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Short-distance contribution to the spectrum of Hawking radiation

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The Hawking effect can be rederived in terms of two-point functions and in such a way that it makes it possible to estimate, within the conventional semiclassical theory, the contribution of ultrashort distances at I^+ to the Planckian spectrum. The analysis shows that, for Schwarzschild astrophysical black holes, the Hawking radiation (for both bosons and fermions) is very robust up to very high frequencies (typically two orders above Hawking's temperature). Below this scale, the contribution of ultrashort distances to the spectrum is negligible. We argue, using a simple model with modified two-point functions, that the above result seems to have a general validity and that it is related to the observer independence of the short-distance behavior of the corresponding two-point function. The above suggests that only at high emission frequencies could an underlying quantum theory of gravity potentially predict significant deviations from Hawking's semiclassical result.

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I. INTRODUCTION

Semiclassical gravity predicts the radiation of quanta by black holes [1, 2]. The emission rate is given by the product of the Planckian factor times the grey-body coefficient $\Gamma_{lmp}(w)$

$$\frac{dN_{lmp}(w)}{dw dt} = \frac{1}{2\pi} \Gamma_{lmp}(w) \frac{1}{e^{2\pi\kappa^{-1}(w-m\Omega_H-q\Phi_H)} \pm 1}, \quad (1)$$

where κ , Ω_H and Φ_H are the surface gravity, angular velocity and the electric potential of the black hole horizon. The signs \pm in the denominator account for the Bose or Fermi statistics, and m , p and q are the corresponding axial angular momentum, helicity, and charge of the radiated particle.

The deep connection of this result with thermodynamics [3] and, in particular, with a generalized second law [4] strongly supports its robustness [5, 6, 7]. However, as stressed in [8], a crucial ingredient in deriving Hawking radiation via semiclassical gravity is the fact that any emitted quanta, even those with very low frequency at future infinity, will suffer a divergent blueshift when propagated backwards in time and measured by a freely falling observer. Also, in the derivation of Fredenhagen and Haag [9], the role of the short-distance behavior of the two-point function is fundamental. All derivations seem to invoke Planck-scale physics. The exponential blueshift effect of the horizon of the black hole could thus be regarded as a magnifying glass that makes visible the ultrashort-distance physics to external observers. According to this reasoning the microscopic structure offered by string theory (or any other underlying theory) could leave some imprint or signal in the emission rate. However, the results of string theory seem to agree with Hawking's prediction. For the emission of low-energy quanta (with wavelength large compared to the black hole radius), and for some particular near-extremal charged black holes, the prediction of string theory [10, 11] coincides with the rate (1). This agreement is complete, despite the fact that the two calculations are very different. For instance, whereas in semiclassical gravity one can naturally split the emission rate into two factors (pure Planckian black-body term and grey-body factor), in the D-brane derivation one gets directly the final answer without the above mentioned splitting.

While the calculation of string theory requires low frequencies for the emitted particles, in the arena of semiclassical gravity the result is valid for all wavelengths, even those smaller than the size of the black hole. The thermodynamic

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picture strongly suggests the robustness of Hawking’s prediction and its interpretation as a low-energy effect, not affected by the particular underlying theory of quantum gravity (see also [12]), and expected to be valid for a large range of frequencies. However, from the perspective of quantum field theory in curved spacetime, it is unclear how to introduce a cutoff in the scheme (parameterizing our ignorance on trans-Planckian physics) in such a way that, for low-energy emitted quanta, the decay rate (1) is kept unaltered. The aim of this work is to study this issue in some detail¹.

In section II we will review the standard derivation of black hole radiation emphasizing the role of ultrahigh frequencies to get the Planckian spectrum. In section III we rederive the Hawking effect in terms of two-point functions, instead of Bogolubov transformations (for a general reference see [13]), for both massless scalar and spin-1/2 fields. The new derivation of the black hole decay rate offers an explicit way to evaluate the contribution of ultrashort (Planck-scale) distances to the thermal Hawking spectrum. This is the subject of section IV. In section V we present a simple model, where the two-point functions are deformed with a Planck-length parameter, to show how the previous results emerge in this new scenario and support their robustness. We point out that a generalized Hadamard condition plays a fundamental role to keep unaltered the bulk of the Hawking effect. Finally, in section VI, we summarize our conclusions and make some speculative comments. In the appendices we give details of some calculations used in the body of the text.

II. BOGOLUBOV COEFFICIENTS AND BLACK HOLE RADIANCE

Let us consider the formation process of a Schwarzschild black hole, as depicted in Fig.1, and a massless real scalar field ϕ propagating in this background. The equation of motion obeyed by the field is $\square\phi = 0$ and the Klein-Gordon scalar product is given by

$$(\phi_1, \phi_2) = -i \int_{\Sigma} d\Sigma^{\mu} (\phi_1 \partial_{\mu} \phi_2^* - \phi_2^* \partial_{\mu} \phi_1) , \quad (2)$$

where Σ is a suitable “initial data” hypersurface. A natural choice for Σ is the past null infinity I^- and therefore one can express the field in a set of modes $u_j^{in}(x)$, which have positive frequency in I^-

$$\phi = \sum_i (a_i^{in} u_i^{in} + a_i^{in\dagger} u_i^{in*}) . \quad (3)$$

Alternatively, we can choose Σ as $\Sigma = I^+ \cup H^+$, where I^+ is the future null infinity and H^+ is the event horizon.

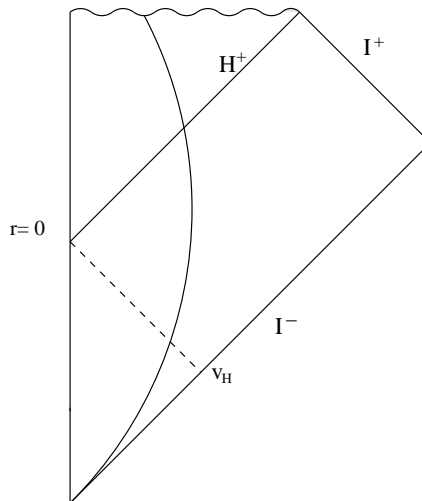


Figure 1: Penrose diagram of a collapsing body producing a Schwarzschild black hole.

¹ We note that this sort of problem has already been addressed in the context of acoustic black holes by modifying the dispersion relation of the wave equation obeyed by sonic disturbances [14, 15]. This is naturally justified as an effect of the atomic microscopic structure of the fluid, and requires a breakdown of Lorentz invariance. The rest frame of the atoms of the fluid plays a privileged role. In this paper we follow an alternative route.

According to this we can then expand the field in an orthonormal set of modes $u_i^{out}(x)$, which have positive frequency with respect to the inertial time at I^+ and have zero Cauchy data in H^+ , together with a set of modes $u_i^{int}(x)$ with null outgoing component at I^+ . Therefore we can write

$$\phi = \sum_i (a_i^{out} u_i^{out} + a_i^{out\dagger} u_i^{out*}) + (a_i^{int} u_i^{int} + a_i^{int\dagger} u_i^{int*}) . \quad (4)$$

The particular choice of modes u_i^{int} does not affect the computation of particle production at I^+ , so we leave them unspecified.

The modes $u_j^{out}(x)$ can be expressed in terms of the basis u_i^{in}

$$u_j^{out}(x) = \sum_i \alpha_{ji} u_i^{in}(x) + \beta_{ji} u_i^{in*}(x) , \quad (5)$$

where the coefficients α_{ji} and β_{ji} are the so-called Bogolubov coefficients and are given by the scalar products

$$\alpha_{ij} = (u_i^{out}, u_j^{in}) , \quad \beta_{ij} = -(u_i^{out}, u_j^{in*}) . \quad (6)$$

The above expansion leads to an analogous relation for the creation and annihilation operators:

$$a_i^{out} = \sum_j (\alpha_{ij}^* a_j^{in} - \beta_{ij}^* a_j^{in\dagger}) . \quad (7)$$

When the coefficients β_{ij} do not vanish the vacuum states $|in\rangle$ and $|out\rangle$, defined as $a_i^{in}|in\rangle = 0$ and $a_i^{out}|out\rangle = 0$, do not coincide and, as a consequence, the number of particles measured in the i^{th} mode by an ‘‘out’’ observer, $N_i^{out} = \hbar^{-1} a_i^{out\dagger} a_i^{out}$, in the state $|in\rangle$ is given by

$$\langle in | N_i^{out} | in \rangle = \sum_k |\beta_{ik}|^2 . \quad (8)$$

Let us now briefly summarize the main steps of Hawking’s derivation. Assuming for simplicity that the background is spherically symmetric we can choose the following basis for the ingoing and outgoing modes

$$u_{wlm}^{in}|_{I^-} \sim \frac{1}{\sqrt{4\pi w}} \frac{e^{-i w v}}{r} Y_l^m(\theta, \phi) , \quad (9)$$

$$u_{wlm}^{out}|_{I^+} \sim \frac{1}{\sqrt{4\pi w}} \frac{e^{-i w u}}{r} Y_l^m(\theta, \phi) . \quad (10)$$

Here $Y_l^m(\theta, \phi)$ are the spherical harmonics. One can evaluate the coefficients $\beta_{wlm, w'l'm'}$ according to previous expressions by making the convenient choice $\Sigma = I^-$

$$\beta_{wlm, w'l'm'} = i \int_{I^-} dv r^2 d\Omega (u_{wlm}^{out} \partial_v u_{w'l'm'}^{in} - u_{w'l'm'}^{in} \partial_v u_{wlm}^{out}) . \quad (11)$$

The angular integration is straightforward and leads to delta functions $\delta_{ll'} \delta_{m, -m'}$ for the β coefficients. The relevant point is to realize that the coefficients can be evaluated and have a unique answer, which turns out to be independent of the details of the collapse, if u_i^{out} represents a late-time wave-packet mode (i.e., centered around an instant u with $u \rightarrow +\infty$ along I^+). When these modes are propagated backwards in time they are largely blueshifted when they approach the event horizon. After passing through the collapsing body they are scattered to I^- in a very small interval just before v_H . To know how they behave on I^- (as needed to evaluate the scalar product with u_{wlm}^{in}) one can apply the geometrical optics approximation since the effective frequency, as measured by freely falling observers, is very large. The (late-time) mode u_{wlm}^{out} , which is of the form (10) at I^+ , evolves and arrives at I^- with the form

$$u_{wlm}^{out}|_{I^-} \sim \frac{t_l(w)}{\sqrt{4\pi w}} \frac{e^{-i w u(v)}}{r} Y_l^m(\theta, \phi) \Theta(v_H - v) , \quad (12)$$

where $t_l(w)$ is the transmission coefficient for the Schwarzschild metric and the relation between null inertial coordinates u at I^+ and v at I^- is typically given by the logarithmic term

$$u = v_H - \kappa^{-1} \ln \kappa |v_H - v| , \quad (13)$$

where, for the Schwarzschild black hole, $\kappa = 1/4M$ and v_H represents the location of the null ray that will form the event horizon H^+ (see Fig. 1). One has then all ingredients to work out the (late-time) Bogolubov coefficients

$$\beta_{wlm,w'l'm'} = \frac{-(-)^m t_l(w)}{2\pi} \sqrt{\frac{w'}{w}} \int_{-\infty}^{v_H} dv e^{-iw(v_H - \kappa^{-1} \ln \kappa |v_H - v|) - iw'v} \delta_{ll'} \delta_{m-m'} . \quad (14)$$

They can be evaluated explicitly

$$\beta_{wlm,w'l'm'} = \frac{-(-)^m t_l(w)}{2\pi\kappa} \sqrt{\frac{w'}{w}} \frac{e^{-i(w+w')v_H}}{(-\kappa^{-1}w'i + \epsilon)^{1+\kappa^{-1}wi}} \Gamma(1 + \kappa^{-1}wi) \delta_{ll'} \delta_{m-m'} , \quad (15)$$

where we have introduced a negative real part ($-\epsilon$) into the exponent of (14) to ensure convergence of the corresponding integrals. To compute the particle production at I^+ one has to evaluate the integral (from now on in this section we shall omit, for simplicity, the subscripts l, m)

$$\int_0^{+\infty} dw' \beta_{w_1 w'} \beta_{w_2 w'}^* . \quad (16)$$

The integration in w' reduces to

$$\int_0^{+\infty} \frac{dw'}{w'} e^{-\kappa^{-1}w_1 i \ln(-\kappa^{-1}w' - i\epsilon)} e^{\kappa^{-1}w_2 i \ln(\kappa^{-1}w' - i\epsilon)} = 2\pi\kappa e^{-\pi\kappa^{-1}w_1} \delta(w_1 - w_2) , \quad (17)$$

from which we finally get

$$\int_0^{+\infty} dw' \beta_{w_1 w'} \beta_{w_2 w'}^* = \frac{|t_l(w_1)|^2}{e^{2\pi\kappa^{-1}w_1} - 1} \delta(w_1 - w_2) , \quad (18)$$

where the coefficient in front of $\delta(w_1 - w_2)$ represents a steady thermal flow of radiation of frequency $w = w_1$

$$\frac{dN_{lm}(w)}{dw dt} \equiv \frac{1}{2\pi} \langle in | N_{wlm}^{out} | in \rangle = \frac{1}{2\pi} \frac{\Gamma_l(w)}{e^{2\pi\kappa^{-1}w} - 1} , \quad (19)$$

and the grey-body factor is given by $\Gamma_l(w) \equiv |t_l(w)|^2$. For a generic collapse the result leads to formula (1).

