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NONPARAMETRIC REGRESSION WITH NON-GAUSSIAN LONG MEMORY

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Abstract. We consider a cointegrated regressor model where the regressor is a fractional Brownian motion with self-similarity index $H_1 \in (0,1)$ and the errors are considered to be the increments of a Hermite process which is a long memory non Gaussian H_2- self-similar process with stationary increments, where $H_2 \in (\frac{1}{2}, 1)$. We prove that the estimator of the regression function is consistent and asymptotically normal.

1. Introduction

We will consider the following regression model

$$y_{t_i^{(n)}} = r(x_{t_i^{(n)}}) + \varepsilon_{t_i^{(n)}}, \qquad i = 1, ..., n; n \geq 1; \tag{1.1}$$

Here $(t_i^{(n)})_{i=1,..,n;n\geq 1}$ are points in the interval $[0,1]$ and r is the function to be estimated based on the observations $(y_{t_i^{(n)}}, x_{t_i^{(n)}})_{i=1,..,n;n\geq 1}$.

Our model is a structural cointegrated regression model in the sense that the regression vector $(x_{t_1^{(n)}}, x_{t_2^{(n)}}, \ldots, .., x_{t_n^{(n)}})$ has dependent components and the same happens for the components of the dependent vector Y.

We will assume the following:

- the regressor x is a fractional Brownian motion with Hurst parameter $H_1 \in (0,1)$.
- the errors ε are the increments of a Hermite process X^{H_2} (the increments will be renormalized to have constant variance equal to 1) which is a self-similar with index $H_2 \in (\frac{1}{2}, 1)$, with stationary increments, possibly non Gaussian process, that exhibits long memory. We refer to the next section for the definition and the basic properties of these processes. We mention that the class of Hermite process includes the fractional Brownian motion with Hurst parameter H_2 which is the only Gaussian Hermite process. This class also includes the so-called Rosenblatt process (see e.g. [11], [3], [10], [13]).
- the regressor x and the error ε are independent.

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- the point at which y and x are observed are equidistant. More precisely we will assume that $t_i^{(n)} = \frac{i}{n}$ for $i = 1, ..., n$ and for every $n \geq 1$.

To summarize, for every $n \geq 1$ and $i = 1, .., n$

$$x_{t_i^{(n)}} = B^{H_1}_{\frac{i}{n}} \text{ and } \varepsilon_{t_i^{(n)}} = n^{H_2} \left(X^{H_2}_{\frac{i+1}{n}} - X^{H_2}_{\frac{i}{n}} \right) \tag{1.2}$$

where B^{H_1} is a fBm with $H_1 \in (0,1)$ and X^{H_2} is a Hermite process with self-similarity index $H_2 \in (\frac{1}{2}, 1)$ independent by B^{H_1}. The factor n^{H_2} is included in order to have constant variance for the error.

Models related to (1.1) have been the object of intensive study in the recent scientific literature. In general the context is different in the sense that the observations are assumed to be at times $1, 2, ...$, and the asymptotic behavior is studied when the number n of the observations goes to infinity. In our situation, the observations are in the interval $[0,1]$ and the limit is taken when the discretization step goes to zero. For related works we refer, among others, to [5] and [6] for the case where x_t is a recurrent Markov chain, to [14] for the case where x_t is a partial sum of a general linear process, and [15] for a more general situation. See also [8] or [9]. An important assumption in the main part of the above references is the fact that ε_i is a martingale difference sequence.

The novelty of our paper comes from the fact that the errors are increments of a Hermite process and they are thus correlated. They are not related with any martingale property and in addition, they are supposed to be in principle non-Gaussian. Also the regressor is a fractional Brownian motion but this was also allowed in [14] or [15]. The fact that we will suppose the observations time to be contained in the interval $[0,1]$ and that we study the limit when the discretization step goes to zero allow to use different techniques, somehow easier in order to obtain the consistency of the estimator for the function r. We recall that the conventional kernel estimate of $r(x)$ is

$$\hat{r}_n(x) = \frac{\sum_{i=0}^{n} K_h(x_i - x) y_i}{\sum_{i=0}^{n} K_h(x_i - x)}$$

where K is a nonnegative real kernel function satisfying $\int_{\mathbb{R}} K(y)dy = 1$ and $\int_{\mathbb{R}} yK(y)dy = 0$ and $K_h(s) = \frac{1}{h} K(\frac{s}{h})$. The bandwidth parameter $h \equiv h_n$ satisfies $h_n \to 0$ as $n \to \infty$. We will assume that

$$h_n := h = n^{-\alpha} \text{ with } 0 < \alpha < 1. \tag{1.3}$$

Our paper is structured as follows. Section 2 contains the definition and the basic properties of the Hermite process. In Section 3 we prove the consistence of the estimator $\widehat{r}_n$ for several choices of the kernel K (the Gaussian kernel, the triangle kernel, the Epachnikov kernel and the quartic kernel). In Section 4 we prove the asymptotic normality but we restrict to the situation when the errors are given by the increments of the fractional motion which is, as mentioned above, the only Gaussian Hermite process. As a final remark, we mention that we denote by $c, cst...$ a generic positive constant that may vary from line to line.

2. Preliminaries: Fractional Brownian Motion and Hermite Processes

The fractional Brownian motion $(B_t^H)_{t\in[0,1]}$ with Hurst parameter $H \in (0,1)$ is a centered Gaussian process starting from zero with covariance function

$$R^H(t,s) := \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \qquad s,t \in [0,1].$$

As we already mentioned in the introduction, B^H is a H-self-similar process with stationary increments. Actually it is the only Gaussian process H-self-similar with stationary increments.

The fractional Brownian process $(B_t^H)_{t\in[0,1]}$ with Hurst parameter $H \in (0,1)$ can be written as

$$B_t^H = \int_0^t K^H(t,s)dW_s, \quad t \in [0,1]$$

where $(W_t, t \in [0,1])$ is a standard Wiener process, the kernel $K^H(t,s)$ has the expression, if $H > \frac{1}{2}$,
$c_H s^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} du$ for $t > s$ (and it vanishes if $s \geq t$), c_H is an explicit positive constant. The above integral is a Wiener integral with respect to the the standard Wiener process W. For $t > s$, the kernel's derivative is $\frac{\partial K^H}{\partial t}(t,s) = c_H \left(\frac{s}{t}\right)^{1/2-H} (t-s)^{H-3/2}$. Fortunately we will not need to use these expressions explicitly, since they will be involved below only in integrals whose expressions are known.

THe Hermite processes appear as limit in the so-called Non Central Limit Theorem (see e.g. [11]). The class of Hermite processes includes the fractional Brownian motion which is the only Gaussian process in this class. Their practical aspects are striking: they provide a wide class of processes from which to model long memory, self-similarity and Hölder-regularity, allowing significant deviation from fBm and other Gaussian processes . Since they are non-Gaussian and self-similar with stationary increments, the Hermite processes can also be an input in models where self-similarity is observed in empirical data which appears to be non-Gaussian.

We will denote by $(X_t^{(q,H)})_{t\in[0,1]}$ the qth Hermite process *with self-similarity parameter* $H \in (1/2,1)$. Here $q \geq 1$ is an integer. The Hermite process can be defined as a multiple integral with respect to the standard Wiener process $(W_t)_{t\in[0,1]}$; details concerning the definition of multiple stochastic integrals can be found in Chapter 1 in [7]. These multiple integrals can be viewed as iterated Itô integrals. Concretely, we have the following definition.

