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Regularization and derivatives of multipole potentials

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A R T I C L E I N F O

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A B S T R A C T

The only harmonic homogeneous functions defined in $\mathbb{R}^n\setminus\{\mathbf{0}\}$ are the harmonic polynomials and the so-called multipole potentials, namely functions of the type $P(\mathbf{x}) = p(\mathbf{x})/|\mathbf{x}|^{2k+n-2}$ for some harmonic polynomial p of degree k. The first aim of this article is to study the distributional regularization of multipole potentials. We show that even though the Hadamard regularization $\mathcal{P}f\left(p(\mathbf{x})/|\mathbf{x}|^{2k+n-2}\right)$ exists for any homogeneous polynomial of degree k, the principal value p.v. $\left(p(\mathbf{x})/|\mathbf{x}|^{2k+n-2}\right)$ exists if and only if p is harmonic; this means that if p is harmonic then for any test function ϕ the *divergent* integral $\int_{\mathbb{R}^n} p(\mathbf{x})\,\phi(\mathbf{x})/|\mathbf{x}|^{2k+n-2}\,\mathrm{d}\mathbf{x}$ can be computed by employing polar coordinates and performing the angular integral first. We also find the first and second order distributional derivatives of these regularizations and, more generally, of the regularizations of functions of the form $P_l(\mathbf{x}) = p(\mathbf{x})/|\mathbf{x}|^{k+l}$. We find many interesting formulas that hold precisely when p is a harmonic polynomial of degree k. In particular, we prove that

$$\overline{\Delta}\text{p.v.}\left(\frac{p(\mathbf{x})}{r^{2k+n-2}}\right) = \frac{(-1)^{k+1}\pi^{n/2}}{2^{k-2}\Gamma\left(\frac{n}{2}+k-1\right)}p(\boldsymbol{\nabla})\,\delta(\mathbf{x}) \ ,$$

generalizing the well known relation $\overline{\Delta}\left(r^{2-n}\right) = (2-n)\,C\delta(\mathbf{x})$, where C is the area of a sphere of radius 1. Actually formulas like this one hold for a homogeneous polynomial p of degree k if and only if p is harmonic.

1. Introduction

There are some functions that are both harmonic and homogeneous, $u(\lambda\mathbf{x}) = \lambda^\alpha u(\mathbf{x})$, $\lambda > 0$. In the whole space $\mathbb{R}^n$ the only possibility is $\alpha = k \in \mathbb{N}$, and in that case u must be a polynomial function, $u \in \mathcal{H}_k$, where we denote by $\mathcal{H}_k$ the set of harmonic homogeneous functions of degree k. Actually one may consider $\mathcal{H}_k$ under three different lights, namely, as a set of polynomials in n variables of degree k, or as a set of polynomial *functions*, perhaps better denoted as $\mathcal{H}_k(\mathbb{R}^n)$, or even as the set of restrictions to the unit

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sphere, $\mathcal{H}_k(\mathbb{S})$. The elements of $\mathcal{H}_k(\mathbb{S})$ are usually called spherical harmonics, while those of $\mathcal{H}_k(\mathbb{R}^n)$ are referred to as solid harmonics. Notice that the restriction map $\mathcal{H}_k(\mathbb{R}^n) \longrightarrow \mathcal{H}_k(\mathbb{S})$ is a bijection because of the maximum principle for harmonic functions. See [1] for the many properties of harmonic polynomials and of harmonic functions in general.

On the other hand, we may consider harmonic homogeneous functions defined in $\mathbb{R}^n \setminus \{\mathbf{0}\}$. One way to obtain such functions is to apply the Kelvin transform [1, Chp. 4], $u \mapsto K[u]$ to the elements $u \in \mathcal{H}_k$. In general, if u is a function defined in a region $\Omega \subset \mathbb{R}^n$, then $v = K[u]$ is a function defined in the conjugated set $\Omega^* = \{\mathbf{x}^* : \mathbf{x} \in \Omega\}$, $\mathbf{x}^* = \mathbf{x}/|\mathbf{x}|^2$, by $v(\mathbf{x}) = |\mathbf{x}|^{2-n} u(\mathbf{x}^*)$; the Kelvin transform sends harmonic functions to harmonic functions and it also sends homogeneous functions to homogeneous functions, so that if $p \in \mathcal{H}_k$ then the function

$$P(\mathbf{x}) = \frac{p(\mathbf{x})}{r^{2k+n-2}}, \tag{1.1}$$

where, as customary, $r = |\mathbf{x}|$, is a harmonic function, homogeneous of degree $-k-n+2$, defined[1] in $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Functions of the form (1.1) are sometimes called *multipole potentials* [18]. It is not hard to see that all harmonic homogeneous functions defined in $\mathbb{R}^n \setminus \{\mathbf{0}\}$ are either harmonic polynomials or multipole potentials of the form (1.1).

The aim of this article is to study several properties of multipole potentials and, more generally, of functions of the type $P_l(\mathbf{x}) = r^{-k-l} p(\mathbf{x})$ where $p \in \mathcal{H}_k$. Since $P = P_{k+n-2}$ has a non-integrable singularity at the origin, unless $k = 0$ or $k = 1$, we need to study the *regularization* of P as a distribution of the space $\mathcal{D}'(\mathbb{R}^n)$; we find that the principal value distribution $\text{p.v.}(P(\mathbf{x})) \in \mathcal{D}'(\mathbb{R}^n)$ given as

$$\langle \text{p.v.}(P(\mathbf{x})), \phi(\mathbf{x})\rangle = \lim_{\varepsilon \to 0} \int\limits_{|\mathbf{x}| \geq \varepsilon} P(\mathbf{x}) \phi(\mathbf{x}) \, \mathrm{d}\mathbf{x}, \quad \phi \in \mathcal{D}(\mathbb{R}^n), \tag{1.2}$$

always exists if $p \in \mathcal{H}_k$. This means that if $p \in \mathcal{H}_k$ one may regularize the *divergent* integral $\int_{\mathbb{R}^n} P(\mathbf{x}) \phi(\mathbf{x}) \, \mathrm{d}\mathbf{x}$ by following the simple rule of "*using polar coordinates and performing the angular integrals first.*" In fact, we show that if p is a general homogeneous polynomial of degree k which is not harmonic, then the principal value does not exist, so that the simple rule of regularization *does not work* and one needs to employ the Hadamard regularization $\mathcal{P}f(P(\mathbf{x}))$.

Next we obtain formulas for the first and second order derivatives of the distributions $\mathcal{P}f(P_l(\mathbf{x}))$, in particular for the distributions $\text{p.v.}(P_{k+n-2}(\mathbf{x}))$. Such derivatives are very important in Mathematical Physics [3,4,15,18] and several special cases have been computed by several authors [13,17,21,22]. Naturally it is rather simple to obtain the ordinary derivatives of $P_l(\mathbf{x})$, that is, the derivatives away from the origin,[2] therefore we pay special attention to the *delta part* of these derivatives. Our computations show that many times the derivatives of fields that do not have a delta part may have a high order delta part, that is, derivatives of the delta function can appear in the derivatives of fields that have no delta function at the origin, as (1.3) already shows; this "apparent paradox" was pointed out by Parker [22], who warns of the mistakes that it can produce.

The distributional derivatives of any order of power potentials were given in [8,9], and are available in several textbooks [10,20], and in principle could be employed to compute the distributional derivatives of $\mathcal{P}f(P_l(\mathbf{x}))$, even if p is not harmonic, but such direct computations become rather complicated and no simple formulas are obtained. Nevertheless, we show that when $p \in \mathcal{H}_k$ the expressions for the derivatives can be simplified in a surprising way, leading to rather nice formulas. In particular we show that[3]

[1] In fact P is defined in $\widetilde{\mathbb{R}^n} \setminus \{\mathbf{0}\}$, the one point compactification of $\mathbb{R}^n$, $\widetilde{\mathbb{R}^n}$, with the origin removed.
[2] In other words, this is the far field behavior.
[3] An overbar denotes a distributional derivative, a notation first introduced by the late Professor Farassat [11].

$$\overline{\Delta}\text{p.v.}\left(\frac{p(\mathbf{x})}{r^{2k+n-2}}\right)=\frac{(-1)^{k+1}\pi^{n/2}}{2^{k-2}\Gamma\left(\frac{n}{2}+k-1\right)}p(\nabla)\,\delta(\mathbf{x})\ , \tag{1.3}$$

a generalization of the well known relation $\overline{\Delta}\left(r^{2-n}\right)=(2-n)\,C\delta(\mathbf{x})$, where C is given by (2.1). Actually we compute all first and second order derivatives of the distribution $\mathcal{P}f\left(P_l(\mathbf{x})\right)$. We arrive at our results by a careful study of the relationship between the two natural inner products in $\mathcal{P}_k$, the space of homogeneous polynomials of degree k, a study which is given in Section 3, and by employing the thick distributional analysis [7,27,29], particularly the formulas for the thick derivatives of homogeneous distributions.

