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Abdessatar Barhoumi

Alberto Lanconelli

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CHARACTERIZATION THEOREMS FOR DIFFERENTIAL OPERATORS ON WHITE NOISE SPACES

ABDESSATAR BARHOUMI AND ALBERTO LANCONELLI

Abstract. We characterize through their action on stochastic exponentials the class of white noise operators which are derivations with respect to both the point-wise and Wick products. We define the class of second order differential operators and second order Wick differential operators and we characterize the white noise operators belonging to both classes. We find that the intersection of these two classes, in the first and second order cases, is identified by a skewness condition on the coefficients of the differential operator. Our technique relies on simple algebraic properties of commutators and on the Gaussian structure of our white noise space. Our approach is suitable to study differential operators of any order.

1. Introduction and Notation

In the last two decades differential operators in white noise analysis have been investigated by several authors. The research has been focused mainly on three issues: characterization theorems for Gross laplacian, number operator, Euler operator and Lévy laplacian [2, 4, 6]; differential operators related to the infinite dimensional rotation group [4, 6, 7]; characterization theorems for derivations and Wick derivations [1, 8]. See also the books [3, 5, 9] and the references quoted there. The main tools of investigation in the above mentioned papers are the symbol transform of an operator, the Fock expansion and integral kernels operators.

It is well known, and very useful in carrying calculations, that annihilation operators are derivations with respect to both the ordinary and Wick products. More precisely, for any $\xi \in S'$ (the space of tempered distributions) and $\varphi, \psi \in (S)$ (the Hida test function space) one has:

$$\begin{aligned} a_\xi(\varphi \cdot \psi) &= a_\xi\varphi \cdot \psi + \varphi \cdot a_\xi\psi, \\ a_\xi(\varphi \diamond \psi) &= a_\xi\varphi \diamond \psi + \varphi \diamond a_\xi\psi. \end{aligned}$$

It is natural to wonder whether only annihilations enjoy both properties. The first main result of the present paper is a characterization of those white noise differential operators that behave as derivations with respect to both the ordinary and Wick products (see Theorem 2.8 below). The operators belonging to this class can be written as the sum of an annihilation operator and a first order differential operator whose coefficients satisfy a skewness condition (in Remark

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2.10 below we relate this last class of operators to the generators of some infinite dimensional rotation groups from [4]). To prove this theorem we use a quite elementary machinery consisting of basic algebraic properties of commutators and the Gaussian structure of the underlying white noise space. Our starting point is the useful observation (whose origin can be traced back to Pincherle [10]) that one can reduce the order of a differential operator by simply taking the commutator with a multiplication operator (see Remark 2.5 below). Through this idea we then define the classes of second order differential operators and second order Wick differential operators and we characterize the intersection of these two classes (see Theorem 3.10 below). Also in this case we obtain a family of differential operators whose coefficients satisfy a skewness condition. This class contains also the Gross laplacian. Our technique is suitable to study differential operators of any order and can be potentially adapted to cover the theory of differential operators in non gaussian white noise spaces.

The paper is organized as follows: at the end of this section we briefly set the notation; Section 2 is devoted to first order differential operators: definitions, characterization theorems and examples; Section 3 is focused on second order differential operators.

We will work in the framework of the White Noise Theory following the standard notation of one of the books [3, 5, 9], which we refer to for detailed information on that topic. In particular, S and S' will denote the Schwartz space of smooth rapidly decreasing functions and the space of tempered distributions over $\mathbb{R}$, respectively. The notation $\langle\cdot,\cdot\rangle$ will be used for the dual pairing between S' and S. We will use the symbols (S) and $(S)^*$ to denote respectively the Hida's test function and distribution spaces constructed from the Hilbert space $L^2(\mathbb{R})$ and the harmonic oscillator $A := -\frac{d^2}{dx^2}+x^2+1$. The notation $\langle\langle\cdot,\cdot\rangle\rangle$ will be used for the dual pairing between $(S)^*$ and (S).

For any $n\in\mathbb{N}$, we will say that $Y\in(S)^*$ belongs to the n-th chaos if Y is of the form $I_n(F_n)$, where $F_n\in S'^{\otimes n}$ is a symmetric distribution and $I_n(F_n)$ stands for the (generalized) n-th order multiple Itô integral of F_n (with respect to the canonical Brownian motion defined on the standard Gaussian white noise space $(S',\mathcal{B}(S'),\mu)$).

For $\xi\in S'$ we will denote $X(\xi):=I_1(\xi)$ and by a_ξ and a_ξ^* the usual annihilation and creation operators, respectively. We observe that $a_\xi, a_\xi^*\in\mathcal{L}((S),(S)^*)$ and that for any $\varphi\in(S)$,

$$(a_\xi+a_\xi^*)\varphi = X(\xi)\cdot\varphi.$$

In particular if $\xi\in S$, then $a_\xi, a_\xi^*\in\mathcal{L}((S),(S))\cap\mathcal{L}((S)^*,(S)^*)$. For $\xi\in S$ we will write ϕ_ξ for the usual stochastic exponential vector which we recall to be an element of (S). The symbol $\diamond$ will denote the Wick product; we remind that (S) and $(S)^*$ are closed under this operation. Moreover for any $\varphi\in(S)$, we have

$$a_\xi^*\varphi = X(\xi)\diamond\varphi.$$

For $A\in\mathcal{L}((S),(S))\cap\mathcal{L}((S)^*,(S)^*)$ and $B\in\mathcal{L}((S),(S)^*)$, we will set

$$[A,B] := AB-BA\in\mathcal{L}((S),(S)^*).$$

2. First Order Differential Operators

We begin with the following.

Definition 2.1. Let $M \in \mathcal{L}((S),(S)^*)$.

- M is a *multiplication operator* if there exists $\Phi \in (S)^*$ such that for any $\varphi \in (S)$ the following equality holds:
$$M\varphi = \Phi \cdot \varphi. \tag{2.1}$$
In this case the operator M will be denoted by M_Φ.
- M is a *Wick multiplication operator* if there exists $\Phi \in (S)^*$ such that for any $\varphi \in (S)$ the following equality holds:
$$M\varphi = \Phi \diamond \varphi. \tag{2.2}$$
In this case the operator M will be denoted by $M_\Phi^\diamond$.