Definition 2.1. The *Hermite process* $(X_t^{(q,H)})_{t\in[0,1]}$ of order $q \geq 1$ and with self-similarity parameter $H \in (\frac{1}{2},1)$ is given, for $t \in [0,1]$, by

$$\begin{aligned} X_t^{(q,H)} &= d(H)\int_0^t ..\int_0^t dW_{y_1}..dW_{y_q} \\ &\quad \left(\int_{y_1\vee..\vee y_q}^t \partial_1 K^{H'}(u,y_1)\ldots\partial_1 K^{H'}(u,y_q)du\right) \qquad (2.1) \end{aligned}$$

where $d(H)$ is a normalizing constant (it ensures that the variance at time one is equal to one), $K^{H'}$ is the usual kernel of the fractional Brownian motion and

$$H' = 1 + \frac{H-1}{q} \Longleftrightarrow (2H'-2)q = 2H-2. \tag{2.2}$$

In order to avoid sophisticated and unnecessary elements from stochastic integration, we prefer to avoid more details on the construction of the above integrals. We refer, as we said, to [7]. What we will actually need, are the properties of the Hermite process.

First we mention that $X_t^{(q,H)}$ is a centered process since it is defined by a multiple stochastic integral. Of fundamental importance is the fact that the covariance of $X^{(q,H)}$ is identical to that of fBm, namely for every $s,t \in [0,1]$

$$\mathbf{E}\left[X_s^{(q,H)} X_t^{(q,H)}\right] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

The basic properties of the Hermite process are listed below:

- the Hermite process $X^{(q,H)}$ is H-self-similar and it has stationary increments.
- the mean square of the increment is given by, for $s,t \in [0,1]$
$$\mathbf{E}\left[\left|X_t^{(q,H)} - X_s^{(q,H)}\right|^2\right] = |t-s|^{2H}; \tag{2.3}$$
as a consequence, it follows will little extra effort from Kolmogorov's continuity criterion that $X^{(q,H)}$ has Hölder-continuous paths of any exponent $\delta < H$.
- it exhibits long-range dependence in the sense that
$$\sum_{n \geq 1} \mathbf{E}\left[X_1^{(q,H)}(X_{n+1}^{(q,H)} - X_n^{(q,H)})\right] = \infty.$$
In fact, the summand in this series is of order n^{2H-2}. This property is identical to that of fBm since the processes share the same covariance structure, and the property is well-known for fBm with $H > 1/2$. Of course, this property holds for the process $X^{(q,H)}$ with time interval $[0,\infty)$.
- for $q=1$, $X^{(1,H)}$ is standard fBm with Hurst parameter H, while for $q \geq 2$ the Hermite process is not Gaussian. In the case $q=2$ this stochastic process is known as *the Rosenblatt process*.

3. The Model and the Consistency of the Estimator

Taking into account our choice for the regressor and the error (1.2) (and changing the index of the observations from $1,...,n$ to $0,...,n-1$) the model (1.1) becomes

$$Y_{i/n} = r(B_{i/n}^{H_1}) + n^{H_2}(X_{i+1/n}^{H_2} - X_{i/n}^{H_2}), \quad 0 \leq i \leq n-1 \text{ and } n \geq 1 \tag{3.1}$$

with B^{H_1} a fractional Brownian motion with Hurst parameter $H_1 \in (0,1)$ and X^{H_2} an independent Hermite process of order $q \geq 1$ with self-similarity parameter

$H_2 \in (\frac{1}{2}, 1)$. As we mentioned, the factor n^{H_2} is added in order to have the constant variance equal to 1 for the error (see (2.3)). We will omit in the sequel the order q in the notation of the process X^{H_2}. We also mention that in the particular case when $q = 1$, X^{H_2} becomes a fractional Brownian motion and then, the self-similarity parameter H_2 is allowed to belong to the whole interval $(0, 1)$. The standard estimator of the function r can we written as

$$\widehat{r}_n(x) = \frac{\sum_{i=0}^{n-1} Y_{i/n} K\left(\frac{x - B^{H_1}_{i/n}}{h}\right)}{\sum_{i=0}^{n-1} K\left(\frac{x - B^{H_1}_{i/n}}{h}\right)}. \tag{3.2}$$

Note that by (3.1) the estimator $\hat{r}_n(x)$ can be decomposed as

$$\begin{aligned} &\hat{r}_n(x) \\ = \;& \frac{\sum_{i=0}^{n-1} K\left(\frac{x-B^{H_1}_{i/n}}{h}\right) r(B^{H_1}_{\frac{i}{n}})}{\sum_{i=0}^{n-1} K\left(\frac{x-B^{H_1}_{i/n}}{h}\right)} + \frac{\sum_{i=0}^{n-1} K\left(\frac{x-B^{H_1}_{i/n}}{h}\right) n^{H_2}\left(X^{H_2}_{\frac{i+1}{n}} - X^{H_2}_{\frac{i}{n}}\right)}{\sum_{i=0}^{n-1} K\left(\frac{x-B^{H_1}_{i/n}}{h}\right)} \\ := \;& T_1^{(n)}(x) + T_2^{(n)}(x) \end{aligned} \tag{3.3}$$

for every $x \in \mathbb{R}$. The regression kernel is supposed to be bounded and positive throughout the paper. Also it satisfies $\int_{\mathbb{R}} K(y)dy = 1$ and $\int_{\mathbb{R}} K^2(y)dy < \infty$.

The purpose of this section is to show that the estimator $\hat{r}_n(x)$ given by (3.2) is consistent, that is, $\hat{r}_n(x)$ converges in probability as $n \to \infty$ to $r(x)$ for every $x \in \mathbb{R}$. We will handle succesively the two terms denoted by $T_1^{(n)}$ and $T_2^{(n)}$ above. Actually we will prove that $T_n^{(1)}(x)$ converges almost surely to $r(x)$ and $T_n^{(2)}(x)$ converges almost surely to 0 when $n \to \infty$. The results will be a consequence of several lemmas.

3.1. The asymptotic behavior of $T_1^{(n)}$. The first step is to study the convergence of the denumerator and the numerator of $T_1^{(n)}$ in (3.3). This convergence will be done in two parts: in a first result we approximate these sequence by other sequences which can be easier handled and then we give the limit of these new sequences.

Lemma 3.1. *Consider a Hölder continuous function $f : \mathbb{R} \to \mathbb{R}$, such that $|f(x) - f(y)| \leq C_f |x - y|^{\gamma_f}$ for every $x, y \in \mathbb{R}$. Let $(B^{H_1}_t)_{t \geq 0}$ be a fractional Brownian motion with Hurst parameter $H_1 \in (0, 1)$. Then, if $\alpha < \frac{H_1 \gamma_f}{1+\gamma_f}$, we get that*

$$R_n := n^{\alpha}\left\{\frac{1}{n}\sum_{i=0}^{n-1} f((x - B^{H_1}_{i/n})n^{\alpha}) - \int_0^1 f((x - B^{H_1}_s)n^{\alpha})ds\right\} \to_{n\to\infty} 0$$

almost surely.

