s) + J_t \tag{2.4}$$

with

$$J_t := \int_0^t \int_{\mathbb{R}} y\tilde{N}(ds,dy).$$

Remark 2.1. Observe that this is a very general stochastic volatility model. First of all, being σ adapted to $\mathcal{F}^W \vee \mathcal{F}^J$, we are allowing jumps both in price and volatility. We can consider the following particular cases:

- If we restrict our model to the case σ adapted only to $\mathcal{F}^W$ we have a generalization of the Bates model in a double sense. On one hand we do not assume any concrete dynamics for the stochastic volatility process and on other hand we are not assuming finite activity nor finite variation on ν.
- If we assume no jumps, that is $\nu = 0$, we have a generalization of the well-known Heston model or other classical stochastic volatility models in the same sense as before.
- If in addition $\rho = 0$ we have a generalization of different non correlated stochastic volatility models as Hull - White, Scott, Stein - Stein or Ball - Roma.
- If we assume no correlation but presence of jumps we cover for example Heston-Kou model, or any uncorrelated model with the addition of Lévy jumps on the price process with any Lévy measure ν
- If σ is constant and we have jumps, we cover the so called exponential Lévy models.
- Finally if we have no jumps and σ is constant we have the classical Osborne-Samuelson-Black-Scholes model.

The following facts define the notation that is going to be used in the paper:

- We denote by $\mathcal{F}^W$, $\mathcal{F}^B$ and $\mathcal{F}^N$ the filtrations generated by the independent processes W, B and J respectively. Note that the filtration generated by J is the same as the filtration by J^0 because the difference of this two processes is deterministic. Moreover, we define $\mathcal{F} = \mathcal{F}^W \vee \mathcal{F}^B \vee \mathcal{F}^N$.
- Recall that the pricing formula for a plain vanilla call with strike price K under a risk neutral measure is given by

  $$V_t = e^{-r(T-t)}\mathbb{E}_t\left[(e^{X_T} - K)_+\right],$$

  where for simplicity we use the notation $\mathbb{E}_t(\cdot) := E(\cdot|\mathcal{F}_t)$.

- The process $v_t := \sqrt{\frac{Y_t}{T-t}}$, with $Y_t := \int_t^T \sigma_s^2 ds$, denotes the future average volatility.
- In the classical Black-Scholes model with constant volatility σ, current log stock price x, time to maturity $T-t$, strike price K, and interest rate r, the function $BS(t,x,\sigma)$ can be written as

$$BS(t,x,\sigma) = e^x \Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-), \tag{2.5}$$

where Φ denotes the cumulative probability function of the standard normal distribution and

$$d_\pm = \frac{x - \log K + r(T-t)}{\sigma\sqrt{T-t}} \pm \frac{\sigma}{2}\sqrt{T-t}.$$

Recall also that the function $BS(t,x,\sigma)$ satisfies

$$\mathcal{L}_{BS}(\sigma) BS(\cdot,\cdot,\sigma) = 0$$

where

$$\mathcal{L}_{BS}(\sigma) = \partial_t + \frac{1}{2}\sigma^2 \partial_{xx}^2 + (r - \frac{1}{2}\sigma^2)\partial_x - r$$

is the Black-Scholes operator, in the log variable, with volatility σ.
- Finally we will write $G(t,x,\sigma) := (\partial_{xx}^2 - \partial_x) BS(t,x,\sigma)$. Recall from Lemma 2 in [3] that for $0 \le t \le s \le T$, $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{F}_T^W \vee \mathcal{F}_T^N$ and $n \ge 0$ there exists a constant $C := C(n,\rho)$ such that

$$|E(\partial_x^n G(s, X_s, v_s)|\mathcal{G}_t)| \le C(\int_t^T \sigma_s^2 ds)^{-\frac{n+1}{2}}.$$

3. A Fast Review of Elements of Malliavin-Skorohod Calculus

3.1. Malliavin calculus for the Wiener process. Nowadays, Malliavin calculus for the Wiener process is a classical topic and a lot of references are available. We refer the reader to [11] and [12]. Here we simply recall some basic definitions and facts necessary for our purpose.