Example 2.2.
- For any $\xi \in S'$ the operator $a_\xi + a_\xi^*$ is a multiplication operator since
$$(a_\xi + a_\xi^*)\varphi = X(\xi) \cdot \varphi.$$
- For any $\xi \in S'$ the operator a_ξ^* is a Wick multiplication operator since
$$a_\xi^*\varphi = X(\xi) \diamond \varphi.$$

Definition 2.3. Let $D \in \mathcal{L}((S),(S)^*)$.

- D is a *derivation* if for any $\varphi, \psi \in (S)$ the following equality holds:
$$D(\varphi \cdot \psi) = (D\varphi) \cdot \psi + \varphi \cdot (D\psi). \tag{2.3}$$
- D is a *Wick derivation* if for any $\varphi, \psi \in (S)$ the following equality holds:
$$D(\varphi \diamond \psi) = (D\varphi) \diamond \psi + \varphi \diamond (D\psi). \tag{2.4}$$

Example 2.4.
- The number operator N is a Wick derivation.
- The Euler operator $\Delta_E := \Delta_G + N$ is a derivation (Δ_G is the Gross laplacian).
- For any $\xi \in S'$ the operator a_ξ is both a derivation and a Wick derivation.

Remark 2.5. By means of the definition of multiplication operator, equation (2.3) can be rewritten as
$$DM_\varphi\psi = M_{D\varphi}\psi + M_\varphi D\psi,$$
or equivalently
$$DM_\varphi\psi - M_\varphi D\psi = M_{D\varphi}\psi,$$
i.e.,
$$[D, M_\varphi]\psi = M_{D\varphi}\psi. \tag{2.5}$$
Therefore D is a derivation if and only if for any $\varphi \in (S)$, one has
$$[D, M_\varphi] = M_{D\varphi}, \tag{2.6}$$
meant as an equality for operators in $\mathcal{L}((S),(S)^*)$.

The same reasoning can be carried for Wick derivations using Wick multiplication operators; namely, D is a Wick derivation if and only if for any $\varphi \in (S)$, one has

$$[D, M^{\diamond}_{\varphi}] = M^{\diamond}_{D\varphi}. \tag{2.7}$$

The next theorem establishes that in order to check whether a given operator is a derivation (or a Wick derivation), it is sufficient to verify equation (2.6) (or (2.7)) only for $\varphi = X(\xi)$, $\xi \in S$.

Theorem 2.6. *Let $D \in \mathcal{L}((S),(S)^*)$. Then*

- *D is a derivation if and only if for any $\xi \in S$, $[D, a_\xi + a^*_\xi] = M_{DX(\xi)}$.*
- *D is a Wick derivation if and only if for any $\xi \in S$, $[D, a^*_\xi] = M^{\diamond}_{DX(\xi)}$.*

Proof. We will only prove the statement concerning Wick derivations. The proof of the other part is obtained by straightforward modifications.

If D is a Wick derivation, then there is nothing to prove (see Remark 2.5). Now suppose that for any $\xi \in S$,

$$[D, a^*_\xi] = M^{\diamond}_{DX(\xi)}.$$

This means that for any $\xi \in S$ and $\varphi \in (S)$ we have

$$[D, a^*_\xi]\varphi = DX(\xi) \diamond \varphi,$$

or equivalently,

$$D(X(\xi) \diamond \varphi) = DX(\xi) \diamond \varphi + X(\xi) \diamond D\varphi.$$

Now let $\xi_1, \xi_2 \in S$; then exploiting the previous equality we get

$$\begin{aligned} D((X(\xi_1) \diamond X(\xi_2)) \diamond \varphi) &= D(X(\xi_1) \diamond (X(\xi_2) \diamond \varphi)) \\ &= DX(\xi_1) \diamond (X(\xi_2) \diamond \varphi) + X(\xi_1) \diamond D(X(\xi_2) \diamond \varphi) \\ &= DX(\xi_1) \diamond X(\xi_2) \diamond \varphi + X(\xi_1) \diamond DX(\xi_2) \diamond \varphi \\ &\quad + X(\xi_1) \diamond X(\xi_2) \diamond D\varphi \\ &= D(X(\xi_1) \diamond X(\xi_2)) \diamond \varphi + (X(\xi_1) \diamond X(\xi_2)) \diamond D\varphi. \end{aligned}$$

It is easy to verify by induction that for any $n \in \mathbb{N}$ and $\xi_1, \cdots, \xi_n \in S$ one has

$$\begin{aligned} D(X(\xi_1) \diamond \cdots \diamond X(\xi_n) \diamond \varphi) &= D(X(\xi_1) \diamond \cdots \diamond X(\xi_n)) \diamond \varphi \\ &\quad + (X(\xi_1) \diamond \cdots \diamond X(\xi_n)) \diamond D\varphi. \end{aligned}$$

Since the set $\{X(\xi_1) \diamond \cdots \diamond X(\xi_n),\ n \in \mathbb{N} \text{ and } \xi_1, \cdots, \xi_n \in S\}$ is total in (S) and since $D \in \mathcal{L}((S),(S)^*)$ we conclude via a density argument that for any $\psi, \varphi \in (S)$,

$$D(\psi \diamond \phi) = D\psi \diamond \varphi + \psi \diamond D\varphi.$$

□

An equivalent formulation of the previous theorem is the following.

Theorem 2.7. *Let $D \in \mathcal{L}((S),(S)^*)$. Then*

- *D is a derivation if and only if $D1 = 0$ and for any $\xi \in S$, $[D, a_\xi + a^*_\xi]$ is a multiplication operator.*

- *D is a Wick derivation if and only if $D1 = 0$ and for any $\xi \in S$, $[D, a_\xi^*]$ is a Wick multiplication operator.*

Proof. As before we will only prove the statement concerning Wick derivations.