Proof. Note that $\sum_{i=0}^{n} f((x - B^{H_1}_{i/n})n^{\alpha})$ can be expressed as an integral in the following way

$$\begin{aligned}
\frac{1}{n}\sum_{i=0}^{n-1} f((x-B^{H_1}_{i/n})n^{\alpha}) &= \sum_{i=0}^{n-1}\int_{\frac{i}{n}}^{\frac{i+1}{n}} f((x-B^{H_1}_{i/n})n^{\alpha})ds\\
&= \sum_{i=0}^{n-1}\int_{\frac{i}{n}}^{\frac{i+1}{n}} f\left((x-B^{H_1}_{\frac{[sn]}{n}})n^{\alpha}\right)ds\\
&= \int_0^1 f\left((x-B^{H_1}_{\frac{[sn]}{n}})n^{\alpha}\right)ds
\end{aligned}$$

and then R_n can be bounded as follows

$$\begin{aligned}
|R_n| &= n^{\alpha}\left|\int_0^1 \left(f((x-B^{H_1}_{\frac{[ns]}{n}})n^{\alpha}) - f((x-B^{H_1}_s)n^{\alpha})\right)ds\right|\\
&\leq C_f n^{\alpha+\alpha\gamma_f}\int_0^1 \left|B^{H_1}_{\frac{[ns]}{n}} - B^{H_1}_s ds\right|^{\gamma_f} ds.
\end{aligned}$$

We will now use the Hölder continuity of the fractional Brownian motion: for each δ such that $0<\delta<H_1$, there exists a random variable $W=W_\delta$ positive such that

$$|B^{H_1}_t - B^{H_1}_s| \leq W|t-s|^{\delta}$$

almost surely for every $s,t\in[0,1]$. Since for every $s\in[0,1]$

$$\left|\frac{[ns]}{n}-s\right| \leq \frac{1}{n},$$

the above stated Hölder continuity of the trajectories of $B^{H_1}_t$ guarantees that for any $0<\delta<H_1$,

$$\left|B^{H_1}_{[ns]/n} - B^{H_1}_s ds\right|^{\gamma_f} \leq W^{\gamma_f} n^{-\delta\gamma_f}$$

and so

$$R_n \leq W^{\gamma_f} C_f n^{\alpha+\alpha\gamma_f-\delta\gamma_f}$$

almost surely. The conclusion is immediate. □

Remark 3.2. As can be seen throughout our paper, the above lemma will be applied to the regression kernel K which will be Lipschitz in the examples we consider. In this case the order γ_f is equal to 1.

We deal now with the numerator of the summand $T_1^{(n)}$. We show that, after suitable normalization, it has the same limit almost surely as another sequence.

Lemma 3.3. *Suppose that the kernel K is Lipschitz continuous and the function r is Hölder continuous of order γ_r. Assume $\alpha<\min(H_1\gamma_r, \frac{H_1}{2})$*

Let L_n be defined by, for every $n\geq 1$

$$\begin{aligned}
L_n &= n^{\alpha}\left\{\frac{1}{n}\sum_{i=0}^{n-1}\left(r(B^{H_1}_{i/n})-r(x)\right)K((x-B^{H_1}_{i/n})n^{\alpha})\right.\\
&\qquad \left. -\int_0^1 \left(r(B^{H_1}_s)-r(x)\right)K((x-B^{H_1}_s)n^{\alpha})\right\}.
\end{aligned}$$

Then $L_n \to 0$ almost surely as $n \to \infty$.

Proof. The sequence L_n is a difference of two sequences

$$L_n = L_{1n} - L_{2n}$$

where

$$L_{1n} = n^{\alpha} \left\{ \frac{1}{n} \sum_{i=0}^{n-1} r(B^{H_1}_{i/n}) K((x - B^{H_1}_{i/n}) n^{\alpha}) - \int_0^1 r(B^{H_1}_s) K((x - B^{H_1}_s) n^{\alpha}) \right\} ds$$

$$L_{2n} = r(x) n^{\alpha} \left\{ \frac{1}{n} \sum_{i=0}^{n-1} K((x - B^{H_1}_{i/n}) n^{\alpha}) - \int_0^1 K((x - B^{H_1}_s) n^{\alpha}) ds \right\}.$$

It is obvious that L_{2n} can be treated as in Lemma 3.1 (with $\gamma_K = 1$) and it is clear that is converges to zero almost surely when $n \to \infty$. Note that L_{1n} can be written as $L_{1n} = M_n + T_n$, with

$$M_n = n^{\alpha} \int_0^1 \left(r(B^{H_1}_{[ns]/n}) - r(B^{H_1}_s) \right) K((x - B^{H_1}_{[ns]/n}) n^{\alpha})\, ds \tag{3.4}$$

and

$$T_n = n^{\alpha} \int_0^1 r(B^{H_1}_s) \left(K((x - B^{H_1}_{[ns]/n}) n^{\alpha}) - K((x - B^{H_1}_s) n^{\alpha}) \right) ds. \tag{3.5}$$

For T_n, we get

$$|T_n| \le n^{\alpha} \left\{ \int_0^1 r^2(B^{H_1}_s)\, ds \right\}^{1/2} \times \left\{ \int_0^1 \left(K((x - B^{H_1}_{[ns]/n}) n^{\alpha}) - K((x - B^{H_1}_s) n^{\alpha}) \right)^2 ds \right\}^{1/2}.$$

Let C_k denotes the Lipschitz constant of the kernel K. Then, for $0 < \delta < H_1$

$$\left| K((x - B^{H_1}_{[ns]/n}) n^{\alpha}) - K((x - B^{H_1}_s) n^{\alpha}) \right|^2 \le C_K^2 n^{2\alpha} \left| B^{H_1}_{[ns]/n} - B^{H_1}_s \right|^2 \le C_K^2 n^{2\alpha} \tilde{W} n^{-2\delta}$$

and so for almost all ω

$$|T_n| \le \tilde{X} n^{2\alpha - \delta},$$

for some random variable $\tilde{X}$.

Working with M_n we can write

$$\begin{aligned} |M_n| &= \left| n^{\alpha} \int_0^1 \left(r(B^{H_1}_{\frac{[ns]}{n}}) - r(B^{H_1}_s) \right) K((x - B^{H_1}_{[ns]/n}) n^{\alpha})\, ds \right| \\ &\le CY n^{\alpha - \gamma_r \delta} \end{aligned}$$

where $0 < \delta < H_1$ and Y is the random variable given by the Hölder continuity of the fractional Brownian motion and C is a positive constant (actually the product of the Lipschitz constant of r and of the constant that which bounds the kernel K). $\square$

At this point we need to introduce the local time of the fractional Brownian motion. For any $t \geq 0$ and $y \in \mathbb{R}$ we define $L^{H_1}(t,y)$ as the density of the occupation measure (see [1], [4])

$$\mu_t(A) = \int_0^t 1_A(B_s^{H_1})ds, \qquad A \in B(\mathbb{R}).$$

The local time $L^{H_1}(t,y)$ satisfies the occupation time formula

$$\int_0^t f(B_s^{H_1})ds = \int_{\mathbb{R}} L^{H_1}(t,y)f(y)dy \tag{3.6}$$

for any measurable function f. The local time is Hölder continuous with respect to t and with respect to y (for the sake of completeness $L^{H_1}(t,y)$ has Hölder continuous paths of order $\gamma < 1 - H_1$ in time and of order $\delta < \frac{1-H_1}{2H_1}$ in the space variable (see Table 2 in [4])). Moreover, it admits a bicontinuous version with respect to (t,y).