2. Preliminaries and notation

We shall employ the notations

$$c_{m,n}=\frac{2\Gamma(m+n/2)\,\pi^{(n-1)/2}}{\Gamma(m+1/2)}=\int_{\mathbb{S}}\omega_j^{2m}\,\mathrm{d}\sigma(\omega)\ ,\quad C=c_{0,n}\,. \tag{2.1}$$

Notice that $c_{0,n}=C=2\pi^{n/2}/\Gamma(n/2)$, is the surface area of the unit sphere $\mathbb{S}$ of $\mathbb{R}^n$.[4]

Sometimes we shall need the delta derivatives $\delta f/\delta x_i$ of a function defined on a hypersurface Σ of $\mathbb{R}^n$ [10, Sect. 2.7]. They are defined as follows: Suppose f is a smooth function defined in Σ and let F be any smooth extension of f to an open neighborhood of Σ in $\mathbb{R}^n$; the derivatives $\partial F/\partial x_j$ will exist, but their restriction to Σ will depend not only on f but also on the extension employed. However, it can be shown that the formulas $\delta f/\delta x_j=(\partial F/\partial x_j-n_j dF/dn)|_{\Sigma}$, where $\mathbf{n}=(n_j)$ is the normal unit vector to Σ and where $dF/dn=n_k\partial F/\partial x_k$ is the derivative of F in the normal direction, define derivatives $\delta f/\delta x_j$, $1\le j\le n$, that depend *only* on f and not on the extension. In general, $\mathbf{n}=(n_j)$ denotes the normal vector to the hypersurface Σ, but when $\Sigma=\mathbb{S}$, the unit sphere, then at $\omega\in\mathbb{S}$ we have $\mathbf{n}=\omega$, and thus one finds in some formulas in the literature n_j while in others one finds ω_j, but of course they are the same.

Let us now recall the notion of the finite part[5] of a limit [10, Section 2.4]. Suppose $\mathfrak{F}$, the basic functions, is a family of strictly positive functions defined for $0<\varepsilon<\varepsilon_0$ such that all of them tend to infinity at 0 and such that, given two different elements $f_1, f_2\in\mathfrak{F}$, then $\lim_{\varepsilon\to0^+}f_1(\varepsilon)/f_2(\varepsilon)$ is either 0 or ∞.

Definition 2.1. Let $G(\varepsilon)$ be a function defined for $0<\varepsilon<\varepsilon_0$ with $|\lim_{\varepsilon\to0^+}G(\varepsilon)|=\infty$. The finite part of the limit of $G(\varepsilon)$ as $\varepsilon\to0^+$ with respect to $\mathfrak{F}$ exists and equals A if we can write[6] $G(\varepsilon)=G_1(\varepsilon)+G_2(\varepsilon)$, where G_1, the *infinite part,* is a linear combination of the basic functions and where G_2, the *finite part*, has the property that the limit $A=\lim_{\varepsilon\to0^+}G_2(\varepsilon)$ exists. We then employ the notation

$$\text{F.p.}_{\mathfrak{F}}\lim_{\varepsilon\to0^+}G(\varepsilon)=A\,. \tag{2.2}$$

The Hadamard finite part limit corresponds to the case when $\mathfrak{F}$ is the family of functions $\varepsilon^{-\alpha}|\ln\alpha|^{\beta}$, where $\alpha>0$ and $\beta\ge0$ or where $\alpha=0$ and $\beta>0$. We then use the simpler notation $\text{F.p.}\lim_{\varepsilon\to0^+}G(\varepsilon)$.

Consider now a function f defined in $\mathbb{R}^n$ that is probably not integrable over the whole space but which is integrable in the region $|\mathbf{x}|>\varepsilon$ for any $\varepsilon>0$. Then the *radial* finite part integral is defined as

[4] Interestingly, integrals of the type $\int_{\mathbb{S}}\omega_1^{2m_1}\cdots\omega_n^{2m_n}\,\mathrm{d}\sigma(\omega)$ have been evaluated independently by several authors, starting with Weyl [25], and, continuing with, among others, [8] and [4].

[5] Hadamard introduced the notion of the finite parts, and the name, when considering the divergent integrals that appear in the fundamental solutions of hyperbolic equations [16].

[6] Such a decomposition is *unique* since any finite number of elements of $\mathfrak{F}$ is linearly independent.

$$\text{F.p.} \int_{\mathbb{R}^n} f(\mathbf{x})\, \mathrm{d}\mathbf{x} = \text{F.p.} \lim_{\varepsilon \to 0^+} \int_{|\mathbf{x}|>\varepsilon} f(\mathbf{x})\, \mathrm{d}\mathbf{x}\,, \tag{2.3}$$

if the finite part limit exists. When the ordinary limit $\lim_{\varepsilon\to 0^+} \int_{|\mathbf{x}|>\varepsilon} f(\mathbf{x})\, \mathrm{d}\mathbf{x}$ exists we call the integral a radial principal value integral and use the notation p.v. $\int_{\mathbb{R}^n} f(\mathbf{x})\, \mathrm{d}\mathbf{x}$; this actually equals the *convergent* integral $\int_0^\infty F(r)\, \mathrm{d}r$, where $F(r) = r^{n-1} \int_{\mathbb{S}} f(r\omega)\, \mathrm{d}\sigma(\omega)$.

We shall employ the ideas of the recently developed thick distributional calculus [7,27–29]. The thick calculus allows one to study spaces where there is one special point, where the laws governing the rest of the space do not apply [2]. If $\mathbf{a}$ is a fixed point of $\mathbb{R}^n$, then the space of test functions with a thick point at $\mathbf{x} = \mathbf{a}$ is the space $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ of all smooth functions ϕ defined in $\mathbb{R}^n \setminus \{\mathbf{a}\}$, with support of the form $K \setminus \{\mathbf{a}\}$, where K is compact in $\mathbb{R}^n$, that admit a strong[7] asymptotic expansion of the form $\phi(\mathbf{a} + \mathbf{x}) = \phi(\mathbf{a} + r\omega) \sim \sum_{j=m}^{\infty} a_j(\omega)\, r^j$, as $\mathbf{x} \to \mathbf{0}$, where m is an integer (positive or negative), and where the a_j are smooth functions of ω, that is, $a_j \in \mathcal{D}(\mathbb{S})$. We call $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ the space of test functions on $\mathbb{R}^n$ with a thick point located at $\mathbf{x} = \mathbf{a}$; we denote $\mathcal{D}_{*,\mathbf{0}}(\mathbb{R}^n)$ as $\mathcal{D}_*(\mathbb{R}^n)$. The space $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ has a natural topology, which makes it a locally convex topological vector space [27].

The space of distributions on $\mathbb{R}^n$ with a thick point at $\mathbf{x} = \mathbf{a}$ is the dual space of $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$. We denote it $\mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n)$, or just as $\mathcal{D}'_*(\mathbb{R}^n)$ when $\mathbf{a} = \mathbf{0}$. Observe that $\mathcal{D}(\mathbb{R}^n)$, the space of standard test functions, is a closed subspace of $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$; if $i : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ is the inclusion map then the dual of the inclusion is the projection operator

$$\Pi : \mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)\,. \tag{2.4}$$

The derivatives of thick distributions are defined in much the same way as the usual distributional derivatives, that is, by duality, namely, if $f \in \mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n)$ then its thick distributional derivative $\partial^* f/\partial x_j$ is defined as $\langle \partial^* f/\partial x_j, \phi\rangle = -\langle f, \partial^* \phi/\partial x_j\rangle$ if $\phi \in \mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$.

Let $g(\omega)$ be a distribution on $\mathbb{S}$. The thick delta function of degree q, denoted as $g\delta_*^{[q]}$, or as $g(\omega)\,\delta_*^{[q]}$, acts on a thick test function $\phi(\mathbf{x})$ as

$$\left\langle g\delta_*^{[q]}, \phi \right\rangle_{\mathcal{D}'_*(\mathbb{R}^n)\times\mathcal{D}_*(\mathbb{R}^n)} = \frac{1}{C} \left\langle g(\omega)\,, a_q(\omega)\right\rangle_{\mathcal{D}'(\mathbb{S})\times\mathcal{D}(\mathbb{S})}\,, \tag{2.5}$$

where $\phi(r\omega) \sim \sum_{j=m}^{\infty} a_j(\omega)\, r^j$, as $r \to 0^+$, and where C is given by (2.1).