For the necessity one has only to check by a simple verification that $D1 = 0$. Now suppose that $[D, a_\xi^*]$ is a Wick multiplication operator. Then for any $\xi \in S$ there exists a $\Phi_\xi \in (S)^*$ such that

$$[D, a_\xi^*]\varphi = M^\diamond_{\Phi_\xi}\varphi, \text{ for all } \varphi \in (S),$$

or equivalently

$$D(\varphi \diamond X(\xi)) = (D\varphi) \diamond X(\xi) + \varphi \diamond \Phi_\xi.$$

Choosing $\varphi = 1$ and using the condition $D1 = 0$ we get

$$\Phi_\xi = DX(\xi).$$

Therefore

$$[D, a_\xi^*] = M^\diamond_{DX(\xi)}.$$

By the previous theorem this means that D is a Wick derivation. □

We are now ready to state and prove the first main result of the present paper.

Theorem 2.8. *Let $D \in \mathcal{L}((S), (S)^*)$. D is both a derivation and a Wick derivation if and only if there exist $F_1 \in S'$ and $F_2 \in S'^{\otimes 2}$, with*

$$\langle F_2, \xi \otimes \eta \rangle + \langle F_2, \eta \otimes \xi \rangle = 0,$$

for all $\xi, \eta \in S$ such that

$$\begin{aligned} D\phi_\xi &= \langle F_1, \xi \rangle \phi_\xi + X(F_2 \otimes_1 \xi) \cdot \phi_\xi \\ &= \langle F_1, \xi \rangle \phi_\xi + X(F_2 \otimes_1 \xi) \diamond \phi_\xi, \end{aligned} \tag{2.8}$$

where $F_2 \otimes_1 \xi$ denotes the unique element in S' such that for any $\eta \in S$, $\langle F_2 \otimes_1 \xi, \eta \rangle = \langle F_2, \xi \otimes \eta \rangle$.

Proof. Suppose that D is both a derivation and a Wick derivation. It is not difficult to see that for any $\xi \in S$, $[D, a_\xi]$ is also an operator of this kind. Hence for any $\xi_1, \xi_2 \in S$ one has

$$\begin{aligned} [D, a_\xi](X(\xi_1) \cdot X(\xi_2)) &= [D, a_\xi](a_{\xi_1} + a_{\xi_1}^*)X(\xi_2) \\ &= [[D, a_\xi], a_{\xi_1} + a_{\xi_1}^*]X(\xi_2) + (a_{\xi_1} + a_{\xi_1}^*)[D, a_\xi]X(\xi_2) \\ &= M_{[D,a_\xi]X(\xi_1)}X(\xi_2) + (a_{\xi_1} + a_{\xi_1}^*)a_\xi DX(\xi_2) \\ &= -M_{a_\xi DX(\xi_1)}X(\xi_2) - X(\xi_1) \cdot a_\xi DX(\xi_2) \\ &= -a_\xi DX(\xi_1) \cdot X(\xi_2) - X(\xi_1) \cdot a_\xi DX(\xi_2). \end{aligned} \tag{2.9}$$

On the other hand, since

$$[D, a_\xi](X(\xi_1) \cdot X(\xi_2)) = [D, a_\xi](X(\xi_1) \diamond X(\xi_2)),$$

and

$$[D, a_\xi](X(\xi_1) \diamond X(\xi_2)) = [D, a_\xi]a_{\xi_1}^* X(\xi_2),$$

we get proceeding as above that,

$$[D,a_\xi](X(\xi_1)\cdot X(\xi_2)) = -a_\xi DX(\xi_1)\diamond X(\xi_2) - X(\xi_1)\diamond a_\xi DX(\xi_2). \qquad (2.10)$$

Comparing (2.9) with (2.10) we deduce that

$$\begin{aligned} a_\xi DX(\xi_1)\cdot X(\xi_2) + X(\xi_1)\cdot a_\xi DX(\xi_2) &= a_\xi DX(\xi_1)\diamond X(\xi_2) \\ &\quad + X(\xi_1)\diamond a_\xi DX(\xi_2), \end{aligned}$$

or equivalently

$$a_{\xi_2}a_\xi DX(\xi_1) + a_{\xi_1}a_\xi DX(\xi_2) = 0,$$

that means

$$a_{\xi_2}DX(\xi_1) + a_{\xi_1}DX(\xi_2) \in \mathbb{R}.$$

If now we repeat the preceding argument replacing $[D,a_\xi](X(\xi_1)\cdot X(\xi_2))$ with $D(X(\xi_1)\cdot X(\xi_2))$ we obtain

$$a_{\xi_2}DX(\xi_1) + a_{\xi_1}DX(\xi_2) = 0.$$

Therefore there exist $F_1 \in S'$ and $F_2 \in S'^{\otimes 2}$ with

$$\langle F_2, \xi\otimes\eta\rangle + \langle F_2, \eta\otimes\xi\rangle = 0 \qquad (2.11)$$

for all $\xi,\eta\in S$, such that,

$$DX(\xi) = \langle F_1,\xi\rangle + X(F_2\otimes_1 \xi).$$

Observe that due to (2.11) we have

$$\phi_\xi\cdot(\langle F_1,\xi\rangle + X(F_2\otimes_1\xi)) = \phi_\xi\diamond(\langle F_1,\xi\rangle + X(F_2\otimes_1\xi)).$$

Let us now assume $D\in\mathcal{L}((S),(S)^*)$ to be of the form

$$D\phi_\xi = \langle F_1,\xi\rangle\phi_\xi + X(F_2\otimes_1\xi)\cdot\phi_\xi.$$

We need to verify that D is derivation. For any $\xi,\eta\in S$ we have

$$\begin{aligned} D(\phi_\xi\phi_\eta) &= D(\phi_{\xi+\eta}e^{\langle\xi,\eta\rangle}) \\ &= e^{\langle\xi,\eta\rangle}(\langle F_1,\xi+\eta\rangle\phi_{\xi+\eta} + X(F_2\otimes_1(\xi+\eta))\cdot\phi_{\xi+\eta}) \\ &= \langle F_1,\xi+\eta\rangle\phi_\xi\phi_\eta + X(F_2\otimes_1(\xi+\eta))\cdot\phi_\xi\phi_\eta \\ &= \langle F_1,\xi\rangle\phi_\xi\phi_\eta + X(F_2\otimes_1(\xi))\cdot\phi_\xi\phi_\eta \\ &\quad + \langle F_1,\eta\rangle\phi_\xi\phi_\eta + X(F_2\otimes_1(\eta))\cdot\phi_\xi\phi_\eta \\ &= D\phi_\xi\cdot\phi_\eta + \phi_\xi\cdot D\phi_\eta. \end{aligned}$$