Lemma 3.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that*

$$\int_{\mathbb{R}} |f(z)||z|^{\delta} dz < \infty \tag{3.7}$$

for some $0 < \delta < \frac{1-H_1}{2H_1}$. Then for every $x \in \mathbb{R}$ the sequence

$$n^{\alpha} \int_0^1 f\left(n^{\alpha}(x - B_s^{H_1}\right) ds$$

converges as $n \to \infty$ to $d_1 L^{H_1}(1,x)$ with

$$d_1 = \int_{\mathbb{R}} f(y)dy. \tag{3.8}$$

Proof. First we apply the occupation time formula (3.6) to get

$$n^{\alpha} \int_0^1 f\left(n^{\alpha}(x - B_s^{H_1})\right) ds = n^{\alpha} \int_{\mathbb{R}} f(n^{\alpha}(x-y))L^{H_1}(1,y)dy$$

and then by the change of variables $n^{\alpha}(x-y) = z$ we obtain

$$n^{\alpha} \int_0^1 f\left(n^{\alpha}(x - B_s^{H_1})\right) ds = \int_{\mathbb{R}} f(z)L^{H_1}\left(1, x - \frac{z}{n^{\alpha}}\right) dz.$$

Finally we note that

$$\begin{aligned} &\left| \int_{\mathbb{R}} f(z)L^{H_1}\left(1, x - \frac{z}{n^{\alpha}}\right) dz - \int_{\mathbb{R}} f(y)dyL^{H_1}(1,x) \right| \\ \leq\ & C(\omega)\frac{1}{n^{\alpha\delta}} \int_{\mathbb{R}} |f(z)|z|^{\delta} dz \end{aligned}$$

for every δ such that $0 < \delta < \frac{1-H_1}{2H_1}$ due to the Hölder continuity of the local time. Clearly the last expression tends to 0 as $n \to \infty$. $\square$

Lemma 3.5. *Assume that the function r is Hölder continuous with exponent γ_r and*

$$\int_{\mathbb{R}} |f(z)z^{\delta}|dz < \infty \tag{3.9}$$

for some δ with $0 < \delta < \gamma_r + \frac{1-H_1}{2H_1}$. Then for every $x \in \mathbb{R}$ the sequence

$$n^{\alpha} \int_0^1 r(B_s^{H_1}) f\left(n^{\alpha}(x - B_s^{H_1})\right) ds$$

converges as $n \to \infty$ to $d_1 r(x) L^{H_1}(1, x)$, with d_1 given by (3.8).

Proof. Again by the occupation time formula (3.6) and making the change of variable $n^{\alpha}(x - y) = z$ we can write

$$n^{\alpha} \int_0^1 r(B_s^{H_1}) f\left(n^{\alpha}(x - B_s^{H_1})\right) ds = \int_{\mathbb{R}} r(x - \frac{z}{n^{\alpha}}) f(z) L^{H_1}(1, x - \frac{z}{n^{\alpha}}) dz$$

and

$$\begin{aligned}
&\left| \int_{\mathbb{R}} dz r(x - \frac{z}{n^{\alpha}}) f(z) L^{H_1}(1, x - \frac{z}{n^{\alpha}},) dz - d_1 r(x) L^{H_1}(1, x) \right| \\
\leq\ & \int_{\mathbb{R}} dz |f(z)| \left| r(x - \frac{z}{n^{\alpha}}) L^{H_1}(1, x - \frac{z}{n^{\alpha}}) - r(x) L^{H_1}(1, x) \right| \\
\leq\ & \int_{\mathbb{R}} dz |f(z)| \left| (r(x - \frac{z}{n^{\alpha}}) - r(x)) L^{H_1}(1, x - \frac{z}{n^{\alpha}}) \right| \\
& + |r(x)| \int_{\mathbb{R}} dz |f(z)| \left| L^{H_1}(1, x - \frac{z}{n^{\alpha}}) - L^{H_1}(1, x) \right|
\end{aligned}$$

The second summand goes to zero using assumption (3.9) as in the proof of Lemma 3.4. Using the Hölder continuity of the local time in the space variable and the Hölder assumption on r, the first summand can be bounded by

$$C(\omega) n^{-\alpha\gamma_r} |x|^{\delta} \int_{\mathbb{R}} |f(z)||z|^{\gamma_r} dz + C(\omega) n^{-\alpha\gamma_r - \alpha\delta} \int_{\mathbb{R}} |f(z)||z^{\delta+\gamma_r}| dz$$

and it converges to zero due to (3.9). □

We can conclude the convergence of the sequence $T_1^{(n)}$ defined by (3.3).

Proposition 3.6. *Assume that the regression function r is Hölder continuous with exponent γ_r. Take $\alpha < \min(\frac{H_1}{2}, H_1\gamma_r)$ and for $h = n^{-\alpha}$, let $T_1^{(n)}$ be given by (3.3), with a Lipschitz kernel K satisfying conditions (3.7) and (3.9). Then for every $x \in \mathbb{R}$*

$$T_1^{(n)}(x) \to_{n\to\infty} r(x)$$

Proof. The proof is a consequence of Lemmas 3.1, 3.3, 3.4 and 3.5. □

Example 3.7. Let us consider some kernels K that satisfies the Lipschitz continuity assumption (see [12]).

- The Gaussian kernel given by

$$K(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, x \in \mathbb{R}. \tag{3.10}$$

- The triangle kernel

$$K(x) = (1 - |x|) 1_{[-1,1]}(x). \tag{3.11}$$

- The Epanechnikov kernel

$$K(x) = \frac{3}{4}(1 - x^2) 1_{[-1,1]}(x). \tag{3.12}$$

- The quartic kernel

$$K(x) = \frac{15}{16}(1 - x^2)^2 1_{[-1,1]}(x). \tag{3.13}$$

Remark 3.8. Except the Gaussian kernel, all the other kernel listed above have compact support $J = [-1, 1]$. Using this fact we can avoid the Hölder continuity assumption for the regression function r. For example, in Lemma 2 the term denote (3.4) can be handled as follows

$$\begin{aligned} M_n &= n^\alpha \int_0^1 \left(r(B^{H_1}_{[ns]/n}) - r(B^{H_1}_s) \right) K((x - B^{H_1}_{[ns]/n}) n^\alpha)\, ds \\ &\leq n^\alpha \int_0^1 \left(r(B^{H_1}_{[ns]/n}) - r(B^{H_1}_s) \right) 1_{(x - n^{-\alpha} \leq B^{H_1}_{[ns]/n} \leq x + n^{-\alpha})} \\ &\leq \sup_{|a-b| \leq \frac{1}{n}} \left| r(B^{H_1}_a) - r(B^{H_1}_b) \right| \int_0^1 ds n^\alpha 1_{(x - n^{-\alpha} \leq B^{H_1}_{[ns]/n} \leq x + n^{-\alpha})} \end{aligned}$$

and if we assume the uniform continuity of r the first summand

$$\sup_{|a-b| \leq \frac{1}{n}} \left| r(B^{H_1}_a) - r(B^{H_1}_b) \right|$$

converges to zero almost surely and the second is almost surely bounded because

$$\mathbf{E} n^\alpha 1_{(x - n^{-\alpha} \leq B^{H_1}_{[ns]/n} \leq x + n^{-\alpha})} = n^\alpha P\left(x - n^{-\alpha} \leq B^{H_1}_{[ns]/n} \leq x + n^{-\alpha} \right) \leq \frac{1}{\sqrt{2\pi}}.$$

But in order to have an unitary approach for all the kernels listen above we prefer to keep the Hölder continuity assumption for the regression function (which in particular holds for linear functions).

3.2. The asymptotic behavior of the term $T_2^{(n)}$. We will handle now the summand denoted by $T_2^{(n)}$ in (3.3). We will treat first the numerator of $T_2^{(n)}$.

