Communications on Stochastic Analysis

Volume 7 | Number 1 Article 4

3-1-2013

Proving existence results in martingale theory using a subsequence principle

Alexander Sokol

Follow this and additional works at: https://digitalcommons.lsu.edu/cosa

Part of the Analysis Commons, and the Other Mathematics Commons

Recommended Citation

Sokol, Alexander (2013) "Proving existence results in martingale theory using a subsequence principle," *Communications on Stochastic Analysis*: Vol. 7 : No. 1 , Article 4.
DOI: 10.31390/cosa.7.1.04
Available at: https://digitalcommons.lsu.edu/cosa/vol7/iss1/4

Communications on Stochastic Analysis
Vol. 7, No. 1 (2013) 65-79

Serials Publications
www.serialspublications.com

PROVING EXISTENCE RESULTS IN MARTINGALE THEORY USING A SUBSEQUENCE PRINCIPLE

ALEXANDER SOKOL

Abstract. New proofs are given of the existence of the compensator of a locally integrable càdlàg adapted process of finite variation and of the existence of the quadratic variation process for a càdlàg local martingale. Both proofs apply a functional analytic subsequence principle. After presenting the proofs, we discuss their application in giving a simplified account of the construction of the stochastic integral of a locally bounded predictable process with respect to a semimartingale.

1. Introduction

Assume given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ satisfying the usual conditions, see [8], Section I.1, for the definition of this and other standard probabilistic concepts. For a locally integrable càdlàg adapted process A with initial value zero and finite variation, the compensator, also known as the dual predictable projection, is the unique locally integrable càdlag predictable process $\Pi_p^* A$ with initial value zero and finite variation such that $A - \Pi_p^* A$ is a local martingale. For a càdlàg local martingale M with initial value zero, the quadratic variation process is the unique increasing càdlàg adapted process $[M]$ with initial value zero such that $M^2 - [M]$ is a local martingale and $\Delta[M] = (\Delta M)^2$. In both cases, uniqueness is up to indistinguishability.

For both the dual predictable projection and the quadratic variation, the proofs of the existence of these processes are among the most difficult in classical martingale theory, see for example [11], [4] or [8] for proofs. In this article, we give new proofs of the existence of these processes. The proofs are facilitated by the following lemma, previously applied in this form to probability theory in [2]. We also give a short proof of the lemma.

Lemma 1.1. *Let (X_n) be sequence of variables bounded in $\mathcal{L}^2$. There exists a sequence (Y_n) such that each Y_n is a convex combination of a finite set of elements in $\{X_n, X_{n+1}, \ldots\}$ and (Y_n) is convergent in $\mathcal{L}^2$.*

Proof. Let α_n be the infimum of EZ^2, where Z ranges through all convex combinations of elements in $\{X_n, X_{n+1}, \ldots\}$, and define $\alpha = \sup_n \alpha_n$. If $Z = \sum_{k=n}^{K_n} \lambda_k X_k$ for some convex weights $\lambda_n, \ldots, \lambda_{K_n}$, we obtain $\sqrt{EZ^2} \leq \sup_n \sqrt{EX_n^2}$, in particular we have $\alpha_n \leq \sup_n EX_n^2$ and so $\alpha \leq \sup_n EX_n^2$ as well, proving that α is

Received 2012-4-24; Communicated by Hui-Hsiung Kuo.
2000 *Mathematics Subject Classification.* Primary 60G07; Secondary 60G44, 60H05.
Key words and phrases. Martingale, compensator, quadratic variation, stochastic integral.

finite. For each n, there is a variable Y_n which is a finite convex combination of elements in $\{X_n, X_{n+1}, \ldots\}$ such that $E(Y_n)^2 \leq \alpha_n + \frac{1}{n}$. Let n be so large that $\alpha_n \geq \alpha - \frac{1}{n}$, and let $m \geq n$, we then obtain

$$\begin{aligned} E(Y_n - Y_m)^2 &= 2EY_n^2 + 2EY_m^2 - E(Y_n + Y_m)^2 \\ &= 2EY_n^2 + 2EY_m^2 - 4E(\tfrac{1}{2}(Y_n + Y_m))^2 \\ &\leq 2(\alpha_n + \tfrac{1}{n}) + 2(\alpha_m + \tfrac{1}{m}) - 4\alpha_n \\ &= 2(\tfrac{1}{n} + \tfrac{1}{m}) + 2(\alpha_m - \alpha_n). \end{aligned} \tag{1.1}$$

As (α_n) is convergent, it is Cauchy. Therefore, the above shows that (Y_n) is Cauchy in $\mathcal{L}^2$, therefore convergent, proving the lemma. □

Lemma 1.1 may be seen as a combination of variants of the following two classical results: Every bounded sequence in a reflexive Banach space contains a weakly convergent subsequence (see Theorem 4.41-B of [13]), and every weakly convergent sequence in a reflexive Banach space has a sequence of convex combinations of its elements converging strongly to the weak limit (see Theorem 3.13 of [12]). In [2], an $\mathcal{L}^1$ version of Lemma 1.1 is used to give a simple proof of the Doob-Meyer theorem, building on the ideas of [6] and [9].

The remainder of the article is organized as follows. In Section 2, we give our proof of the existence of the compensator, and in Section 3, we give our proof of the existence of the quadratic variation. In Section 4, we discuss how these results may be used to give a simplified account of the theory of stochastic integration with respect to semimartingales. In particular, the account proposed excludes the use of: the début theorem, the section theorems and the Doob-Meyer theorem. Appendix A contains auxiliary results which are needed in the main proofs.

2. The Existence of the Compensator

In this section, we will show that for any càdlàg adapted process A with initial value zero and paths of finite variation, locally integrable, there exists a càdlàg predictable process $\Pi_p^* A$ with initial value zero and paths of finite variation, locally integrable, unique up to indistinguishability, such that $A - \Pi_p^* A$ is a local martingale. We refer to $\Pi_p^* A$ as the compensator of A. The proofs will use some basic facts from the general theory of processes, some properties of monotone convergence for càdlàg increasing mappings, and Lemma 1.1. Essential for the results are the results on the limes superior of discrete approximations to the compensator, the proof of this is based on the technique developed in [6] and also applied in [2]. Note that as the existence of the compensator follows directly from the Doob-Meyer theorem, see for example Section I.3b of [5], the interest of the proofs given in this section is that if we restrict our attention to the compensator of a finite variation process instead of a submartingale, the complicated uniform integrability arguments applied in [9] may be done away with, and furthermore we need only an $\mathcal{L}^2$ subsequence principle and not an $\mathcal{L}^1$ subsequence principle as in [2]. We begin by recalling some standard nomenclature and fixing our notation.

