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# THE FOURIER TRANSFORM OF THICK DISTRIBUTIONS

RICARDO ESTRADA, JASSON VINDAS, AND YUNYUN YANG

ABSTRACT. We first construct a space  $\mathcal{W}(\mathbb{R}_c^n)$  whose elements are test functions defined in  $\mathbb{R}_c^n = \mathbb{R}^n \cup \{\infty\}$ , the one point compactification of  $\mathbb{R}^n$ , that have a thick expansion at infinity of special logarithmic type, and its dual space  $\mathcal{W}'(\mathbb{R}_c^n)$ , the space of  $sl$ -thick distributions. We show that there is a canonical projection of  $\mathcal{W}'(\mathbb{R}_c^n)$  onto  $\mathcal{S}'(\mathbb{R}^n)$ . We study several  $sl$ -thick distributions and consider operations in  $\mathcal{W}'(\mathbb{R}_c^n)$ .

We define and study the Fourier transform of thick test functions of  $\mathcal{S}_*(\mathbb{R}^n)$  and thick tempered distributions of  $\mathcal{S}'_*(\mathbb{R}^n)$ . We construct isomorphisms

$$\begin{aligned}\mathcal{F}_* &: \mathcal{S}'_*(\mathbb{R}^n) \longrightarrow \mathcal{W}'(\mathbb{R}_c^n), \\ \mathcal{F}^* &: \mathcal{W}'(\mathbb{R}_c^n) \longrightarrow \mathcal{S}'_*(\mathbb{R}^n),\end{aligned}$$

that extend the Fourier transform of tempered distributions, namely,  $\Pi\mathcal{F}_* = \mathcal{F}\Pi$  and  $\Pi\mathcal{F}^* = \mathcal{F}\Pi$ , where  $\Pi$  are the canonical projections of  $\mathcal{S}'_*(\mathbb{R}^n)$  or  $\mathcal{W}'(\mathbb{R}_c^n)$  onto  $\mathcal{S}'(\mathbb{R}^n)$ .

We determine the Fourier transform of several finite part regularizations and of general thick delta functions.

## 1. INTRODUCTION

The aim of this article is to construct the Fourier transform of *thick tempered distributions* in several variables. Thick distributions were introduced in one variable in [12] and in several variables in [36, 37, 38, 39]. Thick distributions have found applications in understanding problems in several areas, such as quantum field theory [5], engineering [26, 34], the understanding of singularities in mathematical physics as considered in [4] or in [6], or in obtaining formulas for the regularization of multipoles [8, 25] that play a fundamental role in the ideas of the late professor Stora on convergent Feynman amplitudes [24, 33]. They also appear in other problems, as generalizations of Frahm formulas [16] involving discontinuous test functions [17, 37]. Thick distributions are the distributional theory corresponding to the theory given by Blanchet and Faye [2], whose aim is the study of the dynamics of point particles in high post-Newtonian approximations of general relativity [3] and who develop such a scheme in the context of finite parts, pseudo-functions and Hadamard regularization, as studied by Sellier [30, 31]. In this article we consider spaces with one thick point, located at the origin, but it is possible to consider spaces with a finite number of such singular points.

The Fourier transform of one-dimensional thick distributions with one special point at the origin was given in [12]. The transform of thick distributions is shown to belong to a space  $\mathcal{W}'$  of distributions on the space  $\mathbb{R}_c = \mathbb{R} \cup \{\infty\}$ , the one point compactification of

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the real line. Employing this thick Fourier transform it is possible to understand several puzzles, particularly those found in [5].

The theory of thick distributions in higher dimensions [36] is quite different from that in one dimension, because the topology of  $\mathbb{R}^n \setminus \{\mathbf{0}\}$ ,  $n \geq 2$ , is quite unlike that of  $\mathbb{R} \setminus \{0\}$ , since the latter space is disconnected, consisting of two unrelated rays, while the former is connected, all directions of approach to the point  $\mathbf{0}$  are related, and such behavior imposes strong restrictions on the singularities. Therefore the thick Fourier transform in several variables cannot be constructed as a straightforward extension of the transform in one variable; such construction in several variables, the Fourier transform in  $\mathcal{S}'_*(\mathbb{R}^n)$ , is the main aim of this article.

In Section 2 we review some useful results from the theory of thick distributions and then in Section 3 we collect the Fourier transform of several tempered distributions in order to find the asymptotic expansion of the Fourier transform of finite part regularizations of thick test functions. Taking into account the asymptotic behavior of such Fourier transforms we construct a space  $\mathcal{W}(\mathbb{R}^n_c)$  whose elements are test functions defined in  $\mathbb{R}^n_c = \mathbb{R}^n \cup \{\infty\}$ , the one point compactification of  $\mathbb{R}^n$ , that have a thick expansion at infinity of special logarithmic type. We are thus able in Section 4 to define Fourier transform operators  $\mathcal{F}_{*,t}$  and  $\mathcal{F}_t^*$ , topological isomorphism of  $\mathcal{S}_*(\mathbb{R}^n)$  to  $\mathcal{W}(\mathbb{R}^n_c)$  and from  $\mathcal{W}(\mathbb{R}^n_c)$  to  $\mathcal{S}_*(\mathbb{R}^n)$ , respectively; the subscript 't' is used because these are the transforms of test functions.