The verification that D is also a Wick derivation can be achieved by straightforward modifications. □

Remark 2.9. Let $D\in\mathcal{L}((S),(S)^*)$ be of the form

$$D\phi_\xi = X(F_2\otimes_1\xi)\cdot\phi_\xi.$$

From the previous theorem we know that D is both a derivation and a Wick derivation. Let us find its adjoint $D^* \in \mathcal{L}((S),(S)^*)$:

$$\begin{aligned}
\langle\langle D\phi_\xi, \phi_\eta\rangle\rangle &= \langle\langle X(F_2 \otimes_1 \xi)\cdot \phi_\xi, \phi_\eta\rangle\rangle \\
&= \langle\langle X(F_2 \otimes_1 \xi), \phi_{\xi+\eta} e^{\langle \xi,\eta\rangle}\rangle\rangle \\
&= \langle F_2, \xi \otimes (\xi+\eta)\rangle e^{\langle \xi,\eta\rangle} \\
&= \langle F_2, \xi\otimes\xi\rangle e^{\langle \xi,\eta\rangle} + \langle F_2, \xi\otimes\eta\rangle e^{\langle \xi,\eta\rangle} \\
&= \langle F_2, \xi\otimes\eta\rangle e^{\langle \xi,\eta\rangle} \\
&= -\langle F_2, \eta\otimes\xi\rangle e^{\langle \xi,\eta\rangle} \\
&= -\langle\langle D\phi_\eta, \phi_\xi\rangle\rangle.
\end{aligned}$$

Here we used the antisymmetry of F_2, i.e. $\langle F_2, \xi\otimes\eta\rangle = -\langle F_2, \eta\otimes\xi\rangle$ which is equivalent to the condition $\langle F_2, \xi^{\otimes 2}\rangle = 0$ for any $\xi \in S$. In conclusion $D^* = -D$.

Remark 2.10. The class of operators of the form

$$D\phi_\xi = X(F_2 \otimes_1 \xi)\cdot \phi_\xi,$$

considered in the previous remark, coincides with the class of operators of the type $d\Gamma(Y)$, where Y is the infinitesimal generator of a regular one-parameter subgroup of $O(S; \mathcal{L}^2(\mathbb{R}))$ and $d\Gamma(Y)$ denotes its differential second quantization. This class of operators has been characterized in [4], Theorem 4.3. The equivalence of two classes can be easily verified via the symbol transform.

We now define first order (Wick) differential operators.

Definition 2.11. Let $\mathcal{D} \in \mathcal{L}((S),(S)^*)$.

- We say that $\mathcal{D}$ is a *first order differential operator* if
$$\mathcal{D} = D + M,$$
where $D \in \mathcal{L}((S),(S)^*)$ is a derivation and $M \in \mathcal{L}((S),(S)^*)$ is a multiplication operator.
- We say that $\mathcal{D}$ is a *first order Wick differential operator* if
$$\mathcal{D} = D + M^\diamond,$$
where $D \in \mathcal{L}((S),(S)^*)$ is a Wick derivation and $M^\diamond \in \mathcal{L}((S),(S)^*)$ is a Wick multiplication operator.

The next theorem is the analogue of Theorem 2.7 for the case of first order (Wick) differential operators.

Theorem 2.12. *Let $\mathcal{D} \in \mathcal{L}((S),(S)^*)$. Then:*

- *$\mathcal{D}$ is a first order differential operator if and only if for any $\xi \in S$, $[\mathcal{D}, a_\xi + a_\xi^*]$ is a multiplication operator.*
- *$\mathcal{D}$ is a first order Wick differential operator if and only if for any $\xi \in S$, $[\mathcal{D}, a_\xi^*]$ is a Wick multiplication operator.*

Proof. As usual we prove the theorem only for first order Wick differential operators.

Assume that $\mathcal{D}$ is a first order Wick differential operator, i.e. $\mathcal{D} = D + M^\diamond$ where D is a Wick derivation and $M^\diamond$ is a Wick multiplication operator. Then

$$\begin{aligned}[\mathcal{D}, a_\xi^*] &= [D, a_\xi^*] + [M^\diamond, a_\xi^*] \\ &= M_{DX(\xi)} + 0 \\ &= M_{DX(\xi)},\end{aligned}$$

since Wick multiplication operators commute with each other. Now suppose that for any $\xi \in S$, $[\mathcal{D}, a_\xi^*]$ is a Wick multiplication operator and consider the operator $\mathcal{D} - M^\diamond_{\mathcal{D}1} \in \mathcal{L}((S),(S)^*)$. It is straightforward to check that

$$(\mathcal{D} - M^\diamond_{\mathcal{D}1})1 = 0 \quad \text{and} \quad [\mathcal{D} - M^\diamond_{\mathcal{D}1}, a_\xi^*] \text{ is a Wick multiplication operator.}$$

Therefore by Theorem 2.7, $\mathcal{D} - M^\diamond_{\mathcal{D}1}$ is a Wick derivation and hence $\mathcal{D}$ is a first order Wick derivation since

$$\mathcal{D} = (\mathcal{D} - M^\diamond_{\mathcal{D}1}) + M^\diamond_{\mathcal{D}1}.$$

□

3. Second Order Differential Operators

Inspired by Remark 2.5 we give the following definition.

Definition 3.1. Let $\Delta \in \mathcal{L}((S),(S)^*)$.

- We say that Δ is a *second order differential operator* if for any $\varphi \in (S)$, $[\Delta, M_\varphi]$ is a first order differential operator.
- We say that Δ is a *second order Wick differential operator* if for any $\varphi \in (S)$, $[\Delta, M^\diamond_\varphi]$ is a first order Wick differential operator.

As for (Wick) derivations and first order (Wick) differential operators, it is sufficient to check the above condition only on a small class of (Wick) multiplication operators.

Theorem 3.2. *Let $\Delta \in \mathcal{L}((S),(S)^*)$. Then:*

- *Δ is a second order differential operator if and only if for any $\xi \in S$, $[\Delta, a_\xi + a_\xi^*]$ is a first order differential operator.*
- *Δ is a second order Wick differential operator if and only if for any $\xi \in S$, $[\Delta, a_\xi^*]$ is a first order Wick differential operator.*

Proof. We will only give the proof of the second statement of the theorem.