By $\mathcal{A}$, we denote the set of processes which are càdlàg adapted and increasing with initial value zero. For $A \in \mathcal{A}$, the limit A_∞ of A_t for t tending to infinity

always exists in $[0,\infty]$. We say that A is integrable if A_∞ is integrable. The subset of integrable processes in $\mathcal{A}$ is denoted by $\mathcal{A}^i$. For $A \in \mathcal{A}$, we say that A is locally integrable if there exists a localising sequence (T_n) such that $A^{T_n} \in \mathcal{A}^i$. The set of such processes is denoted by $\mathcal{A}^i_\ell$. By $\mathcal{V}$, we denote the set of processes which are càdlàg adapted with initial value zero and has paths of finite variation. For $A \in \mathcal{V}$, V_A denotes the process such that $(V_A)_t$ is the variation of A over $[0,t]$. V_A is then an element of $\mathcal{A}$. For $A \in \mathcal{V}$, we say that A is integrable if V_A is integrable, and we say that A is locally integrable if V_A is locally integrable. The corresponding spaces of stochastic processes are denoted by $\mathcal{V}^i$ and $\mathcal{V}^i_\ell$, respectively. By $\mathbb{D}_+$, we denote the set of nonnegative dyadic rationals, $\mathbb{D}_+ = \{k2^{-n} | k \geq 0, n \geq 0\}$. The space of square-integrable martingales with initial value zero is denoted by $\mathcal{M}^2$. Also, we say that two processes X and Y are indistinguishable if their sample paths are almost surely equal, and in this case, we say that X is a modification of Y and vice versa. We say that a process X is càdlàg if it is right-continuous with left limits, and we say that a process X is càglàd if it is left-continuous with right limits.

Our main goal in this section is to show that for any $A \in \mathcal{V}^i_\ell$, there is a predictable element $\Pi^*_p A$ of $\mathcal{V}^i_\ell$, unique up to indistinguishability, such that $A - \Pi^*_p A$ is a local martingale. To prove the result, we first establish the existence of the compensator for some simple elements of $\mathcal{V}^i_\ell$, namely processes of the type $\xi 1_{[\![T,\infty[\![}$, where T is a stopping time with $T > 0$, ξ is bounded, nonnegative and $\mathcal{F}_T$ measurable and $[\![T,\infty[\![= \{(t,\omega) \in \mathbb{R}_+ \times \Omega \mid T(\omega) \leq t\}$. After this, we apply monotone convergence arguments and localisation arguments to obtain the general existence result.

Lemma 2.1. *Let T be a stopping time with $T > 0$ and let ξ be nonnegative, bounded and $\mathcal{F}_T$ measurable. Define $A = \xi 1_{[\![T,\infty[\![}$. A is then an element of $\mathcal{A}^i$, and there exists a predictable process $\Pi^*_p A$ in $\mathcal{A}^i$ such that $A - \Pi^*_p A$ is a uniformly integrable martingale.*

Proof. Let $t^n_k = k2^{-n}$ for $k, n \geq 0$. We define

$$A^n_t = A_{t^n_k} \text{ for } t^n_k \leq t < t^n_{k+1} \tag{2.1}$$

and

$$B^n_t = \sum_{i=1}^{k+1} E(A_{t^n_i} - A_{t^n_{i-1}} | \mathcal{F}_{t^n_{i-1}}) \text{ for } t^n_k < t \leq t^n_{k+1}, \tag{2.2}$$

and $B^n_0 = 0$. Note that both A^n and B^n have initial value zero, since $T > 0$. Also note that A^n is càdlàg adapted and B^n is càglàd adapted. Put $M^n = A^n - B^n$. Note that M^n is adapted, but not necessarily càdlàg or càglàd. Also note that, with the convention that a sum over an empty index set is zero, it holds that

$$A^n_{t^n_k} = A_{t^n_k} \quad \text{and} \quad B^n_{t^n_k} = \sum_{i=1}^{k} E(A_{t^n_i} - A_{t^n_{i-1}} | \mathcal{F}_{t^n_{i-1}}) \tag{2.3}$$

for $k \geq 0$. Therefore, $(B_{t^n_k})_{k\geq 0}$ is the compensator of the discrete-time increasing process $(A_{t^n_k})_{k\geq 0}$, see Section II.54 of [10], so $(M^n_{t^n_k})_{k\geq 0}$ is a discrete-time martingale with initial value zero. We next show that each element in this sequence of

discrete-time martingales is bounded in $\mathcal{L}^2$, and the limit variables constitute a sequence bounded in $\mathcal{L}^2$ as well, this will allow us to apply Lemma 1.1. To this end, note that since B^n has initial value zero,

$$\begin{aligned}(B^n_{t^n_k})^2 &= 2(B^n_{t^n_k})^2 - \sum_{i=0}^{k-1}(B^n_{t^n_{i+1}})^2 - (B^n_{t^n_i})^2 \\ &= \sum_{i=0}^{k-1} 2B^n_{t^n_k}(B^n_{t^n_{i+1}} - B^n_{t^n_i}) - (B^n_{t^n_{i+1}})^2 + (B^n_{t^n_i})^2 \\ &= \sum_{i=0}^{k-1} 2(B^n_{t^n_k} - B^n_{t^n_i})(B^n_{t^n_{i+1}} - B^n_{t^n_i}) - (B^n_{t^n_{i+1}} - B^n_{t^n_i})^2 \\ &\le \sum_{i=0}^{k-1} 2(B^n_{t^n_k} - B^n_{t^n_i})(B^n_{t^n_{i+1}} - B^n_{t^n_i}). \end{aligned} \tag{2.4}$$

Now let c be a bound for ξ. Applying that $B^n_{t^n_{i+1}}$ is $\mathcal{F}_{t^n_i}$ measurable, the martingale property of $(M^n_{t^n_k})_{k\ge 0}$ and the fact that A and B are increasing and A is bounded by c, we find

$$\begin{aligned}E(B^n_{t^n_k} - B^n_{t^n_i})(B^n_{t^n_{i+1}} - B^n_{t^n_i}) &= E(B^n_{t^n_{i+1}} - B^n_{t^n_i})E(B^n_{t^n_k} - B^n_{t^n_i}|\mathcal{F}_{t^n_i}) \\ &= E(B^n_{t^n_{i+1}} - B^n_{t^n_i})E(A^n_{t^n_k} - A^n_{t^n_i}|\mathcal{F}_{t^n_i}) \\ &\le cE(B^n_{t^n_{i+1}} - B^n_{t^n_i}). \end{aligned} \tag{2.5}$$

All in all, we find $E(B^n_{t^n_k})^2 \le 2c\sum_{i=0}^{k-1} E(B^n_{t^n_{i+1}} - B^n_{t^n_i}) = 2cEB^n_{t^n_k} = 2cEA^n_{t^n_k} \le 2c^2$. Thus $E(M^n_{t^n_k})^2 \le 4E(A^n_{t^n_k})^2 + 4E(B^n_{t^n_k})^2 \le 12c^2$. We conclude that $(M^n_{t^n_k})_{k\ge 0}$ is bounded in $\mathcal{L}^2$, and so convergent almost surely and in $\mathcal{L}^2$ to a limit M^n_∞, and the sequence $(M^n_\infty)_{n\ge 0}$ is bounded in $\mathcal{L}^2$ as well.

By Lemma 1.1, there exists a sequence of naturals (K_n) with $K_n \ge n$ and for each n a finite sequence of reals $\lambda^n_n, \dots, \lambda^n_{K_n}$ in the unit interval summing to one, such that $\sum_{i=n}^{K_n} \lambda^n_i M^i_\infty$ is convergent in $\mathcal{L}^2$ to some variable M_∞. By Theorem II.70.2 of [10], there is $M \in \mathcal{M}^2$ such that $E\sup_{t\ge 0}(M_t - \sum_{i=n}^{K_n} \lambda^n_i M^i_t)^2$ tends to zero, M is a càdlàg version of the process $t \mapsto E(M_\infty|\mathcal{F}_t)$. By picking a subsequence and relabeling, we may further assume that $\sup_{t\ge 0}(M_t - \sum_{i=n}^{K_n} \lambda^n_i M^i_t)^2$ also converges almost surely to zero. Define $B = A - M$, we wish to argue that there is a modification of B satisfying the requirements of the lemma.