We study the dual space  $\mathcal{W}'(\mathbb{R}^n_c)$ , the space of  $sl$ -thick distributions in Section 5. We consider the basic operations in  $\mathcal{W}'(\mathbb{R}^n_c)$ , such as linear changes of variables, derivatives, and multiplication by polynomials. We study several  $sl$ -thick distributions, particularly the finite part regularization at infinity of power functions and thick delta functions at infinity. We are therefore able in Section 6 to define and study the Fourier transform of thick test tempered distributions of  $\mathcal{S}'_*(\mathbb{R}^n)$ . We construct isomorphisms

$$\begin{aligned} \mathcal{F}_* &: \mathcal{S}'_*(\mathbb{R}^n) \longrightarrow \mathcal{W}'(\mathbb{R}^n_c), \\ \mathcal{F}^* &: \mathcal{W}'(\mathbb{R}^n_c) \longrightarrow \mathcal{S}'_*(\mathbb{R}^n), \end{aligned}$$

that extend the Fourier transform of tempered distributions, namely,

$$\Pi_{\mathcal{W}',\mathcal{S}'}\mathcal{F}_* = \mathcal{F}\Pi_{\mathcal{S}'_*,\mathcal{S}'}, \quad \Pi_{\mathcal{S}'_*,\mathcal{S}'}\mathcal{F}^* = \mathcal{F}\Pi_{\mathcal{W}',\mathcal{S}'},$$

where  $\Pi_{\mathcal{W}',\mathcal{S}'}$  and  $\Pi_{\mathcal{S}'_*,\mathcal{S}'}$  are the canonical projections of  $\mathcal{S}'_*(\mathbb{R}^n)$  or  $\mathcal{W}'(\mathbb{R}^n_c)$  onto  $\mathcal{S}'(\mathbb{R}^n)$ . We give the transformation rules for the Fourier transform of derivatives, multiplications, and linear changes of variables, as well as the Fourier inversion formulas. We determine the Fourier transform of several finite part regularizations and of general thick delta functions.

Since we need to employ many spaces, operators, and distributions, following advise of one referee, we also included an appendix that lists the many notations used.

## 2. PRELIMINARIES

We shall use basic facts about distributions and general functional analysis as can be found in the textbooks [14, 21, 22, 29, 32]. However, in this section we recall several recently introduced or not so well known ideas that will be needed in our analysis. We will also fix the notation employed; particularly, we use the Fourier transform

$$F\{f(\mathbf{x}); \mathbf{u}\} = \int_{\mathbb{R}^n} f(\mathbf{x}) e^{i\mathbf{x}\cdot\mathbf{u}} d\mathbf{x}.$$

We will write

$$(2.1) \quad c_{m,n} = \frac{2\Gamma(m+1/2)\pi^{(n-1)/2}}{\Gamma(m+n/2)} = \int_{\mathbb{S}} \omega_j^{2m} d\sigma(\omega), \quad C = c_{0,n}.$$

Notice that  $c_{0,n} = C = 2\pi^{n/2}/\Gamma(n/2)$ , is the surface area of the unit sphere  $\mathbb{S}$  of  $\mathbb{R}^n$ , denoted as  $C_{n-1}$  in [36].

**2.1. Spaces of thick test functions and spaces of thick distributions.** The construction of the space of thick test functions  $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$  and its dual,  $\mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n)$ , the space of thick distributions is as follows [36]. Let  $\mathbf{a}$  be a fixed point of  $\mathbb{R}^n$ . Let  $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$  denote the vector space of all smooth functions  $\phi$  defined in  $\mathbb{R}^n \setminus \{\mathbf{a}\}$ , with support of the form  $K \setminus \{\mathbf{a}\}$ , where  $K$  is compact in  $\mathbb{R}^n$ , that admit a strong asymptotic expansion of the form

$$(2.2) \quad \phi(\mathbf{a} + \mathbf{x}) = \phi(\mathbf{a} + r\mathbf{w}) \sim \sum_{j=m}^{\infty} a_j(\mathbf{w}) r^j, \quad \text{as } \mathbf{x} \rightarrow \mathbf{0},$$

where  $m \in \mathbb{Z}$ . We denote  $\mathcal{D}_{*,\mathbf{0}}(\mathbb{R}^n)$  as  $\mathcal{D}_*(\mathbb{R}^n)$ . The space  $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$  has a natural topology that makes it a complete locally convex topological vector space [36].

**Definition 2.1.** The space of distributions on  $\mathbb{R}^n$  with a thick point at  $\mathbf{x} = \mathbf{a}$  is the dual space of  $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ . We denote it by  $\mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n)$ , or just as  $\mathcal{D}'_*(\mathbb{R}^n)$  when  $\mathbf{a} = \mathbf{0}$ .

In general, we shall denote by  $\Pi$  canonical projections, say from  $E$  to  $F$ , if they exist but as  $\Pi_{E,F}$  when we would like to emphasize the spaces. In particular we will need the projection operator  $\Pi = \Pi_{\mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n), \mathcal{D}'(\mathbb{R}^n)} : \mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R}^n)$ , dual of the inclusion  $i : \mathcal{D}(\mathbb{R}^n) \rightarrow \mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ . Observe that  $\mathcal{D}(\mathbb{R}^n)$ , the space of standard test functions, is a closed subspace of  $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ .

Typical elements of  $\mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n)$  are the finite part regularizations considered in Definition 2.4 and the thick delta functions of order  $q$ ,  $g(\mathbf{w}) \delta_*^{[q]}(\mathbf{x} - \mathbf{a})$  for  $g \in \mathcal{D}'(\mathbb{S})$  given as

$$(2.3) \quad \langle g(\mathbf{w}) \delta_*^{[q]}(\mathbf{x} - \mathbf{a}), \phi \rangle = \frac{1}{C} \langle g(\mathbf{w}), a_q(\mathbf{w}) \rangle,$$

if  $\phi \in \mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$  has the development (2.2). When  $g = 1$  they are called plain thick delta functions.

We refer to [36] for the definition of the basic operations on thick distributions, like derivatives, changes of variables, and multiplication by smooth functions. In general ordinary derivatives are denoted as  $\nabla_i$ , distributional derivatives are denoted as  $\overline{\nabla}_i$ , following [15], while thick distributional derivatives are denoted as  $\nabla_i^*$ .

**2.1.1. Other spaces of thick distributions.** Let  $\mathcal{A}(\mathbb{R}^n)$  be a space of test functions in  $\mathbb{R}^n$  and let  $\mathcal{A}'(\mathbb{R}^n)$  be the corresponding space of distributions<sup>1</sup>. Our aim in this section is to construct the spaces of thick test functions and distributions,  $\mathcal{A}_{*,\mathbf{a}}(\mathbb{R}^n)$  and  $\mathcal{A}'_{*,\mathbf{a}}(\mathbb{R}^n)$ . Our construction will apply in multiple cases. For instance,  $\mathcal{A}(\mathbb{R}^n)$  can be  $\mathcal{E}(\mathbb{R}^n)$ , the space of all smooth functions and thus  $\mathcal{A}'(\mathbb{R}^n)$  becomes  $\mathcal{E}'(\mathbb{R}^n)$ , the space of distributions with compact support; or  $\mathcal{A}(\mathbb{R}^n)$  can be  $\mathcal{S}(\mathbb{R}^n)$ , so that  $\mathcal{A}'(\mathbb{R}^n)$  becomes the space of tempered distributions  $\mathcal{S}'(\mathbb{R}^n)$ . The case of  $\mathcal{K}(\mathbb{R}^n)$  and  $\mathcal{K}'(\mathbb{R}^n)$  played a central role in the asymptotic analysis of thick distributions [39].

<sup>1</sup>In the sense of Zemanian [40]; we assume that  $\mathcal{D}(\mathbb{R}^n) \subset \mathcal{A}(\mathbb{R}^n) \subset \mathcal{E}(\mathbb{R}^n)$  densely and continuously and that differentiation is a continuous map of  $\mathcal{A}(\mathbb{R}^n)$ .

**Definition 2.2.** Let  $\mathcal{A}(\mathbb{R}^n)$  be a space of test functions in  $\mathbb{R}^n$ . The space  $\mathcal{A}_{*,\mathbf{a}}(\mathbb{R}^n)$  consists of those functions  $\phi$  defined in  $\mathbb{R}^n \setminus \{\mathbf{a}\}$  that can be written as  $\phi_1 + \phi_2$ , where  $\phi_1 \in \mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$  and where  $\phi_2 \in \mathcal{A}(\mathbb{R}^n)$ . The topology of  $\mathcal{A}_{*,\mathbf{a}}(\mathbb{R}^n)$  is the finest topology induced by the map  $A : \mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n) \times \mathcal{A}(\mathbb{R}^n) \rightarrow \mathcal{A}_{*,\mathbf{a}}(\mathbb{R}^n)$ ,  $A(\phi_1, \phi_2) = \phi_1 + \phi_2$ . The space of thick distributions  $\mathcal{A}'_{*,\mathbf{a}}(\mathbb{R}^n)$  is the corresponding dual space.

The topology of  $\mathcal{A}_{*,\mathbf{a}}(\mathbb{R}^n)$  can actually be described in several ways. Suppose for instance that  $\rho \in \mathcal{D}(\mathbb{R}^n)$  is a test function that satisfies that  $\rho(\mathbf{x}) = 1$  in a neighborhood of  $\mathbf{x} = \mathbf{a}$ . If  $\|\cdot\|_1$  is a continuous seminorm of  $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$  while  $\|\cdot\|_2$  is a continuous seminorm of  $\mathcal{A}(\mathbb{R}^n)$ , then  $\|\phi\| = \max\{\|\rho\phi\|_1, \|(1-\rho)\phi\|_2\}$  is a continuous seminorm of  $\mathcal{A}_{*,\mathbf{a}}(\mathbb{R}^n)$  and the collection of seminorms so constructed form a basis for the continuous seminorms of  $\mathcal{A}_{*,\mathbf{a}}(\mathbb{R}^n)$ . The elements of  $\mathcal{A}_{*,\mathbf{a}}(\mathbb{R}^n)$  can be described as those smooth functions defined in  $\mathbb{R}^n \setminus \{\mathbf{a}\}$  that show the behavior of thick test functions near  $\mathbf{x} = \mathbf{a}$  while at infinity show the behavior of the elements of  $\mathcal{A}(\mathbb{R}^n)$ . Similar considerations apply to the dual spaces.