Let W be the canonical Wiener process, that is, defined on the space of $\Omega^W := C_0([0,T])$ of continuous functions on $[0,T]$, null at the origin. We consider the family of smooth functionals of type $F = f(W_{t_1}, \ldots, W_{t_n})$ for any $n \ge 0, t_1, \ldots t_n \in [0,T]$ and $f \in C_b^\infty(\mathbb{R}^n)$. Given a smooth functional F we define its Mallavin derivative $D^W F$ as the element of $L^2(\Omega^W \times [0,T])$ given by

$$D_t F = \sum_{i=1}^n \partial_i f(W_{t_1}, \ldots, W_{t_n}) \mathbb{1}_{[0,t_i]}(t).$$

The operator D^W is closed and densely defined in $L^2(\Omega^W)$, and its domain $DomD^W$ is the closure of the smooth functionals with respect the norm

$$||F||_{DomD^W} := \big(E_W(|F|^2) + E_W \int_0^T |D_t^W F|^2 dt\big)^{\frac{1}{2}}.$$

We define δ^W as the dual operator of D^W. Given $u \in L^2(\Omega^W \times [0,T])$, $\delta^W(u)$ is the element of $L^2(\Omega^W)$ characterized by

$$E_W(F\delta^W(u)) = E_W \int_0^T u_t D_t^W F dt$$

for any $F \in DomD^W$.

It is well known that D^W can be interpreted as a directional derivative on the Wiener space and δ^W is an extension of the classical Itô integral.

The following results will be helpful:

- If F, G and $F \cdot G$ belong to $DomD^W$ we have $D^W(F \cdot G) = FD^W G + GD^W F$.
- If $F \in DomD^W$, $u \in Dom\delta^W$ and $F \cdot u \in Dom\delta^W$ then

$$\delta^W(F \cdot u) = F\delta^W(u) - \int_0^T u_t D_t^W F dt.$$

We define the space $\mathbb{L}_W^{1,2} := L^2([0,T]; DomD^W)$, that is the space of processes $u \in L^2([0,T] \times \Omega^W)$ such that $u_t \in DomD^W$ for almost all t and $Du \in L^2(\Omega^W \times [0,T]^2)$. It can be proved that $\mathbb{L}_W^{1,2} \subseteq Dom\delta^W$ and

$$E_W(\delta^W(u)^2) \leq ||u||^2_{\mathbb{L}_W^{1,2}} := E_W(||u||^2_{L^2([0,T])}) + E_W(||D^W u||^2_{L^2([0,T]^2)}).$$

Finally we will denote $\delta_t^W(u) := \delta^W(u \mathbb{1}_{[0,t]})$.

3.2. Malliavin calculus for a pure jump Lévy process. The literature on Malliavin calculus for Lévy processes is more recent and less extended. Here we follow closely [4] and [14]. We refer the reader to these references for proofs of next results. Note that our point of view is slightly different as the point of view of [2] and so, formulas are slightly different.

Let us denote $\mathbb{R}_0 := \mathbb{R} - \{0\}$. Consider the canonical version of the pure jump Lévy process J. It is defined on the space Ω^N given by the finite or infinite sequences of pairs $(t_i, x_i) \in (0,T] \times \mathbb{R}_0$ such that for every $\epsilon > 0$ there is only a finite number of (t_i, x_i) with $|x_i| > \epsilon$. Of course, t_i denotes a jump instant and x_i a jump size.

Consider $\omega^N \in \Omega^N$. Given $(t,x) \in [0,T] \times \mathbb{R}_0$ we can introduce a jump of size x at instant t to ω^N and call the new element $\omega_{t,x}^N := ((t,x), (t_1, x_1)(t_2, x_s), ...)$. For a random variable $F \in L^2(\Omega^N)$, we define $T_{t,x}F(\omega^N) = F(\omega_{t,x}^N)$. This is a well defined operator. See [14] for the details. Finally we define

$$D_{t,x}^N F = \frac{T_{t,x}F(\omega^N) - F(\omega^N)}{x}, \; x \neq 0,$$

and denote by $DomD^N$ its domain.

The operator D^N is closed and densely defined in $L^2(\Omega^N)$ and its domain $DomD^N$ can be characterized by the fact that $F \in DomD^N$ if and only if $DF \in L^2(\Omega \times [0,T] \times \mathbb{R}_0; P \otimes ds \otimes x^2\nu(dx))$.

We define δ^N as the dual operator of D^N. Given $u \in L^2(\Omega^W \times [0,T] \times \mathbb{R}, P \otimes ds \otimes x^2\nu(dx))$, $\delta^N(u)$ is the element of $L^2(\Omega^N)$ characterized by

$$E_N(F\delta^N(u)) = E_N \int_0^T u_{t,x} D_{t,x}^N F x^2 \nu(dx) dt$$

for any $F \in DomD^N$.

Let's denote $\delta_t^N(u) := \delta^N(u\mathbb{1}_{[0,t]})$. As we have seen, D^N is an increment quotient operator and it is also known that δ_t^N is an extension of Itô integral in the sense that

$$\delta_t^N(u\mathbb{1}_{\mathbb{R}_0}) = \int_0^t \int_{\mathbb{R}} u(s,x)x\tilde{N}(ds,dx)$$

for predictable integrands u.