First put $C^n = \sum_{i=n}^{K_n} \lambda^n_i B^i$. Note that C^n is càglàd, adapted and increasing, in particular predictable, and

$$\begin{aligned}\lim_{t\to\infty} C^n_t = \lim_{m\to\infty} C^n_m &= \lim_{m\to\infty} \sum_{i=n}^{K_n} \lambda^n_i B^i_m \\ &= \lim_{m\to\infty} A_m - \sum_{i=n}^{K_n} \lambda^n_i M^i_m = A_\infty - \sum_{i=n}^{K_n} \lambda^n_i M^i_\infty, \end{aligned} \tag{2.6}$$

showing that $C^n \in \mathcal{A}^i$ and that $(C^n_\infty)_{n\geq 0}$ is bounded in $\mathcal{L}^2$. Also note that for each $q \in \mathbb{D}_+$, it holds that $A_q = \lim_{n\to\infty} A^n_q$ almost surely. Therefore,

$$B_q = A_q - M_q = \lim_{n\to\infty} A^n_q - \sum_{i=n}^{K_n} \lambda^n_i M^i_q = \lim_{n\to\infty} \sum_{i=n}^{K_n} \lambda^n_i B^i_q = \lim_{n\to\infty} C^n_q, \tag{2.7}$$

almost surely. From this, we obtain that B is almost surely increasing on $\mathbb{D}_+$. As B is càdlàg, this shows that B is almost surely increasing on all of $\mathbb{R}_+$. Next, we show that $B_t = \limsup_{n\to\infty} C^n_t$ almost surely, simultaneuously for all $t \geq 0$, this will allow us to show that B has a predictable modification. To this end, note that for $t \geq 0$ and $q \in \mathbb{D}_+$ with $q \geq t$, $\limsup_{n\to\infty} C^n_t \leq \limsup_{n\to\infty} C^n_q = B_q$. As B is càdlàg, this yields $\limsup_{n\to\infty} C^n_t \leq B_t$. This holds almost surely for all $t \in \mathbb{R}_+$ simultaneously. Similarly, $\liminf_{n\to\infty} C^n_t \geq B_{t-}$ almost surely, simultaneously for all $t \geq 0$. All in all, we conclude that almost surely, $B_t = \limsup_{n\to\infty} C^n_t$ for all continuity points t of B, simultaneously for all $t \geq 0$. As the jumps of B can be exhausted by a countable sequence of stopping times, we find that in order to show the desired result on the limes superior, it suffices to show for any stopping time S that $B_S = \limsup_{n\to\infty} C^n_S$.

Fixing a stopping time S, we first note that as $0 \leq C^n_S \leq C^n_\infty$, the sequence of variables $(C^n_S)_{n\geq 0}$ is bounded in $\mathcal{L}^2$ and thus in particular uniformly integrable. Therefore, by Lemma A.1, $\limsup_{n\to\infty} EC^n_S \leq E\limsup_{n\to\infty} C^n_S \leq EB_S$. As $\limsup_{n\to\infty} C^n_S \leq B_S$ almost surely, we find that to show $\limsup_{n\to\infty} C^n_S = B_S$ almost surely, it suffices to show that EC^n_S converges to EB_S, and to this end, it suffices to show that EB^n_S converges to EB_S. Now define S_n by putting $S_n = \infty$ whenever $S = \infty$ and $S_n = t^n_k$ whenever $t^n_{k-1} < S \leq t^n_k$. (S_n) is then a sequence of stopping times taking values in $\mathbb{D}_+$ and infinity and converging downwards to S, and

$$B^n_S = \sum_{k=0}^{\infty} B^n_{t^n_{k+1}} 1_{(t^n_k < S \leq t^n_{k+1})} = \sum_{k=0}^{\infty} B^n_{t^n_{k+1}} 1_{(S_n = t^n_{k+1})} = B^n_{S_n}. \tag{2.8}$$

As A is càdlàg and bounded and $A^n_{S_n} = A_{S_n}$, the dominated convergence theorem allows us to obtain

$$\begin{aligned} \lim_{n\to\infty} EB^n_S &= \lim_{n\to\infty} EB^n_{S_n} = \lim_{n\to\infty} EA_{S_n} - EM^n_{S_n} \\ &= \lim_{n\to\infty} EA_S - EM_S = EB_S. \end{aligned} \tag{2.9}$$

Recalling our earlier observations, we may now conclude that $\limsup_{n\to\infty} C^n_t = B_t$ almost surely for all points of discontinuity of B, and so all in all, the result holds almost surely for all $t \in \mathbb{R}_+$ simultaneously.

We now apply this to show that B has a predictable modification. Let F be the almost sure set where $B = \limsup_{n\to\infty} C^n$. Theorem 3.33 of [4] then shows that $1_F C^n$ is a predictable càdlàg process, and $B1_F = \limsup_{n\to\infty} C^n 1_F$. Therefore, $B1_F$ is a predictable càdlàg process, and $B1_F$ is almost surely increasing as well. Now let $\Pi^*_p A$ be a modification of B such that $\Pi^*_p A$ is in $\mathcal{A}^i$. Again using Theorem 3.33 of [4], $\Pi^*_p A$ is predictable since B is predictable, and as $A - \Pi^*_p A$ is a modification of the uniformly integrable martingale $A - B$, we conclude that $\Pi^*_p A$ satisfies all the requirements to be the compensator of A. □

With Lemma 2.1 in hand, the remainder of the proof for the existence of the compensator merely consists of monotone convergence arguments.

Lemma 2.2. *Let A^n be a sequence of processes in $\mathcal{A}^i$ such that $\sum_{n=1}^{\infty} A^n$ converges pointwise to a process A. Assume for each $n \geq 1$ that B^n is a predictable element of $\mathcal{A}^i$ such that $A^n - B^n$ is a uniformly integrable martingale. A is then in $\mathcal{A}^i$, and $\sum_{n=1}^{\infty} B^n$ almost surely converges pointwise to a predictable process $\Pi_p^* A$ in $\mathcal{A}^i$ such that $A - \Pi_p^* A$ is a uniformly integrable martingale.*

Proof. Clearly, A is in $\mathcal{A}^i$. With $B = \sum_{n=0}^{\infty} B^n$, B is a well-defined process with values in $[0, \infty]$, since each B^n is nonnegative. We wish to argue that there is a modification of B which is the compensator of A. First note that as each B^n is increasing and nonnegative, so is B. Also, as $A^n - B^n$ is a uniformly integrable martingale, the optional sampling theorem and two applications of the monotone convergence theorem yields for any bounded stopping time T that

$$EB_T = \lim_{n\to\infty} \sum_{k=1}^{n} EB_T^k = \lim_{n\to\infty} \sum_{k=1}^{n} EA_T^k = EA_T, \tag{2.10}$$

which in particular shows that B almost surely takes finite values. Therefore, by Lemma A.2, we obtain that B is almost surely nonnegative, càdlàg and increasing. Also, by another two applications of the monotone convergence theorem, we obtain for any stopping time T that $EB_T = \lim_{t\to\infty} EB_{T\wedge t} = \lim_{t\to\infty} EA_{T\wedge t} = EA_T$. This holds in particular with $T = \infty$, and therefore, the limit of B_t as t tends to infinity is almost surely finite and is furthermore integrable. Lemma A.2 then also shows that $\sum_{k=1}^{n} B^k$ converges almost surely uniformly to B on $\mathbb{R}_+$.