Naturally one may consider spaces of thick test functions and thick distributions on smooth manifolds. In particular, considering the one point compactification  $\mathbb{R}_c^n = \mathbb{R}^n \cup \{\infty\}$ , that can be identified with a sphere in dimension  $n + 1$ , we obtain  $\mathcal{D}_{*,\infty}(\mathbb{R}_c^n)$ , the space of smooth functions in  $\mathbb{R}^n$  with a thick point at  $\infty$ , namely, smooth functions  $\phi$  such that  $\psi(\mathbf{x}) = \phi(\mathbf{x}/|\mathbf{x}|^2)$  has a thick point at the origin. We can also consider another simple modification of thick test functions, namely, by considering functions whose expansion at the thick point is given not in terms of the asymptotic sequence  $\{r^j\}$  but in terms of another asymptotic sequence. The topology of such spaces can be constructed in a completely analogous fashion. In this article we will need to consider test functions with expansions in terms of the sequence  $\{r^j \ln r, r^j\}$ , the space  $\mathcal{W}_{\text{pre}}(\mathbb{R}^n)$  of the Definition 4.1.

**2.2. Finite parts.** Let us now recall the notion of the finite part of a limit [14, Section 2.4]. Let  $X$  be a topological space, and let  $x_0 \in X$ . Suppose  $\mathfrak{F}$ , the basic functions, is a family of strictly positive functions defined for  $x \in V \setminus \{x_0\}$ , where  $V$  is a neighborhood of  $x_0$ , such that all of them tend to infinity at  $x_0$  and such that, given two different elements  $f_1, f_2 \in \mathfrak{F}$ , then  $\lim_{x \rightarrow x_0} f_1(x)/f_2(x)$  is either 0 or  $\infty$ .

**Definition 2.3.** Let  $G(x)$  be a function defined for  $x \in V \setminus \{x_0\}$  with  $\lim_{x \rightarrow x_0} G(x) = \infty$ . The finite part of the limit of  $G(x)$  as  $x \rightarrow x_0$  with respect to  $\mathfrak{F}$  exists and equals  $A$  if we can write<sup>2</sup>  $G(x) = G_1(x) + G_2(x)$ , where  $G_1$ , the *infinite part*, is a linear combination of the basic functions and where  $G_2$ , the *finite part*, has the property that the limit  $A = \lim_{x \rightarrow x_0} G_2(x)$  exists. We then employ the notation  $\text{F.p.}_{\mathfrak{F}} \lim_{x \rightarrow x_0} G(x) = A$ . The Hadamard finite part limit corresponds to the case when  $x_0 = 0$  and  $\mathfrak{F}$  is the family of functions  $x^{-\alpha} |\ln x|^\beta$ , where  $\alpha > 0$  and  $\beta \geq 0$  or where  $\alpha = 0$  and  $\beta > 0$ , or when  $x_0 = \infty$  and  $\mathfrak{F}$  is the family of functions  $x^\alpha |\ln x|^\beta$ , where  $\alpha > 0$  and  $\beta \geq 0$  or where  $\alpha = 0$  and  $\beta > 0$ . We then use the simpler notations  $\text{F.p.} \lim_{x \rightarrow 0^+} G(x)$  or  $\text{F.p.} \lim_{x \rightarrow \infty} G(x)$ .

Consider now a function  $f$  defined in  $\mathbb{R}^n$  that may or may not be integrable over the whole space but which is integrable in the region  $|\mathbf{x}| > \varepsilon$  for any  $\varepsilon > 0$ . Then the *radial*

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<sup>2</sup>Such a decomposition, if it exists, is *unique* since any finite number of elements of  $\mathfrak{F}$  has to be linearly independent.

finite part integral is defined as

$$\text{F.p.} \int_{\mathbb{R}^n} f(\mathbf{x}) \, d\mathbf{x} = \text{F.p.} \lim_{\varepsilon \rightarrow 0^+} \int_{|\mathbf{x}| > \varepsilon} f(\mathbf{x}) \, d\mathbf{x},$$

if the finite part limit exists. The notion of finite part integrals and its name were introduced by Hadamard [19], who used them in his study of fundamental solutions of partial differential equations.

**Definition 2.4.** If  $g$  is a locally integrable function in  $\mathbb{R}^n \setminus \{\mathbf{0}\}$  such that the radial finite part integral of  $g\phi$  exists for each  $\phi$  belonging to a space of thick test functions  $\mathcal{A}_*(\mathbb{R}^n)$ , then we can define a thick<sup>3</sup> distribution  $\mathcal{P}f\{g(\mathbf{z}); \mathbf{x}\} = \mathcal{P}f(g) \in \mathcal{A}'_*(\mathbb{R}^n)$  as

$$\langle \mathcal{P}f\{g(\mathbf{z}); \mathbf{x}\}, \phi(\mathbf{x}) \rangle = \langle \mathcal{P}f(g), \phi \rangle = \text{F.p.} \int_{\mathbb{R}^n} g(\mathbf{x}) \phi(\mathbf{x}) \, d\mathbf{x}.$$

The notation  $\mathcal{P}f(f(\mathbf{x}))$  was introduced by Schwartz [29, Chp. 2, §2], who called it a pseudofunction, a term that is still in use.

A particularly important case of finite part limits is the finite part of a meromorphic  $f$  function at a pole  $\omega$ , which is exactly the value of the regular part of  $f$ , say  $g$ , at the pole:  $\text{F.p.} \lim_{\lambda \rightarrow \omega} f(\lambda) = g(\omega)$ .