In this case, the following formulas will be helpful:

- If F, G and $F \cdot G$ belong to $DomD^N$ we have $D^N(F \cdot G) = FD^N G + GD^N F + xD^N F D^N G$.
- If $F \in DomD^N$, $u \in Dom\delta^N$ and $u \cdot T_{t,x}F \in Dom\delta^N$ then

$$\delta^N(F \cdot u) = F\delta^N(u) - \int_0^T \int_{\mathbb{R}} u_{t,x} D_{t,x}^N F x^2 \nu(dx)dt - \delta^N(x \cdot u \cdot D^N F).$$

As in the Wiener case we define the space $\mathbb{L}_N^{1,2} := L^2([0,T]\times\mathbb{R}, DomD^N)$, that is the space of processes $u \in L^2([0,T]\times\mathbb{R}\times\Omega^N)$ such that $u_{t,x} \in DomD^N$ for almost all (t,x) and $Du \in L^2(\Omega^N \times ([0,T]\times\mathbb{R})^2)$. It can be proved that $\mathbb{L}_N^{1,2} \subseteq Dom\delta^N$ and $E_N(\delta^N(u)^2) \le ||u||^2_{\mathbb{L}_N^{1,2}} := E_N(||u||^2_{L^2([0,T]\times\mathbb{R})}) + E_N(||D^N u||^2_{L^2(([0,T]\times\mathbb{R})^2)})$.

Definition 3.1. We define the space $\mathbb{L}_{N,-}^{1,2}$ as the subspace of $\mathbb{L}_N^{1,2}$ of processes u such that the left-limits

$$u(s-,y) := \lim_{r\uparrow s, x\uparrow y} u(r,x)$$

and

$$D_{s,y}^{N,-} u(s-,y) := \lim_{r\uparrow s, x\uparrow y} D_{s,y}^N u(r,x)$$

exists $\mathbb{P}_N \otimes ds \otimes x^2\nu(dx)-$a.s. and belong to $L^2(\Omega^N \times [0,T]\times\mathbb{R})$.

Observe that this definition includes processes not depending on y. So Y_{s-} and $D_{s,y}^{N,-} Y_s$ can be considered. On the other hand we can define

$$T_{s,y}^- u(s-,y) := u(s-,y) + yD_{s,y}^{N,-} u(s-,y).$$

The next proposition will be a key point in the paper

Proposition 3.2. *Assume* $u \in \mathbb{L}_{N,-}^{1,2}$ *and* $\int_0^T \int_{\mathbb{R}_0} |u(s-,y)||y|N(ds,dy) \in L^2(\Omega^N)$. *Then, for any* $t \in [0,T]$,

$$T_{s,y}^- u(s-,y) \in Dom\delta_t^N$$

and

$$\int_0^t \int_{\mathbb{R}} u(s-,y)y\tilde{N}(ds,dy) = \delta_t^N(T_{s,y}^- u(s-,y)\mathbb{1}_{\mathbb{R}_0}) + \int_0^t \int_{\mathbb{R}} D_{s,y}^{N,-} u(s-,y)y^2\nu(dy)ds.$$

Proof. The proof is analogous to Proposition 3.4 in [2] □

Remark 3.3.

(1) The space $\mathbb{L}_{N,-}^{1,2}$ could be changed by an analogous one with right limits with respect to the space variable y.

(2) If u is adapted to the filtration generated by N we have

$$T^{-}_{s,y}u(s-,y) = u(s-,y),$$

and

$$D^{N,-}_{s,y}u(s-,y) = 0.$$

Hence, in this case,

$$\int_0^t \int_{\mathbb{R}} u(s-,y)y\tilde{N}(ds,dy) = \delta^N_t(u(s-,y)\mathbb{1}_{\mathbb{R}_0}).$$

3.3. A canonical space for our model. We will consider our price model defined on the product of the canonical spaces of processes W, B and J. We will write $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\Omega = \Omega^W \times \Omega^B \times \Omega^N,$$

$$\mathcal{F} = \mathcal{F}^W \times \mathcal{F}^B \times \mathcal{F}^N$$

and

$$\mathbb{P} = \mathbb{P}_W \times \mathbb{P}_B \times \mathbb{P}_N.$$

If we write the canonical processes as $\bar{W}$, $\bar{B}$ and $\bar{J}$ and the elements of Ω as

$$\omega := (\omega^W, \omega^B, \omega^N),$$

processes W, B and J in the model have to be interpreted as

$$W(\omega) := \bar{W}(\omega^W), \quad B(\omega) := \bar{B}(\omega^B), \quad J(\omega) := \bar{J}(\omega^N).$$