We now let $\Pi_p^* A$ be a nonnegative càdlàg increasing adapted modification of B. Then $\Pi_p^* A$ is in $\mathcal{A}^i$, and $E(\Pi_p^* A)_T = EA_T$ for all stopping times T, so by Theorem II.77.6 of [10], $A - \Pi_p^* A$ is a uniformly integrable martingale. Also, $\sum_{k=1}^{n} B^k$ almost surely converges uniformly to $\Pi_p^* A$ on $\mathbb{R}_+$. In order to complete the proof, it remains to show that $\Pi_p^* A$ is predictable. To this end, note that by uniform convergence, Lemma A.3 shows that $\Delta(\Pi_p^* A)_T = \lim_n \sum_{k=1}^{n} \Delta B_T^k$ for any stopping time T. As B^k is predictable, we find by Theorem 3.33 of [4] that if T is totally inacessible, $\Delta(\Pi_p^* A)_T$ is zero almost surely, and if T is predictable, $\Delta(\Pi_p^* A)_T$ is $\mathcal{F}_{T-}$ measurable. Therefore, Theorem 3.33 of [4] shows that $\Pi_p^* A$ is predictable. □

Theorem 2.3. *Let $A \in \mathcal{V}_\ell^i$. There exists a predictable process $\Pi_p^* A$ in $\mathcal{V}_\ell^i$, unique up to indistinguishability, such that $A - \Pi_p^* A$ is a local martingale.*

Proof. We first consider uniqueness. If $A \in \mathcal{V}_\ell^i$ and B and C are two predictable processes in $\mathcal{V}_\ell^i$ such that $A - B$ and $A - C$ both are local martingales, we find that $B - C$ is a predictable local martingale with paths of finite variation. By Theorem 6.3 of [4], uniqueness follows.

As for existence, Lemma 2.1 establishes existence for the case where $A = \xi 1_{[\![T,\infty[\![}$ where ξ is nonnegative, bounded and $\mathcal{F}_T$ measurable. Using Lemma 2.2, this extends to the case where $\xi \in \mathcal{L}^1(\mathcal{F}_T)$. For general $A \in \mathcal{A}^i$, there exists by Theorem 3.32 of [4] a sequence of stopping times (T_n) covering the jumps of A.

Put $A^d = \sum_{n=1}^{\infty} \Delta A_{T_n} 1_{[\![T_n,\infty[\![}$. As $A \in \mathcal{A}^i$, A^d is a well-defined element of $\mathcal{A}^i$, and $A - A^d$ is a continuous element of $\mathcal{A}^i$. As we have existence for each $\Delta A_{T_n} 1_{[\![T_n,\infty[\![}$, Lemma 2.2 allows us to obtain existence for A. Existence for $A \in \mathcal{V}^i$ is then obtained by decomposing $A = A^+ - A^-$, where $A^+, A^- \in \mathcal{A}^i$, and extends to $A \in \mathcal{V}^i_\ell$ by a localisation argument. □

From the characterisation in Theorem 2.3, the usual properties of the compensator such as linearity, positivity, idempotency and commutation with stopping, can then be shown.

3. The Existence of the Quadratic Variation

In this section, we will prove the existence of the quadratic variation process for a local martingale by a reduction to the cases of bounded martingales and martingales of integrable variation, applying Lemma 1.1 to obtain existence for bounded martingales. Apart from Lemma 1.1, the proofs will also use the fundamental theorem of local martingales as well as some properties of martingales with finite variation. Our method of proof is direct and is simpler than the methods employed in for example [7] or [5], where the quadratic covariation is defined through the integration-by-parts formula and requires the construction and properties of the stochastic integral.

Lemma 3.1. *Let M be a bounded martingale with initial value zero. There exists a process $[M]$ in $\mathcal{A}^i$, unique up to indistinguishability, such that $M^2 - [M] \in \mathcal{M}^2$ and $\Delta[M] = (\Delta M)^2$. We call $[M]$ the quadratic variation process of M.*

Proof. We first consider uniqueness. Assume that A and B are two processes in $\mathcal{A}^i$ such that $M^2 - A$ and $M^2 - B$ are in $\mathcal{M}^2$ and $\Delta A = \Delta B = (\Delta M)^2$. In particular, $A - B$ is a continuous element of $\mathcal{M}^2$ and has paths of finite variation, so Theorem 6.3 of [4] shows that $A - B$ is almost surely zero, such that A and B are indistinguishable. This proves uniqueness. Next, we consider the existence of the process. Let $t^n_k = k2^{-n}$ for $n, k \geq 0$, we then find

$$\begin{aligned} M_t^2 &= \sum_{k=1}^{\infty} M^2_{t\wedge t^n_k} - M^2_{t\wedge t^n_{k-1}} \\ &= 2\sum_{k=1}^{\infty} M_{t\wedge t^n_{k-1}}(M_{t\wedge t^n_k} - M_{t\wedge t^n_{k-1}}) + \sum_{k=1}^{\infty}(M_{t\wedge t^n_k} - M_{t\wedge t^n_{k-1}})^2, \end{aligned} \tag{3.1}$$

where the terms in the sum are zero from a point onwards, namely for such k that $t^n_{k-1} \geq t$. Define $N^n_t = 2\sum_{k=1}^{\infty} M_{t\wedge t^n_{k-1}}(M_{t\wedge t^n_k} - M_{t\wedge t^n_{k-1}})$. Our plan for the proof is to show that (N^n) is a bounded sequence in $\mathcal{M}^2$. This will allow us to apply Lemma 1.1 in order to obtain some $N \in \mathcal{M}^2$ which is the limit of appropriate convex combinations of the (N^n). We then show that by putting $[M]$ equal to a modification of $M^2 - N$, we obtain a process with the desired qualities.

We first show that N^n is a martingale by applying Theorem II.77.6 of [10]. Clearly, N^n is càdlàg and adapted with initial value zero, and so it suffices to prove that N^n_T is integrable and that $EN^n_T = 0$ for all bounded stopping times T. To this end, note that as M is bounded, there is $c > 0$ such that $|M_t| \leq c$ for all

$t \geq 0$. Then N_T^n is clearly integrable, as it is the sum of finitely many terms each bounded by $4c^2$, and we have

$$\begin{aligned} EN_T^n &= E\sum_{k=1}^{\infty} M_{T\wedge t_{k-1}^n}(M_{T\wedge t_k^n} - M_{T\wedge t_{k-1}^n}) \qquad (3.2)\\ &= \sum_{k=1}^{\infty} EM_{t_{k-1}^n}^T(M_{t_k^n}^T - M_{t_{k-1}^n}^T) = \sum_{k=1}^{\infty} EM_{t_{k-1}^n}^T E(M_{t_k^n}^T - M_{t_{k-1}^n}^T|\mathcal{F}_{t_{k-1}^n}), \end{aligned}$$

where the interchange of summation and expectation is allowed, as the only nonzero terms in the sum are for those k such that $t_{k-1}^n \leq t$, and there are only finitely many such terms. As M^T is a martingale, $E(M_{t_k^n}^T - M_{t_{k-1}^n}^T|\mathcal{F}_{t_{k-1}^n}) = 0$ by optional sampling, so the above is zero and N^n is a martingale by Theorem II.77.6 of [10]. Next, we show that N^n is bounded in $\mathcal{L}^2$. Fix $k \geq 1$, we first consider a bound for the second moment of $N_{t_k^n}^n$. To obtain this, note that for $i < j$,