**Example 2.5.** Let  $x_0 > 0$  and let  $\varphi$  be a continuous function in  $[x_0, \infty)$ , that satisfies the asymptotic relation  $\varphi(x) = Ax^\beta + Bx^\beta \ln x + o(x^{-\infty})$  as  $x \rightarrow \infty$ . The finite part integral  $F(\lambda) = \text{F.p.} \int_{x_0}^{\infty} x^\lambda \varphi(x) \, dx$  exists for all  $\lambda \in \mathbb{C}$ , and  $F$  will be a meromorphic function, with a double pole at  $\lambda = -\beta - 1$ , with singular part  $B(\lambda + \beta + 1)^{-2} - A(\lambda + \beta + 1)^{-1}$ , and the finite part is

$$(2.4) \quad \text{F.p.} \lim_{\lambda \rightarrow -\beta - 1} F(\lambda) = \text{F.p.} \int_{x_0}^{\infty} x^{-\beta - 1} \varphi(x) \, dx.$$

Notice that we have two very different finite part limits in (2.4) and in this case they give the same result; in fact, that is usually true with radial finite part integrals [39] but not otherwise [15, 35].

### 3. SOME FOURIER TRANSFORMS

We need the Fourier transform of several distributions in  $\mathbb{R}^n$  for later use, especially transforms of the type  $\mathcal{F}\{\mathcal{P}f(r^{-N})a(\mathbf{w}); \mathbf{u}\}$  where  $\mathbf{x} = r\mathbf{w}$  are polar coordinates, and where  $a$  is a smooth function defined on the unit sphere  $\mathbb{S}$ . The formulas for such transforms are available [28] but we preferred to present a self consistent approach to their derivation since these ideas will be useful when considering the Fourier transform of thick distributions.

We start with the case when  $a = 1$ . In this case it is well known that

$$(3.1) \quad \mathcal{F}\{r^\lambda; \mathbf{u}\} = \frac{\pi^{n/2} 2^{\lambda+n} \Gamma\left(\frac{\lambda+n}{2}\right) s^{-\lambda-n}}{\Gamma\left(-\frac{\lambda}{2}\right)},$$

whenever  $\lambda \neq -n, -n-2, -n-4, \dots$  [14, 21, 22, 29]. Here  $\mathbf{u} = s\mathbf{v}$  are polar coordinates. Observe that  $\mathcal{F}\{r^\lambda; \mathbf{u}\}$  is in fact analytic at  $\lambda = 0, 2, 4, \dots$ , so that the right side can be

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<sup>3</sup>The same notation is employed for standard distributions.

computed as a limit –employing (3.2) –, namely,

$$\mathcal{F}\{r^{2q}; \mathbf{u}\} = \lim_{\lambda \rightarrow 2q} \frac{\pi^{n/2} 2^{\lambda+n} \Gamma\left(\frac{\lambda+n}{2}\right) s^{-\lambda-n}}{\Gamma\left(-\frac{\lambda}{2}\right)} = (2\pi)^n (-1)^q \nabla^{2q} \delta(\mathbf{u}) .$$

This is of course the result we would obtain if we use that  $\mathcal{F}\{1; \mathbf{u}\} = (2\pi)^n \delta(\mathbf{u})$  and that  $\mathcal{F}\{r^{2q} f(\mathbf{x}); \mathbf{u}\} = (-1)^q \nabla^{2q} \mathcal{F}\{f(\mathbf{x}); \mathbf{u}\}$ . Next, let us now find  $\mathcal{F}\{\mathcal{P}f(r^{-n-2m}); \mathbf{u}\}$  for  $m = 0, 1, 2, \dots$ . We have

$$(3.2) \quad r^\lambda = \frac{c_{m,n} \nabla^{2m} \delta(\mathbf{x})}{(2m)! (\lambda + 2m + n)} + \mathcal{P}f\left(\frac{1}{r^{n+2m}}\right) + O(\lambda + 2m + n) ,$$

as  $\lambda \rightarrow -(2m + n)$ , so that we obtain the finite part limit  $\text{F.p.} \lim_{\lambda \rightarrow -(2m+n)} r^\lambda = \mathcal{P}f(r^{-n-2m})$ . Therefore  $\mathcal{F}\{\mathcal{P}f(r^{-n-2m}); \mathbf{u}\}$  equals

$$\text{F.p.} \lim_{\lambda \rightarrow -(2m+n)} \mathcal{F}\{r^\lambda; \mathbf{u}\} = \text{F.p.} \lim_{\lambda \rightarrow -(2m+n)} \frac{\pi^{n/2} 2^{\lambda+n} \Gamma\left(\frac{\lambda+n}{2}\right) s^{-\lambda-n}}{\Gamma\left(-\frac{\lambda}{2}\right)} .$$

This finite part limit is actually already computed in the first edition of [29]. It follows easily from the following lemma [23, 39].