3. Two inner products

There are two natural inner products defined in the space $\mathcal{P}_k$ of homogeneous polynomials of degree k in n variables. One is defined in terms of the coefficients as

$$\{p, q\} = \sum_{|\alpha|=k} \alpha! a_\alpha \bar{b}_\alpha\,, \tag{3.1}$$

if $p(\mathbf{x}) = \sum_{|\alpha|=k} a_\alpha \mathbf{x}^\alpha$ and $q(\mathbf{x}) = \sum_{|\alpha|=k} b_\alpha \mathbf{x}^\alpha$. Notice that $\{p, q\}$ actually equals the following constant function, $\{p, q\} = p(\boldsymbol{\nabla})\,\overline{q(\mathbf{x})}$, where $\boldsymbol{\nabla} = (\partial/\partial x_i)_{i=1}^n$ is the gradient. The other product is given as

$$(p, q) = \frac{1}{C} \int_{\mathbb{S}} p(\omega)\, \overline{q(\omega)}\, \mathrm{d}\sigma(\omega) = \frac{1}{C} \left\langle p(\omega)\,, \overline{q(\omega)} \right\rangle_{\mathcal{D}'(\mathbb{S})\times\mathcal{D}(\mathbb{S})}\,, \tag{3.2}$$

[7] Observe that we require the asymptotic development of $\phi(\mathbf{x})$ as $\mathbf{x} \to \mathbf{a}$ to be "strong". This means [10, Chapter 1] that for any differentiation operator $(\partial/\partial\mathbf{x})^{\mathbf{p}} = (\partial^{p_1}...\partial^{p_n})/\partial x_1^{p_1}...\partial x_n^{p_n}$, the asymptotic development of $(\partial/\partial\mathbf{x})^{\mathbf{p}}\,\phi(\mathbf{x})$ as $\mathbf{x} \to \mathbf{a}$ exists and is equal to the term-by-term differentiation of $\sum_{j=m}^{\infty} a_j(\mathbf{w})\, r^j$.

where $\mathrm{d}\sigma$ is surface measure on $\mathbb{S}$ and C is given by (2.1). Both products can also be considered in the space $\mathcal{P}_{\leq k} = \bigoplus_{j=0}^{k} \mathcal{P}_k$ of polynomials of degree k at the most.

Clearly the two products are different. In fact, if $\alpha \neq \beta$ then $\{\mathbf{x}^\alpha, \mathbf{x}^\beta\} = 0$, while if all the entries in $\alpha + \beta$ are even then $(\mathbf{x}^\alpha, \mathbf{x}^\beta) > 0$. The interesting thing is that *sometimes* the two inner products are very closely related: Indeed, it is proved in [1, Thm. 5.4] that if p and q are both harmonic, $p, q \in \mathcal{H}_k$ then

$$\{p, q\} = n(n+2)\cdots(n+2k-2)\,(p, q)\ , \tag{3.3}$$

and we shall see that (3.3) holds if just one of the two polynomials is harmonic. Actually, more is true.

Lemma 3.1. *If $u \in \mathcal{H}_k$ then*

$$\Delta^m\left(r^{2m}u\right) = 2^m m!\,(n+2k)\cdots(n+2k+2m-2)\,u\,. \tag{3.4}$$

Proof. If $m = 1$ then $\Delta\left(r^2 u\right) = \Delta\left(r^2\right)u + 4x_i u_{,i} = 2nu + 4ku = 2(n+2k)\,u$, as required. The formula then follows by iteration. $\square$

Let us now recall the identity [12], $\{p, r^2 q\} = \{\Delta p, q\}$, valid if $p \in \mathcal{P}_k$ and $q \in \mathcal{P}_{k-2}$. It yields the ensuing formulas.

Lemma 3.2. *If $p \in \mathcal{H}_k$ and $q \in \mathcal{H}_k$, then*

$$\left\{r^{2m}p, r^{2m}q\right\} = 2^m m!\,n(n+2)\cdots(n+2k+2m-2)\,(p, q)\ , \tag{3.5}$$

while if $p \in \mathcal{H}_k$ and $q \in \mathcal{H}_{k-2l+2m}$, where $l \neq m$ then

$$\left\{r^{2m}p, r^{2l}q\right\} = (p, q) = 0\,. \tag{3.6}$$

Proof. Since $\left\{r^{2m}p, r^{2l}q\right\} = \left\{\Delta^l\left(r^{2m}p\right), q\right\}$, we immediately obtain $\left\{r^{2m}p, r^{2l}q\right\} = 0$ if $l > m$, and thus if $l \neq m$; since $(p, q) = 0$ if p and q are spherical harmonics of different order, (3.6) follows. On the other hand, $\left\{r^{2m}p, r^{2m}q\right\} = \left\{\Delta^m\left(r^{2m}p\right), q\right\}$, so that (3.5) is obtained by combining (3.3) and (3.4). $\square$

Every polynomial $p \in \mathcal{P}_k$ can be written as [1,12]

$$p = u_0 + r^2 u_2 + \cdots + r^{2m}u_{2m}\,, \tag{3.7}$$

where $k - 2m = 0$ or 1, and where $u_{2j} \in \mathcal{H}_{k-2j}$. This is an orthogonal decomposition with respect to *both* inner products, as follows from (3.6). If we now employ (3.5), and collect results, we obtain the next proposition.

Proposition 3.3. *Let $p \in \mathcal{H}_k$ and $q \in \mathcal{P}_{k+2m}$. Then*

$$\left\{r^{2m}p, q\right\} = W_{n,k,m}\left(r^{2m}p, q\right)\ , \tag{3.8}$$

where $W_{n.k,m} = 2^{2m+k}m!\Gamma(k+m+n/2)/\Gamma(n/2)$, that is $W_{n.0,0} = 1$ while if $k + m > 0$,

$$W_{n.k,m} = 2^m m!\,n(n+2)\cdots(n+2k+2m-2)\ . \tag{3.9}$$

Notice that we may rewrite (3.8) in the following way,

$$\frac{1}{C}\int_{\mathbb{S}} p(\omega)\, q(\omega)\, \mathrm{d}\sigma(\omega) = \frac{1}{W_{n,k,m}}\Delta^m p(\boldsymbol{\nabla})\, q(\mathbf{x}) \,, \tag{3.10}$$

if $p \in \mathcal{H}_k$ and $q \in \mathcal{P}_{k+2m}$, the function on the right side of this formula being a constant function.

Example 3.4. An interesting particular case of (3.10) is the following. Suppose $p \in \mathcal{H}_k$. Write $r^{2m}p = \sum_{|\alpha|=k+2m} a_\alpha \mathbf{x}^\alpha$. If $|\beta| = k+2m$ then

$$a_\beta = \frac{W_{n,k,m}}{C\,\beta!}\int_{\mathbb{S}} p(\omega)\, \omega^\beta\, \mathrm{d}\sigma(\omega) \,. \tag{3.11}$$

Example 3.5. If $p \in \mathcal{H}_k$ then, in general, $x_i p$ is not harmonic, and thus formulas (3.8) and (3.10) do not hold if we replace p by $x_i p$. Indeed, if $p \in \mathcal{H}_k$ and $q \in \mathcal{P}_{k+2m+1}$ then

$$(\omega_i p, q) = (p, \omega_i q) = \frac{1}{W_{n,k,m+1}}\left\{r^{2m+2}p, x_i q\right\} \,.$$

But

$$\left\{r^{2m+2}p, x_i q\right\} = \left\{\nabla_i\left(r^{2m+2}p\right), q\right\} = \left\{(2m+2)\, x_i r^{2m}p + r^{2m+2}p_{,i}, q\right\} \,,$$

so that we obtain

$$(\omega_i p, q) = \frac{1}{W_{n,k,m+1}}\left[(2m+2)\,\Delta^m p(\boldsymbol{\nabla})\,\nabla_i + \Delta^{m+1}p_{,i}(\boldsymbol{\nabla})\right]\overline{q}(\mathbf{x}) \,. \tag{3.12}$$

Similarly, if $p \in \mathcal{H}_k$ and $q \in \mathcal{P}_{k+2m}$ then

$$(\omega_i\omega_j p, q) = \frac{1}{W_{n,k,m+1}}\left\{A_{ij}, q\right\} = \frac{1}{W_{n,k,m+1}}A_{ij}(\boldsymbol{\nabla})\,\overline{q}(\mathbf{x}) \,, \tag{3.13}$$

where

$$\begin{aligned} A_{ij} &= 2m\,(2m+2)\, r^{2m-2}x_i x_j p \\ &\quad + (2m+2)\, r^{2m}\delta_{ij}p + (2m+2)\, r^{2m}\left(x_i p_{,j} + x_j p_{,i}\right) + r^{2m+2}p_{,ij} \,. \end{aligned} \tag{3.14}$$

Here and in similar formulas the derivatives of p are denoted as $p_{,i}$, $p_{,ij}$, and so on. Notice that these formulas hold not only for $m \in \mathbb{N}$, but if $m = -1$ as well, yielding

$$(\omega_i p, q) = \frac{1}{W_{n,k,0}}p_{,i}(\boldsymbol{\nabla})\,\overline{q}(\mathbf{x}) \,, \qquad p \in \mathcal{H}_k \,, q \in \mathcal{P}_{k-1} \,, \tag{3.15}$$

and

$$(\omega_i\omega_j p, q) = \frac{1}{W_{n,k,0}}p_{,ij}(\boldsymbol{\nabla})\,\overline{q}(\mathbf{x}) \,, \qquad p \in \mathcal{H}_k \,, q \in \mathcal{P}_{k-2} \,. \tag{3.16}$$

Naturally, if we put $i = j$ and sum over the repeated index in (3.13) we recover (3.10) since $W_{n,k,m+1} = 2\,(m+1)\,(n+2m+2k)\,W_{n,k,m}$.