Let us show that the sequence

$$n^{\alpha-1} \sum_{i=0}^{n-1} K\left(n^\alpha (x - B^{H_1}_{\frac{i}{n}}) \right) n^{H_2} \left(X^{H_2}_{\frac{i+1}{n}} - X^{H_2}_{\frac{i}{n}} \right)$$

converges to zero in $L^2(\Omega)$ as $n \to \infty$. It is trivial that the term with $i = 0$ converges to zero for $\alpha < 1$. Therefore it suffices to check that the following converges to zero in L^2

$$A_n := n^{\alpha-1}\sum_{i=1}^{n-1} K\left(n^{\alpha}(x-B^{H_1}_{\frac{i}{n}})\right) n^{H_2}\left(X^{H_2}_{\frac{i+1}{n}} - X^{H_2}_{\frac{i}{n}}\right) \tag{3.14}$$

The L^2 norm of A_n can be expressed as

$$\begin{aligned}\mathbf{E}A_n^2 &= n^{2\alpha-2+2H_2}\sum_{i,j=1}^{n-1}\mathbf{E}K\left(n^{\alpha}(x-B^{H_1}_{\frac{i}{n}})\right)K\left(n^{\alpha}(x-B^{H_1}_{\frac{j}{n}})\right)\\ &\quad\times\mathbf{E}\left(X^{H_2}_{\frac{i+1}{n}} - X^{H_2}_{\frac{i}{n}}\right)\left(X^{H_2}_{\frac{j+1}{n}} - X^{H_2}_{\frac{j}{n}}\right)\\ &= n^{2\alpha-2}\sum_{i,j=1}^{n-1}\mathbf{E}K\left(n^{\alpha}(x-B^{H_1}_{\frac{i}{n}})\right)K\left(n^{\alpha}(x-B^{H_1}_{\frac{j}{n}})\right)\rho_{H_2}(i-j),\end{aligned}$$

where we denoted by

$$\rho_{H_2}(x) = \frac{1}{2}\left(|x+1|^{2H_2} + |x-1|^{2H_2} - 2|x|^{2H_2}\right). \tag{3.15}$$

The sequence $\mathbf{E}A_n^2$ can be decomposed into two parts: a diagonal part containing the terms with $i=j$ and a non-diagonal part containing the terms with $i\neq j$

$$\begin{aligned}\mathbf{E}A_n^2 &= n^{2\alpha-2}\sum_{i=1}^{n-1}\mathbf{E}K^2\left(n^{\alpha}(x-B^{H_1}_{\frac{i}{n}})\right)\\ &\quad+n^{2\alpha-2}\sum_{i,j=1;i\neq j}^{n-1}\mathbf{E}K\left(n^{\alpha}(x-B^{H_1}_{\frac{i}{n}})\right)K\left(n^{\alpha}(x-B^{H_1}_{\frac{j}{n}})\right)\rho_{H_2}(i-j)\\ &:= a_n^{(1)} + a_n^{(2)}.\end{aligned} \tag{3.16}$$

In order to compute explicitly the expectation appearing in the above formula we will consider particular choices for the kernel K. We will concentrate on the situations when K is the Gaussian kernel and the triangle kernel.

3.2.1. *The case of the Gaussian kernel.* The expectation of the product

$$\mathbf{E}K\left(n^{\alpha}(x-B^{H_1}_{\frac{i}{n}})\right)K\left(n^{\alpha}(x-B^{H_1}_{\frac{j}{n}})\right)$$

can be computed using the Gaussian law of the process B^{H_1}. We will distinguish the cases $i=j$ and $i\neq j$. If $i=j$ we have

$$\begin{aligned}\mathbf{E}K^2\left(n^{\alpha}(x-B^{H_1}_{\frac{i}{n}})\right) &= \frac{1}{2\pi}\mathbf{E}e^{-n^{2\alpha}\left(x-B^{H_1}_{\frac{i}{n}}\right)^2}\\ &= \frac{1}{2\pi}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dy e^{-n^{2\alpha}(x-y)^2}e^{-\frac{y^2}{2\left(\frac{i}{n}\right)^{2H_1}}}\\ &= \frac{1}{2\pi}\frac{1}{\sqrt{2\pi}}e^{-n^{2\alpha}x^2}\int_{\mathbb{R}} dy e^{-\frac{1}{2}y^2\left(2n^{2\alpha}+\left(\frac{n}{i}\right)^{2H_1}\right)}e^{2n^{2\alpha}xy}.\end{aligned}$$

Let us use the notation

$$a_{n,i} := 2n^{2\alpha} + \left(\frac{n}{i}\right)^{2H_1}.$$

We can write

$$\begin{aligned}
\mathbf{E}K^2\left(n^\alpha(x - B^{H_1}_{\frac{i}{n}})\right) &= \frac{1}{2\pi}\frac{1}{\sqrt{2\pi}}e^{-n^{2\alpha}x^2}\int_{\mathbb{R}} dye^{-\frac{1}{2}y^2 a_{n,i}}e^{2n^{2\alpha}xy} \\
&= \frac{1}{2\pi}\frac{1}{\sqrt{2\pi}}e^{-n^{2\alpha}x^2}\int_{\mathbb{R}} dye^{-\frac{a_{n,i}}{2}\left(y-\frac{2xyn^{2\alpha}}{a_{n,i}}\right)^2}e^{\frac{2n^{4\alpha}x^2}{a_{n,i}}} \\
&= \frac{1}{2\pi}\frac{1}{\sqrt{a_{n,i}}}e^{-x^2n^{2\alpha}\left(1-2\frac{n^{2\alpha}}{a_{n,i}}\right)}.
\end{aligned}$$

We notice that $a_{n,i} \geq 2n^{2\alpha}$ for every n, i so $1 - 2\frac{n^{2\alpha}}{a_{n,i}} \geq 0$. This implies that the quantity $e^{-x^2n^{2\alpha}\left(1-\frac{n^{2\alpha}}{a_{n,i}}\right)}$ can be bounded by 1. Thus

$$a_n^{(1)} \leq n^{2\alpha-2}\sum_{i=1}^{n-1}\frac{1}{\sqrt{a_{n,i}}} \leq cn^{\alpha-1}$$

and this goes to zero if $\alpha < 1$ which is always true.

Remark 3.9. The convergence of the diagonal term can be proven using Lemmas 3.1 and 3.4. Using these two results it follows that $n^{\alpha-1}\sum_{i=0}^{n-1} K^2\left(n^\alpha(x - B^{H_1}_{\frac{i}{n}})\right)$ converges to a non-trivial limit when $\alpha < H_1$ when the function K^2 is Lipschitz continuous. This is for example the case of the Gaussian kernel. This implies that $n^{2\alpha-2}\sum_{i=0}^{n-1} K^2\left(n^\alpha(x - B^{H_1}_{\frac{i}{n}})\right)$ because $\alpha < 1$. We prefer to keep the above computations because they give the convergence to zero of the diagonal term in L^2 and under the weaker condition $\alpha < 1$ instead of $\alpha < H_1$.

Consider now the case $i \neq j$. The vector $\left(B^{H_1}_{\frac{i}{n}}, B^{H_1}_{\frac{j}{n}}\right)$ is Gaussian with covariance matrix given by $\frac{1}{n^{2H_1}}\Gamma$ where

$$\Gamma = \begin{pmatrix} i^{2H_1} & R^{H_1}(i,j) \\ R^{H_1}(i,j) & j^{2H_1}\end{pmatrix}$$

which implies that the density of the vector is

$$\frac{n^{2H_1}}{2\pi\sqrt{\det\Gamma}}e^{-\frac{n^{2H_1}}{2\det\Gamma}\left(j^{2H_1}u^2-2R_{H_1}(u,v)uv+i^{2H_1}v^2\right)}$$