One implication is trivial. Suppose now that for any $\xi \in S$, $[\Delta, a_\xi^*]$ is a first order differential operator and let us prove that for any $\xi_1, \xi_2 \in S$, $[\Delta, a_{\xi_2}^* a_{\xi_1}^*]$ is also a first order Wick differential operator (observe that the operator $a_{\xi_2}^* a_{\xi_1}^*$ corresponds to the Wick multiplication operator $M^\diamond_{X(\xi_2)\diamond X(\xi_1)}$):

$$\begin{aligned}[\Delta, a_{\xi_2}^* a_{\xi_1}^*] &= [\Delta, a_{\xi_2}^*] a_{\xi_1}^* + a_{\xi_2}^* [\Delta, a_{\xi_1}^*] \\ &= [[\Delta, a_{\xi_2}^*], a_{\xi_1}^*] + a_{\xi_1}^* [\Delta, a_{\xi_2}^*] + a_{\xi_2}^* [\Delta, a_{\xi_1}^*] \\ &= B_1 + B_2 + B_3\end{aligned}$$

where we set

$$\begin{aligned} B_1 &:= [[\Delta, a^*_{\xi_2}], a^*_{\xi_1}] \\ B_2 &:= a^*_{\xi_1}[\Delta, a^*_{\xi_2}] \\ B_3 &:= a^*_{\xi_2}[\Delta, a^*_{\xi_1}]. \end{aligned}$$

Since by assumption $[\Delta, a^*_{\xi_2}]$ is a first order Wick differential operator, using Theorem 2.12 we deduce that B_1 is a Wick multiplication operator. Moreover B_2 and B_3 are first order Wick differential operators. In fact, considering for instance B_2, since $[\Delta, a^*_{\xi_2}]$ is a first order Wick differential operator, we can write $[\Delta, a^*_{\xi_2}] = D + M^\diamond$ for some Wick derivation D and Wick multiplication operator $M^\diamond$ (both of course depending on ξ_2); by a simple verification, one sees that $a^*_{\xi_1}D$ is also a Wick derivation and that $a^*_{\xi_1}M^\diamond$ is also a Wick multiplication operator. In this way, B_2 is a first order Wick differential operator. Therefore $[\Delta, a^*_{\xi_2}a^*_{\xi_1}]$ is a first order Wick differential operator, being a sum of operators of this kind.

Following the same reasoning one can easily prove by induction that for any $n \in \mathbb{N}$ and $\xi_1, ..., \xi_n \in S$, $[\Delta, a^*_{\xi_n} \cdots a^*_{\xi_1}]$ is a first order Wick differential operator. Since the set $\{X(\xi_1) \diamond \cdots \diamond X(\xi_n),\ n \in \mathbb{N} \text{ and } \xi_1, ..., \xi_n \in S\}$ is total in (S_ρ) for any $\rho \in [0,1]$ and since $\Delta \in \mathcal{L}((S), (S)^*)$ we conclude via a density argument that for any $\varphi \in (S)$, $[\Delta, M^\diamond_\varphi]$ is a first order Wick differential operator. □

Corollary 3.3. *Let $\Delta \in \mathcal{L}((S), (S)^*)$. Then:*

- *Δ is a second order differential operator if and only if for any $\xi, \eta \in S$, $[[\Delta, a_\xi + a^*_\xi], a_\eta + a^*_\eta]$ is a multiplication operator.*
- *Δ is a second order Wick differential operator if and only if for any $\xi, \eta \in S$, $[[\Delta, a^*_\xi], a^*_\eta]$ is a Wick multiplication operator.*

Proof. It follows immediately from Theorem 3.2 and Theorem 2.12. □

Definition 3.4. Let $\Delta \in \mathcal{L}((S), (S)^*)$.

- We say that Δ is a *pure second order differential operator* if Δ is a second order differential operator and for any $\xi \in S$, $\Delta 1 = \Delta X(\xi) = 0$.
- We say that Δ is a *pure second order Wick differential operator* if Δ is a second order Wick differential operator and for any $\xi \in S$, $\Delta 1 = \Delta X(\xi) = 0$.

Example 3.5.

- Let $N \in \mathcal{L}((S), (S))$ be the number operator. Then N is a second order differential operator but not a pure one. In fact:

$$\begin{aligned} [N, a_\xi + a^*_\xi] &= -a_\xi + a^*_\xi \\ &= -2a_\xi + (a_\xi + a^*_\xi) \end{aligned}$$

which is a first order differential operator. However:

$$N1 = 0 \text{ and } NX(\xi) = X(\xi).$$

- $N^{\diamond 2} := N^2 - N$ is a pure second order Wick derivation. In fact:

$$[N^2 - N, a^*_\xi] = 2a^*_\xi N \text{ and } (N^2 - N)1 = (N^2 - N)X(\xi) = 0.$$

- The Gross laplacian Δ_G is both a pure second order differential operator and a pure second order Wick differential operator. In fact:

$$[\Delta_G, a_\xi + a_\xi^*] = [\Delta_G, a_\xi^*] = 2a_\xi,$$

which is both a derivation and a Wick derivation. Moreover,

$$\Delta_G 1 = \Delta_G X(\xi) = 0.$$

Theorem 3.6. *Let $\Delta \in \mathcal{L}((S),(S)^*)$. Then*

- *Δ is a pure second order differential operator if and only if $\Delta 1 = 0$ and for any $\xi \in S$, $[\Delta, a_\xi + a_\xi^*]$ is a derivation.*
- *Δ is a pure second order Wick differential operator if and only if $\Delta 1 = 0$ and for any $\xi \in S$, $[\Delta, a_\xi^*]$ is a Wick derivation.*

Remark 3.7. If $[\Delta, a_\xi + a_\xi^*]$ is a derivation, it is not in general true that $[\Delta, M_\varphi]$ is a derivation for any $\varphi \in (S)$. In fact, choosing for instance Δ to be the Gross laplacian and $\varphi = X(\xi)^2$, we obtain

$$\begin{aligned}
[\Delta, (a_\xi + a_\xi^*)^2] &= [\Delta, a_\xi + a_\xi^*](a_\xi + a_\xi^*) + (a_\xi + a_\xi^*)[\Delta, a_\xi + a_\xi^*] \\
&= 2a_\xi(a_\xi + a_\xi^*) + (a_\xi + a_\xi^*)2a_\xi \\
&= 4a_\xi^2 + 2(a_\xi a_\xi^* + a_\xi^* a_\xi).
\end{aligned}$$

Therefore, since $(4a_\xi^2 + 2(a_\xi a_\xi^* + a_\xi^* a_\xi))1 = 2|\xi|^2$, it is not a derivation.
The Wick analogue of this observation is also true.