4. Regularization

We would now like to consider the regularization[8] in $\mathcal{D}'(\mathbb{R}^n)$ of homogeneous functions of the type

$$P(\mathbf{x}) = \frac{p(\mathbf{x})}{r^{2k+n-2}}, \tag{4.1}$$

where p is a homogeneous polynomial of degree k, that belong to $\mathcal{D}'(\mathbb{R}^n \setminus \{\mathbf{0}\})$. In general, if f is a function or distribution defined in $\mathbb{R}^n \setminus \{\mathbf{0}\}$, then it may or may not be possible to extend it to a distribution in the whole space, and, even if possible, the extension is not unique. Furthermore there is no canonical procedure that works in *all* cases [5].

The simplest regularization procedure is the *spherical principal value regularization,* defined as[9]

$$\text{p.v.}(f) = \lim_{\varepsilon\to 0} f(\mathbf{x}) H(r-\varepsilon), \tag{4.2}$$

where H is the Heaviside function. Naturally, for a given $f \in \mathcal{D}'(\mathbb{R}^n \setminus \{\mathbf{0}\})$ the product $f(\mathbf{x}) H(r-\varepsilon)$ does not have to be defined, and the limit might not exist; therefore, in general, one would not expect the existence of the principal value. Interestingly, when the spherical principal value exists then one may employ the simple procedure explained in the Physics literature, namely, *"in order to regularize the integral* $\int_{\mathbb{R}^n} f(\mathbf{x})\phi(\mathbf{x})\,\mathrm{d}\mathbf{x}$, *one just needs to use polar coordinates and perform the angular integrals first."* It goes without saying that this simple method fails if the spherical principal value does not exist. Many times the limit in (4.2) does not exist but the Hadamard finite part limit of Definition 2.1 exists, and this gives the distribution

$$\mathcal{P}f(f(\mathbf{x})) = \text{F.p.}\lim_{\varepsilon\to 0} f(\mathbf{x}) H(r-\varepsilon). \tag{4.3}$$

The notation $\mathcal{P}f(f(\mathbf{x}))$ was introduced by Schwartz [24, Chp. 2, §2], who called it a *pseudofunction,* a term that many still use. For instance, if $\alpha \ge n$ then $\mathcal{P}f(r^{-\alpha})$ exists but the principal value does not. The distribution $\mathcal{P}f(P(\mathbf{x})) = p(\mathbf{x})\mathcal{P}f\left(r^{-(2k+n-2)}\right)$, where $P(\mathbf{x})$ is given in (4.1) will always exist, but as we shall see, if $p \in \mathcal{H}_k$ then actually more is true, since the principal value p.v.$(P(\mathbf{x}))$ exists as well.

Example 4.1. The principal value p.v.$\left(\left(3x_ix_j - r^2\delta_{ij}\right)/r^5\right)$ exists in $\mathbb{R}^3$, as will follow from Proposition 4.3, but we have

$$\text{p.v.}\left(\frac{3x_ix_j - r^2\delta_{ij}}{r^5}\right) \ne \mathcal{P}f\left(\frac{3x_ix_j}{r^5}\right) - \delta_{ij}\mathcal{P}f\left(\frac{1}{r^3}\right), \tag{4.4}$$

since the principal values of the distributions on the right do not exist.

We need a simple auxiliary result at this point.

Lemma 4.2. *Let $p \in \mathcal{P}_k$. Then $p \in \mathcal{H}_k$ if and only if $(p,q) = 0$ for all polynomials of degree less than k.*

Proof. If $p \in \mathcal{H}_k$ then clearly $(p,q) = 0$ for all polynomials of degree less than k, since the restriction of q to $\mathbb{S}$ can be expressed as a linear combination of homogeneous harmonic polynomials of degree less than k and homogeneous harmonic polynomials of different degrees are orthogonal for the inner product $(\ ,\ )$.

[8] Regularization methods are considered in the texts on distributions [10,14,19,20]. See also [6] and [23].

[9] One should call the procedure (4.2) a *spherical* principal value, since the use of the variable r means that $f(\mathbf{x}) H(r-\varepsilon)$ is the distribution where f has been replaced by 0 inside a *ball* of radius ε. The results when solids of other shapes are removed could be very different [11,17,26].

Conversely, if $p \in \mathcal{P}_k \setminus \mathcal{H}_k$, then we can write $p = u_0 + r^2 u_2 + \cdots + r^{2m} u_{2m}$, where $k - 2m = 0$ or 1, where $u_{2j} \in \mathcal{H}_{k-2j}$, and where $u_{2j} \neq 0$ for some $j > 0$; but this gives

$$\begin{aligned}(p, u_{2j}) = \left(p, r^{2j} u_{2j}\right) &= \frac{1}{W_{n,k-2j,j}} \left\{p, r^{2j} u_{2j}\right\} = \frac{1}{W_{n,k-2j,j}} \left\{\Delta^j p, u_{2j}\right\} \\ &= \frac{2^j j!\, (n+2k) \cdots (n+2k+2j-2)}{W_{n,k-2j,j}} \left\{u_{2j}, u_{2j}\right\} \neq 0 . \quad \square\end{aligned}$$

We thus obtain the following result.

Proposition 4.3. *Let $p \in \mathcal{P}_k$. Then $p \in \mathcal{H}_k$ if and only if the principal value*

$$\text{p.v.} \left(\frac{p(\mathbf{x})}{r^{2k+n-2}}\right) = \lim_{\varepsilon \to 0} \frac{p(\mathbf{x})}{r^{2k+n-2}} H(r - \varepsilon) , \tag{4.5}$$

exists in $\mathcal{D}'(\mathbb{R}^n)$.

Proof. Let $\phi \in \mathcal{D}(\mathbb{R}^n)$. Suppose its support is contained in the ball $|\mathbf{x}| \le A$. Let $q(\mathbf{x}) = \sum_{j=0}^{k-1} q_j(\mathbf{x})$ be its Taylor polynomial of degree $k-1$, where $q_j \in \mathcal{P}_j$. Then if $\varepsilon < A$ we have

$$\begin{aligned}\left\langle \frac{p(\mathbf{x})}{r^{2k+n-2}} H(r-\varepsilon), \phi(\mathbf{x}) \right\rangle &= \int\limits_{\varepsilon \le |\mathbf{x}| \le A} \frac{p(\mathbf{x})\, \phi(\mathbf{x})}{r^{2k+n-2}}\, \mathrm{d}\mathbf{x} \\ &= \int\limits_{\varepsilon \le |\mathbf{x}| \le A} \frac{p(\mathbf{x}) \left(\phi(\mathbf{x}) - q(\mathbf{x})\right)}{r^{2k+n-2}}\, \mathrm{d}\mathbf{x} + \int\limits_{\varepsilon \le |\mathbf{x}| \le A} \frac{p(\mathbf{x})\, q(\mathbf{x})}{r^{2k+n-2}}\, \mathrm{d}\mathbf{x} ,\end{aligned}$$

and since $(p, q_j) = 0$ if $k - j$ is odd,

$$\int\limits_{\varepsilon \le |\mathbf{x}| \le A} \frac{p(\mathbf{x})\, q(\mathbf{x})}{r^{2k+n-2}}\, \mathrm{d}\mathbf{x} = \sum_{l=1}^{[\![k/2]\!]} (p, q_{k-2l}) \int_{\varepsilon}^{A} r^{-2l+1} \mathrm{d}r , \tag{4.6}$$

so that

$$\left\langle \frac{p(\mathbf{x})}{r^{2k+n-2}} H(r-\varepsilon), \phi(\mathbf{x}) \right\rangle = G_{\text{Infinite}}(\varepsilon) + G_{\text{Finite}}(\varepsilon) , \tag{4.7}$$

where $\lim_{\varepsilon \to 0} G_{\text{Finite}}(\varepsilon)$ exists, and where

$$G_{\text{Infinite}}(\varepsilon) = -(p, q_{k-2}) \ln \varepsilon + \sum_{l=2}^{[\![k/2]\!]} \frac{(p, q_{k-2l})}{(2l-2)\, \varepsilon^{2l-2}} . \tag{4.8}$$

It follows that the limit in (4.7) exist for all test functions $\phi \in \mathcal{D}(\mathbb{R}^n)$ if and only if $G_{\text{Infinite}}(\varepsilon) = 0$, and because of (4.8) this is equivalent to $(p, q_j) = 0$ for all $q_j \in \mathcal{P}_j$, $0 \le j \le k-1$. The Lemma 4.2 gives therefore that the principal value exists precisely when $p \in \mathcal{H}_k$. $\square$

When $k = 2$, then $p \in \mathcal{P}_2$ is harmonic if and only if $\int_{\mathbb{S}} p(\omega)\, \mathrm{d}\sigma(\omega) = 0$, so we obtain the following simple consequence of the above proposition.

Example 4.4. If $p \in \mathcal{P}_2$ then the principal value p.v. $\left(p(\mathbf{x}) / r^{n+2}\right)$ exists if and only if $\int_{\mathbb{S}} p(\omega)\, \mathrm{d}\sigma(\omega) = 0$.