4. An Itô Formula for Lévy Process

Consider processes X and Y defined in the first section. Recall that X is an adapted process with jumps and Y is a continuous and non adapted process. For a suitable function F we introduce the following notation that will be used in the rest of the paper:

- $$\Delta_x F(s, X_{s-}, Y_s) := F(s, X_{s-} + x, Y_s) - F(s, X_{s-}, Y_s)$$

- $$\Delta^2_{xx} F(s, X_{s-}, Y_s) := F(s, X_{s-} + x, Y_s) - F(s, X_{s-}, Y_s) - x\partial_x F(s, X_{s-}, Y_s).$$

We have the following Itô formula that will be useful for our purposes:

Theorem 4.1. *Assume $\sigma^2 \in \mathbb{L}^{1,2}_W \bigcap \mathbb{L}^{1,2}_N$. Let $F \in C_b^{1,2,2}([0,T]\times\mathbb{R}\times[0,\infty))$. Then we have*

$$\begin{aligned}
F(t,X_t,Y_t) &= F(0,X_0,Y_0) + \int_0^t \partial_s F(s,X_s,Y_s)ds \\
&+ \int_0^t \partial_x F(s,X_s,Y_s)(r - \frac{\sigma_s^2}{2} - c_2)ds + \delta_t^{W,B}\left(\partial_x F(s,X_{s-},Y_s)\sigma_s\right) \\
&- \int_0^t \partial_y F(s,X_s,Y_s)\sigma_s^2 ds + \rho\int_0^t \partial^2_{xy}F(s,X_s,Y_s)\Lambda_s ds \\
&+ \frac{1}{2}\int_0^t \partial^2_{xx}F(s,X_s,Y_s)\sigma_s^2 ds \\
&+ \int_0^t\int_{\mathbb{R}} \Delta^2_{xx}F(s,X_{s-},Y_s)\nu(dx)ds \\
&+ \delta_t^N\left(T^- \frac{\Delta_x F(s,X_{s-},Y_s)}{x}\mathbb{1}_{\mathbb{R}_0}(x)\right) \\
&+ \int_0^t\int_{\mathbb{R}} D^- \frac{\Delta_x F(s,X_{s-},Y_s)}{x}x^2\nu(dx)ds
\end{aligned}$$

where $\delta^{W,B}$ is the Skorohod integral with respect to the Wiener process

$$\rho W_s + \sqrt{1-\rho^2}B_s$$

and $\Lambda_s = \left(\int_s^T D_s^W \sigma_r^2 dr\right)\sigma_s$.

Proof. Fix first of all $\epsilon > 0$, and consider the process

$$\begin{aligned}
X_t^\epsilon &:= x + (r-c_2)t - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s(\rho dW_s + \sqrt{1-\rho^2}dB_s) \\
&+ \int_0^t\int_{|x|>\epsilon} x\tilde{N}(ds,dx)
\end{aligned}$$

This process has a finite number of jumps and converges a.s. and in L^2 to X_t. Denote by T_i^ϵ the jump instants, and write $T_0^\epsilon := 0$. Then

$$\begin{aligned}
&F(T_{i+1}^\epsilon, X^\epsilon_{T^\epsilon_{i+1}}, Y_{T^\epsilon_{i+1}}) - F(T_i^\epsilon, X^\epsilon_{T_i^\epsilon}, Y_{T_i^\epsilon}) = \int_{T_i^\epsilon}^{T_{i+1}^\epsilon -} dF(s, X_s^\epsilon, Y_s) \\
&+ F(T_{i+1}^\epsilon, X^\epsilon_{T^\epsilon_{i+1}}, Y_{T^\epsilon_{i+1}}) - F(T_{i+1}^\epsilon, X^\epsilon_{T^\epsilon_{i+1}-}, Y_{T^\epsilon_{i+1}}).
\end{aligned}$$