Proof. We treat only the case of pure second order Wick differential operators. One implication is trivial. Suppose now that $\Delta 1 = 0$ and that for any $\xi \in S$, $[\Delta, a_\xi^*]$ is a Wick derivation. Since a Wick derivation is a first order Wick differential operator, we get by Theorem 3.2 that Δ is a second order Wick differential operator. Moreover from $[\Delta, a_\xi^*]1 = 0$ we deduce that $\Delta X(\xi) = 0$. ☐

The next two theorems will be crucial in proving the second main result of this paper.

Theorem 3.8. *Let $\Delta \in \mathcal{L}((S),(S)^*)$. The following statements are equivalent:*
i) Δ is a pure second order differential operator;
ii) For any $\xi \in S$, $\Delta 1 = \Delta X(\xi) = 0$ and for $n \geq 2$,

$$\Delta X(\xi)^n = \frac{n(n-1)}{2} X(\xi)^{n-2} \cdot \Delta X(\xi)^2; \tag{3.1}$$

iii) For any $\xi \in S$,

$$\Delta \phi_\xi = \frac{1}{2} \phi_\xi \cdot \Delta X(\xi)^2. \tag{3.2}$$

Proof. *i)* $\Longrightarrow$ *ii)* Let Δ be a pure second order differential operator. Then by definition $\Delta 1 = \Delta X(\xi) = 0$ for any $\xi \in S$. We are now going to prove (3.1) by induction. For $n = 2$ the statement is trivially satisfied and let us assume that equation (3.1) holds for a given $n \in \mathbb{N}$. Hence,

$$\begin{aligned}
\Delta X(\xi)^{n+1} &= \Delta(a_\xi + a_\xi^*)X(\xi)^n \\
&= [\Delta, a_\xi + a_\xi^*]X(\xi)^n + (a_\xi + a_\xi^*)\Delta X(\xi)^n \\
&= nX(\xi)^{n-1} \cdot [\Delta, a_\xi + a_\xi^*]X(\xi) \\
&\quad + (a_\xi + a_\xi^*)\frac{n(n-1)}{2}X(\xi)^{n-2} \cdot \Delta X(\xi)^2 \\
&= nX(\xi)^{n-1} \cdot \Delta X(\xi)^2 + (a_\xi + a_\xi^*)\frac{n(n-1)}{2}X(\xi)^{n-2} \cdot \Delta X(\xi)^2 \\
&= nX(\xi)^{n-1} \cdot \Delta X(\xi)^2 + \frac{n(n-1)}{2}X(\xi)^{n-1} \cdot \Delta X(\xi)^2 \\
&= \frac{(n+1)n}{2}X(\xi)^{n-1} \cdot \Delta X(\xi)^2,
\end{aligned}$$

where in the third equality we used the fact that $[\Delta, a_\xi + a_\xi^*]$ is a derivation and the inductive hypothesis while in the fourth equality we argued as

$$\begin{aligned}
[\Delta, a_\xi + a_\xi^*]X(\xi) &= \Delta(a_\xi + a_\xi^*)X(\xi) - (a_\xi + a_\xi^*)\Delta X(\xi) \\
&= \Delta X(\xi)^2,
\end{aligned}$$

since $\Delta X(\xi) = 0$.
ii) $\Longrightarrow$ *iii)* It is a simple verification:

$$\begin{aligned}
\Delta\phi_\xi &= e^{-\frac{|\xi|^2}{2}}\Delta\sum_{n\geq 0}\frac{1}{n!}X(\xi)^n \\
&= e^{-\frac{|\xi|^2}{2}}\sum_{n\geq 2}\frac{1}{n!}\Delta X(\xi)^n \\
&= e^{-\frac{|\xi|^2}{2}}\sum_{n\geq 2}\frac{1}{n!}\frac{n(n-1)}{2}X(\xi)^{n-2} \cdot \Delta X(\xi)^2 \\
&= \frac{e^{-\frac{|\xi|^2}{2}}}{2}\Delta X(\xi)^2 \cdot \sum_{n\geq 2}\frac{1}{(n-2)!}X(\xi)^{n-2} \\
&= \frac{1}{2}\phi_\xi \cdot \Delta X(\xi)^2.
\end{aligned}$$

iii) $\Longrightarrow$ *i)* Choosing $\xi = 0$ in (3.2) we obtain $\Delta 1 = 0$. We now have to prove that for any $\eta \in S$, $[\Delta, a_\eta + a_\eta^*]$ is a derivation. We begin by replacing ξ with $\xi + t\eta$ in (3.2) where $t \in \mathbb{R}$ and $\eta \in S$ and differentiating at $t = 0$ the obtained equation; since

$$\frac{d}{dt}\phi_{\xi+t\eta}\Big|_{t=0} = a_\eta^*\phi_\xi,$$

we get

$$\Delta a_\eta^*\phi_\xi = \frac{1}{2}a_\eta^*\phi_\xi \cdot \Delta X(\xi)^2 + \phi_\xi \cdot \Delta X(\xi)X(\eta).$$