It is important to remark at this point that a basic step to exactly obtain the Planckian spectrum is (17), which crucially requires an unbounded integration in all frequencies w' . In fact, if we introduce an ultraviolet cutoff Λ for w' we should replace (17) by

$$\int_0^{+\Lambda} \frac{dw'}{w'} e^{-\kappa^{-1}w_1 i \ln(-\kappa^{-1}w' - i\epsilon)} e^{\kappa^{-1}w_2 i \ln(\kappa^{-1}w' - i\epsilon)} = e^{-\pi\kappa^{-1}w_1} 2\pi \delta_\sigma[\kappa^{-1}(w_1 - w_2)] , \quad (20)$$

where we have defined

$$\delta_\sigma[\kappa^{-1}(w_1 - w_2)] = \frac{\sin \left[\frac{\kappa^{-1}(w_1 - w_2)}{\sigma} \right]}{\pi \kappa^{-1}(w_1 - w_2)} \quad (21)$$

$$\sigma = \frac{1}{\ln[\kappa^{-1}\Lambda]} \quad (22)$$

Note that in the limit as σ goes to zero δ_σ turns into Dirac's delta function and we recover (17). The new expression is, however, qualitatively different from the previous one. To evaluate the new emission rate requires making use of normalized wave-packet modes. Introducing the standard ones [1]

$$u_{jnlm}^{out} = \frac{1}{\sqrt{\epsilon}} \int_{j\epsilon}^{(j+1)\epsilon} dw e^{2\pi i w n / \epsilon} u_{wlm}^{out} , \quad (23)$$

where $j \geq 0$ and n are integers, representing wave-packets peaked around the retarded time $u_n = 2\pi n / \epsilon$ and centered, with width ϵ , around the frequency $w_j \equiv (j + 1/2)\epsilon$; and, accordingly, defining

$$\beta_{jn,w'} = \frac{1}{\sqrt{\epsilon}} \int_{j\epsilon}^{(j+1)\epsilon} dw e^{2\pi i w n / \epsilon} \beta_{ww'} , \quad (24)$$

the emission rate results (see appendix A):

$$\langle in|N_{jn}^{out,\sigma}|in\rangle \approx \frac{|t_l(w_j)|^2}{e^{2\pi\kappa^{-1}w_j} - 1} \frac{\sin\left[\left(\frac{2\pi n}{\epsilon} - v_H\right)\frac{\pi\kappa\sigma}{2}\right]}{\left[\left(\frac{2\pi n}{\epsilon} - v_H\right)\frac{\pi\kappa\sigma}{2}\right]} \quad (25)$$

From this expression² we see that the rate of emitted particles depends on the retarded time $u_n = 2\pi n/\epsilon$ and decays with time for any small but nonzero value of $\sigma = 1/\ln[\Lambda/\kappa]$. Only when Λ goes to infinity (no high frequency cutoff) do we recover the steady thermal flux of radiation.

In conclusion, the above discussion shows that the radiation is now time-dependent and decays for all finite values of Λ . The decay in time would also occur at low frequencies, where string theory agrees with Hawking's prediction. Therefore, as expected from conventional arguments, the mathematical role of the ultrahigh frequencies is very important for the late-time behavior. Nevertheless, since they only enter as virtual quanta, their actual status is unclear [16]. A derivation of the Hawking effect based on quantities defined on the asymptotically flat region, where physical observations are made, would be preferable. This turns out to be possible if, instead of working with Bogolubov coefficients, one uses two-point functions. They are defined in the I^+ region where a Planck-length cutoff in "distances" can be naturally introduced. This is the task of next sections.

III. TWO-POINT FUNCTIONS AND BLACK HOLE RADIANCE

This section will be devoted to rederive Hawking radiation by means of two-point functions. Intuitively the idea is simple. In the conventional analysis in terms of Bogolubov coefficients, we first perform the integration in distances (to compute the scalar product required for the β coefficients) and leave to the end the integration in frequencies w' . In contrast, we can invert the order and perform first the integration in frequencies (which naturally leads to introduce the two-point function of the matter field) and perform the integration in distances at the end.

Let us rewrite the basic expression (8), or more precisely, the expectation values of the operator $N_{i_1 i_2}^{out} \equiv \hbar^{-1} a_{i_1}^{out\dagger} a_{i_2}^{out}$, as follows

$$\begin{aligned} \langle in|N_{i_1 i_2}^{out}|in\rangle &= \sum_k \beta_{i_1 k} \beta_{i_2 k}^* = - \sum_k (u_{i_1}^{out}, u_k^{in*})(u_{i_2}^{out*}, u_k^{in}) = \\ &= \sum_k \left(\int_{\Sigma} d\Sigma_1^\mu u_{i_1}^{out}(x_1) \overleftrightarrow{\partial}_\mu u_k^{in}(x_1) \right) \left(\int_{\Sigma} d\Sigma_2^\nu u_{i_2}^{out*}(x_2) \overleftrightarrow{\partial}_\nu u_k^{in*}(x_2) \right). \end{aligned} \quad (26)$$

If we now consider the sum in modes before making the integrals of the two scalar products, and take into account that

$$\langle in|\phi(x_1)\phi(x_2)|in\rangle = \hbar \sum_k u_k^{in}(x_1) u_k^{in*}(x_2), \quad (27)$$

we obtain a simple expression for the particle production number in terms of the two-point function

$$\langle in|N_{i_1 i_2}^{out}|in\rangle = \hbar^{-1} \int_{\Sigma} d\Sigma_1^\mu d\Sigma_2^\nu [u_{i_1}^{out}(x_1) \overleftrightarrow{\partial}_\mu][u_{i_2}^{out*}(x_2) \overleftrightarrow{\partial}_\nu] \langle in|\phi(x_1)\phi(x_2)|in\rangle. \quad (28)$$

In the above expression the two-point function should be then interpreted in the distributional sense. The " $i\epsilon$ -prescription" (see eq.(35) below) is therefore assumed for the two-point distribution $\langle in|\phi(x_1)\phi(x_2)|in\rangle$ and it verifies the Hadamard condition³ [5, 17]. Alternatively, taking into account the trivial identity $\langle out|a_{i_1}^{out\dagger} a_{i_2}^{out}|out\rangle = 0$ we can rewrite the above expression as [18]

$$\langle in|N_{i_1 i_2}^{out}|in\rangle = \hbar^{-1} \int_{\Sigma} d\Sigma_1^\mu d\Sigma_2^\nu [u_{i_1}^{out}(x_1) \overleftrightarrow{\partial}_\mu][u_{i_2}^{out*}(x_2) \overleftrightarrow{\partial}_\nu] \langle in| : \phi(x_1)\phi(x_2) : |in\rangle, \quad (29)$$

where $\langle in| : \phi(x_1)\phi(x_2) : |in\rangle \equiv \langle in|\phi(x_1)\phi(x_2)|in\rangle - \langle out|\phi(x_1)\phi(x_2)|out\rangle$. Now the Hadamard condition for both $|in\rangle$ and $|out\rangle$ states ensures that $\langle in| : \phi(x_1)\phi(x_2) : |in\rangle$ is a smooth function.

² We note that the oscillatory behavior in (25) is an artifact of the particular way we have introduced the cutoff. If the cutoff is introduced in a different way, see appendix A, the oscillatory term disappears but the decay with time is maintained as $\sim e^{-[(2\pi n/\epsilon - v_H)(\pi\kappa\sigma/2)]^2}$.

³ The two-point distribution should have, for all physical states, a short-distance structure similar to that of the ordinary vacuum state in Minkowski space: $(2\pi)^{-2}(\sigma + 2i\epsilon t + \epsilon^2)^{-1}$, where $\sigma(x_1, x_2)$ is the squared geodesic distance.

A. Thermal spectrum for a scalar field

Let us now apply this scheme in the formation process of a spherically symmetric black hole and restrict the “out” region to I^+ . The “in” region is, as usual, defined by I^- . At I^+ we can consider the conventional radial plane-wave modes

$$u_{wlm}^{out}(t, r, \theta, \phi)|_{I^+} \sim u_w^{out}(u) \frac{Y_l^m(\theta, \phi)}{r}, \quad (30)$$

where $u_w^{out}(u) = \frac{e^{-i w u}}{\sqrt{4\pi w}}$. We shall now evaluate the matrix coefficients $\langle in|N_{i_1 i_2}^{out}|in\rangle$ where $i_{1,2} \equiv (w_{1,2}, l_{1,2}, m_{1,2})$. Taking as the initial value hypersurface I^- and integrating by parts we obtain

$$\begin{aligned} \langle in|N_{i_1 i_2}^{out}|in\rangle &= \frac{4}{\hbar} \int_{I^-} r_1^2 dv_1 d\Omega_1 \int_{I^-} r_2^2 dv_2 d\Omega_2 u_{w_1}^{out} u_{w_2}^{out*} \times \\ &\frac{Y_{l_1}^{m_1}(\theta_1, \phi_1)}{r_1} \frac{Y_{l_2}^{m_2*}(\theta_2, \phi_2)}{r_2} \partial_{v_1} \partial_{v_2} \langle in|\phi(x_1)\phi(x_2)|in\rangle. \end{aligned} \quad (31)$$

The two-point function above can be now expanded at I^- as

$$\langle in|\phi(x_1)\phi(x_2)|in\rangle = \hbar \int_0^\infty dw \sum_{l,m} \frac{e^{-i w v_1}}{\sqrt{4\pi w}} \frac{Y_l^m(\theta_1, \phi_1)}{r_1} \frac{e^{i w v_2}}{\sqrt{4\pi w}} \frac{Y_l^{m*}(\theta_2, \phi_2)}{r_2}. \quad (32)$$

Recall that the radial part of the late-time “out” modes u_{wlm}^{out} , when they are propagated backward in time and reach I^- , takes the form

$$u_w^{out}|_{I^-} \sim t_l(w) \frac{e^{-i w u(v)}}{\sqrt{4\pi w}} \Theta(v_H - v) \quad (33)$$

where $u(v) \approx v_H - \kappa^{-1} \ln \kappa(v_H - v)$. Performing now angular integrations and taking into account that

$$\partial_{v_1} \partial_{v_2} \int_0^\infty dw \frac{e^{-i w (v_1 - v_2)}}{4\pi w} = -\frac{1}{4\pi} \frac{1}{(v_1 - v_2 - i\epsilon)^2}, \quad (34)$$

we get

$$\langle in|N_{i_1 i_2}^{out}|in\rangle = -\frac{t_{l_1}(w_1)t_{l_2}^*(w_2)}{4\pi^2 \sqrt{w_1 w_2}} \int_{-\infty}^{v_H} dv_1 dv_2 \frac{e^{-i w_1 u(v_1) + i w_2 u(v_2)}}{(v_1 - v_2 - i\epsilon)^2} \delta_{l_1 l_2} \delta_{m_1 m_2}, \quad (35)$$

where the limit $\epsilon \rightarrow 0^+$ is understood. Alternatively, since we are interested in quantities measured at $\Sigma = I^+$, we could use this latter hypersurface to carry out the calculations. In this case, the expression for the particle production rate becomes

$$\langle in|N_{i_1 i_2}^{out}|in\rangle = -\frac{t_{l_1}(w_1)t_{l_2}^*(w_2)}{4\pi^2 \sqrt{w_1 w_2}} \int_{-\infty}^\infty du_1 du_2 \frac{\frac{du}{du}(u_1) \frac{dv}{du}(u_2)}{[v(u_1) - v(u_2) - i\epsilon]^2} e^{-i w_1 u_1 + i w_2 u_2} \delta_{l_1 l_2} \delta_{m_1 m_2}, \quad (36)$$

and leads to

$$\langle in|N_{i_1 i_2}^{out}|in\rangle = \frac{-t_{l_1}(w_1)t_{l_2}^*(w_2)}{4\pi^2 \sqrt{w_1 w_2}} \int_{-\infty}^{+\infty} du_1 du_2 \frac{(\frac{\kappa}{2})^2 e^{-i w_1 u_1 + i w_2 u_2}}{[\sinh \frac{\kappa}{2}(u_1 - u_2 - i\epsilon)]^2} \delta_{l_1 l_2} \delta_{m_1 m_2}. \quad (37)$$