Thus

$$\begin{aligned}
&\mathbf{E}K\left(n^\alpha(x - B^{H_1}_{\frac{i}{n}})\right)K\left(n^\alpha(x - B^{H_1}_{\frac{j}{n}})\right) \\
&= \frac{n^{2H_1}}{2\pi\sqrt{\det\Gamma}}\frac{1}{2\pi} \\
&\quad\times\int_{\mathbb{R}^2} dudve^{-\frac{1}{2}n^{2\alpha}(x-u)^2}e^{-\frac{1}{2}n^{2\alpha}(x-v)^2}e^{-\frac{1}{2}\frac{n^{2H_1}}{\det\Gamma}\left(j^{2H_1}u^2-2R_{H_1}(i,j)uv+i^{2H_1}v^2\right)} \\
&= \frac{n^{2H_1}}{2\pi\sqrt{\det\Gamma}}\frac{1}{2\pi}e^{-n^{2\alpha}x^2}\int_{\mathbb{R}^2} dudve^{-\frac{1}{2}u^2b_{n,j}}e^{-\frac{1}{2}v^2b_{n,i}}e^{n^{2\alpha}xu}e^{n^{2\alpha}xv}e^{\frac{R_{H_1}(i,j)uvn^{2H_1}}{\det\Gamma}}
\end{aligned}$$

where we denoted

$$b_{n,i} = n^{2\alpha} + \frac{n^{2H_1} i^{2H_1}}{\det \Gamma}.$$

We will get

$$\begin{aligned}
&\mathbf{E}K\left(n^{\alpha}(x - B^{H_1}_{\frac{i}{n}})\right) K\left(n^{\alpha}(x - B^{H_1}_{\frac{j}{n}})\right)\\
=\;& \frac{n^{2H_1}}{2\pi\sqrt{\det\Gamma}} \frac{1}{2\pi} e^{-n^{2\alpha}x^2} \int_{\mathbb{R}} du e^{-\frac{1}{2}u^2 b_{n,j}} e^{n^{2\alpha}xu}\\
&\times \int_{\mathbb{R}} dv e^{-\frac{1}{2}b_{n,i}\left[v^2 - \frac{2v}{b_{n,i}}(n^{2\alpha}x + \frac{R_{H_1}(i,j)un^{2H_1}}{\det\Gamma})\right]}\\
=\;& \frac{n^{2H_1}}{2\pi\sqrt{\det\Gamma}} \frac{1}{2\pi} e^{-n^{2\alpha}x^2} \int_{\mathbb{R}} du e^{-\frac{1}{2}u^2 b_{n,j}} e^{n^{2\alpha}xu}\\
&\times \int_{\mathbb{R}} dv e^{-\frac{1}{2}b_{n,i}\left[v - \frac{1}{b_{n,i}}(n^{2\alpha}x + \frac{R_{H_1}(i,j)un^{2H_1}}{\det\Gamma})\right]^2} e^{\frac{1}{2b_{n,i}}(n^{2\alpha}x + \frac{R_{H_1}(i,j)un^{2H_1}}{\det\Gamma})^2}\\
=\;& \frac{n^{2H_1}}{2\pi\sqrt{\det\Gamma}} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{b_{n,i}}} e^{-n^{2\alpha}x^2}\\
&\int_{\mathbb{R}} du e^{\frac{1}{2b_{n,i}}(n^{2\alpha}x + \frac{R_{H_1}(i,j)un^{2H_1}}{\det\Gamma})^2} e^{-\frac{1}{2}u^2 b_{n,j}} e^{n^{2\alpha}xu}\\
=\;& \frac{n^{2H_1}}{2\pi\sqrt{\det\Gamma}} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{b_{n,i}}} e^{-n^{2\alpha}x^2} e^{\frac{1}{2b_{n,i}}n^{4\alpha}x^2}\\
&\int_{\mathbb{R}} du e^{-\frac{1}{2}u^2 c_{n,i,j}} e^{xun^{2\alpha}(1 + \frac{R_{H_1}(i,j)n^{2H_1}}{b_{n,i}\det\Gamma})}
\end{aligned}$$

with

$$c_{n,i,j} = b_{n,j} - \frac{R^2_{H_1}(i,j)n^{4H_1}}{b_{n,i}(\det\Gamma)^2}.$$

So

$$\begin{aligned}
&\mathbf{E}K\left(n^{\alpha}(x - B^{H_1}_{\frac{i}{n}})\right) K\left(n^{\alpha}(x - B^{H_1}_{\frac{j}{n}})\right)\\
=\;& \frac{n^{2H_1}}{2\pi\sqrt{\det\Gamma}} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{b_{n,i}}} e^{-n^{2\alpha}x^2} e^{\frac{1}{2b_{n,i}}n^{4\alpha}x^2} e^{-\frac{1}{2c_{n,i,j}}x^2 n^{4\alpha}\left(1 + \frac{R_{H_1}(i,j)n^{2H_1}}{b_{n,i}\det\Gamma}\right)^2}\\
&\times \int_{\mathbb{R}} e^{-\frac{1}{2}c_{n,i,j}\left[u - \frac{xn^{2\alpha}}{c_{n,i,j}}\left(1 + \frac{R_{H_1}(i,j)n^{2H_1}}{b_{n,i}\det\Gamma}\right)\right]^2} du\\
=\;& \frac{n^{2H_1}}{2\pi\sqrt{\det\Gamma}} \frac{1}{\sqrt{c_{n,i,j}}} \frac{1}{\sqrt{b_{n,i}}} e^{-n^{2\alpha}x^2} e^{\frac{1}{2b_{n,i}}n^{4\alpha}x^2} e^{-\frac{1}{2c_{n,i,j}}x^2 n^{4\alpha}\left(1 + \frac{R_{H_1}(i,j)n^{2H_1}}{b_{n,i}\det\Gamma}\right)^2}
\end{aligned}$$

$$e^{-n^{2\alpha}x^2} e^{\frac{1}{2b_{n,i}}n^{4\alpha}x^2} e^{-\frac{1}{2c_{n,i,j}}x^2 n^{4\alpha}\left(1 + \frac{R_{H_1}(i,j)n^{2H_1}}{b_{n,i}\det\Gamma}\right)^2} \le 1$$

(this follows from the fact that $b_{n,i}$ is bigger than $n^{2\alpha}$). Then a standard calculation shows that

$$c_{n,i,j} = \frac{(n^{2\alpha}\det\Gamma + n^{2H_1}j^{2H_1})(n^{2\alpha}\det\Gamma + n^{2H_1}i^{2H_1}) - R^2_{H_1}(i,j)n^{4H_1}}{\det\Gamma(n^{2\alpha}\det\Gamma + n^{2H_1}i^{2H_1})}$$

and

$$\begin{aligned}&\frac{1}{\sqrt{b_{n,i}c_{n,i,j}}}\\ &= \frac{\det\Gamma}{\sqrt{(n^{2\alpha}\det\Gamma + n^{2H_1}j^{2H_1})(n^{2\alpha}\det\Gamma + n^{2H_1}i^{2H_1}) - R^2_{H_1}(i,j)n^{4H_1}}}.\end{aligned}$$

Consequently

$$\begin{aligned}&\mathbf{E}K\left(n^{\alpha}(x - B^{H_1}_{\frac{i}{n}})\right)K\left(n^{\alpha}(x - B^{H_1}_{\frac{j}{n}})\right)\\ &\leq \frac{n^{2H_1}}{2\pi}\frac{\sqrt{\det\Gamma}}{\sqrt{(n^{2\alpha}\det\Gamma + n^{2H_1}j^{2H_1})(n^{2\alpha}\det\Gamma + n^{2H_1}i^{2H_1}) - R^2_{H_1}(i,j)n^{4H_1}}}\\ &= \frac{1}{2\pi}\frac{\sqrt{\det\Gamma}}{\sqrt{(n^{2(\alpha-H_1)}\det\Gamma + j^{2H_1})(n^{2(\alpha-H_1)}\det\Gamma + i^{2H_1}) - R^2_{H_1}(i,j)}}\end{aligned}$$

At this time we can use the estimation in [2] to see that

$$\sum_{i,j=0;i\neq j}^{n-1}\mathbf{E}K\left(n^{\alpha}(x - B^{H_1}_{\frac{i}{n}})\right)K\left(n^{\alpha}(x - B^{H_1}_{\frac{j}{n}})\right)|i-j|^{2H_2-2} \leq cn^{-\frac{3\alpha}{2}+2H_2}$$

using the fact that $\rho_{H_2}(n)$ behaves as $H_2(2H_2-1)n^{2H-2}$ for n close to infinity. This implies that

$$a_n^{(2)} \leq cn^{2\alpha-2}n^{-\frac{3\alpha}{2}+2H_2} = n^{\frac{\alpha}{2}-2+2H_2}$$

which converges to zero under condition (3.17).