**Lemma 3.1.** *Let  $k \in \mathbb{N}$ . We have that as  $\lambda \rightarrow -k$ ,*

$$\Gamma(\lambda) = \frac{(-1)^k}{k! (\lambda + k)} + \frac{(-1)^k \psi(k+1)}{k!} + O(\lambda + k) ,$$

where  $\psi(\lambda) = \Gamma'(\lambda) / \Gamma(\lambda)$  is the digamma function so that  $\psi(k+1) = \sum_{j=1}^k 1/j - \gamma$ ,  $\gamma$  being Euler's constant. If  $k = 0, 1, 2, \dots$ , and  $f$  is analytic in a neighborhood of  $-k$ ,

$$\text{F.p.} \lim_{\lambda \rightarrow -k} \Gamma(\lambda) f(\lambda) = \frac{(-1)^k \psi(k+1)}{k!} f(-k) + \frac{(-1)^k}{k!} f'(-k) .$$

The ensuing result is therefore obtained.

**Lemma 3.2.** *If  $m = 0, 1, 2, \dots$  then*

$$\mathcal{F}\left\{\mathcal{P}f\left(\frac{1}{r^{n+2m}}\right); \mathbf{u}\right\} = \frac{(-1)^m \pi^{n/2}}{m! \Gamma\left(\frac{n}{2} + m\right)} \left(\frac{s}{2}\right)^{2m} \left\{\psi(m+1) + \psi\left(\frac{n}{2} + m\right) - 2 \ln\left(\frac{s}{2}\right)\right\}$$

Our next task is to find the Fourier transform of distributions of the form  $\mathcal{P}f(r^{-N}) a(\mathbf{w})$  when  $a = \mathbf{Y}_k$  is a spherical harmonic<sup>4</sup> of degree  $k$ .

**Lemma 3.3.** *If  $\mathbf{Y}_k \in \mathcal{H}_k$  and  $\lambda \neq -n - k, -n - k - 2, -n - k - 4, \dots$*

$$(3.3) \quad \mathcal{F}\{r^\lambda \mathbf{Y}_k(\mathbf{w}); s\mathbf{v}\} = \frac{i^k \pi^{n/2} 2^{\lambda+n} \Gamma\left(\frac{k+n+\lambda}{2}\right)}{\Gamma\left(\frac{k-\lambda}{2}\right)} s^{-(\lambda+n)} \mathbf{Y}_k(\mathbf{v}) .$$

*Proof.* Notice that for a general  $k$ , the Fourier transform of  $r^\lambda \mathbf{Y}_k(\mathbf{w})$ , a homogeneous distribution of degree  $\lambda$ , is homogeneous of degree  $-(\lambda + n)$ , so that the Funk-Hecke formula [18, 20] as presented in [9] yields that

$$\mathcal{F}\{r^\lambda \mathbf{Y}_k(\mathbf{w}); s\mathbf{v}\} = C_{k,\lambda} s^{-(\lambda+n)} \mathbf{Y}_k(\mathbf{v}) ,$$

for some constants  $C_{k,\lambda}$  that depend on  $k$  and  $\lambda$  but not otherwise on  $\mathbf{Y}_k$ . Thus, it suffices to show that for each  $k$  (3.3) holds for just *one* spherical harmonic of degree  $k$ . We use

<sup>4</sup>We denote this as  $\mathbf{Y}_k \in \mathcal{H}_k$ . For more on spherical harmonics, see [1, 27].

induction on  $k$ . If  $k = 0$  then (3.3) is exactly (3.1). Let us assume it true for  $k$  and let us prove it for  $k + 1$ . Indeed, we take  $Y_{k+1}(\mathbf{x}) = Y_k(\tilde{\mathbf{x}}) x_n$  where  $\mathbf{x} = (\tilde{\mathbf{x}}, x_n)$ , so that

$$\begin{aligned} \mathcal{F} \{ r^\lambda Y_{k+1}(\mathbf{w}); s\mathbf{v} \} &= -i \frac{\partial}{\partial u_n} \left( \frac{i^k \pi^{n/2} 2^{\lambda-1+n} \Gamma\left(\frac{k+n+\lambda-1}{2}\right)}{\Gamma\left(\frac{k-\lambda+1}{2}\right)} s^{1-\lambda-n} Y_k(\mathbf{v}) \right) \\ &= \frac{-i^{k+1} \pi^{n/2} 2^{\lambda-1+n} \Gamma\left(\frac{k+n+\lambda-1}{2}\right)}{\Gamma\left(\frac{k-\lambda+1}{2}\right)} \frac{\partial}{\partial u_n} (Y_k(\mathbf{u}) s^{-\lambda+1-k-n}) \\ &= \frac{i^{k+1} \pi^{n/2} 2^{\lambda+n} \Gamma\left(\frac{k+1+n+\lambda}{2}\right)}{\Gamma\left(\frac{k+1-\lambda}{2}\right)} s^{-\lambda-n} Y_{k+1}(\mathbf{v}), \end{aligned}$$

as required.  $\square$

Since  $\mathcal{F} \{ r^\lambda Y_k(\mathbf{w}); s\mathbf{v} \}$  is analytic at  $\lambda = k + 2q$ ,  $q = 0, 1, 2, \dots$ , at this value of  $\lambda$  (3.3) is the limit of the expression as  $\lambda \rightarrow k + 2q$ . In fact, using the product formula

$$Y_k(\mathbf{u}) \nabla^{2m} \delta(\mathbf{u}) = \frac{(-1)^k 2^k m!}{(m-k)!} Y_k(\nabla) \nabla^{2m-2k} \delta(\mathbf{u}), \quad m \geq k,$$

from [10, Prop. 3.3], we indeed obtain

$$\begin{aligned} (3.4) \quad \mathcal{F} \{ r^{k+2q} Y_k(\mathbf{w}); s\mathbf{v} \} &= \lim_{\lambda \rightarrow k+2q} \frac{i^k \pi^{n/2} 2^{\lambda+n} \Gamma\left(\frac{k+n+\lambda}{2}\right)}{\Gamma\left(\frac{k-\lambda}{2}\right) s^{\lambda+n}} Y_k(\mathbf{v}) \\ &= \frac{(-i)^k (-1)^q (2\pi)^n (k+q)!}{k!q!} Y_k(\nabla) \nabla^{2q} \delta(\mathbf{u}). \end{aligned}$$

If we now use the Lemma 3.1 as before, we obtain the following formula.