This last expression is more convenient for computational purposes. Since the function in the integral depends only on the difference $z \equiv u_2 - u_1$, the integral in $u_2 + u_1$ can be performed immediately and leads to a delta function in frequencies

$$\langle in|N_{i_1 i_2}^{out}|in\rangle = -\frac{t_{l_1}(w_1)t_{l_2}^*(w_2)\delta(w_1 - w_2)}{2\pi \sqrt{w_1 w_2}} \int_{-\infty}^{+\infty} dz e^{-i \frac{w_1 + w_2}{2} z} \frac{(\frac{\kappa}{2})^2 \delta_{l_1 l_2} \delta_{m_1 m_2}}{[\sinh \frac{\kappa}{2}(z - i\epsilon)]^2}, \quad (38)$$

Performing the integration in z we recover the Planckian spectrum and the particle production rate

$$\langle in|N_{wlm}^{out}|in\rangle = \frac{|t_l(w)|^2}{e^{2\pi \kappa^{-1} w} - 1}. \quad (39)$$

This derivation of black hole radiation is somewhat parallel to the one given in [9]. The emphasis is in the two-point function of the quantum state, instead of the usual treatment in terms of Bogolubov transformations. It is worth noting that (35) displays an apparent sensitivity to ultrashort distances due to the highly oscillatory behavior of the modes in a small region before v_H . A similar conclusion can be obtained from (38) when $z \rightarrow 0$. The sensitivity to short distances is, however, less apparent if we repeat the above calculations using the expression (29) instead of (28). In this case, we find

$$\langle in|N_{i_1 i_2}^{out}|in\rangle = -\frac{t_{l_1}(w_1)t_{l_2}^*(w_2)}{4\pi^2\sqrt{w_1 w_2}} \int_{-\infty}^{v_H} dv_1 dv_2 e^{-iw_1 u(v_1)+iw_2 u(v_2)} \times \left[\frac{1}{(v_1 - v_2)^2} - \frac{\frac{du}{dv}(v_1)\frac{du}{dv}(v_2)}{[u(v_1) - u(v_2)]^2} \right] \delta_{l_1 l_2} \delta_{m_1 m_2} , \quad (40)$$

where we have dropped the $i\epsilon$ terms since they are now redundant. Note that the short-distance divergence of $1/(v_1 - v_2)^2$ in (40) is exactly cancelled by $\frac{\frac{du}{dv}(v_1)\frac{du}{dv}(v_2)}{[u(v_1) - u(v_2)]^2}$ for any smooth choice of the function $u(v)$. This cancellation is a consequence of the Hadamard condition that verify both “in” and “out” vacuum states. It is also important to remark that the above formula exhibits the absence of particle production under conformal-type (Möbius) transformations

$$v = \frac{au + b}{cu + d} \quad (41)$$

where $ab - cd = 1$.⁴

If the calculation is performed using $\Sigma = I^+$, one finds

$$\langle in|N_{i_1 i_2}^{out}|in\rangle = -\frac{t_{l_1}(w_1)t_{l_2}^*(w_2)}{4\pi^2\sqrt{w_1 w_2}} \int_{I^+} du_1 du_2 e^{-iw_1 u_1 + iw_2 u_2} \times \left[\frac{\frac{dv}{du}(u_1)\frac{dv}{du}(u_2)}{(v(u_1) - v(u_2))^2} - \frac{1}{[u_1 - u_2]^2} \right] \delta_{l_1 l_2} \delta_{m_1 m_2} , \quad (42)$$

which leads to

$$\langle in|N_{i_1 i_2}^{out}|in\rangle = -\frac{t_{l_1}(w_1)t_{l_2}^*(w_2)\delta(w_1 - w_2)}{2\pi\sqrt{w_1 w_2}} \int_{-\infty}^{+\infty} dz e^{-i\frac{w_1 + w_2}{2}z} \times \left[\frac{(\frac{\kappa}{2})^2}{(\sinh \frac{\kappa}{2}z)^2} - \frac{1}{z^2} \right] \delta_{l_1 l_2} \delta_{m_1 m_2} . \quad (43)$$

The integral in distances z also leads, as expected, to the Hawking formula⁵

$$\begin{aligned} \langle in|N_{wlm}^{out}|in\rangle &= -\frac{|t_l(w)|^2}{2\pi w} \int_{-\infty}^{+\infty} dz e^{-iwz} \left[\frac{(\frac{\kappa}{2})^2}{(\sinh \frac{\kappa}{2}z)^2} - \frac{1}{z^2} \right] \\ &= \frac{|t_l(w)|^2}{e^{2\pi w\kappa^{-1}} - 1} . \end{aligned} \quad (44)$$

B. Thermal spectrum for a $s = 1/2$ field

In this subsection we shall extend the analysis of the scalar field to a fermionic $s = 1/2$ field. For simplicity we take a massless Dirac field, obeying the wave equation

$$\gamma^\mu \nabla_\mu \psi = 0 , \quad (45)$$

⁴ This leads, immediately, to the expected result that there is no particle production under Lorentz transformations.

⁵ For Kerr-Newman black holes the calculation is similar up to a shift in the wave function e^{-iwz} , which should be now replaced by $e^{-i(w-m\Omega_H - q\Phi_H)z}$, as an effect of wave propagation through the corresponding potential barrier.

where $\gamma^\mu = V_a^\mu \gamma^a$ are the curved space counterparts of the Dirac matrices γ^a (see appendix B for calculations omitted in this section). The Klein-Gordon scalar product (2) is now replaced by

$$(\psi_1, \psi_2) = \int_{\Sigma} d\Sigma^\mu \bar{\psi}_1 \gamma_\mu \psi_2 . \quad (46)$$

Therefore the expression for the expectation values (28) is replaced by

$$\langle in | N_{i_1 i_2}^{out} | in \rangle = \hbar^{-1} \int_{\Sigma} d\Sigma_1^\mu d\Sigma_2^\nu [\bar{u}_{i_2}^{out}(x_2) \gamma_\nu]_b [\gamma_\mu u_{i_1}^{out}(x_1)]^a \langle in | \bar{\psi}_a(x_1) \psi^b(x_2) | in \rangle . \quad (47)$$

At I^+ we can consider the normalized radial plane-wave modes⁶

$$u_{w\kappa_j m_j}^{out}(t, r, \theta, \phi)|_{I^+} \sim \frac{e^{-iwu}}{\sqrt{4\pi r}} \begin{pmatrix} \eta(\hat{r})_{\kappa_j}^{m_j} \\ (\hat{r}\vec{\sigma})\eta(\hat{r})_{\kappa_j}^{m_j} \end{pmatrix} , \quad (48)$$

where $\eta(\hat{r})_{\kappa_j}^{m_j}$ are two-component spinor harmonics (see appendix B). Note that the angular momentum quantum number j is uniquely determined by the relation $\kappa_j = \pm(j + 1/2)$. The above modes, when propagated backwards in time and reach I^- , turn into

$$u_{w\kappa_j m_j}^{out}(t, r, \theta, \phi)|_{I^-} \sim t_{\kappa_j}(w) \frac{e^{-iwu(v)}}{\sqrt{4\pi r}} \begin{pmatrix} \eta(\hat{r})_{\kappa_j}^{m_j} \\ -(\hat{r}\vec{\sigma})\eta(\hat{r})_{\kappa_j}^{m_j} \end{pmatrix} \Theta(v_H - v) \sqrt{\frac{du(v)}{dv}} , \quad (49)$$

where the last term $\sqrt{du(v)/dv}$ appears due to the fermionic character of the field⁷. Proceeding as in the bosonic case we can expand the two-point function as

$$\langle in | \bar{\psi}_a(x_1) \psi^b(x_2) | in \rangle = \hbar \sum_k \bar{v}_{k,a}^{in}(x_1) v_k^{in,b}(x_2) \quad (50)$$

where v_k^{in} are negative-energy solutions which in I^- take the form

$$v_k^{in} \rightarrow v_{w\kappa_j m}^{in}(t, r, \theta, \phi)|_{I^-} \sim \frac{e^{i w v}}{\sqrt{4\pi r}} \begin{pmatrix} \eta(\hat{r})_{\kappa_j}^{m_j} \\ -(\hat{r}\vec{\sigma})\eta(\hat{r})_{\kappa_j}^{m_j} \end{pmatrix} , \quad (51)$$

Performing first the angular integrations and taking into account the orthonormality relations of the spinor harmonics $\eta(\hat{r})_{\kappa_j}^{m_j}$, the above formulas get simplified and become

$$\begin{aligned} \langle in | N_{i_1 i_2}^{out} | in \rangle &= -i \frac{t_{\kappa_{j_1}}(w_1) t_{\kappa_{j_2}}^*(w_2)}{4\pi^2} \delta_{m_{j_1} m_{j_2}} \delta_{\kappa_{j_1} \kappa_{j_2}} \times \\ &\times \int_{-\infty}^{v_H} dv_1 dv_2 \sqrt{\frac{du(v_1)}{dv} \frac{du(v_2)}{dv}} \frac{e^{-i w_1 u(v_1) + i w_2 u(v_2)}}{(v_1 - v_2 - i\epsilon)} \end{aligned} \quad (52)$$

As in the bosonic case, we rewrite this expression as an integral over I^+

$$\begin{aligned} \langle in | N_{i_1 i_2}^{out} | in \rangle &= -i \frac{t_{\kappa_{j_1}}(w_1) t_{\kappa_{j_2}}^*(w_2)}{4\pi^2} \delta_{m_{j_1} m_{j_2}} \delta_{\kappa_{j_1} \kappa_{j_2}} \times \\ &\times \int_{-\infty}^{\infty} du_1 du_2 e^{-i w_1 u_1 + i w_2 u_2} \frac{(\frac{\kappa}{2})}{\sinh[\frac{\kappa}{2}(u_1 - u_2 - i\epsilon)]} \end{aligned} \quad (53)$$

We can also split the integral in a product of a function dependent on $u_2 + u_1$ and another function which depends on $z \equiv u_2 - u_1$. The former leads to a delta function in frequencies and we are left with

$$\begin{aligned} \langle in | N_{i_1 i_2}^{out} | in \rangle &= -\frac{i}{2\pi} t_{\kappa_{j_1}}(w_1) t_{\kappa_{j_2}}^*(w_2) \delta_{m_{j_1} m_{j_2}} \delta_{\kappa_{j_1} \kappa_{j_2}} \delta(w_1 - w_2) \times \\ &\times \int_{-\infty}^{+\infty} dz e^{-i \frac{w_1 + w_2}{2} z} \frac{(\frac{\kappa}{2})}{\sinh[\frac{\kappa}{2}(z - i\epsilon)]} , \end{aligned} \quad (54)$$

⁶ On physical grounds we should use left-handed spinors $u_{L,w\kappa_j m_j}^{out} \equiv \frac{1}{\sqrt{2}}(u_w^{out} |_{\kappa_j m_j} - u_w^{out} |_{-\kappa_j m_j})$. The final result is not changed. See appendix B.

⁷ The vierbein fields needed to properly write the field equation in a curved spacetime have been trivially fixed in the asymptotic flat regions, so its transformation law under change of coordinates is then translated to the spinor itself (see also appendix B).