On the stochastic interval $[T_j^\epsilon, T_{j+1}^\epsilon[$ we can apply the anticipative Itô formula for continuous process presented in [5] and proceed as in [3]. Then we have that

$$\partial_x F(s, X_{s^-}, Y_s)\sigma_s \mathbb{1}_{[0,t]}(s) \in Dom\delta^{W,B}$$

and

$$\begin{aligned}F(t,X_t^\epsilon,Y_t) &= F(0,X_0,Y_0)+\int_0^t \partial_s F(s,X_s^\epsilon,Y_s)ds\\ &+\int_0^t \partial_x F(s,X_s^\epsilon,Y_s)(r-\frac{\sigma_s^2}{2}-c_2)ds+\delta_t^{W,B}\left(\partial_x F(s,X_{s-}^\epsilon,Y_s)\sigma_s\right)\\ &-\int_0^t\int_{\{|x|>\epsilon\}}\partial_x F(s,X_s^\epsilon,Y_s)x\nu(dx)ds-\int_0^t \partial_y F(s,X_s^\epsilon,Y_s)\sigma_s^2 ds\\ &+\rho\int_0^t \partial_{xy}^2 F(s,X_s^\epsilon,Y_s)\Lambda_s ds+\frac{1}{2}\int_0^t \partial_{xx}^2 F(s,X_s^\epsilon,Y_s)\sigma_s^2 ds\\ &+\sum_i [F(T_i^\epsilon,X_{T_i^\epsilon}^\epsilon,Y_{T_i^\epsilon})-F(T_i^\epsilon,X_{T_i^\epsilon-}^\epsilon,Y_{T_i^\epsilon})].\end{aligned}$$

We can write

$$\sum_i [F(T_i^\epsilon,X_{T_i^\epsilon}^\epsilon,Y_{T_i^\epsilon})-F(T_i^\epsilon,X_{T_i^\epsilon-}^\epsilon,Y_{T_i^\epsilon})]=\int_0^t\int_{|x|>\epsilon}\Delta_x F(s,X_{s-},Y_s)N(ds,dx).$$

Then

$$\begin{aligned}&\sum_i [F(T_i^\epsilon,X_{T_i^\epsilon}^\epsilon,Y_{T_i^\epsilon})-F(T_i^\epsilon,X_{T_i^\epsilon-}^\epsilon,Y_{T_i^\epsilon})]-\int_0^t\int_{|x|>\epsilon}\partial_x F(s,X_s^\epsilon,Y_s)x\nu(dx)ds\\ &=\int_0^t\int_{|x|>\epsilon}\Delta_x F(s,X_{s-}^\epsilon,Y_s)\tilde N(ds,dx)+\int_0^t\int_{|x|>\epsilon}\Delta_{xx}^2 F(s,X_{s-}^\epsilon,Y_s)\nu(dx)ds.\end{aligned}$$

Observe that this equality is the crucial step of the proof. Only introducing $\Delta_{xx}^2 F(s,X_{s-}^\epsilon,Y_s)$ we become able to apply succesfully the dominated convergence theorem, even if Y has no jumps.

Using Proposition 3.2 we have

$$\begin{aligned}&\int_0^t\int_{|x|>\epsilon}\Delta_x F(s,X_{s-}^\epsilon,Y_s)\tilde N(ds,dx)\\ &=\delta_t^N(T_{s,x}^-\frac{\Delta_x F(s,X_{s-}^\epsilon,Y_s)}{x}\mathbb{1}_{\{|x|>\epsilon\}})\\ &\quad+\int_0^t\int_{|x|>\epsilon}D_{s,x}^{N,-}\frac{\Delta_x F(s,X_{s-}^\epsilon,Y_s)}{x}x^2\nu(dx)ds. \qquad (4.1)\end{aligned}$$

Using mean value theorem and the fact that first and second derivatives of F are bounded we have

$$|T_{s,x}^-\frac{\Delta_x F(s,X_{s-}^\epsilon,Y_s)}{x}|=|\frac{\Delta_x F(s,X_{s-}^\epsilon,T_{s,x}^-Y_s)}{x}|\le C,$$

$$|D_{r,y}^{N,-}\frac{\Delta_x F(s,X_{s-}^\epsilon,T_{s,x}^-Y_s)}{x}|\le C|D_{r,y}^{N,-}T_{s,x}^-Y_s|=C\int_s^T|D_{r,y}^{N,-}T_{s,x}^-\sigma_u^2|du$$

and

$$|D_{s,x}^{N,-}\frac{\Delta_x F(s,X_{s-}^\epsilon,Y_s)}{x}|\le C|D_{s,x}^{N,-}Y_s|=C\int_s^T|D_{s,x}^{N,-}\sigma_r^2|dr,$$

for a generic constant C.

So, using the norm of $\mathbb{L}_N^{1,2}$, the hypotheses on σ^2 and the dominated convergence theorem the right hand side of (4.1) converges when ϵ goes to 0.

The other terms converge also by the dominated convergence theorem, and the Itô formula follows. □

5. The Hull and White Formula

In this section we use the anticipating Itô formula proved in the previous section to find a Hull-White type formula for our general Lévy model.