$$\begin{aligned} &E(M_{t_{i-1}^n}(M_{t_i^n} - M_{t_{i-1}^n}))(M_{t_{j-1}^n}(M_{t_j^n} - M_{t_{j-1}^n}))\\ &= E(M_{t_{i-1}^n}(M_{t_i^n} - M_{t_{i-1}^n})E(M_{t_{j-1}^n}(M_{t_j^n} - M_{t_{j-1}^n})|\mathcal{F}_{t_i^n})\\ &= E(M_{t_{i-1}^n}(M_{t_i^n} - M_{t_{i-1}^n})M_{t_{j-1}^n}E(M_{t_j^n} - M_{t_{j-1}^n}|\mathcal{F}_{t_i^n}), \qquad (3.3) \end{aligned}$$

which is zero, as $E(M_{t_j^n} - M_{t_{j-1}^n}|\mathcal{F}_{t_i^n}) = 0$, and by the same type of argument, we obtain $E(M_{t_i^n} - M_{t_{i-1}^n})(M_{t_j^n} - M_{t_{j-1}^n}) = 0$. In other words, the variables are pairwisely orthogonal, and so

$$\begin{aligned} E(N_{t_k^n}^n)^2 &= E\left(\sum_{i=1}^{k} M_{t_{i-1}^n}(M_{t_i^n} - M_{t_{i-1}^n})\right)^2 = \sum_{i=1}^{k} E\left(M_{t_{i-1}^n}(M_{t_i^n} - M_{t_{i-1}^n})\right)^2\\ &\leq c^2\sum_{i=1}^{k} E(M_{t_i^n} - M_{t_{i-1}^n})^2 = c^2 E\left(\sum_{i=1}^{k} M_{t_i^n} - M_{t_{i-1}^n}\right)^2\\ &= c^2 EM_{t_k^n}^2, \qquad (3.4) \end{aligned}$$

which yields $\sup_{t\geq 0} E(N_t^n)^2 = \sup_{k\geq 1} E(N_{t_k^n}^n)^2 \leq \sup_{k\geq 1} c^2 EM_{t_k^n}^2 \leq 4c^2 EM_\infty^2$, and this is finite. Thus, $N^n \in \mathcal{M}^2$, and $E(N_\infty^n)^2 = \lim_t E(N_t^n)^2 \leq 4c^2 EM_\infty^2$, so $(N_\infty^n)_{n\geq 1}$ is bounded in $\mathcal{L}^2$.

Now, by Lemma 1.1, there exists a sequence of naturals (K_n) with $K_n \geq n$ and for each n a finite sequence of reals $\lambda_n^n, \ldots, \lambda_{K_n}^n$ in the unit interval summing to one, such that $\sum_{i=n}^{K_n} \lambda_i^n N_\infty^i$ is convergent in $\mathcal{L}^2$ to some variable N_∞. It then holds that there is $N \in \mathcal{M}^2$ such that $E\sup_{t\geq 0}(N_t - \sum_{i=n}^{K_n} \lambda_i^n N_t^i)^2$ tends to zero. By picking a subsequence and relabeling, we may assume without loss of generality that we also have almost sure convergence. Define $A = M^2 - N$, we claim that there is a modification of A satisfying the criteria of the theorem.

To prove this, first note that as M^2 and N are càdlàg and adapted, so is A. We want to show that A is almost surely increasing and that $\Delta A = (\Delta M)^2$ almost surely. We first consider the jumps of A. To prove that $\Delta A = (\Delta M)^2$ almost surely, it suffices to show that $\Delta A = (\Delta M_T)^2$ almost surely for any bounded stopping time T. Let T be any bounded stopping time. Since $\sup_{t\geq 0}(N_t - \sum_{i=n}^{K_n} \lambda_i^n N_t^i)^2$

converges to zero almost surely, we find

$$\begin{aligned} A_T = M_T^2 - N_T &= \lim_{n\to\infty} \sum_{i=n}^{K_n} \lambda_i^n (M_T^2 - N_T^i) \\ &= \lim_{n\to\infty} \sum_{i=n}^{K_n} \lambda_i^n \sum_{k=1}^{\infty} (M_{T\wedge t_k^i} - M_{T\wedge t_{k-1}^i})^2, \end{aligned} \tag{3.5}$$

almost surely. Similarly,

$$\Delta A_T = \lim_{n\to\infty} \sum_{i=n}^{K_n} \lambda_i^n \sum_{k=1}^{\infty} (M_{t\wedge t_k^i} - M_{t\wedge t_{k-1}^i})^2 - (M_{(t\wedge t_k^i)-} - M_{(t\wedge t_{k-1}^i)-})^2, \tag{3.6}$$

understanding that $M_{(t\wedge t_k^i)-}$ is the limit of $M_{s\wedge t_k^i}$ with s tending to t strictly from below, and similarly for $M_{(t\wedge t_{k-1}^i)-}$. Fix $i, k \geq 0$. By inspection, if $t \leq t_{k-1}^i$ or $t > t_k^i$, it holds that $(M_{t\wedge t_k^i} - M_{t\wedge t_{k-1}^i})^2 - (M_{(t\wedge t_k^i)-} - M_{(t\wedge t_{k-1}^i)-})^2$ is zero. In the case where t is such that $t_{k-1}^i < t \leq t_k^i$, we instead obtain

$$(M_{t\wedge t_k^i} - M_{t\wedge t_{k-1}^i})^2 = (M_t - M_{t_{k-1}^i})^2 \tag{3.7}$$

$$(M_{(t\wedge t_k^i)-} - M_{(t\wedge t_{k-1}^i)-})^2 = (M_{t-} - M_{t_{k-1}^i})^2, \tag{3.8}$$

so that with $s(t,i)$ denoting the unique t_{k-1}^i such that $t_{k-1}^i < t \leq t_k^i$, we have

$$\begin{aligned} \Delta A_T &= \lim_{n\to\infty} \sum_{i=n}^{K_n} \lambda_i^n \big((M_T - M_{s(T,i)})^2 - (M_{T-} - M_{s(T,i)})^2\big) \\ &= \lim_{n\to\infty} \sum_{i=n}^{K_n} \lambda_i^n \big(M_T^2 - 2M_T M_{s(T,i)} - M_{T-}^2 + 2M_{T-} M_{s(T,i)}\big) \\ &= (\Delta M_T)^2 + 2\Delta M_T \lim_{n\to\infty} \sum_{i=n}^{K_n} \lambda_i^n (M_{T-} - M_{s(T,i)}). \end{aligned} \tag{3.9}$$

Now, we always have $|s(T,i) - T| \leq 2^{-i}$ and $s(T,i) < T$. Therefore, given $\varepsilon > 0$, there is $n \geq 1$ such that for all $i \geq n$, $|M_{T-} - M_{s(T,i)}| \leq \varepsilon$. As the $(\lambda_i^n)_{n\leq i\leq K_n}$ are convex weights, we obtain for n this large that $|\sum_{i=n}^{K_n} \lambda_i^n (M_{T-} - M_{s(T,i)})| \leq \varepsilon$. This allows us to conclude that $\sum_{i=n}^{K_n} \lambda_i^n (M_{T-} - M_{s(T,i)})$ converges pointwise to zero, and so $\Delta A_T = (\Delta M_T)^2$ almost surely. Since this holds for any arbitrary stopping time, we now obtain $\Delta A = (\Delta M)^2$ up to indistinguishability.