Performing now the integration in z

$$\frac{-i}{2\pi} \int_{-\infty}^{+\infty} dz e^{-iwz} \frac{\left(\frac{\kappa}{2}\right)}{\sinh\left[\frac{\kappa}{2}(z - i\epsilon)\right]} = \frac{1}{e^{2\pi w \kappa^{-1}} + 1}, \quad (55)$$

we recover the Planckian spectrum, with the Dirac-Fermi statistics, and the corresponding particle production rate

$$\langle in | N_{w m_j \kappa_j}^{out} | in \rangle = \frac{|t_{\kappa_j}(w)|^2}{e^{2\pi \kappa^{-1} w} + 1}. \quad (56)$$

Analogously as in the bosonic case, if we use the normal-ordering prescription instead of the $i\epsilon$ one we find

$$\begin{aligned} \langle in | N_{i_1 i_2}^{out} | in \rangle &= -i \frac{t_{\kappa_{j_1}}(w_1) t_{\kappa_{j_2}}(w_2)^*}{4\pi^2} \int_{-\infty}^{v_H} dv_1 dv_2 \sqrt{\frac{du(v_1)}{dv} \frac{du(v_2)}{dv}} \times \\ &e^{-iw_1 u(v_1) + iw_2 u(v_2)} \left[\frac{1}{[v_1 - v_2]} - \frac{\sqrt{\frac{du(v_1)}{dv} \frac{du(v_2)}{dv}}}{[u(v_1) - u(v_2)]} \right] \delta_{\kappa_{j_1} \kappa_{j_2}} \delta_{m_{j_1} m_{j_2}}. \end{aligned} \quad (57)$$

Changing the integration surface to I^+ we get

$$\begin{aligned} \langle in | N_{i_1 i_2}^{out} | in \rangle &= -i \frac{t_{\kappa_{j_1}}(w_1) t_{\kappa_{j_2}}(w_2)^*}{4\pi^2} \int_{I^+} du_1 du_2 \sqrt{\frac{dv(u_1)}{du} \frac{dv(u_2)}{du}} \times \\ &e^{-iw_1 u_1 + iw_2 u_2} \left[\frac{\sqrt{\frac{dv(u_1)}{du} \frac{dv(u_2)}{du}}}{[v(u_1) - v(u_2)]} - \frac{1}{[u_1 - u_2]} \right] \delta_{\kappa_{j_1} \kappa_{j_2}} \delta_{m_{j_1} m_{j_2}}. \end{aligned} \quad (58)$$

Note again that the short-distance divergence of the two-point function of the “out” state $\frac{1}{[u_1 - u_2]}$ is exactly cancelled by the corresponding one of the “in” state $\frac{\sqrt{\frac{dv(u_1)}{du} \frac{dv(u_2)}{du}}}{[v(u_1) - v(u_2)]}$, since both vacua are Hadamard states. After some algebra we get

$$\langle in | N_{w \kappa_j m_j}^{out} | in \rangle = -i \frac{|t_{\kappa_j}(w)|^2}{2\pi} \int_{-\infty}^{+\infty} dz e^{-iwz} \left[\frac{\left(\frac{\kappa}{2}\right)}{\sinh\left(\frac{\kappa}{2}z\right)} - \frac{1}{z} \right], \quad (59)$$

and taking into account

$$\frac{-i}{2\pi} \int_{-\infty}^{+\infty} dz e^{-iwz} \left[\frac{\left(\frac{\kappa}{2}\right)}{\sinh\left(\frac{\kappa}{2}z\right)} - \frac{1}{z} \right] = \frac{1}{e^{2\pi w \kappa^{-1}} + 1}, \quad (60)$$

we newly recover the fermionic thermal spectrum.

IV. SHORT-DISTANCE CONTRIBUTION TO THE PLANCKIAN SPECTRUM

A. Bosons

We have seen in the previous section that it is possible to rederive the Hawking effect in terms of two-point functions. Either via expressions (28), (36) or, equivalently, via expressions (29),(42). Both prescriptions are equivalent and lead to the Planckian spectrum modulated by grey-body factors. The advantage of the final expression (44) is that it offers an explicit evaluation of the contribution of distances to the Planckian spectrum. To be more explicit, the integral⁸

$$I^B(w\kappa^{-1}, \alpha\kappa) = -\frac{1}{2\pi w} \int_{-\alpha}^{+\alpha} dz e^{-iwz} \left[\frac{\left(\frac{\kappa}{2}\right)^2}{\left(\sinh\frac{\kappa}{2}z\right)^2} - \frac{1}{z^2} \right], \quad (61)$$

⁸ We have intentionally omitted the grey-body factors $|t_l(w)|^2$ in (61) because they are irrelevant for the discussion of this section.

can be interpreted as the contribution of short-distances $z \in [-\alpha, \alpha]$ to the (bosonic) thermal spectrum when α is close to the Planck length l_P . One could, alternatively, be tempted to propose, according to (36), the integral

$$I_{i\epsilon}^B(w\kappa^{-1}, \alpha\kappa) \equiv \frac{-1}{2\pi w} \int_{-\alpha}^{+\alpha} dz e^{-iwz} \frac{(\frac{\kappa}{2})^2}{[\sinh \frac{\kappa}{2}(z - i\epsilon)]^2} \quad (62)$$

as a legitimate expression to account for the short-distance contributions. However, this interpretation is not physically sound. In the absence of a black hole, when there is no radiation at all, the above expression becomes

$$\frac{-1}{2\pi w} \int_{-\alpha}^{+\alpha} dz e^{-iwz} \frac{1}{(z - i\epsilon)^2}. \quad (63)$$

For $\alpha \rightarrow +\infty$ this expression vanishes, as expected due to the absence of radiation. However, for finite α it is non-vanishing. In contrast, the proposed expression (61), does not suffer from this weird behavior, due to the presence of the second term.

In conclusion, the calculation of black hole radiation using the prescription (29) offers the possibility to re-evaluate Hawking radiation by removing the range of distances where physics can be dominated by an underlying theory beyond field theory. We shall now work out explicitly the short-distance contribution to Hawking radiation to see whether it is fundamental or not in order to obtain the thermal spectrum. The integral (61) can be worked out analytically

$$\begin{aligned} I^B(w\kappa^{-1}, \alpha\kappa) = & -\frac{Si(\alpha w)}{\pi} - \frac{\kappa}{4\pi w} \{e^{i\alpha w} (F[1, -iw\kappa^{-1}, 1 - iw\kappa^{-1}, e^{-\alpha\kappa}] \\ & - F[1, iw\kappa^{-1}, 1 + iw\kappa^{-1}, e^{\alpha\kappa}]) + e^{-i\alpha w} (F[1, iw\kappa^{-1}, 1 + iw\kappa^{-1}, e^{-\alpha\kappa}] \\ & - F[1, -iw\kappa^{-1}, 1 - iw\kappa^{-1}, e^{\alpha\kappa}])\} + \frac{1}{2\pi\alpha w} \cos(\alpha w) \left[\alpha\kappa \frac{(1 + e^{\alpha\kappa})}{(e^{\alpha\kappa} - 1)} - 2 \right] \end{aligned} \quad (64)$$

where F is a hypergeometric function and $Si(x) = \int_0^x dt \frac{\sin t}{t}$. To get some insight about the properties of this formula, we find useful to expand it in powers of $w\kappa^{-1}$ and $\alpha\kappa$. The expansion in $\alpha\kappa$ assumes that the microscopic length scale $\alpha \sim l_P$ is much smaller than the typical emission wavelength $\sim \kappa^{-1}$ of the black hole, whose (macroscopic) temperature is $T_H = \kappa/2\pi$. For a Solar-mass black hole $\alpha\kappa \sim 10^{-40}$ and for a primordial black hole of 10^{15} g $\alpha\kappa \sim 10^{-21}$. The expansion in $w\kappa^{-1}$ means that we are looking at frequencies below the typical emission frequency, $w_{typical} \sim T_H$, of the black hole. The result is as follows

$$\begin{aligned} I^B(w\kappa^{-1}, \alpha\kappa) = & \left(\frac{1}{12\pi} \alpha\kappa - \frac{1}{720\pi} (\alpha\kappa)^3 + O[(\alpha\kappa)^5] \right) \frac{\kappa}{w} \\ & - \left(\frac{1}{72\pi} (\alpha\kappa)^3 + O[(\alpha\kappa)^5] \right) \frac{w}{\kappa} + (O[(\alpha\kappa)^5]) \left(\frac{w}{\kappa} \right)^3 + \dots \end{aligned} \quad (65)$$

From this expansion we conclude that the contribution of short distances to the spectrum is completely negligible in the very low energy regime $w/\kappa \ll 1$ since

$$\lim_{w\kappa^{-1} \rightarrow 0} \frac{I^B(w\kappa^{-1}, \alpha\kappa)}{(e^{2\pi w\kappa^{-1}} - 1)^{-1}} = \frac{\alpha\kappa}{6} \ll 1. \quad (66)$$

Moreover, due to the smallness of $\alpha\kappa$, we find that $I^B(w_{typical}\kappa^{-1}, \alpha\kappa)$ can be well approximated by (65) even for frequencies close to the typical emission frequency, which leads to

$$\frac{I^B(w_{typical}\kappa^{-1}, \alpha\kappa)}{(e^{2\pi w_{typical}\kappa^{-1}} - 1)^{-1}} \sim 0.3 \alpha\kappa \ll 1. \quad (67)$$

Again, since $\alpha\kappa \ll 1$, we find a negligible contribution at $w_{typical} \sim T_H$. To be precise, for a Schwarzschild black hole of three solar masses, when α is around the Planck length $l_P = 1.6 \times 10^{-33} \text{ cm}$, the relative contribution to the Planckian distribution $\frac{I^B(w\kappa^{-1}, \alpha\kappa)}{(e^{2\pi w\kappa^{-1}} - 1)^{-1}}$ is, for $w = w_{typical}$, of order $10^{-38}\%$. For primordial black holes, $M \sim 10^{15}$ g, the relative contribution is still insignificant: $10^{-19}\%$. Even more, using the expansion (65) we easily get

$$\frac{I^B(w\kappa^{-1}, \alpha\kappa)}{(e^{2\pi w\kappa^{-1}} - 1)^{-1}} \approx \frac{\alpha\kappa(e^{2\pi w\kappa^{-1}} - 1)}{12\pi w\kappa^{-1}}, \quad (68)$$

and we find that, for a black hole of three solar masses, we need to look at the high frequency region, $w/w_{typical} \approx 96$, to find that the contribution of Planck distances $I^B(w\kappa^{-1}, l_P\kappa)$ is of order of the total spectrum itself⁹. For primordial black holes we find $w/w_{typical} \approx 52$. The same numerical estimates can be found using the exact analytical expressions.

We can also naturally ask about the contribution to the spectrum of large distances. This question is immediately answered using our analytical expression (64). The contribution of distances up to $\alpha = 20r_g$, where r_g is the gravitational radius, represents 90% of the thermal peak at $w_{typical}$. For $\alpha = 200r_g$ we obtain 99.7% and for $\alpha = 2 \times 10^4 r_g$ the percentage is around 99.99998%.

Summarizing, we have provided a quantitative estimate of how much of Hawking radiation is actually due to Planckian distances. It turns out that the contribution of ultrashort distances is negligible and thermal radiation is very robust up to frequencies of order $96T_H$ (for Schwarzschild black holes of three solar masses) or $52T_H$ (for primordial black holes). In parallel and dual to this, the contribution of large distances is also insignificant.