Theorem 5.1. *Assume $\sigma^2 \in \mathbb{L}_W^{1,2} \bigcap \mathbb{L}_N^{1,2}$. We have*

$$\begin{aligned}
V_t &= \mathbb{E}_t(BS(t, X_t, v_t)) \\
&+ \frac{\rho}{2}\mathbb{E}_t(\int_t^T e^{-r(s-t)}\partial_x G(s, X_s, v_s)\Lambda_s ds) \\
&- \mathbb{E}_t(\int_t^T \int_{\mathbb{R}} e^{-r(s-t)}\partial_x BS(s, X_s, v_s)(e^y - 1 - y)\nu(dy)ds) \\
&+ \mathbb{E}_t(\int_t^T \int_{\mathbb{R}} e^{-r(s-t)}\Delta_{yy}^2 BS(s, X_{s-}, v_s)\nu(dy)ds) \\
&+ \mathbb{E}_t(\int_t^T \int_{\mathbb{R}} e^{-r(s-t)}D_{s,y}^{N,-}\Delta_y BS(s, X_{s-}, v_s)y\nu(dy)ds).
\end{aligned}$$

Proof. We know that $V_T = (e^{X_T} - K)_+ = BS(T, X_T, v_T)$. So, we can write

$$e^{-rt}V_t = \mathbb{E}_t\left(e^{-rT}BS(T, X_T, v_T)\right).$$

We want to apply the previous Itô formula to the function $e^{-rs}BS(s, X_s, v_s)$. This function is not bounded and has no bounded derivatives. So we will use an approximated version. Let

$$v_t^\delta := \sqrt{\frac{Y_t + \delta}{T - t}}$$

for a fixed small $\delta > 0$ and

$$BS_n(t, x, \sigma) := BS(t, x, \sigma)\phi(\frac{x}{n})$$

where $\phi \in C_b^2[0, \infty)$ such that $\phi(x) = 1$ for all $x < 1$ and $\phi(x) = 0$ for all $x > 2$ and $\phi(x) \in [0, 1]$ for $x \in [1, 2]$. Now we can apply the previous Itô formula to the function

$$F_{n,\delta}(s, x, y) := e^{-rs}BS_n\left(s, x, \sqrt{\frac{y + \delta}{T - s}}\right),$$

that belongs to $C_b^{1,2,2}([0,T]\times\mathbb{R}\times[0,\infty))$, on the interval $[t,T]$. We have

$$\begin{aligned}
e^{-rT}BS_n(T,X_T,v_T^\delta) &= e^{-rt}BS_n(t,X_t,v_t^\delta)\\
&+\int_t^T e^{-rs}\mathcal{L}_{BS}(\sigma_s)BS_n(s,X_s,v_s^\delta)ds\\
&-\frac{1}{2}\int_t^T e^{-rs}\partial_\sigma BS_n(s,X_s,v_s^\delta)\frac{(\sigma_s^2-(v_s^\delta)^2)}{v_s^\delta(T-s)}ds\\
&-c_2\int_t^T e^{-rs}\partial_x BS_n(s,X_s,v_s^\delta)ds\\
&+\delta^{W,B}(e^{-rs}\partial_x BS_n(s,X_s,v_s^\delta)\sigma_s\mathbb{1}_{[t,T]}(s))\\
&+\frac{\rho}{2}\int_t^T e^{-rs}\partial_{\sigma x}^2 BS_n(s,X_s,v_s^\delta)\frac{1}{v_s^\delta(T-s)}\Lambda_s ds\\
&+\int_t^T\int_{\mathbb{R}} e^{-rs}\Delta_{yy}^2 BS_n(s,X_{s^-},v_s^\delta)\nu(dy)ds\\
&+\delta^N\left(e^{-rs}T_{s,y}^-\frac{\Delta_y BS_n(s,X_{s^-},v_s^\delta)}{y}\mathbb{1}_{[0,T]}\right)\\
&+\int_t^T\int_{\mathbb{R}} e^{-rs}D_{s,y}^{N,-}\frac{\Delta_y BS_n(s,X_{s^-},v_s^\delta)}{y}y^2\nu(dy)ds.
\end{aligned}$$

Notice that

$$\mathcal{L}_{BS}(\sigma_s)BS_n(s,X_s,v_s^\delta)=(\mathcal{L}_{BS}(\sigma_s)BS(s,X_s,v_s^\delta))\phi(\frac{X_s}{n})+A_n(s),$$

where

$$\begin{aligned}
&A_n(s)\\
&=\frac{\sigma_s^2}{n}\left[\partial_x BS(s,X_s,v_s^\delta)\phi^{'}(\frac{X_s}{n})+\frac{1}{2}BS(s,X_s,v_s^\delta)(\frac{1}{n}\phi^{''}(\frac{X_s}{n})-\phi^{'}(\frac{X_s}{n}))\right]\\
&\quad+\frac{r}{n}BS(s,X_s,v_s^\delta)\phi^{'}(\frac{X_s}{n}).
\end{aligned}$$