Next, we show that A almost surely increases. Let $\mathbb{D}_+ = \{k2^{-n} | k \geq 0, n \geq 1\}$, then $\mathbb{D}_+$ is dense in $\mathbb{R}_+$. Let $p, q \in \mathbb{D}_+$ with $p \leq q$, we will show that $A_p \leq A_q$ almost surely. There exists $j \geq 1$ and naturals $n_p \leq n_q$ such that $p = n_p 2^{-j}$ and $q = n_q 2^{-j}$. We know that $A_p = \lim_{n\to\infty} \sum_{i=n}^{K_n} \lambda_i^n \sum_{k=1}^{\infty} (M_{p\wedge t_k^i} - M_{p\wedge t_{k-1}^i})^2$, and analogously for A_q. For $i \geq j$, $p \wedge t_k^i = n_p 2^{-j} \wedge k2^{-i} = (n_p 2^{i-j} \wedge k)2^{-i}$, and

analogously for $q \wedge t_k^i$. Therefore, we obtain that almost surely,

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{i=n}^{K_n} \lambda_i^n \sum_{k=1}^{\infty} (M_{p\wedge t_k^i} - M_{p\wedge t_{k-1}^i})^2 \\
&= \lim_{n\to\infty} \sum_{i=n}^{K_n} \lambda_i^n \sum_{k=1}^{n_p 2^{i-j}} (M_{t_k^i} - M_{t_{k-1}^i})^2 \\
&\leq \lim_{n\to\infty} \sum_{i=n}^{K_n} \lambda_i^n \sum_{k=1}^{n_q 2^{i-j}} (M_{t_k^i} - M_{t_{k-1}^i})^2 \\
&= \lim_{n\to\infty} \sum_{i=n}^{K_n} \lambda_i^m \sum_{k=1}^{\infty} (M_{q\wedge t_k^i} - M_{q\wedge t_{k-1}^i})^2, \qquad (3.10)
\end{aligned}$$

allowing us to make the same calculations in reverse and conclude $A_p \leq A_q$ almost surely. As $\mathbb{D}_+$ is countable, we conclude that A is inceasing on $\mathbb{D}_+$ almost surely, and as A is càdlàg, we conclude that A is increasing almost surely. Furthermore, as we have that $A_\infty = M_\infty^2 - N_\infty$ and both M_∞^2 and N_∞ are integrable, we conclude that A_∞ is integrable.

Finally, let F be the null set where A is not increasing. Put $[M] = A1_{F^c}$. As all null sets are in $\mathcal{F}_t$ for $t \geq 0$, $[M]$ is adapted as A is adapted. Furthermore, $[M]$ is càdlàg, increasing and $[M]_\infty$ exists and is integrable. As $M^2 - [M] = N + A1_F$, where $A1_F$ is almost surely zero and therefore in $\mathcal{M}^2$, we now have constructed a process $[M]$ which is in $\mathcal{A}^i$ such that $M^2 - [M]$ is in $\mathcal{M}^2$ and $\Delta[M] = (\Delta M)^2$ up to indistinguishability. This concludes the proof. □

Theorem 3.2. *Let M be a local martingale with initial value zero. There exists $[M] \in \mathcal{A}$ such that $M^2 - [M]$ is a local martingale with initial value zero and $\Delta[M] = (\Delta M)^2$.*

Proof. We first consider the case where $M = M^b + M^i$, where M^b and M^i both are local martingales with initial value zero, M^b is bounded and M^i is of integrable variation. In this case, $\sum_{0<s\leq t}(\Delta M_t^i)^2$ is absolutely convergent for any $t \geq 0$, and we may therefore define a process A^i in $\mathcal{A}$ by putting $A_t^i = \sum_{0<s\leq t}(\Delta M_t^i)^2$. As M^b is bounded, $\sum_{0<s\leq t} \Delta M_t^b \Delta M_t^i$ is almost surely absolutely convergent as well, and so we may define a process A^x in $\mathcal{V}$ by putting $A_t^x = \sum_{0<s\leq t} \Delta M_t^b \Delta M_t^i$. Finally, by Lemma 3.1, there exists a process $[M^b]$ in $\mathcal{A}^i$ such that $(M^b)^2 - [M^b]$ is in $\mathcal{M}^2$ and $\Delta[M^b] = (\Delta M^b)^2$. We put $A_t = [M^b]_t + 2A^x + A^i$ and claim that there is a modification of A satisfying the criteria in the theorem.

To this end, first note that A clearly is càdlàg adapted of finite variation, and for $0 \leq s \leq t$, we have $[M^b]_t \geq [M^b]_s + \sum_{s<u\leq t}(\Delta M_u^b)^2$ almost surely, so that we obtain $A_t - A_s \geq \sum_{s<u\leq t}(\Delta M_u^b + \Delta M_u^i)^2$ almost surely, showing that A is almost surely increasing. To show that $M^2 - A$ is a local martingale, note that

$$M^2 - A = (M^b)^2 - [M^b] + 2(M^b M^i - A^x) + (M^i)^2 - A^i. \qquad (3.11)$$

Here, $(M^b)^2 - [M^b]$ is in $\mathcal{M}^2$ by Theorem 3.1, in particular a local martingale. By the integration-by-parts formula, we have $(M^i)_t^2 - A_t^i = 2\int_0^t M_{s-}^i \, \mathrm{d}M_s^i$, where

the integral is well-defined as M_{s-} is bounded on compacts. Using Theorem 6.5 of [4], the integral process $\int_0^t M^i_{s-}\,\mathrm{d}M^i_s$ is a local martingale, and so $(M^i)^2 - A^i$ is a local martingale. Therefore, in order to obtain that $M^2 - A$ is a local martingale, it suffices to show that $M^b M^i - A^x$ is a local martingale. By Theorem 5.32 of [4], $M^b_t M^i_t - \int_0^t M^b_s\,\mathrm{d}M^i_s$ is a local martingale, so it suffices to show that $\int_0^t M^b_s\,\mathrm{d}M^i_s - A^x_t$ is a local martingale. As ΔM^b is bounded, it is integrable, and so we have

$$\int_0^t M^b_s\,\mathrm{d}M^i_s = \int_0^t \Delta M^b_s\,\mathrm{d}M^i_s + \int_0^t M^b_{s-}\,\mathrm{d}M^i_s = A^x_t + \int_0^t M^b_{s-}\,\mathrm{d}M^i_s. \tag{3.12}$$

As $\int_0^t M^b_{s-}\,\mathrm{d}M^i_s$ is a local martingale, again by Theorem 6.5 of [4], we finally conclude that $M^b M^i - A^x$ is a local martingale. Thus, $M^2 - A$ is a local martingale. This proves existence in the case where $M = M^b + M^i$, where M^b is bounded and M^i has integrable variation.