Lemma 3.10. *Suppose K is the Gaussian kernel and*

$$\alpha < 4 - 4H_2. \tag{3.17}$$

Let A_n be given by (3.14). Then

$$A_n \to_{n\to\infty} 0 \text{ in } L^2(\Omega).$$

Remark 3.11. There is another immediate way to treat the term $a_n^{(2)}$. Actually if we bound the Gaussian kernel by $\frac{1}{\sqrt{2\pi}}$ we obtain that

$$a_n^{(2)} \leq cn^{2\alpha-2-2H_2}$$

which leads to the condition $\alpha < 1 - H_2$. The computations in the proof of the above lemma allows to obtain four times more space for the bandwidth parameter α.

Let us now state the result concerning the convergence of the term $T_2^{(n)}$ in (3.3).

Proposition 3.12. *Suppose that the kernel K is the Gaussian kernel (3.10) and the bandwidth parameter α satisfies*

$$\alpha < \min\left(\frac{H_1}{2}, 4-4H_2\right). \tag{3.18}$$

Then $T_2^{(n)} \to_{n\to\infty} 0$ in probability.

Proof. We have

$$T_2^{(n)} = \frac{n^{\alpha-1}\sum_{i=0}^{n-1} K\left(\frac{x-B^{H_1}_{i/n}}{h}\right) n^{H_2}\left(X^{H_2}_{\frac{i+1}{n}} - X^{H_2}_{\frac{i}{n}}\right)}{n^{\alpha-1}\sum_{i=0}^{n-1} K\left(\frac{x-B^{H_1}_{i/n}}{h}\right)}.$$

Lemma 3.1 implies the convergence of the denumerator to a non zero constant almost surely while Lemma 3.10 gives the convergence of the denumerator to zero in $L^2(\Omega)$. The conclusion is obtained easily. □

Theorem 3.13. *Suppose r is Hölder with exponent γ_r and K is the Gaussian kernel. Assume*

$$\alpha < \min\left(\frac{H_1}{2}, 4-4H_2, H_1\gamma_r\right).$$

Then for every $x\in\mathbb{R}$

$$\hat{r}(x) \to_{n\to\infty} r(x)$$

in probability.

Proof. The proof follows from Proposition 3.6, Proposition 3.12 and the decomposition (3.3). □

3.2.2. *The case of the triangle, Epanichnikov and quartic kernel kernel.* We assume in this part that K is given by (3.11), (3.12) or (3.13). We will prove an analogous of Proposition 2.

Proposition 3.14. *Suppose that the kernel K is either (3.11), (3.12) or (3.13) and the bandwidth parameter α satisfies*

$$\alpha < \min\left(\frac{H_1}{2}, 2-2H_2\right). \tag{3.19}$$

Then $T_2^{(n)} \to_{n\to\infty} 0$ in probability.

Proof. Recall that the L^2 norm of the numerator of $T_2^{(n)}$ given in (3.16). The term denoted by $a_n^{(1)}$ can be easily handled. Indeed

$$\begin{aligned} a_n^{(1)} &= n^{2\alpha-2}\sum_{i=0}^{n-1}\mathbf{E} g^2(n^\alpha(x-B^{H_1}_{\frac{i}{n}}))1_{[-1,1]}(n^\alpha(x-B^{H_1}_{\frac{i}{n}})) \\ &\leq Cn^{2\alpha-2}\sum_{i=0}^{n-1}\mathbf{E}1_{[-1,1]}(n^\alpha(x-B^{H_1}_{\frac{i}{n}})) \end{aligned}$$

where $g(x) = 1 - |x|$ in the case of triangle kernel, $g(x) = \frac{3}{4}(1-x^2)$ in the case of the Epachnikov kernel and $g(x) = \frac{15}{16}(1-x^2)^2$ in the case of the quartic kernel. Since

$$\mathbf{E}1_{[-1,1]}(n^\alpha(x - B^{H_1}_{\frac{i}{n}})) = \frac{1}{\sqrt{2\pi\left(\frac{i}{n}\right)^{2H_1}}}\int_{x-\frac{1}{n^\alpha}}^{x+\frac{1}{n^\alpha}} e^{-\frac{u^2}{2\left(\frac{i}{n}\right)^{2H_1}}}$$

$$\leq \frac{C}{\sqrt{2\pi\left(\frac{i}{n}\right)^{2H_1}}} n^{-\alpha} \tag{3.20}$$

we have

$$a_n^{(1)} \leq Cn^{\alpha-2}\sum_{i=0}^{n-1}\frac{1}{\sqrt{2\pi\left(\frac{i}{n}\right)^{2H_1}}} \leq Cn^{\alpha-1}$$

which goes to zero for $\alpha < 1$.

Let us now compute the term $a_n^{(2)}$. Let us now compute the term $a_n^{(2)}$. We have

$$\begin{aligned} a_n^{(2)} &= n^{2\alpha-2}\sum_{i,j=0;i\neq j}^{n-1}\mathbf{E}g(n^\alpha(x - B^{H_1}_{\frac{i}{n}}))g(n^\alpha(x - B^{H_1}_{\frac{j}{n}})) \\ &\quad \times 1_{[-1,1]}(n^\alpha(x - B^{H_1}_{\frac{i}{n}}))1_{[-1,1]}(n^\alpha(x - B^{H_1}_{\frac{j}{n}}))\rho_{H_2}(i-j) \\ &\leq cst.n^{2\alpha-2}\sum_{i,j=0;i\neq j}^{n-1}\mathbf{E}1_{[-1,1]}(n^\alpha(x - B^{H_1}_{\frac{i}{n}}))\rho_{H_2}(i-j) \end{aligned}$$

where we bounded the quantity $g(n^\alpha(x-B^{H_1}_{\frac{i}{n}}))g(n^\alpha(x-B^{H_1}_{\frac{j}{n}}))1_{[-1,1]}(n^\alpha(x-B^{H_1}_{\frac{j}{n}}))$ by a constant (the constant is $1, \frac{3}{4}, \frac{1}{16}$ in the three cases respectively). Then

$$\begin{aligned} a_n^{(2)} &\leq cst.n^{2\alpha-2}\sum_{i=0}^{n-1}\mathbf{E}1_{[-1,1]}(n^\alpha(x - B^{H_1}_{\frac{i}{n}}))\sum_{j=0}^{n-1}\rho_{H_2}(i-j) \\ &= cst.n^{2\alpha-2}\sum_{i=0}^{n-1}\mathbf{E}1_{[-1,1]}(n^\alpha(x - B^{H_1}_{\frac{i}{n}}))\sum_{j=0}^{n-1}\rho_{H_2}(i-j). \end{aligned}$$

By (3.20) we get

$$\begin{aligned} a_n^{(2)} &\leq cst.n^{\alpha-2}\sum_{i=0}^{n-1}\frac{1}{\sqrt{2\pi\left(\frac{i}{n}\right)^{2H_1}}}\sum_{j=0}^{n-1}\rho_{H_2}(i-j) \\ &\leq cst.n^{\alpha-2+2H_2} \end{aligned}$$

and this converges to zero when $\alpha < 2 - 2H_2$. □

Theorem 3.15. *Suppose that the assumptions in Theorem 3.13 hold with (3.19) instead of (3.18). Then in the case when K is the triangle, Epachnikov or quartic kernel, the estimator $\widehat{r}_n(x)$ converges in probability as $n \to \infty$ to $r(x)$ for every $x \in \mathbb{R}$.*

Proof. We obtain the result using Propositions 3.6 and 3.14. □

At this point let us make some comments. The proofs of the convergence of $T_1^{(n)}$ uses the Lipschitz continuity of the kernel considered. We mention that that is possible because we situate ourselves in the discretization context, that means to have observation equidistant between $[0,1]$. This technique leads to some restriction for the interval where the bandwidth parameter α is allowed to belong. On the other hand, using our proof is technically less sophisticated and easier to understand that in e.g. [2], [14] or [15]. It is plausible that the conditions imposed on the bandwidth parameter α can be improved with a different approach.