**Lemma 3.4.** *If  $Y_k \in \mathcal{H}_k$  and  $m = 0, 1, 2, \dots$  then*

$$\begin{aligned} &\mathcal{F} \left\{ \mathcal{P}f \left( \frac{1}{r^{n+k+2m}} \right) Y_k(\mathbf{w}); s\mathbf{v} \right\} = \\ &\frac{(-1)^m i^k \pi^{n/2}}{m! \Gamma\left(\frac{n}{2} + k + m\right)} \left(\frac{s}{2}\right)^{2m+k} \left\{ \psi(1+m) + \psi\left(\frac{n}{2} + k + m\right) - 2 \ln\left(\frac{s}{2}\right) \right\} Y_k(\mathbf{v}). \end{aligned}$$

Let now  $a$  be a smooth function on the sphere,  $a \in \mathcal{D}(\mathbb{S})$ . Then we can write it in terms of spherical harmonics as

$$a(\mathbf{w}) = \sum_{m=0}^{\infty} Y_m(\mathbf{w}),$$

where  $Y_m = Y_m\{a\} \in \mathcal{H}_m$  are given as  $Y_m(\mathbf{w}) = \int_{\mathbb{S}} Z_m(\mathbf{w}, \mathbf{v}) a(\mathbf{v}) d\sigma(\mathbf{v})$ ; here  $Z_m(\mathbf{w}, \mathbf{v})$  is the reproducing kernel of  $\mathcal{H}_m$ , namely [1, Thm. 5.38]

$$(n+2m-2) \sum_{q=0}^{\lfloor m/2 \rfloor} (-1)^q \frac{n(n+2) \cdots (n+2m-2q-4)}{2^q q! (m-2q)!} (\mathbf{w} \cdot \mathbf{v})^{m-2q}.$$

We thus obtain the following.



**Proposition 3.5.** *If  $\beta \neq 0, 1, 2, \dots$  then*

$$\mathcal{F} \left\{ \mathcal{P}f \left( \frac{1}{r^{n+\beta}} \right) a(\mathbf{w}); \mathbf{u} \right\} = \mathcal{P}f(s^\beta) \mathcal{K}_\beta \{a(\mathbf{w}); \mathbf{v}\},$$

where  $\mathcal{K}_\beta \{a(\mathbf{w}); \mathbf{v}\} = \langle K_\beta(\mathbf{w}, \mathbf{v}), a(\mathbf{w}) \rangle_{\mathbf{w}}$ , and

$$(3.5) \quad K_\beta(\mathbf{w}, \mathbf{v}) = \sum_{m=0}^{\infty} \kappa_{\beta,m} Z_m(\mathbf{w}, \mathbf{v}), \quad \kappa_{\beta,m} = \frac{i^m \pi^{n/2} 2^{-\beta} \Gamma\left(\frac{m-\beta}{2}\right)}{\Gamma\left(\frac{m+n+\beta}{2}\right)}.$$

The operator  $\mathcal{K}_\beta$  is analytic for  $\beta \neq 0, 1, 2, \dots$ ; for  $\beta = q \in \mathbb{N}$  we have the next formula.

**Proposition 3.6.** *If  $q = 0, 1, 2, \dots$  then*

$$\mathcal{F} \left\{ \mathcal{P}f \left( \frac{1}{r^{n+q}} \right) a(\mathbf{w}); \mathbf{u} \right\} = s^q (\mathcal{K}_q \{a(\mathbf{w}); \mathbf{v}\} + \mathcal{L}_q \{a(\mathbf{w}); \mathbf{v}\} \ln s),$$

where  $\mathcal{K}_q \{a(\mathbf{w}); \mathbf{v}\} = \langle K_q(\mathbf{w}, \mathbf{v}), a(\mathbf{w}) \rangle_{\mathbf{w}}$ ,

$$K_q(\mathbf{w}, \mathbf{v}) = \sum_{m=0}^{\infty} \kappa_{q,m} Z_m(\mathbf{w}, \mathbf{v}),$$

the constants  $\kappa_{q,m}$  being given by (3.5) if  $m \neq q, q-2, \dots$  and as

$$\kappa_{q,q-2m} = \frac{i^q \pi^{n/2} 2^{-q}}{m! \Gamma\left(\frac{n}{2} + q - m\right)} \left\{ \psi(1+m) + \psi\left(\frac{n}{2} + q - m\right) + 2 \ln 2 \right\},$$

for  $0 \leq m \leq \lfloor q/2 \rfloor$ . On the other hand,  $\mathcal{L}_q \{a(\mathbf{w}); \mathbf{v}\} = \langle L_q(\mathbf{w}, \mathbf{v}), a(\mathbf{w}) \rangle_{\mathbf{w}}$ ,

$$(3.6) \quad L_q(\mathbf{w}, \mathbf{v}) = \sum_{m=0}^{\lfloor q/2 \rfloor} \lambda_{q,q-2m} Z_{q-2m}(\mathbf{w}, \mathbf{v}), \quad \lambda_{q,q-2m} = \frac{-i^q 2^{-q+1} \pi^{n/2}}{m! \Gamma\left(\frac{n}{2} + q - m\right)}.$$