It is interesting to repeat the same calculations with the $i\epsilon$ -prescription. As we have already stressed with this prescription one cannot expect a meaningful result. The outcome is completely different. The contribution of distances in the interval $z \in [-\alpha, +\alpha]$ is now

$$I_{i\epsilon \rightarrow 0}^B(w\kappa^{-1}, \alpha\kappa) = \frac{e^{\alpha\kappa(1-iw\kappa^{-1})} + e^{i\alpha w}}{2\pi w\kappa^{-1}(e^{\alpha\kappa} - 1)} + \frac{1}{2\pi(i + w\kappa^{-1})} \{e^{\alpha\kappa(1-iw\kappa)} \times \\ F[1, 1 - iw\kappa^{-1}, 2 - iw\kappa^{-1}, e^{\alpha\kappa}] - e^{-\alpha\kappa(1-iw\kappa)} \times \\ F[1, 1 - iw\kappa^{-1}, 2 - iw\kappa^{-1}, e^{-\alpha\kappa}]\} \quad (69)$$

Here, even in the very low energy regime, the contribution of short distances is not negligible. In fact, it is much bigger than the thermal spectrum itself. To see this we can approximate $I_{i\epsilon}^B(w\kappa^{-1}, \alpha\kappa)$ as

$$I_{i\epsilon}^B(w\kappa^{-1}, \alpha\kappa) = \left(\frac{1}{\pi\alpha\kappa} + \frac{\alpha\kappa}{12\pi} - \frac{(\alpha\kappa)^3}{720\pi} + O((\alpha\kappa)^5) \right) \frac{\kappa}{w} - \frac{1}{2} \\ + \left(\frac{\alpha\kappa}{2\pi} - \frac{(\alpha\kappa)^3}{72\pi} + O((\alpha\kappa)^5) \right) \frac{w}{\kappa} - \left(\frac{(\alpha\kappa)^3}{72\pi} + O((\alpha\kappa)^5) \right) \left(\frac{w}{\kappa} \right)^3 + \dots \quad (70)$$

Note that in this case the dominant term is of order $1/\alpha\kappa$, therefore

$$\lim_{\kappa^{-1}w \rightarrow 0} \frac{I_{i\epsilon}^B(w\kappa^{-1}, \alpha\kappa)}{(e^{2\pi w\kappa^{-1}} - 1)^{-1}} = \frac{2}{\alpha\kappa} \gg 1. \quad (71)$$

A similar behavior can be found for $w \approx w_{typical}$.

We illustrate the difference between both calculations in Fig.2. With the normal-ordering prescription the short-distance contribution is small, in contrast with the $i\epsilon$ -prescription. We have chosen a large surface gravity and different values of α to better show the effect in the drawings. We clearly observe that, although both prescriptions lead to the thermal result when $\alpha \rightarrow +\infty$, they do the job in very different ways.

⁹ The exponential behavior in frequencies of the ratio (68) explains why potential deviations from thermality arise at frequencies much lower than $w \sim 1/l_P$.

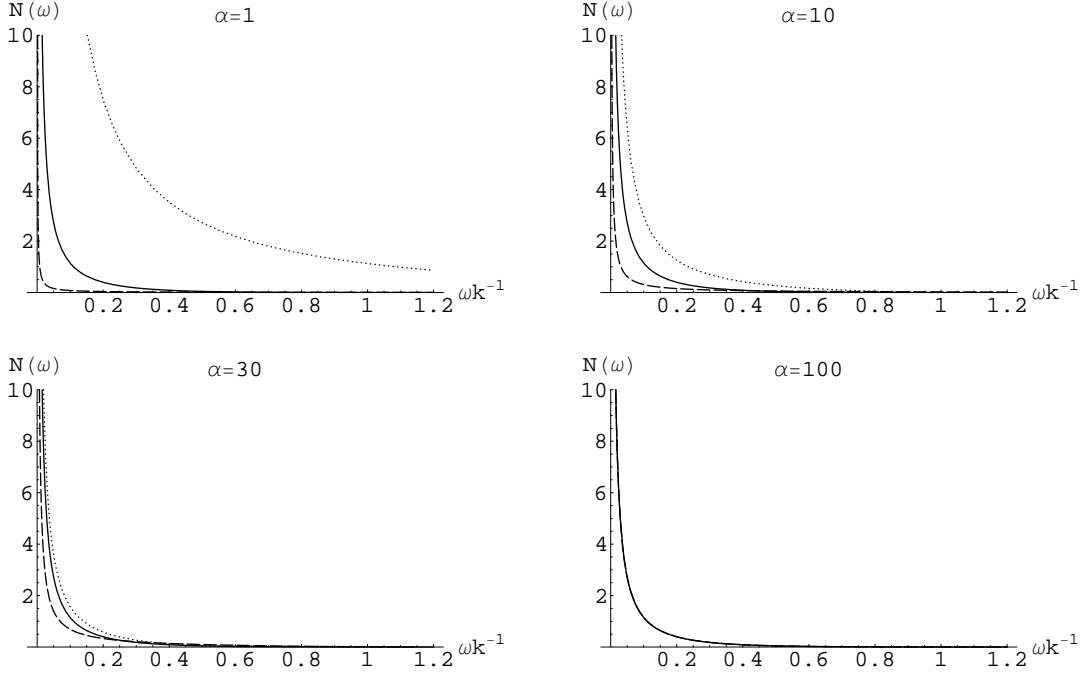


Figure 2: Plot comparing the Planckian spectrum $N(w, \kappa) = (e^{2\pi w \kappa^{-1}} - 1)^{-1}$ (solid line) with the contributions I^B (dashed line) and $I_{i\epsilon}^B$ (dotted line) coming from distances $|z| \lesssim \alpha$ according to the normal-ordering prescription and the $i\epsilon$ prescription, respectively. We have taken $\kappa = 0.1$ and $\alpha = 1, 10, 30$ and 100 (in Planck units), respectively.

B. Fermions

We shall extend the previous analysis to fermions. The integral involved is

$$\begin{aligned}
 I^F(w\kappa^{-1}, \alpha\kappa) &\equiv \frac{-i}{2\pi} \int_{-\alpha}^{+\alpha} dz e^{-i\omega z} \left[\frac{(\frac{\kappa}{2})}{(\sinh \frac{\kappa}{2} z)^2} - \frac{1}{z} \right] = \\
 &\frac{Si(\alpha\omega)}{\pi} + \frac{1}{2\pi(1+4w^2\kappa^{-2})} \{ (-i+2w\kappa^{-1})(e^{-\alpha\kappa/2+i\alpha\omega} \times \\
 &F[1, \frac{1}{2} - iw\kappa^{-1}, \frac{3}{2} - iw\kappa^{-1}, e^{-\alpha\kappa}] - F[1, \frac{1}{2} - iw\kappa^{-1}, \frac{3}{2} - iw\kappa^{-1}, e^{\alpha\kappa}] \times \\
 &e^{\alpha\kappa/2-i\alpha\omega} + (i+2w\kappa^{-1})(e^{-\alpha\kappa/2-i\alpha\omega} F[1, \frac{1}{2} + iw\kappa^{-1}, \frac{3}{2} + iw\kappa^{-1}, e^{-\alpha\kappa}] \\
 &- e^{\alpha\kappa/2+i\alpha\omega} F[1, \frac{1}{2} + iw\kappa^{-1}, \frac{3}{2} + iw\kappa^{-1}, e^{\alpha\kappa}]) \} . \quad (72)
 \end{aligned}$$

See Fig. 3 for a graphical representation. Taking into account that $\alpha\kappa \ll 1$ we can expand $I^F(w\kappa^{-1}, \alpha\kappa)$ as

$$I^F(w\kappa^{-1}, \alpha\kappa) = \left(\frac{(\alpha\kappa)^3}{72\pi} + O[(\alpha\kappa)^5] \right) \frac{w}{\kappa} + O[(\alpha\kappa)^5] \left(\frac{w}{\kappa} \right)^3 + \dots \quad (73)$$

Note that the term proportional to κ/w , appearing in the bosonic case, has disappeared. Therefore, for very low frequencies

$$\frac{I^F(w\kappa^{-1}, \alpha\kappa)}{(e^{2\pi w \kappa^{-1}} + 1)^{-1}} \sim \frac{(\alpha\kappa)^3 w}{36\pi \kappa} \ll 1 . \quad (74)$$

This shows that the contribution of ultrashort distances is negligible, like in the bosonic case. Moreover, for typical Hawking frequencies, $w_{\text{typical}} = T_H$, we have

$$\frac{I^F(w_{\text{typical}}\kappa^{-1}, \alpha\kappa)}{(e^{2\pi w_{\text{typical}}\kappa^{-1}} + 1)^{-1}} \sim 3 \cdot 10^{-3} (\alpha\kappa)^3 \ll 1 . \quad (75)$$

This rate is again very small, but the above expressions unravel the fact that the short-distance contribution for fermions seems to be smaller than that of bosons. For the latter the contribution of short distances is proportional to the first power of $\kappa\alpha$ while for fermions it is the third power.

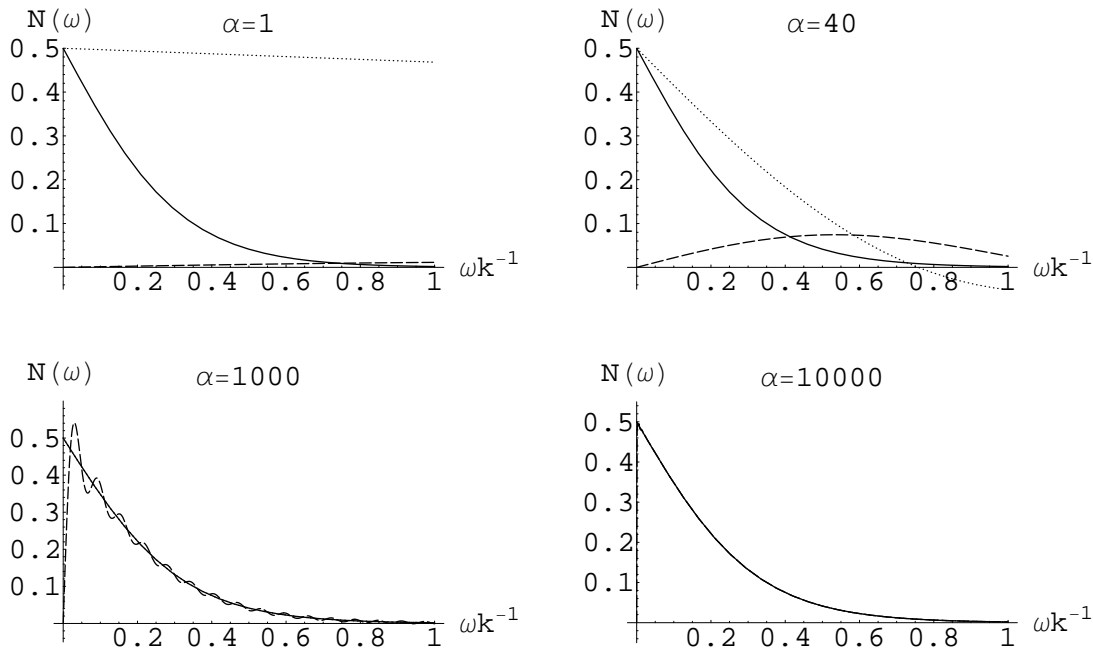


Figure 3: Plot comparing the Dirac-Fermi distribution (solid line) $N(w, \kappa) = (e^{2\pi w\kappa^{-1}} + 1)^{-1}$ with the contribution I^F coming from distances $|z| < \alpha$ according to the normal-ordering prescription (dashed line). For completeness we have also plotted the result obtained with the $i\epsilon$ -prescription (dotted line). We have taken $\kappa = 0.1$ and $\alpha = 1, 40, 10^3$ and 10^4 (in Planck units), respectively.

Finally, let us give numerical estimates for relevant astrophysical black holes using the expansion (73)

$$\frac{I^F(w\kappa^{-1}, \alpha\kappa)}{(e^{2\pi w\kappa^{-1}} + 1)^{-1}} \approx \frac{\alpha^3 w \kappa^2 (e^{2\pi w\kappa^{-1}} + 1)}{72\pi}. \quad (76)$$

For a black hole of three solar masses, the relative contribution to the total Planckian spectrum is, for $w = w_{\text{typical}}$, of order $10^{-118}\%$ and one must go to frequencies of order $w/w_{\text{typical}} \approx 270$ to find contributions $I^F(w\kappa^{-1}, l_P\kappa)$ of the same order as the total spectrum. For primordial black holes, $M \sim 10^{15}$ g, the relative contribution is $10^{-62}\%$ at w_{typical} and we have to reach frequencies of order $142w_{\text{typical}}$ to get a short-distance contribution of order of the thermal distribution. In addition to the conclusions stressed in the bosonic case, namely the robustness of Hawking thermal radiation for wavelengths of order of the size of the black hole, we have a new result. Fermions seem to be less sensitive to ultrashort distance physics than (spinless) bosons.