4. Asymptotic Normality

We study here the asymptotic normality of the estimator (3.2). More concretely, we will prove that after suitable normalization, the difference $\hat{r}_n(x)-r(x)$ converges as $n\to\infty$ to a normal random variable. We will actually restrict to the following situation: the error are supposed to be the increments of the fractional Brownian motion with Hurst parameter $H_2\in(0,1)$.

We will start with the following lemma.

Lemma 4.1. *Let A_n be given by (3.14) with $X^{H_2}=B^{H_2}$ and let us denote by*

$$B_n := n^{\alpha-1}\sum_{i=0}^{n-1} K\left(n^{\alpha}(x-B^{H_1}_{\frac{i}{n}})\right).$$

Assume that $\alpha<\frac{H_1}{2}$. Then the vector $(n^{-(\alpha-1)/2}A_n, B_n)$ converges in distribution to the random couple

$$(d_2 Z(L^{H_1}(1,x))^{\frac{1}{2}}, d_1 L^{H_1}(1,x))$$

where L^{H_1} is the local time of the fractional Brownian motion B^{H_1}, Z is a standard normal random variable independent by B^{H_1}, d_1 is given by (3.8) and

$$d_2=\int_{\mathbb{R}} K^2(y)dy. \tag{4.1}$$

Remark 4.2. We already proved that B_n converges in probability the the local time L^{H_1} and following the ideas in [2] we can show that A_n converges in law to $Z(L^{H_1}(1,x))^{\frac{1}{2}}$. But in order to obtain our result we need joint convergence of (A_n,B_n).

Proof. We will compute the characteristic function of the vector

$$(n^{-(\alpha-1)/2}A_n, B_n).$$

Take $\lambda_1,\lambda_2\in\mathbb{R}$.Then

$$\begin{aligned}\mathbf{E}e^{i\lambda_1 n^{-\frac{\alpha-1}{2}}A_n+i\lambda_2 B_n} &= \mathbf{E}\left(\mathbf{E}e^{i\lambda_1 n^{-\frac{\alpha-1}{2}}A_n+i\lambda_2 B_n}/B^{H_1}\right)\\ &= \mathbf{E}e^{i\lambda_2 B_n}\mathbf{E}\left(e^{i\lambda_1 n^{-\frac{\alpha-1}{2}}A_n+i\lambda_2 B_n}/B^{H_1}\right).\end{aligned}$$

Using again the independence of B^{H_1} and B^{H_2}

$$\mathbf{E}\left(e^{i\lambda_1 n^{-\frac{\alpha-1}{2}}A_n+i\lambda_2 B_n}/B^{H_1}\right)=e^{-\frac{\lambda_1^2}{2}n^{\alpha-1}\sum_{i,j=0}^{n-1}K\left(n^{\alpha}(x-B^{H_1}_{\frac{i}{n}})\right)\rho_{H_2}(i-j)}$$

with ρ_{H_2} defined by (3.15) and therefore

$$\begin{aligned}\mathbf{E}e^{i\lambda_1 n^{-\frac{\alpha-1}{2}}A_n+i\lambda_2 B_n} &= \mathbf{E}e^{i\lambda_2 B_n}e^{-\frac{\lambda_1^2}{2}n^{\alpha-1}\sum_{i=0}^{n-1}K^2\left(n^{\alpha}(x-B^{H_1}_{\frac{i}{n}})\right)}\\ &\quad\times e^{-\frac{\lambda_1^2}{2}n^{\alpha-1}\sum_{i,j=0;i\neq j}^{n-1}K\left(n^{\alpha}(x-B^{H_1}_{\frac{i}{n}})\right).\rho_{H_2}(i-j)} \qquad (4.2)\end{aligned}$$

Using exactly the same calculations as in the proof of Theorem 2 in [2], we can show that

$$e^{-\frac{\lambda_1^2}{2}n^{\alpha-1}\sum_{i,j=0;i\neq j}^{n-1}K\left(n^{\alpha}(x-B^{H_1}_{\frac{i}{n}})\right)\rho_{H_2}(i-j)}=1+C_n$$

where

$$\mathbf{E}|C_n|\to_{n\to\infty}0.$$

Using the above relation and (4.2) we have

$$\lim_{n\to\infty}\mathbf{E}e^{i\lambda_1 n^{-\frac{\alpha-1}{2}}A_n+i\lambda_2 B_n}=\lim_{n\to\infty}\mathbf{E}e^{i\lambda_2 B_n}e^{-\frac{\lambda_1^2}{2}n^{\alpha-1}\sum_{i=0}^{n-1}K^2\left(n^{\alpha}(x-B^{H_1}_{\frac{i}{n}})\right)}$$

which converges (due to Lemma 3.1 and to the fact that K^2 is a Lipschitz functions for our choice of K) to

$$\mathbf{E}e^{i\lambda_2 d_1 L^{H_1}(1,x)-id_2\frac{\lambda_1^2}{2}L^{H_1}(1,x)}$$

which is exactly the characteristic function of the vector $(d_2 Z(L^{H_1}(1,x))^{\frac{1}{2}}, d_1 L^{H_1}(1,x))$. □

We can state our result concerning the asymptotic normality of the estimator.

Theorem 4.3. *We have that*

$$\left(n^{\alpha-1}\sum_{i=0}^{n-1}K\left(n^{\alpha}(x-B^{H_1}_{\frac{i}{n}})\right)\right)^{\frac{1}{2}}(\hat{r}(x)-r(x))\to_{n\to\infty}N(0,d_1).$$

Proof. By (3.3)

$$\begin{aligned}&\left(n^{\alpha-1}\sum_{i=0}^{n-1}K\left(n^{\alpha}(x-B^{H_1}_{\frac{i}{n}})\right)\right)^{\frac{1}{2}}(\hat{r}(x)-r(x))\\ &= (B_n)^{\frac{1}{2}}\frac{n^{-\frac{\alpha-1}{2}}A_n}{B_n}+\left(n^{\alpha-1}\sum_{i=0}^{n-1}K\left(n^{\alpha}(x-B^{H_1}_{\frac{i}{n}})\right)\right)^{\frac{1}{2}}\\ &\quad\times\frac{\sum_{i=0}^{n-1}K\left(n^{\alpha}(x-B^{H_1}_{\frac{i}{n}})\right)\left(r\left(B^{H_1}_{\frac{i}{n}}\right)-r(x)\right)}{\sum_{i=0}^{n-1}K\left(n^{\alpha}(x-B^{H_1}_{\frac{i}{n}})\right)}.\end{aligned}$$

The first summand converges in distribution to the desired limit. It suffices to show that the second converges to zero in probability. This second summand above is equal to

$$\frac{n^{\alpha-1}\sum_{i=0}^{n-1}K\left(n^{\alpha}(x-B^{H_1}_{\frac{i}{n}})\right)\left(r\left(B^{H_1}_{\frac{i}{n}}\right)-r(x)\right)}{B_n^{\frac{1}{2}}}$$

and the result follows if we proof that

$$n^{\alpha-1}\sum_{i=0}^{n-1} K\left(n^{\alpha}(x - B^{H_1}_{\frac{i}{n}})\right)\left(r\left(B^{H_1}_{\frac{i}{n}}\right) - r(x)\right)$$

converges to zero in probability as $n \to \infty$. But this is a consequence of Lemma 3.3. □

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