In general if $f \in \mathcal{D}'(\mathbb{R}^n \setminus \{\mathbf{0}\})$ is homogeneous of some degree α, then the pseudofunction $\mathcal{P}f(f(\mathbf{x}))$ does not have to be homogeneous of degree α. For instance, $\mathcal{P}f\left(r^{-n-2k}\right)$ is *not* homogeneous of degree $-n-2k$ in $\mathcal{D}'(\mathbb{R}^n)$ for $k = 0, 1, 2, \ldots$. However, if the principal value $F(\mathbf{x}) = \text{p.v.}(f(\mathbf{x}))$ exists, then[10] it is homogeneous of degree α, since if $\lambda > 0$

$$F(\lambda \mathbf{x}) = \lim_{\varepsilon \to 0} f(\lambda \mathbf{x}) H(|\lambda \mathbf{x}| - \varepsilon) = \lambda^\alpha \lim_{\varepsilon \to 0} f(\mathbf{x}) H(r - \varepsilon/\lambda) = \lambda^\alpha F(\mathbf{x}) .$$

Therefore we obtain the following on the homogeneity of p.v. $\left(p(\mathbf{x})/r^{2k+n-2}\right)$.

Proposition 4.5. *If $p \in \mathcal{H}_k$ then $\mathcal{P}f\left(p(\mathbf{x})/r^{2k+n-2}\right) = \text{p.v.}\left(p(\mathbf{x})/r^{2k+n-2}\right)$ is homogeneous of degree $-k-n-2$. If $p \in \mathcal{P}_k \setminus \mathcal{H}_k$ then $\mathcal{P}f\left(p(\mathbf{x})/r^{2k+n-2}\right)$ is not homogeneous.*

One may also consider the regularization of (4.1) in the space of thick distributions $\mathcal{D}'_*(\mathbb{R}^n)$. It turns out that in this space we always need to consider the Hadamard regularization, since the principal value *never* exists. We shall use the same notation, $\mathcal{P}f(f(\mathbf{x}))$ in both spaces, $\mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{D}'_*(\mathbb{R}^n)$, since this should not cause any confusion.

4.1. The delta part of a distribution

In general it is not possible to separate the contribution to a distribution from a given point; to talk about the "delta part at $\mathbf{x}_0$" of *all* distributions does not make sense. However, *sometimes,* we can actually separate the delta part.

Definition 4.6. Let $f_0 \in \mathcal{D}'(\mathbb{R}^n \setminus \{\mathbf{0}\})$ be a distribution defined in the complement of the origin. Suppose the pseudofunction $\mathcal{P}f(f_0(\mathbf{x}))$ exists in $\mathcal{D}'(\mathbb{R}^n)$ (respectively in $\mathcal{D}'_*(\mathbb{R}^n)$). Let $f \in \mathcal{D}'(\mathbb{R}^n)$ (respectively in $f \in \mathcal{D}'_*(\mathbb{R}^n)$) be any regularization of f_0. Then the delta part at 0 of f is the distribution $f - \mathcal{P}f(f_0(\mathbf{x}))$, whose support is the origin.[11]

It must be emphasized that even though $\mathcal{P}f(f_0(\mathbf{x}))$ is in a way *the* natural regularization of f_0, other regularizations appear also very naturally, as we illustrate in the following examples.

Example 4.7. Consider the distribution $\mathcal{P}f\left(r^{-k}\right)$ in $\mathbb{R}^n$. Then the distributional derivative $\left(\overline{\partial}/\partial x_i\right)\mathcal{P}f\left(r^{-k}\right)$ is a regularization of $-kx_i r^{-k-2}$, the ordinary derivative of r^{-k}; however [9, (3.16)] if $k - n = 2m$ is an even positive integer, then

$$\frac{\overline{\partial}}{\partial x_i}\mathcal{P}f\left(r^{-k}\right) = \mathcal{P}f\left(-kx_i r^{-k-2}\right) - \frac{c_{m,n}}{(2m)!k}\nabla_i \Delta^m \delta(\mathbf{x}) , \tag{4.9}$$

where $c_{m,n}$ is given by (2.1). Therefore, $\left(-c_{m,n}/(2m)!k\right)\nabla_i \Delta^m \delta(\mathbf{x})$ is the delta part of the distribution $\left(\overline{\partial}/\partial x_i\right)\mathcal{P}f\left(r^{-k}\right)$ in $\mathcal{D}'(\mathbb{R}^n)$. In the space $\mathcal{D}'_*(\mathbb{R}^n)$, now for any integer $k \in \mathbb{Z}$, the delta part of the thick derivative $\left(\partial^*/\partial x_i\right)\mathcal{P}f\left(r^{-k}\right)$ is given [27, Thm. 7.1] as $Cn_i\delta_*^{[-k-n+1]}$. Naturally, when $k - n = 2m \geq 0$, the projection of the thick delta part is precisely the distributional delta part, and this agrees with [27, (7.7)].

Example 4.8. If $\lambda > 0$ is fixed, then $\lambda^k \mathcal{P}f\left(H(x)/(\lambda x)^k\right)$ is a regularization of $H(x)/x^k$ in $\mathcal{D}'(\mathbb{R})$, but [10, (2.93)]

[10] This argument does not work for $\mathcal{P}f(f(\mathbf{x}))$, in general, because $\text{F.p.}\lim_{\varepsilon \to 0} F(c\varepsilon)$ and $\text{F.p.}\lim_{\varepsilon \to 0} F(\varepsilon)$ might be different (take $F(\varepsilon) = \ln \varepsilon$, for instance).

[11] Notice that this delta part is in fact a *spherical* delta part.

$$\mathcal{P}f\left(\frac{H(x)}{(\lambda x)^k}\right) = \frac{1}{\lambda^k}\mathcal{P}f\left(\frac{H(x)}{x^k}\right) + \frac{(-1)^{k-1}\ln\lambda\,\delta^{(k-1)}(x)}{(k-1)!\lambda^k}, \tag{4.10}$$

so that the delta part of $\lambda^k\mathcal{P}f\left(H(x)/(\lambda x)^k\right)$ is $(-1)^{k-1}\ln\lambda\,\delta^{(k-1)}(x)/(k-1)!$. Similarly, in $\mathbb{R}^n$ for $n\geq 2$, and for $m\in\mathbb{N}$, the distribution $\lambda^{n+2m}\mathcal{P}f\left(|\lambda\mathbf{x}|^{-n-2m}\right)$ is a regularization of r^{-n-2m} and in $\mathcal{D}'(\mathbb{R}^n)$ its delta part is $\ln\lambda\, c_{m,n}\boldsymbol{\nabla}^{2m}\delta(\mathbf{x})/(2m)!$, while in $\mathcal{D}'_*(\mathbb{R}^n)$ its delta part is $\ln\lambda\, C\delta_*^{[2m]}$, as follows from [27, (5.13), (5.14)].

Example 4.9. The function $\sin r^{-k}$ is locally integrable in $\mathbb{R}^n$, and thus it gives a well defined regular distribution in $\mathcal{D}'(\mathbb{R}^n)$. If $k>n$, then the distributional derivative $\left(\overline{\partial}/\partial x_i\right)\sin r^{-k}$ is another well defined distribution, but its delta part at the origin is *not* defined, since $\mathcal{P}f\left((\partial/\partial x_i)\sin r^{-k}\right)$ does not exist.

When f_0 is a smooth function defined in $\mathbb{R}^n\setminus\{\mathbf{0}\}$ such that the Hadamard regularization exists at the origin, and $f\in\mathcal{D}'(\mathbb{R}^n)$ is a regularization of f_0, then we call f_0 the *ordinary part* of f. Thus, for instance, $-kx_ir^{-k-2}$ is the ordinary part of $\left(\overline{\partial}/\partial x_i\right)\mathcal{P}f\left(r^{-k}\right)$.

5. Derivatives of homogeneous distributions

We now shall present the distributional derivatives of the regularization $\mathcal{P}f(P_l(\mathbf{x}))$ of the functions of the type

$$P_l(\mathbf{x}) = \frac{p(\mathbf{x})}{r^{k+l}} = \frac{p(\omega)}{r^l}, \quad p\in\mathcal{H}_k, \tag{5.1}$$

that are homogeneous in $\mathbb{R}^n\setminus\{\mathbf{0}\}$, particularly when $l=k+n-2$, so that P_{k+n-2} is harmonic in $\mathbb{R}^n\setminus\{\mathbf{0}\}$. Naturally our main interest is in the delta part of these distributional derivatives, since obtaining the ordinary part is quite straightforward. Actually $\overline{\Delta}\mathcal{P}f(P_{k+n-2}(\mathbf{x}))$ is a distribution concentrated at the origin, that is, it is just the delta part.