V. MODIFYING THE TWO-POINT FUNCTIONS AT SHORT-DISTANCES

In the previous section, we have investigated the contribution to the Hawking spectrum coming from distances $z < l_P$ at I^+ assuming that the physical laws are not modified at such scales. We found that potential deviations from thermality only manifest themselves at high frequencies. One can legitimately wonder, however, why we looked at distances at I^+ instead of at I^- , where the sensitivity of the “in” state to short distances is more apparent. In fact, imposing naively a cutoff at I^- has dramatic effects on the radiation due to the enormous redshift caused by the horizon (see section II). We were motivated to impose the cutoff at I^+ in order to find agreement with the view offered by string theory. The purpose of this section is to shed light on the roles played by distances at I^+ and I^- by using a simple model with a modified two-point function. We shall investigate the potential effects on the radiation due to the modified short-distance behavior of the matter field, supposedly coming from unknown physics at the

Planck scale. We shall see how our model maintains the robustness of the Hawking thermal spectrum at I^+ , while at the same time being insensitive to sub-Planckian distances at I^- .

Let us assume that the standard two-point function for the spin zero “in” and “out” vacuum states at I^- and I^+ , respectively, gets modified by new physics at very short distances and becomes

$$\begin{aligned} G^{in}|_{I^-} &\equiv -\frac{1}{4\pi} \frac{1}{(v_1 - v_2)^2} \rightarrow G_\alpha^{in}|_{I^-} \equiv -\frac{1}{4\pi} \frac{1}{(v_1 - v_2)^2 + \alpha^2} \\ G^{out}|_{I^+} &\equiv -\frac{1}{4\pi} \frac{1}{(u_1 - u_2)^2} \rightarrow G_\alpha^{out}|_{I^+} \equiv -\frac{1}{4\pi} \frac{1}{(u_1 - u_2)^2 + \alpha^2}, \end{aligned} \quad (77)$$

where α is a parameter of order of the Planck length: $\alpha \sim l_P$. With this modification the expression (40) for the black hole particle production becomes (we omit the transmission coefficients $t_l(w)$ and the angular delta functions $\delta_{l_1 l_2} \delta_{m_1 m_2}$ since they are also irrelevant for the discussion of this section)

$$\begin{aligned} \langle in|N_{i_1 i_2}^{out}|in\rangle &= 4 \int_{-\infty}^{v_H} dv_1 dv_2 u_{w_1}^{out}(v_1) u_{w_2}^{out*}(v_2) \times \\ &\quad \left[-\frac{1}{4\pi} \frac{1}{(v_1 - v_2)^2 + \alpha^2} - \frac{dv}{dv}(v_1) \frac{du}{dv}(v_2) G_\alpha^{out}|_{I^-} \right], \end{aligned} \quad (78)$$

where u_w^{out} and $G_\alpha^{out}|_{I^-}$ are understood to be the “out” modes and the “out” two-point function, respectively, propagated back to I^- . Since, according to the standard derivation, the propagation to I^- implies a strong blueshift, the u_w^{out} modes and G_α^{out} might manifest some dependence on the particular details of the modified theory, which are unknown to us. Thus, we see no simple way to estimate the form of the u_w^{out} modes at I^- . For this reason, it is preferable to evaluate the particle production as an integral on I^+ , as in (42),

$$\begin{aligned} \langle in|N_{i_1 i_2}^{out}|in\rangle &= 4 \int_{I^+} du_1 du_2 u_{w_1}^{out}(u_1) u_{w_2}^{out*}(u_2) \times \\ &\quad \times \left[\frac{dv_1}{du_1} \frac{dv_2}{du_2} G_\alpha^{in}|_{I^+} + \frac{1}{4\pi} \frac{1}{(u_1 - u_2)^2 + \alpha^2} \right], \end{aligned} \quad (79)$$

where $G_\alpha^{in}|_{I^+}$ is understood to be the “in” two-point function propagated to I^+ . In this region we can use the standard form of the “out” modes $u_{w_1}^{out}(u_1) = \frac{e^{-i w_1 u_1}}{\sqrt{4\pi w}}$ since we are considering emission frequencies much lower than the Planck frequency $w_P \sim 1/l_P$. We still have to unravel the evolution of G_α^{in} to evaluate the above expression. The modified short-distance physics near the horizon could dramatically modify the evolution of the two-point function, so that its form at I^+ could be rather different from the standard one $G^{in}|_{I^-}$. However, we can make the reasonable assumption that the propagation to I^+ is affected by new physics in such a way that the short-distance behavior of $\frac{dv_1}{du_1} \frac{dv_2}{du_2} G_\alpha^{in}$ at I^+ is identical to that of the two-point function for the “out” state

$$\lim_{u_1 \rightarrow u_2} \frac{dv}{du}(u_1) \frac{dv}{du}(u_2) G_\alpha^{in}|_{I^+} \sim \lim_{u_1 \rightarrow u_2} G_\alpha^{out}(u_1, u_2)|_{I^+}. \quad (80)$$

The above condition can be seen as a natural generalization of the Hadamard condition, i.e., universality of the short-distance behavior for all quantum states. The Hadamard condition, which plays a pivotal role in the algebraic formulation of QFT in curved spacetime [5], ensures the regularity of expression (29) to evaluate the Hawking radiation. Let us see now how (80) constraints the evolution of G_α^{in} from I^- to I^+ . Note that G_α^{in} can be rewritten as

$$G_\alpha^{in} = \frac{G^{in}}{1 + \alpha^2 G^{in}}, \quad (81)$$

where G^{in} is the unmodified two-point function. Since, at late times, G^{in} evolves according to geometrical optics approximation

$$\tilde{G}^{in}|_{I^+} \equiv \frac{dv}{du}(u_1) \frac{dv}{du}(u_2) G^{in}|_{I^+} = -\frac{1}{4\pi} \frac{\frac{dv_1}{du_1} \frac{dv_2}{du_2}}{(v(u_1) - v(u_2))^2}, \quad (82)$$

expression (81) suggests the following evolution for G_α^{in}

$$\tilde{G}_\alpha^{in}|_{I^+} \equiv \frac{dv}{du}(u_1) \frac{dv}{du}(u_2) G_\alpha^{in}|_{I^+} = -\frac{1}{4\pi} \frac{\frac{dv(u_1)}{du} \frac{dv(u_2)}{du}}{(v_1 - v_2)^2 + \alpha^2 \frac{dv_1}{du_1} \frac{dv_2}{du_2}}. \quad (83)$$

This expression guarantees immediately the Hadamard condition (80). We should stress, however, that the evolution of the modified two-point function itself is not equivalent, at least for very small point separations $(u_2 - u_1)^2 \sim \alpha^2$, to the ray tracing (or geometrical optics approximation), which would produce instead (91) (see later) and violate the Hadamard condition. For larger separations $(u_2 - u_1)^2 \gg \alpha^2$ the propagation agrees, as it must, with standard relativistic field theory and is driven by the large redshift (implying then the usual geometrical optics approximation). Plugging this expression in (79) we obtain

$$\begin{aligned} \langle in | N_{i_1 i_2}^{out} | in \rangle &= \frac{-1}{4\pi^2 \sqrt{\omega_1 \omega_2}} \int_{I_+} du_1 du_2 e^{-i(w_1 u_1 - w_2 u_2)} \\ &\times \left[\frac{\frac{dv(u_1)}{du} \frac{dv(u_2)}{du}}{(v_1 - v_2)^2 + \alpha^2 \frac{dv(u_1)}{du} \frac{dv(u_2)}{du}} - \frac{1}{(u_1 - u_2)^2 + \alpha^2} \right]. \end{aligned} \quad (84)$$

It is worth noting that the modified term

$$-4\pi \tilde{G}_\alpha^{in} |_{I_+} \equiv \frac{dv_1}{du_1} \frac{dv_2}{du_2} \frac{1}{(v_1 - v_2)^2 + \alpha^2 \frac{dv_1}{du_1} \frac{dv_2}{du_2}}, \quad (85)$$

which can also be regarded as a transformation law under the change $v = v(u)$, guarantees the absence of particle production under the same group of symmetry transformations (Möbius rescalings) as those of the theory with $\alpha = 0$.¹⁰ Assuming that the geometry remains classical (the black hole scale κ is well above the Planck scale α), we can use in (84) the expression $v(u) = v_H - \kappa^{-1} e^{-\kappa u}$, which represents the relation between the “in” and “out” inertial coordinates. Performing then the integration in $u_2 + u_1$, we are left with ($z \equiv u_2 - u_1$)

$$\langle in | N_{wlm}^{out} | in \rangle = -\frac{1}{2\pi w} \int_{-\infty}^{+\infty} dz e^{-i w z} \left[\frac{(\frac{\kappa}{2})^2}{(\sinh \frac{\kappa}{2} z)^2 + (\frac{\kappa}{2})^2 \alpha^2} - \frac{1}{z^2 + \alpha^2} \right]. \quad (86)$$

Finally, performing the integration in the complex plane, the particle production rate becomes

$$\langle in | N_{wlm}^{out} | in \rangle = \frac{1}{(e^{2\pi w \kappa^{-1}} - 1)} \frac{1}{2w\alpha \sqrt{1 - \alpha^2 \kappa^2/4}} (e^{w\kappa^{-1}\theta} - e^{w\kappa^{-1}(2\pi-\theta)}) + \frac{e^{-w\alpha}}{2\alpha w} \quad (87)$$

where

$$\theta = \arctan \frac{\alpha \kappa \sqrt{1 - \alpha^2 \kappa^2/4}}{(1 - \alpha^2 \kappa^2/2)}. \quad (88)$$

The thermal Planckian spectrum is smoothly recovered in the limit $\alpha \rightarrow 0$. Moreover, for $\alpha \sim l_P$, the deviation to the thermal spectrum is negligible for small values of $w\kappa^{-1}$. This deviation can be expanded as

$$\frac{\langle in | N_{wlm}^{out} | in \rangle}{(e^{2\pi w \kappa^{-1}} - 1)^{-1}} \approx 1 - \frac{\alpha \kappa (e^{2\pi w \kappa^{-1}} - 1)}{16w\kappa^{-1}}. \quad (89)$$

For astrophysical black holes, $\kappa\alpha \ll 1$, the second factor is negligible for frequencies up to $\sim 10^2 w_{typical}$, in complete agreement with the results obtained in section IV (compare, for instance, with (68)).

The above discussion shows that, despite the apparent sensitivity of Hawking radiation to high energy physics (see section II), a Planck-scale modification of the two-point function does not necessarily imply a substantial change of the Planckian spectrum. This is so, at least, if the modified two-point function obeys a modified Hadamard-type condition. The simplest realization of this condition turns out to be equivalent to the preservation of the powerful conformal (Möbius) symmetry existing in the unmodified theory. This seems an unavoidable requirement if the corrections to the Planckian spectrum are to be in agreement with the results of string theory in the low-frequency limit $w \rightarrow 0$. The effect of the generalized Hadamard condition is to constrain the short-distance behavior of the propagated “in” two-point function, $\tilde{G}_\alpha^{in} |_{I_+}$, in such a way that it remains close to $-1/(4\pi\alpha^2)$ through its evolution

¹⁰ Even more, it is what exactly guarantees the invariance of the production rate, up to a shift on the emission frequency, under a radial boost with rapidity ξ : $u \rightarrow \bar{u} = e^\xi u, v \rightarrow \bar{v} = e^{-\xi} v$.

to I^+ , despite the large blueshift. In fact, $\tilde{G}_\alpha^{in}|_{I^+}$ is an observer-independent quantity in the limit $x_1 \rightarrow x_2$, i.e., it tends to $-1/(4\pi\alpha^2)$ for any function $v = v(u)$. Note in passing that this condition is somewhat related to approaches to quantum gravity aimed at deforming Lorentz symmetry while keeping the principle of relativity [19].