Therefore,

$$\begin{aligned}
[\Delta, a_\eta + a_\eta^*]\phi_\xi &= \Delta(a_\eta + a_\eta^*)\phi_\xi - (a_\eta + a_\eta^*)\Delta\phi_\xi \\
&= \langle \xi, \eta\rangle \Delta\phi_\xi + \frac{1}{2}a_\eta^*\phi_\xi \cdot \Delta X(\xi)^2 + \phi_\xi \cdot \Delta X(\xi)X(\eta) \\
&\quad -X(\eta)\cdot \frac{1}{2}\phi_\xi \cdot \Delta X(\xi)^2 \\
&= \langle \xi, \eta\rangle \Delta\phi_\xi - \frac{1}{2}a_\eta\phi_\xi \cdot \Delta X^2(\xi) + \phi_\xi \cdot \Delta X(\xi)X(\eta) \\
&= \langle \xi, \eta\rangle (\Delta\phi_\xi - \frac{1}{2}\phi_\xi \Delta X(\xi)^2) + \phi_\xi \cdot \Delta X(\xi)X(\eta) \\
&= \phi_\xi \cdot \Delta X(\xi)X(\eta) \\
&= \phi_\xi \cdot \Delta(a_\eta + a_\eta^*)X(\xi) \\
&= \phi_\xi \cdot [\Delta, a_\eta + a_\eta^*]X(\xi).
\end{aligned}$$

Comparing the first and the last members of this chain of equalities we conclude that

$$[\Delta, a_\eta + a_\eta^*]\phi_\xi = \phi_\xi \cdot [\Delta, a_\eta + a_\eta^*]X(\xi),$$

i.e. $[\Delta, a_\eta + a_\eta^*]$ is a derivation. □

Theorem 3.9. *Let $\Delta \in \mathcal{L}((S),(S)^*)$. The following statements are equivalent:*
i) Δ is a pure second order Wick differential operator;
ii) For any $\xi \in S$, $\Delta 1 = \Delta X(\xi) = 0$ and for $n \geq 2$,

$$\Delta X(\xi)^{\diamond n} = \frac{n(n-1)}{2} X(\xi)^{\diamond(n-2)} \diamond \Delta X(\xi)^{\diamond 2}; \tag{3.3}$$

iii) For any $\xi \in S$,

$$\Delta\phi_\xi = \frac{1}{2}\phi_\xi \diamond \Delta X(\xi)^{\diamond 2}. \tag{3.4}$$

Proof. It is a straightforward adaptation of the proof of Theorem 3.8. □

We now come to our second main result.

Theorem 3.10. *Let $\Delta \in \mathcal{L}((S),(S)^*)$. Δ is both a pure second order differential operator and a pure second order Wick differential operator if and only if there exist $G_2 \in S'^{\otimes 2}$ and $G_3 \in S'^{\otimes 3}$ with*

$$\langle G_3, \xi\otimes\eta\otimes\theta\rangle + \langle G_3, \theta\otimes\xi\otimes\eta\rangle + \langle G_3, \eta\otimes\theta\otimes\xi\rangle = 0 \tag{3.5}$$

for all $\xi, \eta, \theta \in S$, such that

$$\begin{aligned}
\Delta\phi_\xi &= \frac{1}{2}\phi_\xi \cdot (\langle G_2, \xi^{\otimes 2}\rangle + X(G_3 \otimes_2 \xi^{\otimes 2})) \qquad (3.6)\\
&= \frac{1}{2}\phi_\xi \diamond (\langle G_2, \xi^{\otimes 2}\rangle + X(G_3 \otimes_2 \xi^{\otimes 2})),
\end{aligned}$$

where $G_3 \otimes_2 \xi^{\otimes 2}$ denotes the unique element in S' such that for any $\eta \in S$, $\langle G_3 \otimes_2 \xi^{\otimes 2}, \eta\rangle = \langle G_3, \xi^{\otimes 2} \otimes \eta\rangle$.

Remark 3.11. Let $\Delta \in \mathcal{L}((S),(S)^*)$. Then for any $\xi, \eta \in S$,

$$[[\Delta, a_\xi], a_\eta^*] = [[\Delta, a_\eta^*], a_\xi].$$

(It can be easily checked via a direct verification). In particular, if Δ is a pure second order (Wick) differential operator, then $[\Delta, a_\xi]$ is also a pure second order (Wick) differential operator.

Proof. First of all observe that the second equality in (3.6) follows from condition (3.5) and the identity:

$$\phi_\xi \cdot X(\eta) = \phi_\xi \diamond X(\eta) + \langle \xi, \eta \rangle \phi_\xi,$$

for all $\xi, \eta \in S$. If $\Delta \in \mathcal{L}((S),(S)^*)$ is of the form (3.6), then replacing ξ with $t\xi$, $t \in \mathbb{R}$, and differentiating twice at $t = 0$ one gets

$$\Delta X(\xi)^2 = \langle G_2, \xi^{\otimes 2} \rangle + X(G_3 \otimes_2 \xi^{\otimes 2}).$$

Therefore by Theorem 3.8 and Theorem 3.9 we deduce that Δ is a pure second order differential operator and a pure second order Wick differential operator.

Now suppose Δ to be both a pure second order differential operator and a pure second order Wick differential operator. According to the two previous theorem we have to characterize the action of Δ on the second chaos. According to the previous remark for any $\xi \in S$, $[\Delta, a_\xi]$ is also both a pure second order differential operator and a pure second order Wick differential operator. This means that for any $\eta \in S$, $[[\Delta, a_\xi], a_\eta + a_\eta^*]$ is a derivation and $[[\Delta, a_\xi], a_\eta^*]$ is a Wick derivation. Therefore for any $\xi_1, \xi_2, \xi_3 \in S$ we can write,

$$\begin{aligned}
[\Delta, a_\xi](X(\xi_1) \cdot X(\xi_2) \cdot X(\xi_3)) &= [\Delta, a_\xi](a_{\xi_1} + a_{\xi_1}^*)(X(\xi_2) \cdot X(\xi_3)) \\
&= [[\Delta, a_\xi], a_{\xi_1} + a_{\xi_1}^*](X(\xi_2) \cdot X(\xi_3)) \\
&\quad + (a_{\xi_1} + a_{\xi_1}^*)[\Delta, a_\xi](X(\xi_2) \cdot X(\xi_3)) \\
&= X(\xi_3) \cdot [[\Delta, a_\xi], a_{\xi_1} + a_{\xi_1}^*]X(\xi_2) \\
&\quad + X(\xi_2) \cdot [[\Delta, a_\xi], a_{\xi_1} + a_{\xi_1}^*]X(\xi_3) \\
&\quad + (a_{\xi_1} + a_{\xi_1}^*)[\Delta, a_\xi](X(\xi_2) \cdot X(\xi_3)) \\
&= X(\xi_3) \cdot [\Delta, a_\xi](X(\xi_1) \cdot X(\xi_2)) \\
&\quad + X(\xi_2) \cdot [\Delta, a_\xi](X(\xi_1) \cdot X(\xi_3)) \\
&\quad + X(\xi_1) \cdot [\Delta, a_\xi](X(\xi_2) \cdot X(\xi_3)) \\
&= X(\xi_3) \cdot a_\xi \Delta(X(\xi_1) \cdot X(\xi_2)) \\
&\quad + X(\xi_2) \cdot a_\xi \Delta(X(\xi_1) \cdot X(\xi_3)) \\
&\quad + X(\xi_1) \cdot a_\xi \Delta(X(\xi_2) \cdot X(\xi_3)),
\end{aligned}$$