Finally, we consider the case of a general local martingale M with initial value zero. By Theorem III.29 of [8], $M = M^b + M^i$, where M^b is locally bounded and M^i has paths of finite variation. With (T_n) a localising sequence for both M^b and M^i, our previous results then show the existence of a process $A^n \in \mathcal{A}$ such that $(M^{T_n})^2 - A^n$ is a local martingale and $\Delta A^n = (\Delta M^{T_n})^2$. By uniqueness, we may define $[M]$ by putting $[M]_t = A^n_t$ for $t \le T_n$. We then obtain that $[M] \in \mathcal{A}$, $M^2 - [M]$ is a local martingale and $\Delta[M] = (\Delta M)^2$, and the proof is complete. □

4. Discussion

The results given in Sections 2 and 3 yield comparatively simple proofs of the existence of the compensator and the quadratic variation, two technical concepts essential to martingale theory in general and stochastic calculus in particular. We will now discuss how these proofs may be used to give a simplified account of the development of the basic results of stochastic integration theory. Specifically, the question we ask is the following: How can one, starting from basic continuous-time martingale theory, construct the stochastic integral of a locally bounded predictable process with respect to a semimartingale, as simply as possible?

Since the publication of one of the first complete accounts of the general theory of stochastic integration in [3], several others have followed, notably [4], [11], [7], [5] and [8], each contributing with simplified and improved proofs. The accounts in [4] and [11] make use of the predictable projection to prove the Doob-Meyer theorem, and to obtain the uniqueness of this projection, they apply the difficult section theorems. In [7] and [8], this dependence is removed, using the methods of, among others, [9] and [1], respectively. In general, however, the methods in [7] and [8] are not entirely comparable, as [7] follows the traditional path of starting with continuous-time martingale theory, developing some general theory of processes, and finally constructing the stochastic integral for semimartingales, while [8] begins by defining a semimartingale as a "good integrator" in a suitable sense, and develops the theory from there, in the end proving through the Bichteler-Dellacherie theorem that the two methods are equivalent. The developement of the stochastic integral we will suggest below follows in the tradition seen in [7].

We suggest the following path to the construction of the stochastic integral:

(1) Development of the predictable σ-algebra and predictable stopping times, in particular the equivalence between, in the notation of [11], being "previsible" and being "announceable".
(2) Development of the main results on predictable processes, in particular the characterization of predictable càdlàg processes as having jumps only at predictable times, and having the jump at a predictable time T being measurable with respect to the σ-algebra $\mathcal{F}_{T-}$.
(3) Proof of the existence of the compensator, leading to the fundamental theorem of local martingales, meaning the decomposition of any local martingale into a locally bounded and a locally integrable variation part. Development of the quadratic variation process using these results.
(4) Construction of the stochastic integral using the fundamental theorem of local martingales and the quadratic variation process.

The proofs given in Sections 2 and 3 help make this comparatively short path possible. We now comment on each of the points above, and afterwards compare the path outlined with other accounts of the theory.

As regards point (1), the equivalence between a stopping time being previsible (having a predictable graph) and being announceable (having an announcing sequence) is proved in [11] as part of the PFA theorem, which includes the introduction of $\mathcal{F}_{T-}$. However, the equivalence between P (previsibility) and A (accessibility) may be done without any reference to $\mathcal{F}_{T-}$, and this makes for a pleasant separation of concerns.

The main result in point (2), the characterization of predictable càdlàg processes, can be found for example as Theorem 3.33 of [4].

The existence of the compensator in point (3) may now be obtained as in Section 2, and the fundamental theorem of local martingales may then be proven as in the proof of Theorem III.29 of [8]. After this, the existence of the quadratic variation may be obtained as in Section 3. Note that the traditional method for obtaining the quadratic variation is either as the remainder term in the integration-by-parts formula (as in [5]), or through a localisation to $\mathcal{M}^2$ and a decomposition into continuous and purely discontinuous parts (as in [11]). Our method employs a localisation to bounded martingales and allows for a rather direct proof.

Finally, in point (4), these results may be combined to obtain the existence of the stochastic integral of a locally bounded predictable process with respect to a semimartingale using the fundamental theorem of local martingales and a modification of the methods given in Chapter IX of [4].

As for comparisons of the approach outlined above with other approaches, for example [7], the main benefit of the above approach would be that the development of the compensator is obtained in a very simple manner, in particular not necessitating a decomposition into predictable and totally inaccessible parts, and without any reference to "naturality". Note, however, that the expulsion of "naturality" from the proof of the Doob-Meyer theorem in [9] already was obtained in [6] and [2]. In any case, focusing attention on finite variation processes instead of a general supermartingales simplifies matters considerably. Furthermore, developing the quadratic variation directly using the fundamental theorem of local martingales allows for a very direct construction of the stochastic integral, while

the method given in [7] first develops a preliminary integral for local martingales which are locally in $\mathcal{M}^2$.

Appendix A. Auxiliary Results

Lemma A.1. *Let (X_n) be a sequence of uniformly integrable variables. It then holds that*

$$\limsup_{n\to\infty} EX_n \leq E \limsup_{n\to\infty} X_n. \tag{A.1}$$

Proof. Since (X_n) is uniformly integrable, it holds that $\lim_{\lambda\to\infty} \sup_n EX_n 1_{(X_n>\lambda)}$ is zero. Let $\varepsilon > 0$ be given, we may then pick λ so large that $EX_n 1_{(X_n>\lambda)} \leq \varepsilon$ for all n. Now, the sequence $(\lambda - X_n 1_{(X_n\leq\lambda)})_{n\geq 1}$ is nonnegative, and Fatou's lemma therefore yields

$$\begin{aligned}\lambda - E\limsup_{n\to\infty} X_n 1_{(X_n\leq\lambda)} &= E\liminf_{n\to\infty}(\lambda - X_n 1_{(X_n\leq\lambda)})\\ &\leq \liminf_{n\to\infty} E(\lambda - X_n 1_{(X_n\leq\lambda)})\\ &= \lambda - \limsup_{n\to\infty} EX_n 1_{(X_n\leq\lambda)}.\end{aligned} \tag{A.2}$$

The terms involving the limes superior may be infinite and are therefore a priori not amenable to arbitrary arithmetic manipulation. However, by subtracting λ and multiplying by minus one, we may still obtain

$$\limsup_{n\to\infty} EX_n 1_{(X_n\leq\lambda)} \leq E\limsup_{n\to\infty} X_n 1_{(X_n\leq\lambda)}. \tag{A.3}$$

As we have ensured that $EX_n 1_{(X_n>\lambda)} \leq \varepsilon$ for all n, this yields

$$\limsup_{n\to\infty} EX_n \leq \varepsilon + E\limsup_{n\to\infty} X_n 1_{(X_n\leq\lambda)} \leq \varepsilon + E\limsup_{n\to\infty} X_n, \tag{A.4}$$

and as $\varepsilon > 0$ was arbitrary, the result follows. □

Lemma A.2. *Let (f_n) be a sequence of nonnegative increasing càdlàg mappings from $\mathbb{R}_+$ to $\mathbb{R}$. Assume that $\sum_{n=1}^{\infty} f_n$ converges pointwise to some mapping f from $\mathbb{R}_+ \to \mathbb{R}$. Then, the convergence is uniform on compacts, and f is a nonnegative increasing càdlàg mapping. If $f(t)$ has a limit as t tends to infinity, the convergence is uniform on $\mathbb{R}_+$.*

Proof. Fix $t \geq 0$. For $m \geq n$, we have

$$\sup_{0\leq s\leq t}\left|\sum_{k=1}^{m} f_k(s) - \sum_{k=1}^{n} f_k(s)\right| = \sup_{0\leq s\leq t}\sum_{k=n+1}^{m} f_k(s) = \sum_{k=n+1}^{m} f_k(t), \tag{A.5}$$

which tends to zero as m and n tend to infinity. Therefore, $(\sum_{k=1}^{n} f_k)$ is uniformly Cauchy on $[0,t]$, and so has a càdlàg limit on $[0,t]$. As this limit must agree with the pointwise limit, we conclude that $\sum_{k=1}^{n} f_k$ converges uniformly on compacts to f, and therefore f is nonnegative, increasing and càdlàg.