We can use the following relations

$$\partial_\sigma BS(s,x,\sigma)\frac{1}{\sigma(T-s)}=(\partial_{xx}^2-\partial_x)BS(s,x,\sigma),$$

and

$$\mathcal{L}_{BS}(\sigma_s)=\mathcal{L}_{BS}(v_s^\delta)+(1/2)(\sigma_s^2-(v_s^\delta)^2)(\partial_{xx}^2-\partial_x),$$

then

$$
\begin{aligned}
& e^{-rT} BS_n(T, X_T, v_T^\delta) \\
= \quad & e^{-rt} BS_n(t, X_t, v_t^\delta) + \int_t^T e^{-rs} A_n(s) ds \\
& - c_2 \int_t^T e^{-rs} \partial_x BS_n(s, X_s, v_s^\delta) ds \\
& + \delta^{W,B}(e^{-rs} \partial_x BS_n(s, X_s, v_s^\delta) \sigma_s 1_{[t,T]}(s)) \\
& + \frac{\rho}{2} \int_t^T e^{-rs} \left[\partial_x G(s, X_s, v_s^\delta) \phi_n(\frac{X_s}{n}) + G(s, X_s, v_s^\delta) \frac{1}{n} \phi^{'}(\frac{X_s}{n}) \right] \Lambda_s ds \\
& + \int_t^T \int_{\mathbb{R}} e^{-rs} \Delta_{yy}^2 BS_n(s, X_{s^-}, v_s^\delta) \nu(dy) ds \\
& + \delta^N \left(e^{-rs} T_{s,y}^{-} \frac{\Delta_y BS_n(s, X_{s^-}, v_s^\delta)}{y} \mathbb{1}_{\mathbb{R}_0}(y) \mathbb{1}_{[t,T]}(s) \right) \\
& + \int_t^T \int_{\mathbb{R}} e^{-rs} D_{s,y}^{N,-} \frac{\Delta_y BS_n(s, X_{s^-}, v_s^\delta)}{y} y^2 \nu(dy) ds.
\end{aligned}
$$

Now taking conditional expectations with respect to t, and using the fact that Skorohod integrals have zero expectation, we obtain

$$
\begin{aligned}
& \mathbb{E}_t(e^{-rT} BS_n(T, X_T, v_T^\delta)) \\
= \quad & \mathbb{E}_t(e^{-rt} BS_n(t, X_t, v_t^\delta)) + \mathbb{E}_t(\int_t^T e^{-rs} A_n(s) ds) \\
& - c_2 \mathbb{E}_t(\int_t^T e^{-rs} \partial_x BS_n(s, X_s, v_s^\delta) ds) \\
& + \frac{\rho}{2} \mathbb{E}_t(\int_t^T e^{-rs} (\partial_x G(s, X_s, v_s^\delta) \phi_n(\frac{X_s}{n}) + G(s, X_s, v_s^\delta) \frac{1}{n} \phi^{'}(\frac{X_s}{n})) \Lambda_s ds) \\
& + \mathbb{E}_t(\int_t^T \int_{\mathbb{R}} e^{-rs} \Delta_{yy}^2 BS_n(s, X_{s^-}, v_s^\delta) \nu(dy) ds) \\
& + \mathbb{E}_t(\int_t^T \int_{\mathbb{R}} e^{-rs} D_{s,y}^{N,-} \frac{\Delta_y BS_n(s, X_{s^-}, v_s^\delta)}{y} y^2 \nu(dy) ds).
\end{aligned}
$$

Letting first $n \uparrow \infty$, then $\delta \downarrow 0$ and using properties of function G and the dominated convergence theorem we obtain the result □

Remark 5.2. Observe that in fact we can write

$$
\begin{aligned}
V_t = & \mathbb{E}_t(BS(t, X_t, v_t)) + \frac{\rho}{2} \mathbb{E}_t(\int_t^T e^{-r(s-t)} \partial_x G(s, X_s, v_s) \Lambda_s ds) \\
& + \mathbb{E}_t \left\{ \int_t^T \int_{\mathbb{R}} e^{-r(s-t)} \left[\Delta_y BS(s, X_{s^-}, v_s) - (e^y - 1) \partial_x BS(s, X_{s^-}, v_s) \right] \nu(dy) ds \right\} \\
& + \mathbb{E}_t(\int_t^T \int_{\mathbb{R}} e^{-r(s-t)} D_{s,y}^{N,-} \Delta_y BS(s, X_{s^-}, v_s) y \nu(dy) ds)
\end{aligned}
$$

and all the terms are well defined.