Our approach is based on the thick distributional calculus, since the thick derivatives of *general* homogeneous functions are available [29, Prop. 3.3 and Prop. 3.5]. Therefore the distributional derivatives are obtained by projection onto $\mathcal{D}'(\mathbb{R}^n)$, since [27, Prop. 5.9]

$$\Pi\left(\boldsymbol{\nabla}_i^*(f)\right) = \overline{\boldsymbol{\nabla}}_i(\Pi(f)) . \tag{5.2}$$

Even though one can do this for general homogeneous functions, the expressions for the projections can be simplified and yield a rather nice formula in the case of $P_l(\mathbf{x})$; in fact, if we assume $p\in\mathcal{P}_k$, but p not harmonic, the formulas become much more involved.

5.1. Projection formulas

We now give several projection formulas of thick distributions that involve harmonic polynomials. Let us recall, first, the general formula for the projection of $\Pi\left(g(\omega)\,\delta_*^{[l]}\right)$ if $g\in\mathcal{D}'(\mathbb{S})$, namely [27, (4.9)]

$$\Pi\left(g(\omega)\,\delta_*^{[l]}\right) = \frac{(-1)^l}{C}\sum_{|\alpha|=l}\frac{\langle g(\omega),\omega^\alpha\rangle_{\mathcal{D}'(\mathbb{S})\times\mathcal{D}(\mathbb{S})}}{\alpha!}\boldsymbol{\nabla}^\alpha\delta(\mathbf{x}) . \tag{5.3}$$

No extra simplification is to be expected for a general g, but if $g=p\in\mathcal{H}_k$, then we have the ensuing projection results.

Proposition 5.1. *Let* $p \in \mathcal{H}_k$. *Then*

$$\Pi \left(p(\omega)\, \delta_*^{[l]} \right) = 0 \,, \quad \text{if } l - k \neq 0, 2, 4, \dots \,, \tag{5.4}$$

$$\Pi \left(p(\omega)\, \delta_*^{[k+2m]} \right) = \frac{(-1)^k}{W_{n,k,m}} p(\boldsymbol{\nabla})\, \boldsymbol{\nabla}^{2m} \delta(\mathbf{x}) \ . \tag{5.5}$$

In particular,

$$\Pi \left(p(\omega)\, \delta_*^{[k]} \right) = \frac{(-1)^k}{W_{n,k,0}} p(\boldsymbol{\nabla})\, \delta(\mathbf{x}) \ . \tag{5.6}$$

Proof. Formula (5.4) follows since $(p, \omega^\alpha) = 0$ if $|\alpha| = l$ and $l - k$ is not an even positive[12] integer. To establish (5.5) we proceed as follows. Let $\phi \in \mathcal{D}(\mathbb{R}^n)$; let $\sum_{j=0}^{\infty} q_j$ be its Taylor expansion, where $q_j \in \mathcal{P}_j$. Then

$$\left\langle \Pi \left(p(\omega)\, \delta_*^{[k+2m]} \right), \phi \right\rangle_{\mathcal{D}'(\mathbb{R}^n) \times \mathcal{D}(\mathbb{R}^n)} = \left\langle p(\omega)\, \delta_*^{[k+2m]}, \phi \right\rangle_{\mathcal{D}'_*(\mathbb{R}^n) \times \mathcal{D}_*(\mathbb{R}^n)}$$
$$= \frac{1}{C} \int_{\mathbb{S}} p(\omega)\, q_{k+2m}(\omega)\, \mathrm{d}\sigma(\omega) = \frac{1}{W_{n,k,m}} \Delta^m p(\boldsymbol{\nabla})\, q_{k+2m}(\mathbf{x})$$
$$= \frac{1}{W_{n,k,m}} \Delta^m p(\boldsymbol{\nabla})\, \phi(\mathbf{x}) \Big|_{\mathbf{x}=\mathbf{0}} \,,$$

that is

$$\left\langle \Pi \left(p(\omega)\, \delta_*^{[k+2m]} \right), \phi \right\rangle_{\mathcal{D}'(\mathbb{R}^n) \times \mathcal{D}(\mathbb{R}^n)} = \left\langle \frac{(-1)^k}{W_{n,k,m}} p(\boldsymbol{\nabla})\, \boldsymbol{\nabla}^{2m} \delta(\mathbf{x}), \phi(\mathbf{x}) \right\rangle_{\mathcal{D}'(\mathbb{R}^n) \times \mathcal{D}(\mathbb{R}^n)} \,,$$

as required. An alternative derivation employs (3.11) and (5.3). □

Notice that we may take $p = 1$ in the above projection formulas, and this yields $\Pi \left(\delta_*^{[l]} \right) = 0$, if $l \neq 0, 2, 4, \dots$, and

$$\Pi \left(\delta_*^{[2m]} \right) = \frac{1}{W_{n,0,m}} \boldsymbol{\nabla}^{2m} \delta(\mathbf{x}) = \frac{\Gamma\left(m + \frac{1}{2}\right) \Gamma\left(\frac{n}{2}\right)}{\Gamma\left(m + \frac{n}{2}\right) \Gamma\left(\frac{1}{2}\right) (2m)!} \boldsymbol{\nabla}^{2m} \delta(\mathbf{x}) \,, \tag{5.7}$$

that agrees with [27, (6.5)]. Similarly, we could take $p = x_i$, which gives $\Pi \left(\delta_*^{[l]} \right) = 0$, if $l \neq 1, 3, 5, \dots$, and [27, (7.7)]

$$\Pi \left(\omega_i \delta_*^{[1+2m]} \right) = \frac{-1}{W_{n,1,m}} \nabla_i \boldsymbol{\nabla}^{2m} \delta(\mathbf{x}) = \frac{-\Gamma\left(m + \frac{1}{2}\right) \Gamma\left(\frac{n}{2}\right)}{(2m+2)\, \Gamma\left(m + \frac{n}{2}\right) \Gamma\left(\frac{1}{2}\right) (2m)!} \nabla_i \boldsymbol{\nabla}^{2m} \delta(\mathbf{x}) \ . \tag{5.8}$$

Formula (5.5) cannot be applied if p is not harmonic, but we may employ formulas (3.12) and (3.13) and an argument similar to the proof of the Proposition 5.1 to obtain the following results.

[12] We employ the term x positive to mean $x \geq 0$; if $x > 0$ we say that x is strictly positive.

Proposition 5.2. *Let $p \in \mathcal{H}_k$. Then $\Pi\left(\omega_i p(\omega)\,\delta_*^{[l]}\right) = 0$ if $l - k \neq -1, 1, 3, 5, \ldots$, while*

$$\Pi\left(\omega_i p(\omega)\,\delta_*^{[k+2m+1]}\right) = \frac{(-1)^{k+1}}{W_{n,k,m+1}}\left((2m+2)\nabla_i p(\boldsymbol{\nabla})\,\boldsymbol{\nabla}^{2m} + p_{,i}(\boldsymbol{\nabla})\,\boldsymbol{\nabla}^{2m+2}\right)\delta(\mathbf{x})\ . \tag{5.9}$$

Also $\Pi\left(\omega_i\omega_j p(\omega)\,\delta_*^{[l]}\right) = 0$ if $l - k \neq -2, 0, 2, 4, 6, \ldots$, and

$$\Pi\left(\omega_i\omega_j p(\omega)\,\delta_*^{[k+2m+2]}\right) = \frac{(-1)^{k+2}}{W_{n,k,m+1}} A_{ij}(\boldsymbol{\nabla})\,\delta(\mathbf{x})\ , \tag{5.10}$$

where the polynomial $A_{ij} \in \mathcal{P}_{k+2}$ is given by (3.14).

Formulas (5.9) and (5.10) hold not only if $m \in \mathbb{N}$, but for $m = -1$ as well, namely,

$$\Pi\left(\omega_i p(\omega)\,\delta_*^{[k-1]}\right) = \frac{(-1)^{k+1}}{W_{n,k,0}} p_{,i}(\boldsymbol{\nabla})\,\delta(\mathbf{x})\ , \tag{5.11}$$

$$\Pi\left(\omega_i\omega_j p(\omega)\,\delta_*^{[k-2]}\right) = \frac{(-1)^{k+2}}{W_{n,k,0}} p_{,ij}(\boldsymbol{\nabla})\,\delta(\mathbf{x})\ . \tag{5.12}$$

Notice also that (5.9) allows us to generalize the projection formula $\Pi\left(\omega_i\omega_j\delta_*^{[0]}\right) = (\delta_{ij}/n)\,\delta(\mathbf{x})$ [27] to

$$\Pi\left(\omega_i\omega_j\delta_*^{[2m]}\right) = \frac{1}{W_{n,1,m}}\left(2m\nabla_i\nabla_j\nabla^{2m-2} + \delta_{ij}\nabla^{2m}\right)\delta(\mathbf{x})\ . \tag{5.13}$$

5.2. First order derivatives of $\mathcal{P}f\,(P_l(\mathbf{x}))$

The thick first order derivatives of $\mathcal{P}f\,(P_l(\mathbf{x}))$ can be obtained from [29, Prop. 3.3]. There is no delta part unless $l \in \mathbb{Z}$, and in this case

$$\boldsymbol{\nabla}_i^*\mathcal{P}f\left(\frac{p(\mathbf{x})}{r^{k+l}}\right) = \mathcal{P}f\left(\frac{r^2 p_{,i}(\mathbf{x}) - (k+l)\,x_i p(\mathbf{x})}{r^{k+l+2}}\right) + Cn_i p\delta_*^{[1-n+l]}. \tag{5.14}$$

Use of the Proposition 5.2 then yields the ensuing result.