To conclude, we note that if the deformed two-point function at I^-

$$G_\alpha^{in}|_{I^-} \equiv -\frac{1}{4\pi} \frac{1}{(v_1 - v_2)^2 + \alpha^2}, \quad (90)$$

is naively propagated (i.e., by ray tracing) to I^+ as

$$\tilde{G}_\alpha^{in}|_{I^+} = -\frac{1}{4\pi} \frac{\frac{dv_1}{du_1} \frac{dv_2}{du_2}}{(v(u_1) - v(u_2))^2 + \alpha^2}, \quad (91)$$

where $\tilde{G}_\alpha^{in}|_{I^+} \equiv \frac{dv}{du}(u_1) \frac{dv}{du}(u_2) G_\alpha^{in}|_{I^+}$, the particle production rate is now time-dependent and the thermal spectrum is lost for any nonvanishing α .

VI. CONCLUSIONS AND FINAL COMMENTS

It is highly non-trivial [8] to truncate Hawking's derivation of black hole radiance to account for unknown physics at the Planck scale. A simple estimate of the contribution of virtual high frequencies apparently shows that they are essential to produce the thermal outcome. One can then change strategy and try to evaluate the contribution of Planckian physics in position space, which requires a rederivation of the Hawking calculation in terms of two-point functions, as we have explicitly shown in section III. When these two-point functions are treated in the distributional sense, with the usual $i\epsilon$ prescription, one reproduces exactly the thermal result. However, one can equivalently handle the divergence of the two-point function by trivially taking normal ordering. The consistency of this procedure is guaranteed by the Hadamard condition: the short-distance behavior is universal for all physical states. The advantage of this second option is that it offers a natural way to evaluate the contribution of short distances at I^+ to Hawking radiation.

We have found that the contribution of short-distances at low frequencies $w \ll \kappa$ is negligible. Our analysis allows us to go further and investigate the short-distance contribution for frequencies of order the Hawking temperature T_H and beyond. We find that the contribution of ultrashort distances is also negligible for frequencies of order T_H . In fact, for a black hole of three solar masses we need to look at high frequencies, $w/w_{\text{typical}} \approx 96$ (for bosons) or $w/w_{\text{typical}} \approx 270$ (for fermions), to find that the contribution of Planck distances is of order of the total spectrum itself. This suggests that Hawking thermal radiation is very robust, as it has been confirmed in completely different analyses based on black hole analogues; in string theory (for large wavelength) in near-extremal charged black holes; and also in some models of canonical quantum gravity [20].

One can legitimately ask why, in section IV, we evaluate distances at I^+ , instead of just at I^- , where the sensitivity of the “in” state to high energy scales is more apparent, as we showed in section II. Our heuristic motivation is based on the view offered by string theory, where the Hawking radiation is obtained as the result of collisions between open string excitations. In that approach, the standard large blueshift of low-energy gravity theory does not seem to play the pivotal role that it does in the pure semiclassical treatment. The fact that we consider the fundamental Planck scale at I^+ does not immediately guarantee that the Hawking radiation is kept unaltered from Planck-scale physics. As we show in section IV, with the standard $i\epsilon$ -prescription the short-distance contribution to Hawking radiation is not negligible. In contrast, with the normal-ordering prescription the bulk of the Hawking effect is maintained at low frequencies, in agreement with the results of string theory.

In addition to the above arguments we have approached the problem in section V in a different way. We have considered an explicit modification of the two-point function at the Planck scale. Motivated by the crucial role played by the Hadamard condition in the ordinary relativistic theory, we have assumed that the short-distance behavior of the modified theory should also satisfy a sort of generalized Hadamard condition (*universal* short-distance behavior). The simplest realization of this idea turns out to be equivalent to the preservation of the powerful conformal (Möbius) symmetry existing in the unmodified theory. Armed with this condition, the contribution to the particle production rate of the “in” and “out” two-point functions in (84) is similar when they are compared in the same ultrashort range of distances, despite the large blueshift horizon effect. As a result, the two contributions compensate each other and lead

to an emission spectrum very insensitive to trans-Planckian physics. The generalized Hadamard condition, therefore, seems to be necessary to maintain the bulk of the Hawking effect. Moreover, it is in this context that the apparent tension between I^+ and I^- to measure separations is eliminated since in both we find the same finite short distance limit.

A last comment is now in order. In the string theory analysis one has, at least, two relevant parameters: the surface gravity κ and the radius r_g of the supersymmetric, charged black hole. The surface gravity is assumed to be small, in comparison with the inverse of the size of the black hole, i.e., $\kappa \ll 1/r_g$. The emission frequency can reach κ , but can never reach $1/r_g$ (or become larger) to guarantee the validity of the string theory calculation. Obviously the analysis of string theory excludes astrophysical black holes of the Schwarzschild type (for which $\kappa \sim 1/r_g$). Our results, however, suggest that one could also expect string theory to predict in this case, in some subtle way, agreement with Hawking's results for frequencies around $1/r_g$ and, at least, a few orders beyond. This is so because we do not observe any significant contribution to the thermal spectrum coming from the short-distance region, where new physics could arise, up to such high frequencies. This fact offers a very non-trivial challenge for any quantum theory of gravity having computational rules very different from those of semiclassical gravity (as in string theory or background-independent approaches), since when $w \sim 1/r_g$ the grey-body factors $\Gamma_i(w)$ cannot be computed analytically. They are only known numerically [21], as a result of solving field wave equations in the black hole background. String theory manages to account for the greybody factors in the low-energy regime, where they admit an analytic expression. In fact, for all spherically symmetric black holes, the low-energy absorption cross section is proportional to the area of the horizon [22, 23]. But for typical Hawking frequencies the grey-body factors remain elusive for any analytic treatment. Reobtaining them from such a different computation would be extremely impressive.

Appendix A

We will complete here the steps missing in the derivation that led to the emission rate (25). Using the wave packets (23) we can express it as

$$\begin{aligned} \langle in | N_{j_1 n_1, j_2 n_2}^{out, \sigma} | in \rangle &= \int_0^\Lambda dw' \beta_{j_1 n_1, w'} \beta_{j_2 n_2, w'}^* = \\ & \frac{1}{\epsilon} \int_{j_1 \epsilon}^{(j_1+1)\epsilon} dw_1 \int_{j_2 \epsilon}^{(j_2+1)\epsilon} dw_2 e^{2\pi i w_1 n_1 / \epsilon} e^{-2\pi i w_2 n_2 / \epsilon} \int_0^\Lambda dw' \beta_{w_1 w'} \beta_{w_2 w'}^* . \end{aligned} \quad (92)$$

Using (20) we get

$$\begin{aligned} \langle in | N_{j_1 n_1, j_2 n_2}^{out, \sigma} | in \rangle &= \frac{1}{\epsilon} \int_{j_1 \epsilon}^{(j_1+1)\epsilon} dw_1 \int_{j_2 \epsilon}^{(j_2+1)\epsilon} dw_2 e^{i \frac{2\pi w_1 n_1}{\epsilon}} e^{-i \frac{2\pi w_2 n_2}{\epsilon}} \\ & t_l(w_1) t_l^*(w_2) \frac{e^{-i(w_1 - w_2)v_H}}{2\pi \sqrt{w_1 w_2}} e^{-\pi \kappa^{-1} w_1} e^{-i \kappa^{-1}(w_1 - w_2)} \\ & \Gamma(1 + i \kappa^{-1} w_1) \Gamma(1 - i \kappa^{-1} w_2) \delta_\sigma[\kappa^{-1}(w_1 - w_2)] . \end{aligned} \quad (93)$$

This integral can be estimated explicitly when the width ϵ of the frequency interval $[j\epsilon, (j+1)\epsilon]$ is assumed, as usual, small. In this case, the integral is essentially as follows

$$\langle in | N_{j_1 n_1, j_2 n_2}^{out, \sigma} | in \rangle \approx \delta_{j_1 j_2} \frac{|t_l(w_j)|^2 |\Gamma(1 + i \kappa^{-1} w_j)|^2}{2\pi w_j} e^{-\pi \kappa^{-1} w_j} e^{\frac{2\pi(n_1 - n_2)w_j}{\epsilon}} I_{n_1 n_2}(\sigma) \quad (94)$$

where

$$I_{n_1 n_2}(\sigma) = \frac{1}{\epsilon} \int_{-\epsilon/2}^{\epsilon/2} dx_1 \int_{-\epsilon/2}^{\epsilon/2} dx_2 e^{i[\frac{2\pi n_1}{\epsilon} - v_H]x_1 - i[\frac{2\pi n_2}{\epsilon} - v_H]x_2 - \pi \kappa^{-1}(x_1 + x_2)/2} \delta_\sigma[\kappa^{-1}(x_1 - x_2)] \quad (95)$$

and $x_{1,2} \equiv w_{1,2} - (j + 1/2)\epsilon$. The factor $\delta_{j_1 j_2}$ in (94) is due to the role of δ_σ , which selects frequencies on a very narrow band of order $|w_1 - w_2| \sim \kappa \sigma$. For this reason, it is also convenient to introduce a new variable $y = x_1 - x_2$ and rewrite $I_{n_1 n_2}(\sigma)$ as follows

$$I_{n_1 n_2}(\sigma) \approx \frac{1}{\epsilon} \int_{-\epsilon/2}^{\epsilon/2} dx_1 e^{\frac{2\pi(n_1 - n_2)x_1}{\epsilon}} \int_{x_1 - \epsilon/2}^{x_1 + \epsilon/2} dy e^{i[\frac{2\pi n_2}{\epsilon} - v_H]y} \delta_\sigma[\kappa^{-1}y] . \quad (96)$$

In writing this we have neglected the term $e^{\pi\kappa^{-1}(x_1+x_2)/2}$ which is almost constant (unity) over the integral. We can now estimate the integral over y having in mind that δ_σ is very well approximated by a square step of width $\pi\kappa\sigma$ and height $1/(\pi\sigma)$ centered at $y = 0$. This means that the main contribution comes from the interval $[-\frac{\pi\kappa\sigma}{2}, \frac{\pi\kappa\sigma}{2}]$. This fact makes the outcome of the integral independent of x_1 , which also allows us to perform the integral in x_1 . Putting all together we find

$$I_{n_1 n_2}(\sigma) \approx \kappa \delta_{n_1 n_2} \frac{\sin\left[\left(\frac{2\pi n_2}{\epsilon} - v_H\right) \frac{\pi\kappa\sigma}{2}\right]}{\left[\left(\frac{2\pi n_2}{\epsilon} - v_H\right) \frac{\pi\kappa\sigma}{2}\right]} \quad (97)$$

Plugging this result back into (94) we find (25).

We will now briefly consider the effect of introducing the cutoff in frequencies in a different way. The cutoff was introduced in (20) in the form

$$\int_{-\infty}^{\infty} d\log[w/\kappa] e^{-i\kappa^{-1}(w_1-w_2)\log[w/\kappa]} \rightarrow \int_{-\log[\Lambda/\kappa]}^{\log[\Lambda/\kappa]} d\log[w/\kappa] e^{-i\kappa^{-1}(w_1-w_2)\log[w/\kappa]} \quad (98)$$

We will now consider the change

$$\int_{-\log[\Lambda/\kappa]}^{\log[\Lambda/\kappa]} d\lambda e^{-i\kappa^{-1}(w_1-w_2)\lambda} \rightarrow \int_{-\infty}^{\infty} d\lambda e^{-i\kappa^{-1}(w_1-w_2)\lambda} e^{-(\lambda/\tilde{\Lambda})^2} \quad (99)$$

where $\tilde{\Lambda}$ must be of order $\sim \log[\Lambda/\kappa]$. This modification leads to a redefinition of δ_σ

$$\delta_{\tilde{\sigma}}[\kappa^{-1}(w_1 - w_2)] = \frac{\exp\left(-\left[\frac{\kappa^{-1}(w_1-w_2)}{2\tilde{\sigma}}\right]^2\right)}{2\tilde{\sigma}\sqrt{\pi}} \quad (100)$$

which in the limit $2\tilde{\sigma} \rightarrow 0$ also becomes Dirac's delta function. One can then proceed as above and define the corresponding function $I_{n_1 n_2}(\tilde{\sigma})$, which this time can be evaluated extending up to infinity the limits of integration over the variable $y = x_1 - x_2$. This leads to

$$I_{n_1 n_2}(\tilde{\sigma}) = \kappa \delta_{n_1 n_2} e^{-\left[\left(\frac{2\pi n_2}{\epsilon} - v_H\right) \kappa \tilde{\sigma}\right]^2} \quad (101)$$

The corresponding emission rate is now

$$\langle in | N_{j_1 n_1, j_2 n_2}^{out, \sigma} | in \rangle = \delta_{j_1 j_2} \delta_{n_1 n_2} \frac{|t_l(w_j)|^2}{e^{2\pi\kappa^{-1}w_j} - 1} e^{-\left[\left(\frac{2\pi n_2}{\epsilon} - v_H\right) \kappa \tilde{\sigma}\right]^2} \quad (102)$$

This expression is always positive definite and exhibits the same decay rate as (25) if we identify $2\tilde{\sigma}$ with σ , which in fact is the right choice for the definition of (100).