Observe that we cannot split the third term in two terms because in the general case

$$\mathbb{E}_t(\int_t^T \int_{\mathbb{R}} e^{-r(s-t)} \Delta_y BS(s, X_{s^-}, v_s)\nu(dy)ds)$$

and

$$\mathbb{E}_t(\int_t^T \int_{\mathbb{R}} e^{-r(s-t)}(e^y - 1)\partial_x BS(s, X_{s^-}, v_s)\nu(dy)ds)$$

are not convergent.

Moreover

$$\Delta_x BS(s, X_{s^-}, v_s) - (e^y - 1)\partial_x BS(s, X_{s^-}, v_s) = \sum_{j=2}^{\infty} \frac{y^j}{j!}(\partial_x^j - \partial_x) BS(s, X_{s^-}, v_s).$$

Remark 5.3. Observe that if in the previous theorem we assume $\int_{\mathbb{R}} |y|\nu(dy) < \infty$, that is, finite variation, we obtain

$$\begin{aligned}
V_t &= \mathbb{E}_t(BS(t, X_t, v_t)) \\
&+ \frac{\rho}{2}\mathbb{E}_t(\int_t^T e^{-r(s-t)} \partial_x G(s, X_s, v_s)\Lambda_s ds) \\
&- \mathbb{E}_t(\int_t^T \int_{\mathbb{R}} e^{-r(s-t)}(e^y - 1)\partial_x BS(s, X_s, v_s)\nu(dy)ds) \\
&+ \mathbb{E}_t(\int_t^T \int_{\mathbb{R}} e^{-r(s-t)} T_{s,y}^{-} \Delta_y BS(s, X_{s^-}, v_s)\nu(dy)ds),
\end{aligned}$$

that is exactly the formula obtained in [2] for the finite activity case.

The key fact is that the third term on the right hand side of the formula of the previous remark only can be splitted in the finite variation case, but not in general. After splitting this third term in two terms, the first part is joined with the last term via the operator T^- and the second one becomes the third term of the new finite variation version of the formula.

Observe that in particular we are showing that the formula obtained in [2] in the finite activity case is also valid in the infinite activity and finite variation case.

Remark 5.4. If the volatility process is independent from price jumps, we have $D_{s,y}^{N,-} u(s-, y) = 0$ and we obtain

$$\begin{aligned}
V_t &= \mathbb{E}_t(BS(t, X_t, v_t)) \\
&+ \frac{\rho}{2}\mathbb{E}_t(\int_t^T e^{-r(s-t)} \partial_x G(s, X_s, v_s)\Lambda_s ds) \\
&+ \mathbb{E}_t\left\{ \int_t^T \int_{\mathbb{R}} e^{-r(s-t)} \left[\Delta_y BS(s, X_{s^-}, v_s) - (e^y - 1)\partial_x BS(s, X_{s^-}, v_s)\right] \nu(dy)ds \right\}
\end{aligned}$$

that generalizes the formula in [3]. As in the previous remark, only in the finite variation case we can recuperate exactly the formula in [3].

This formula covers Bates model and any correlated model with any type of Lévy jumps in the price process.

Remark 5.5. If moreover, the volatility process is independent from the price process, that is, $\rho = 0$, we obtain

$$V_t = \mathbb{E}_t(BS(t, X_t, v_t)) + \mathbb{E}_t\left\{ \int_t^T \int_{\mathbb{R}} e^{-r(s-t)}[\Delta_y BS(s, X_{s^-}, v_s) - (e^y - 1)\partial_x BS(s, X_{s^-}, v_s)]\nu(dy)ds \right\}.$$

This covers all the so called uncorrelated models plus jumps (Heston-Kou model for example) and in the particular case of constant volatility, the so called exponential Lévy models. In the jump part we can consider infinite activity jumps as CGMY model (for $Y \geq 0$) or Meixner model for example.

Remark 5.6. All these formulas for the different mentioned particular cases, that is for different concrete models for the stochastic volatility process and for different selections of the Lévy measure, give detailed pricing formulas that can be useful in practice. The Heston case is analyzed with detail in [1]. For the finite activity jump case, some detailed formulas are obtained in [2]. The analysis of some infinite activity particular cases and the development of other consequences of the formulas presented here, for example for the short time behaviour of the implied volatility, are in progress and will be part of a forthcoming paper.

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