It remains to consider the case where $f(t)$ has a limit $f(\infty)$ as t tends to infinity. In this case, we find that $\lim_t f_n(t) \leq \lim_t f(t) = f(\infty)$, so $f_n(t)$ has a limit $f_n(\infty)$

as t tends to infinity as well. Fixing $n \geq 1$, we have

$$\sum_{k=1}^{n} f_k(\infty) = \sum_{k=1}^{n} \lim_{t\to\infty} f_k(t) = \lim_{t\to\infty} \sum_{k=1}^{n} f_k(t) \leq \lim_{t\to\infty} f(t) = f(\infty). \tag{A.6}$$

Therefore, $(f_k(\infty))$ is absolutely summable. As we have

$$\sup_{t\geq 0} \left| \sum_{k=1}^{m} f_k(t) - \sum_{k=1}^{n} f_k(t) \right| = \sup_{t\geq 0} \sum_{k=n+1}^{m} f_k(t) = \sum_{k=n+1}^{m} f_k(\infty), \tag{A.7}$$

we find that $(\sum_{k=1}^{n} f_k)$ is uniformly Cauchy on $\mathbb{R}_+$, and therefore uniformly convergent. As the limit must agree with the pointwise limit, we conclude that f_n converges uniformly to f on $\mathbb{R}_+$. This concludes the proof. □

Lemma A.3. *Let (f_n) be a sequence of bounded càdlàg mappings from $\mathbb{R}_+$ to $\mathbb{R}$. If (f_n) is Cauchy in the uniform norm, there is a bounded càdlàg mapping f from $\mathbb{R}_+$ to $\mathbb{R}$ such that $\sup_{t\geq 0} |f_n(t) - f(t)|$ tends to zero. In this case, it holds that $\sup_{t\geq 0} |f_n(t-) - f(t-)|$ and $\sup_{t\geq 0} |\Delta f_n(t) - \Delta f(t)|$ tends to zero as well.*

Proof. Assume that (f_n) is Cauchy in the uniform norm. This implies that $(f_n(t))_{n\geq 1}$ is Cauchy for any $t \geq 0$, therefore convergent. Let $f(t)$ be the limit. Now note that as (f_n) is Cauchy in the uniform norm, (f_n) is bounded in the uniform norm, and therefore $\sup_{t\geq 0} |f(t)| \leq \sup_{n\geq 1} \sup_{t\geq 0} |f_n(t)|$, so f is bounded as well. In order to obtain uniform convergence, let $\varepsilon > 0$. Let k be such that for $m, n \geq k$, $\sup_{t\geq 0} |f_n(t) - f_m(t)| \leq \varepsilon$. Fix $t \geq 0$, we then obtain for $n \geq k$ that

$$|f(t) - f_n(t)| = \lim_{m} |f_m(t) - f_n(t)| \leq \varepsilon. \tag{A.8}$$

Therefore, $\sup_{t\geq 0} |f(t) - f_n(t)| \leq \varepsilon$, and so f_n converges uniformly to f.

We now show that f is càdlàg. Let $t \geq 0$, we will show that f is right-continuous at t. Take $\varepsilon > 0$ and take n so that $\sup_{t\geq 0} |f(t) - f_n(t)| \leq \varepsilon$. Let $\delta > 0$ be such that $|f_n(t) - f_n(s)| \leq \varepsilon$ for $s \in [t, t+\delta]$, then

$$|f(t) - f(s)| \leq |f(t) - f_n(t)| + |f_n(t) - f_n(s)| + |f_n(s) - f_n(t)| \leq 3\varepsilon \tag{A.9}$$

for such s. Therefore, f is right-continuous at t. Now let $t > 0$, we claim that f has a left limit at t. First note that for n and m large enough, it holds for any $t > 0$ that $|f_n(t-) - f_m(t-)| \leq \sup_{t\geq 0} |f_n(t) - f_m(t)|$. Therefore, the sequence $(f_n(t-))_{n\geq 1}$ is Cauchy, and so convergent to some limit $\xi(t)$. Now let $\varepsilon > 0$ and take n so that $\sup_{t\geq 0} |f(t) - f_n(t)| \leq \varepsilon$ and $|f_n(t-) - \xi(t)| \leq \varepsilon$. Let $\delta > 0$ be such that $t - \delta \geq 0$ and such that whenever $s \in [t-\delta, t)$, $|f_n(s) - f_n(t-)| \leq \varepsilon$. Then

$$|f(s) - \xi(t)| \leq |f(s) - f_n(s)| + |f_n(s) - f_n(t-)| + |f_n(t-) - \xi(t)| \leq 3\varepsilon \tag{A.10}$$

for any such s. Therefore, f has a left limit at t. This shows that f is càdlàg.

Finally, we have for any $t > 0$ and any sequence (s_n) converging strictly upwards to t that $|f(t-) - f_n(t-)| = \lim_m |f(s_m) - f_n(s_m)| \leq \sup_{t\geq 0} |f(t) - f_n(t)|$, so we conclude that $\sup_{t\geq 0} |f(t-) - f_n(t-)|$ converges to zero as well. As a consequence, we also obtain that $\sup_{t\geq 0} |\Delta f(t) - \Delta f_n(t)|$ converges to zero. □

References

1. Bass, Richard F.: The Doob-Meyer decomposition revisited, *Canad. Math. Bull.* **39** (1996) 138–150.
2. Beiglböck, M., Schachermayer, W., and Veliyev, B.: A short proof of the Doob-Meyer theorem, *Stochastic Processes and their Applications* **122** (2012) 1204–1209.
3. Dellacherie, C. and Meyer, P.-A.: *Probabilities and Potential B: Theory of Martingales*, Elsevier, 1982.
4. He, S.-W., Wang, J.-G., and Yan, J.-A.: *Semimartingale Theory and Stochastic Calculus*, Science Press, 1992.
5. Jacod, J. and Shiryaev, A.: *Limit Theorems for Stochastic Processes*, Springer, 2003.
6. Jakubowski, A.: An almost sure approximation for the predictable process in the Doob-Meyer decomposition theorem, in: *Séminaire de Probabilités XXXVIII* (2005) 158–164, Springer.
7. Kallenberg, O.: *Foundations of Modern Probability*, Springer, 2002.
8. Protter, P.: *Stochastic Integration and Differential Equations*, Springer, 2005.
9. Rao, M.: On decomposition theorems of Meyer, *Math. Scand.* **24** (1969), 66–78.
10. Rogers, L. C. G. and Williams, D.: *Diffusions, Markov Processes and Martingales*, volume 1, Cambridge University Press, 2000.
11. Rogers, L. C. G. and Williams, D.: *Diffusions, Markov Processes and Martingales*, volume 2, Cambridge University Press, 2000.
12. Rudin, W.: *Functional Analysis*, McGraw-Hill, 1973.
13. Taylor, A. E.: *Functional Analysis*, Wiley, 1958.

Alexander Sokol: Institute of Mathematics, University of Copenhagen, 2100 Copenhagen, Denmark

E-mail address: `alexander@math.ku.dk`

URL: `http://www.math.ku.dk/~alexander`