where in the fourth equality we used the fact that $[\Delta, a_\xi]$ annihilates the first chaos. Observe in addition that

$$X(\xi_1) \cdot X(\xi_2) \cdot X(\xi_3) = X(\xi_1) \diamond X(\xi_2) \diamond X(\xi_3) + \text{terms in the first chaos.}$$

Therefore,

$$[\Delta, a_\xi](X(\xi_1) \cdot X(\xi_2) \cdot X(\xi_3)) = [\Delta, a_\xi](X(\xi_1) \diamond X(\xi_2) \diamond X(\xi_3)).$$

From the identity

$$X(\xi_1)\diamond X(\xi_2)\diamond X(\xi_3)=a^*_{\xi_1}(X(\xi_2)\diamond X(\xi_3)),$$

and the fact that $[[\Delta,a_\xi],a^*_\eta]$ is a Wick derivation, we obtain proceeding as above that

$$\begin{aligned}[\Delta,a_\xi](X(\xi_1)\cdot X(\xi_2)\cdot X(\xi_3)) &= [\Delta,a_\xi](X(\xi_1)\diamond X(\xi_2)\diamond X(\xi_3))\\ &= X(\xi_3)\diamond a_\xi\Delta(X(\xi_1)\cdot X(\xi_2))\\ &\quad +X(\xi_2)\diamond a_\xi\Delta(X(\xi_1)\cdot X(\xi_3))\\ &\quad +X(\xi_1)\diamond a_\xi\Delta(X(\xi_2)\cdot X(\xi_3)).\end{aligned}$$

Here we used that fact that $\Delta(X(\xi_1)\diamond X(\xi_2))=\Delta(X(\xi_1)\cdot X(\xi_2))$. A comparison between the last and previous identities for $[\Delta,a_\xi](X(\xi_1)\cdot X(\xi_2)\cdot X(\xi_3))$ gives

$$\begin{gathered}X(\xi_3)\cdot a_\xi\Delta(X(\xi_1)\cdot X(\xi_2))+X(\xi_2)\cdot a_\xi\Delta(X(\xi_1)\cdot X(\xi_3))\\ +X(\xi_1)\cdot a_\xi\Delta(X(\xi_2)\cdot X(\xi_3))=X(\xi_3)\diamond a_\xi\Delta(X(\xi_1)\cdot X(\xi_2))\\ +X(\xi_2)\diamond a_\xi\Delta(X(\xi_1)\cdot X(\xi_3))+X(\xi_1)\diamond a_\xi\Delta(X(\xi_2)\cdot X(\xi_3)),\end{gathered}$$

or equivalently,

$$\begin{aligned}a_{\xi_3}a_\xi\Delta(X(\xi_1)\cdot X(\xi_2)) &+ a_{\xi_2}a_\xi\Delta(X(\xi_1)\cdot X(\xi_3))\\ &+ a_{\xi_1}a_\xi\Delta(X(\xi_2)\cdot X(\xi_3))=0.\end{aligned}$$

This gives,

$$a_\xi(a_{\xi_3}\Delta(X(\xi_1)\cdot X(\xi_2))+a_{\xi_2}\Delta(X(\xi_1)\cdot X(\xi_3))+a_{\xi_1}\Delta(X(\xi_2)\cdot X(\xi_3)))=0.$$

We have therefore proved that for any $\xi_1,\xi_2,\xi_3\in S$,

$$a_{\xi_3}\Delta(X(\xi_1)\cdot X(\xi_2))+a_{\xi_2}\Delta(X(\xi_1)\cdot X(\xi_3))+a_{\xi_1}\Delta(X(\xi_2)\cdot X(\xi_3))\in\mathbb{R}.$$

More precisely, if we repeat the argument so far utilized to study the quantity

$$[\Delta,a_\xi](X(\xi_1)\cdot X(\xi_2)\cdot X(\xi_3)),$$

on the element

$$\Delta(X(\xi_1)\cdot X(\xi_2)\cdot X(\xi_3)),$$

we will obtain

$$a_{\xi_3}\Delta(X(\xi_1)\cdot X(\xi_2))+a_{\xi_2}\Delta(X(\xi_1)\cdot X(\xi_3))+a_{\xi_1}\Delta(X(\xi_2)\cdot X(\xi_3))=0.$$

From the above identity we deduce that for there exist $G_2\in S'^{\otimes 2}$ and $G_3\in S'^{\otimes 3}$ with

$$\langle G_3,\xi_1\otimes\xi_2\otimes\xi_3\rangle+\langle G_3,\xi_3\otimes\xi_1\otimes\xi_2\rangle+\langle G_3,\xi_2\otimes\xi_3\otimes\xi_1\rangle=0 \tag{3.7}$$

for all $\xi_1,\xi_2,\xi_3\in S$, such that

$$\Delta(X(\xi_1)\cdot X(\xi_2))=\langle G_2,\xi_1\otimes\xi_2\rangle+X(G_3\otimes_2(\xi_1\otimes\xi_2)).$$

The proof is complete. □

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Abdessatar Barhoumi: Department of Mathematics, Higher School of Sciences and Technologies, Sousse University. Sousse, Tunisia

E-mail address: abdessatar.barhoumi@ipein.rnu.tn

Alberto Lanconelli: Dipartimento di Matematica, Universita' degli Studi di Bari, Via E. Orabona, 4, 70125 Bari, Italia

E-mail address: alberto.lanconelli@uniba.it