Proposition 5.3. *Let $p \in \mathcal{H}_k$. If $l \neq k + 2m + n - 2$, $m \in \mathbb{N}$, then*

$$\overline{\boldsymbol{\nabla}}_i\mathcal{P}f\left(\frac{p(\mathbf{x})}{r^{k+l}}\right) = \mathcal{P}f\left(\frac{r^2 p_{,i}(\mathbf{x}) - (k+l)\,x_i p(\mathbf{x})}{r^{k+l+2}}\right)\ , \tag{5.15}$$

while if $m \in \mathbb{N}$

$$\overline{\boldsymbol{\nabla}}_i\mathcal{P}f\left(\frac{p(\mathbf{x})}{r^{2k+2m+n-2}}\right) = \mathcal{P}f\left(\frac{r^2 p_{,i}(\mathbf{x}) - (n+2k+2m-2)\,x_i p(\mathbf{x})}{r^{2k+2m+n}}\right) + \frac{(-1)^{k+1}C}{W_{n,k,m}}\left(2m\nabla_i p(\boldsymbol{\nabla})\,\boldsymbol{\nabla}^{2m-2} + p_{,i}(\boldsymbol{\nabla})\,\boldsymbol{\nabla}^{2m}\right)\delta(\mathbf{x})\ . \tag{5.16}$$

Notice that when $k = 0$, so that $p = 1$, we recover the formula [9, (3.16)], namely,

$$\overline{\boldsymbol{\nabla}}_i\mathcal{P}f\left(\frac{1}{r^{2m+n}}\right) = -(n+2m)\,\mathcal{P}f\left(\frac{x_i}{r^{2m+n+2}}\right) - \frac{\pi^{n/2}}{2^{2m}m!\,\Gamma\left(\frac{n}{2}+m+1\right)}\boldsymbol{\nabla}_i\boldsymbol{\nabla}^{2m}\delta(\mathbf{x})\ . \tag{5.17}$$

Notice also the case $m = 0$, that yields for $k \geq 1$

$$\overline{\nabla}_i \mathcal{P}f\left(\frac{p(\mathbf{x})}{r^{2k+n-2}}\right) = \mathcal{P}f\left(\frac{r^2 p_{,i}(\mathbf{x}) - (n+2k-2)\, x_i p(\mathbf{x})}{r^{2k+n}}\right) + \frac{(-1)^{k+1} C}{W_{n,k,0}} p_{,i}(\nabla)\, \delta(\mathbf{x}) \ . \tag{5.18}$$

In particular,

$$\overline{\nabla}_i \mathcal{P}f\left(\frac{x_j}{r^n}\right) = \mathcal{P}f\left(\frac{r^2 \delta_{ij} - n x_i x_j}{r^{n+2}}\right) + \frac{C}{n} \delta_{ij} \delta(\mathbf{x}) \ . \tag{5.19}$$

5.3. The Laplacian of $\mathcal{P}f\,(P_l\,(\mathbf{x}))$

The ordinary Laplacian of P_l is easily obtained to be

$$\Delta P_l = (l+k)(l-k-n+2)\, P_{l+2} \ . \tag{5.20}$$

Hence if we employ [29, Prop. 3.5] we obtain the thick distributional Laplacian of $\mathcal{P}f\,(P_l\,(\mathbf{x}))$ as $\Delta^* \mathcal{P}f\,(P_l) = (l+k)(l-k-n+2)\, \mathcal{P}f\,(P_{l+2})$ if $l \notin \mathbb{Z}$, while if l is an integer, then

$$\Delta^* \mathcal{P}f\,(P_l) = (l+k)(l-k-n+2)\, \mathcal{P}f\,(P_{l+2}) - C\,(2l-n+2)\, p(\omega)\, \delta_*^{[l-n+2]}. \tag{5.21}$$

Therefore, use of Proposition 5.1 yields the ensuing result on the distributional Laplacian of $\mathcal{P}f\,(P_l)$.

Proposition 5.4. *Let $p \in \mathcal{H}_k$. If $l \neq k + 2m + n - 2$, $m \in \mathbb{N}$, then*

$$\overline{\Delta} \mathcal{P}f\,(P_l) = (l+k)(l-k-n+2)\, \mathcal{P}f\,(P_{l+2}) \ , \tag{5.22}$$

while

$$\overline{\Delta} \mathcal{P}f\left(\frac{p(\mathbf{x})}{r^{2k+2m+n-2}}\right) = 2m\,(2k+2m+n-2)\, \mathcal{P}f\left(\frac{p(\mathbf{x})}{r^{2k+2m+n}}\right) + \frac{(-1)^{k+1} C\,(n+2k+4m-2)}{W_{n,k,m}} p(\nabla)\, \nabla^{2m} \delta(\mathbf{x}) \ . \tag{5.23}$$

Notice that when $k = 0$, so that $p = 1$, we recover the distributional Laplacian of $\mathcal{P}f\left(r^{-2m-n-2}\right)$ as given in [8,9] and in [19, pg. 248]:

$$\begin{aligned} \overline{\Delta} \mathcal{P}f\left(\frac{1}{r^{2m+n-2}}\right) &= \mathcal{P}f\left(\frac{2m\,(n+2m-2)}{r^{2m+n}}\right) - \frac{C\,(n+4m-2)}{W_{n,0,m}} \nabla^{2m} \delta(\mathbf{x}) \\ &= \mathcal{P}f\left(\frac{2m\,(n+2m-2)}{r^{2m+n}}\right) - \frac{(n+4m-2)\, \pi^{n/2}}{2^{2m-1} m! \Gamma(m+n/2)} \nabla^{2m} \delta(\mathbf{x}) \ . \end{aligned} \tag{5.24}$$

The particular case $m = 0$ of (5.23) deserves special mention.

Proposition 5.5. *If $p \in \mathcal{H}_k$ then*

$$\overline{\Delta} \text{p.v.}\left(\frac{p(\mathbf{x})}{r^{2k+n-2}}\right) = \frac{(-1)^{k+1} C\,(n+2k-2)}{W_{n,k,0}} p(\nabla)\, \delta(\mathbf{x}) \ . \tag{5.25}$$

Observe that the distribution p.v. $\left(p(\mathbf{x})/r^{2k+n-2}\right)$ is harmonic in $\mathbb{R}^n \setminus \{\mathbf{0}\}$ and homogeneous of degree $-(k+n-2)$ in the whole space $\mathbb{R}^n$, and this information is enough to conclude that its Laplacian is of the form $q(\boldsymbol{\nabla})\delta(\mathbf{x})$ for some homogeneous polynomial of degree k, $q \in \mathcal{P}_k$. The Proposition 5.5 yields much more, since it says that actually $q = Mp$ for some constant M. In fact, one may simplify the formula for M given in (5.25) to obtain the well known [12] identity $\overline{\Delta}\text{p.v.}\left(r^{2-n}1\right) = C(2-n)\delta(\mathbf{x})$, and for $k > 0$,

$$\overline{\Delta}\text{p.v.}\left(\frac{p(\mathbf{x})}{r^{2k+n-2}}\right) = \frac{(-1)^{k+1}C}{W_{n,k-1,0}}p(\boldsymbol{\nabla})\delta(\mathbf{x}) = \frac{(-1)^{k+1}\pi^{n/2}}{2^{k-2}\Gamma\left(\frac{n}{2}+k-1\right)}p(\boldsymbol{\nabla})\delta(\mathbf{x}) \ . \tag{5.26}$$

Example 5.6. When $k = 1$ then $p(\mathbf{x}) = \mathbf{a}\cdot\mathbf{x} = a_i x_i$, where $\mathbf{a} = (a_i)_{i=1}^n$ is an arbitrary vector, and thus we obtain the Laplacian of the dipole p.v. $(a_i x_i/r^n)$,

$$\overline{\Delta}\text{p.v.}\left(\frac{a_i x_i}{r^n}\right) = C a_i \nabla_i \delta(\mathbf{x}) \ . \tag{5.27}$$

The identity (5.27) can also be derived as follows,

$$\begin{aligned}\overline{\Delta}\text{p.v.}\left(\frac{a_i x_i}{r^n}\right) &= -\frac{a_i}{n-2}\overline{\Delta}\text{p.v.}\left(\overline{\boldsymbol{\nabla}}_i\left(\frac{1}{r^{n-2}}\right)\right) = -\frac{a_i}{n-2}\overline{\boldsymbol{\nabla}}_i\overline{\Delta}\left(\frac{1}{r^{n-2}}\right)\\ &= -\frac{a_i}{n-2}\boldsymbol{\nabla}_i\left(-C(n-2)\delta(\mathbf{x})\right) = C a_i \boldsymbol{\nabla}_i \delta(\mathbf{x}) \ .\end{aligned}$$