Appendix B

We will proceed now to solve the massless Dirac equation in a curved background with spherical symmetry. The equation to solve is¹¹

$$\gamma^\mu \nabla_\mu \psi = 0 \quad (103)$$

where $\gamma^\mu = \gamma^a V_a^\mu(x)$ satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ and $V_a^\mu V_b^\nu \eta^{ab} = g^{\mu\nu}$ represent the *vierbeins*¹². Note that $\nabla_\mu \psi = (\partial_\mu - \Gamma_\mu)\psi$ where $\Gamma_\mu = -\frac{1}{4}\gamma^b \gamma^c V_b^\nu \nabla_\mu V_{\nu c}$ represents the spin connection. We will take the curved space

¹¹ For earlier references see [24], and for a more advanced treatment (no needed for the purposed of this paper) see [25].

¹² Due to our convention for the metric signature the γ^a matrices should verify the conditions $(\gamma^0)^2 = -I$, $(\gamma^i)^2 = I$. However, to agree with the standard notation for Dirac matrices in this appendix we have flipped the metric signature to $(+, -, -, -)$. This is, however, irrelevant for the computations carried out in the body of this paper.

line element $ds^2 = e^{2\rho}dx^+dx^- - r^2d\Omega^2$, with $dx^+dx^- = dt^2 - dr^{*2}$, and $\eta_{ab} = \text{diag}(1, -1, -1, -1)$. Introducing the ansatz $\psi = \frac{e^{-\rho/2}}{r}\Phi$ and making the simplest choice for vierbeins (i.e., to be parallel to the unit vectors in t, r^*, θ, ϕ directions), the Dirac equation (103) becomes

$$\gamma^a V_a^i \partial_i \Phi + \frac{1}{r} \left[\frac{\gamma^2}{\sin^{1/2} \theta} \partial_\theta \sin^{1/2} \theta + \frac{\gamma^3}{\sin \theta} \partial_\phi \right] \Phi = 0 \quad (104)$$

where the index i runs over the non-angular variables. Since for $x^\pm = t \pm r^*$ we have $\gamma^a V_a^i \partial_i = e^{-\rho}[\gamma^0 \partial_t + \gamma^1 \partial_{r^*}]$, (104) can be written in the more familiar form

$$\partial_t \Phi = -\gamma^0 \gamma^1 \left[\partial_{r^*} + \frac{e^\rho}{r} \left(\frac{\gamma^2 \gamma^1}{\sin^{1/2} \theta} \partial_\theta \sin^{1/2} \theta + \frac{\gamma^3 \gamma^1}{\sin \theta} \partial_\phi \right) \right] \Phi \quad (105)$$

The angular part of this equation can be reexpressed as $e^\rho \gamma^0 K/r$:

$$\partial_t \Phi = -\gamma^0 \gamma^1 \left[\partial_{r^*} - \frac{e^\rho}{r} \gamma^0 K \right] \Phi \quad (106)$$

where the operator K

$$K = \gamma^0 \left(\frac{\gamma^1 \gamma^2}{\sin^{1/2} \theta} \partial_\theta \sin^{1/2} \theta + \frac{\gamma^1 \gamma^3}{\sin \theta} \partial_\phi \right) \quad (107)$$

commutes with the Dirac equation as well as \vec{J}^2 and J_3 and, therefore, its eigenvalues can be used to characterize the angular part $\chi_{m_j \kappa_j}$ of the modes: $K \chi_{m_j \kappa_j} = (-\kappa_j) \chi_{m_j \kappa_j}$, with $\kappa_j^2 = (j + \frac{1}{2})^2$. Moreover the eigenfunctions $\chi_{m_j \kappa_j}$ admit the following decomposition $\chi_{m_j \kappa_j} = c^+ \chi_{m_j \kappa_j}^+ + c^- \chi_{m_j \kappa_j}^-$, with

$$\chi_{m_j \kappa_j}^+ = \begin{bmatrix} \eta(\hat{r})_{\kappa_j}^{m_j} \\ 0 \\ 0 \end{bmatrix} \quad (108)$$

$$\chi_{m_j \kappa_j}^- = \begin{bmatrix} 0 \\ \eta(\hat{r})_{-\kappa_j}^{m_j} \end{bmatrix} \quad (109)$$

Therefore, in a stationary spacetime, $\rho = \rho(r)$, a general solution can then be expressed as

$$\psi_{w \kappa_j m_j}(x) = \frac{e^{-\rho/2} e^{-i\omega t}}{r} \begin{bmatrix} G_{w \kappa_j}(r) \eta(\hat{r})_{\kappa_j}^{m_j} \\ -i F_{w \kappa_j}(r) \sigma^1 \eta(\hat{r})_{\kappa_j}^{m_j} \end{bmatrix} \quad (110)$$

where we have used that $\sigma^1 \eta(\hat{r})_{\kappa_j}^{m_j} = \eta(\hat{r})_{-\kappa_j}^{m_j}$ and the functions $F_{w \kappa_j}(r^*)$ and $G_{w \kappa_j}(r^*)$ satisfy the following equations (see also [23])

$$\partial_{r^*} G_{w \kappa_j} = -\frac{e^\rho}{r} \kappa_j G_{w \kappa_j} + w F_{w \kappa_j} \quad (111)$$

$$\partial_{r^*} F_{w \kappa_j} = \frac{e^\rho}{r} \kappa_j F_{w \kappa_j} - w G_{w \kappa_j} \quad (112)$$

Adding the time-dependent part, the above equations lead to plane-wave solutions $\sim e^{-i\omega(t \pm r^*)} = e^{-i\omega x^\pm}$ for all κ_j as $r \rightarrow \infty$.

We note that the form of the eigenfunctions $\chi_{m_j \kappa_j}$ can be worked out immediately if the vierbeins are chosen to be parallel to unit vectors in the standard t, x, y, z directions¹³. The bispinors $\eta_{m_j \kappa_j}$ can be constructed, as it is

¹³ With this orientation for the vierbeins K can be written as $K = \gamma^0(I + 2\vec{\sigma} \cdot \vec{L})$, which is the standard form of this operator in Minkowski space.

well-known, using the Clebsch-Gordon rules for addition of angular momentum in terms of spherical harmonics and two-component spinors, and the result is

$$\eta(\hat{r})_{\kappa_j < 0}^{m_j} = \begin{bmatrix} \sqrt{\frac{j+m_j}{2j}} Y_{j-1/2}^{m_j-1/2}(\theta, \phi) \\ \sqrt{\frac{j-m_j}{2j}} Y_{j-1/2}^{m_j+1/2}(\theta, \phi) \end{bmatrix} \quad (113)$$

and

$$\eta(\hat{r})_{\kappa_j > 0}^{m_j} = \begin{bmatrix} \sqrt{\frac{j+1-m_j}{2j+2}} Y_{j+1/2}^{m_j-1/2}(\theta, \phi) \\ -\sqrt{\frac{j+1+m_j}{2j+2}} Y_{j+1/2}^{m_j+1/2}(\theta, \phi) \end{bmatrix} \quad (114)$$

With the modes given in (110) conveniently normalized, the quantized Dirac field can be expanded in modes as

$$\psi(x) = \sum_{\kappa_j m_j} \int dw \left[a_{w\kappa_j m_j} u_{w\kappa_j m_j}(x) + b_{w\kappa_j m_j}^\dagger v_{w\kappa_j m_j}(x) \right] \quad (115)$$

where $u_{w\kappa_j m_j}(x)$ and $v_{w\kappa_j m_j}(x)$ represent positive and negative energy solutions respectively. On the other hand, since we are dealing with massless spinors, it is necessary, on physical grounds, to use states with well defined helicity. In particular, left-handed spinors can be obtained from (110) by projecting with $P_L = \frac{1}{2}(I - \gamma^5)$, where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. We will therefore be working with the (normalized) modes $\psi_{w\kappa_j m_j}^L = \frac{1}{\sqrt{2}}(\psi_{w|\kappa_j| m_j} - \psi_{w-|\kappa_j| m_j})$.

We will now carry out the calculations that lead to (52) (adapted now for chiral spinors). First thing to note is that the propagated backwards mode (49) contains a term of the form $\sqrt{du(v)/dv}$. A simple way to realize why this term arises is that it is necessary to ensure the invariance of the scalar product under time evolution. Putting aside backscattering effects, the Dirac scalar product for out modes can be written, equivalently, as

$$\int_{I^+} d\Omega du r^2 \bar{u}^{out} \gamma_+ u^{out} = \int_{I^-} d\Omega r^2 dv \bar{u}^{out} \gamma_- u^{out} . \quad (116)$$

The above equality requires, up to relative signs in the spinor components, that

$$u^{out}(v)|_{I^+} = \sqrt{du(v)/dv} \Theta(v_H - v) u^{out}(u)|_{I^-} . \quad (117)$$

Note that the factor $e^{-\rho/2}$ in (110) also signals this behavior. Since the spinor $\psi(x)$ does behave as a scalar under general changes of coordinates, it follows that the functions F and G must somehow compensate the change in $e^{-\rho/2}$ under conformal transformations.

Let us now focus on the integration over the angular variables prior to (52). This integration can be readily performed if we put the result of (50) into (47). We then find

$$\begin{aligned} \langle in | N_{i_1 i_2} | in \rangle &= \sum_k \int_{I^-} dv_2 r_2^2 d\Omega_2 \left(\bar{u}_{i_2}^{out, L}(x_2) \frac{[\gamma^0 - \gamma^1]}{2} v_k^{in, L}(x_2) \right) \times \\ &\times \int_{I^-} dv_1 r_1^2 d\Omega_1 \left(\bar{v}_k^{in, L}(x_1) \frac{[\gamma^0 - \gamma^1]}{2} u_{i_1}^{out, L}(x_1) \right) \end{aligned} \quad (118)$$

where the indices i_1, i_2 and k denote (w, j, m_j) . Using the modes of eqs.(49) and (51) it is immediate to verify that

$$\begin{aligned} \int d\Omega_2 \bar{u}_{i_2}^{out, L}(x_2) \frac{[\gamma^0 - \gamma^1]}{2} v_k^{in, L}(x_2) &= \frac{t_{j_2}^*(w_2)}{2\pi r_2^2} \sqrt{\frac{du(v)}{dv}} \Theta(v_H - v) \times \\ &\times e^{i w_2 u(v_2) + i w v_2} \delta_{m_{j_2} m_k} \delta_{j_2 j_k} \end{aligned} \quad (119)$$

where we have used that $\int d\Omega \eta_{\kappa_j}^{m_j \dagger}(\hat{r}) \eta_{\kappa_{j'}}^{m_{j'}}(\hat{r}) = \delta_{m_j m_{j'}} \delta_{\kappa_j \kappa_{j'}}$. An analogous calculation applies to the second factor in (118). Plugging these results back into (118) we obtain

$$\begin{aligned} \langle in | N_{i_1 i_2} | in \rangle &= \delta_{m_{j_1} m_{j_2}} \delta_{j_1 j_2} \frac{t_{j_1}(w_1) t_{j_2}^*(w_2)}{4\pi^2} \int_{-\infty}^{v_H} dv_1 dv_2 \\ &\sqrt{\frac{du(v_1)}{dv} \frac{du(v_2)}{dv}} e^{-i w_1 u(v_1) + i w_2 u(v_2)} \int_0^\infty dw e^{-i w (v_1 - v_2)} \end{aligned} \quad (120)$$

There remains to perform the integration in w , which yields

$$\int_0^\infty dw e^{-iw(v_1-v_2)} = \lim_{\epsilon \rightarrow 0} \frac{-i}{(v_1 - v_2 - i\epsilon)} \quad (121)$$

and leads to the sought-after result.

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