A third alternative is to use [9, (3.19)], the n dimensional analog of Frahm formulas [13],

$$\begin{aligned}\overline{\Delta}\text{p.v.}\left(\frac{a_i x_i}{r^n}\right) &= -\frac{a_i\overline{\boldsymbol{\nabla}}_j}{n-2}\left(\overline{\boldsymbol{\nabla}}_i\overline{\boldsymbol{\nabla}}_j\left(\frac{1}{r^{n-2}}\right)\right) = -a_i\overline{\boldsymbol{\nabla}}_j\left(\text{p.v.}\left(\frac{n x_i x_j - \delta_{ij}r^2}{r^{n+2}}\right) - \frac{C\delta_{ij}}{n}\delta(\mathbf{x})\right)\\ &= \overline{\boldsymbol{\nabla}}_j\left(\text{p.v.}\left(\frac{n(a_i x_i)x_j - a_j r^2}{r^{n+2}}\right)\right) - \frac{C}{n}a_i\overline{\boldsymbol{\nabla}}_i\delta(\mathbf{x}) \ ,\end{aligned}$$

and thus we get the “apparent paradox” mentioned by Parker [22], since in order to obtain (5.27) the derivatives of p.v. $\left(\left(n(a_i x_i)x_j - a_j r^2\right)/r^{n+2}\right)$, a distribution with zero delta part, should have a delta part equal to a derivative of the delta function (and, of course, this is not a paradox, since it is true, as (5.16) shows).

Example 5.7. For $k = 2$ then $p(\mathbf{x}) = a_{ij}x_i x_j$, where $A = (a_{ij})_{i,j=1}^n$ is a symmetric matrix with null trace, $\text{tr}(A) = a_{ii} = 0$, and hence

$$\overline{\Delta}\text{p.v.}\left(\frac{a_{ij}x_i x_j}{r^{n+2}}\right) = -\frac{C}{n}a_{ij}\nabla_i\nabla_j\delta(\mathbf{x}) \qquad \text{if } a_{ii} = 0 \ . \tag{5.28}$$

Actually for a general matrix A with arbitrary trace we can write $a_{ij} = (\text{tr}(A)/n)\delta_{ij} + (a_{ij} - (\text{tr}(A)/n)\delta_{ij})$, and use (5.24) with $m = 1$, to obtain

$$\overline{\Delta}\mathcal{P}f\left(\frac{a_{ij}x_i x_j}{r^{n+2}}\right) = \mathcal{P}f\left(\frac{2\,\text{tr}(A)}{r^{n+2}}\right) - \frac{C}{n}a_{ij}\nabla_i\nabla_j\delta(\mathbf{x}) - \frac{\text{tr}(A)\,C}{n}\boldsymbol{\nabla}^2\delta(\mathbf{x}) \ . \tag{5.29}$$

5.4. General second order derivatives

The distributional derivatives $\overline{\nabla}_i\overline{\nabla}_j\mathcal{P}f(P_l)$ can be obtained by iteration of (5.15) and (5.16) or, as we now explain, by projecting the formula for the thick derivatives [29, Prop. 3.5]. The thick derivatives have a non-zero delta part only if $l \in \mathbb{Z}$, and in this case,

$$\nabla_i^* \nabla_j^* \mathcal{P}f\left(P_l\right) = \mathcal{P}f\left(\nabla_i \nabla_j P_l\right) + C\left(\delta_{ij} p - (2l+2) p n_i n_j + n_i \frac{\delta p}{\delta x_j} + n_j \frac{\delta p}{\delta x_i}\right) \delta_*^{[2-n+l]}, \tag{5.30}$$

or since $\delta p/\delta x_i = p,_i - n_i dp/dn = p,_i - k n_i p$,

$$\nabla_i^* \nabla_j^* \mathcal{P}f\left(P_l\right) = \mathcal{P}f\left(\nabla_i \nabla_j P_l\right) + C\left(\delta_{ij} p - (2l + +2k+2) p n_i n_j + n_i p,_j + n_j p,_i\right) \delta_*^{[2-n+l]}. \tag{5.31}$$

Notice that the ordinary part $\nabla_i \nabla_j P_l$ is given by

$$\frac{r^4 p,_{ij}(\mathbf{x}) - r^2 (k+l)\left(\delta_{ij} p(\mathbf{x}) + x_i p,_j(\mathbf{x}) + x_j p,_i(\mathbf{x})\right) + (k+l)(k+l+2) x_i x_j p(\mathbf{x})}{r^{k+l+4}} . \tag{5.32}$$

Employing the Propositions 5.1 and 5.2 we obtain that the distributional derivatives $\overline{\nabla}_i \overline{\nabla}_j \mathcal{P}f\left(P_l\right)$ have a non-zero delta part precisely when $l = k + 2m + n - 2$ for some m with $m + 1 \in \mathbb{N}$, and in this case we obtain the ensuing formulas.

Proposition 5.8. *Let $p \in \mathcal{H}_k$. Then the delta part of $\overline{\nabla}_i \overline{\nabla}_j \mathcal{P}f\left(p(\mathbf{x})/r^{k+2m+n-2}\right)$ is given as*

$$\begin{aligned}\Big\{ Z^I p,_{ij}(\boldsymbol{\nabla}) \nabla^{2m+2} + Z^{II}\left(p,_i(\boldsymbol{\nabla}) \nabla_j + p,_i(\boldsymbol{\nabla}) \nabla_j\right) \nabla^{2m} \\ + Z^{II} \delta_{ij} p(\boldsymbol{\nabla}) \nabla^{2m} + Z^{III} \nabla_i \nabla_j p(\boldsymbol{\nabla}) \nabla^{2m-2} \Big\} \delta(\mathbf{x}) \ ,\end{aligned} \tag{5.33}$$

for $m + 1 \in \mathbb{N}$, where

$$Z^I = \frac{2(-1)^k C}{W_{n,k,m+1}} \ , \tag{5.34}$$

$$Z^{II} = \frac{(-1)^{k+1}(n+2k+2m-2) C}{W_{n,k,m}(n+2k+2m)} \ , \tag{5.35}$$

$$Z^{III} = \frac{(-1)^{k+1} 4m (n+2k+2m-1) C}{W_{n,k,m}(n+2k+2m)} \ . \tag{5.36}$$

Proof. We employ (5.31) with $l = k + 2m + n - 2$ and the projection formulas (5.5), (5.9), and (5.10) and simplify. □

Observe that when we put $i = j$ and sum in (5.33) we recover the delta part of (5.23).

It is interesting that the formulas for the distributional Laplacian of $p(\mathbf{x})/r^{k+2m+n-2}$ give a non-zero delta part when $m \in \mathbb{N}$, but one may obtain a non-zero delta part for $\overline{\nabla}_i \overline{\nabla}_j \mathcal{P}f\left(p(\mathbf{x})/r^{k+2m+n-2}\right)$, namely $Z^I p,_{ij}(\boldsymbol{\nabla}) \delta(\mathbf{x})$, even for $m = -1$. For example, $\overline{\Delta}\text{p.v.}\left(x_1 x_2/r^n\right)$ does not have a delta part, but the delta part of $\overline{\nabla}_2 \overline{\nabla}_1 \text{p.v.}\left(x_1 x_2/r^n\right)$ is $2C\delta(\mathbf{x})/W_{n,2,0} = 2C\delta(\mathbf{x})/n(n+2)$. It is instructive to derive this last expression by iteration of (5.15) and (5.16); indeed,

$$\overline{\nabla}_1 \text{p.v.}\left(\frac{x_1 x_2}{r^n}\right) = \mathcal{P}f\left(\frac{r^2 x_2 - n x_1^2 x_2}{r^{n+2}}\right) . \tag{5.37}$$

We cannot apply (5.16) to compute $\overline{\nabla}_2$ of this expression since the third order polynomial $r^2 x_2 - n x_1^2 x_2$ is not harmonic, but we can write

$$r^2 x_2 - n x_1^2 x_2 = \frac{2 r^2 x_2}{n+2} + p_3 \ , \ \text{where } p_3 = -n\left(x_1^2 x_2 - \frac{1}{n+2} r^2 x_2\right) \in \mathcal{H}_3 \ , \tag{5.38}$$

and now employ (5.16) to obtain the delta part of $\overline{\nabla}_2\overline{\nabla}_1\text{p.v.}\,(x_1x_2/r^n)$ as $(2/(n+2))$ times the delta part of $\overline{\nabla}_2\text{p.v.}\,(x_2/r^n)$, which (5.19) gives as $(2/(n+2))$ times $C\delta\,(\mathbf{x})\,/n$, plus the delta part of $\overline{\nabla}_2\mathcal{P}f\,\left(p_3\,(\mathbf{x})\,/r^{n+2}\right)$, which is 0. Summarizing, the delta part of $\overline{\nabla}_2\overline{\nabla}_1\text{p.v.}\,(x_1x_2/r^n)$ is $2C\delta\,(\mathbf{x})\,/n\,(n+2)$, in agreement with (5.33).

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