**3.1. The operators  $\mathcal{K}_\beta$ .** Notice that  $\mathcal{K}_\beta$  is the analytic continuation of an integral operator  $a \rightsquigarrow \int_{\mathbb{S}} K_\beta(\mathbf{w}, \mathbf{v}) a(\mathbf{w}) d\sigma(\mathbf{w})$ , namely, if  $\Re \beta > 0$  then employing polar coordinates we obtain

$$\mathcal{F} \{r^{-n-\beta} a(\mathbf{w}); \mathbf{u}\} = s^\beta \Gamma(-\beta) e^{-i\pi\beta/2} \int_{\mathbb{S}} a(\mathbf{w}) (\mathbf{w} \cdot \mathbf{v} + i0)^\beta d\sigma(\mathbf{w}),$$

since in dimension 1 [22]  $\mathcal{F} \{x_+^{-1-\beta}; t\} = \Gamma(-\beta) e^{-i\pi\beta/2} (t+i0)^\beta$  if  $\beta \neq 0, 1, 2, \dots$ . Thus,

$$(3.7) \quad K_\beta(\mathbf{w}, \mathbf{v}) = \Gamma(-\beta) e^{-i\pi\beta/2} (\mathbf{w} \cdot \mathbf{v} + i0)^\beta,$$

a distributional kernel for  $\beta \neq 0, 1, 2, \dots$  that becomes an integral operator if  $\Re \beta > 0$ .

Observe that the distribution  $(t+i0)^\beta$  is an entire function of  $\beta$ . The singularity of  $K_\beta(\mathbf{w}, \mathbf{v})$  at  $\beta = q \in \mathbb{N}$  is produced by the term  $\Gamma(-\beta)$ . The formula of the Proposition 3.6 can therefore be derived by computing the finite part of the limit of  $K_\beta(\mathbf{w}, \mathbf{v}) s^\beta$  as  $\beta \rightarrow q$ , because the Lemma 3.1 gives

$$\text{F.p.} \lim_{\lambda \rightarrow -q} \Gamma(\lambda) a^\lambda = \frac{(-1)^q}{q!} (\psi(q+1) + \ln a) a^{-q}.$$

Hence, since  $(\mathbf{w} \cdot \mathbf{v} + i0)^q = (\mathbf{w} \cdot \mathbf{v})^q$ ,

$$(3.8) \quad K_q(\mathbf{w}, \mathbf{v}) = \frac{(-1)^q e^{-i\pi q/2}}{q!} \left( \psi(q+1) + \ln \left( \frac{e^{i\pi/2}}{(\mathbf{w} \cdot \mathbf{v} + i0)} \right) \right) (\mathbf{w} \cdot \mathbf{v})^q,$$

and

$$(3.9) \quad L_q(\mathbf{w}, \mathbf{v}) = \frac{(-1)^q e^{-i\pi q/2}}{q!} (\mathbf{w} \cdot \mathbf{v})^q.$$

It is also interesting to observe the form of  $K_{-m}(\mathbf{w}, \mathbf{v})$  for  $m = 1, 2, 3, \dots$ ,

$$(3.10) \quad \begin{aligned} K_{-m}(\mathbf{w}, \mathbf{v}) &= (m-1)! i^m (\mathbf{w} \cdot \mathbf{v} + i0)^{-m} \\ &= (m-1)! i^m (\mathbf{w} \cdot \mathbf{v})^{-m} - \pi (-i)^{m+1} \delta^{(m-1)}(\mathbf{w} \cdot \mathbf{v}) \\ &= -2\pi (-i)^{m+1} \delta^{+(m-1)}(\mathbf{w} \cdot \mathbf{v}), \end{aligned}$$

where  $\delta^{+(m-1)}(x)$  is the Heisenberg delta function [14, (2.61), (2.63)].

If  $\beta \neq 0, 1, 2, \dots$ , the coefficients  $\kappa_{\beta, m}$  never vanish for  $\beta \neq -n - q$ ,  $q = 0, 1, 2, \dots$ , but they could vanish for some  $m$  when  $\beta = -n - q$ , so that the operator  $\mathcal{K}_\beta$  is an isomorphism of  $\mathcal{D}(\mathbb{S})$  for  $\beta \neq -n - q$ , but  $\mathcal{K}_{-n-q}(\mathcal{D}(\mathbb{S}))$  is a subspace of finite codimension of  $\mathcal{D}(\mathbb{S})$ . The Fourier inversion formula yields the inverses of the operators  $\mathcal{K}_\beta$  for  $\beta \in \mathbb{C} \setminus \mathbb{Z}$  or for  $\beta \in \{1 - n, 2 - n, \dots, -1\}$  as

$$(3.11) \quad \mathcal{K}_\beta^{-1} \{A(\mathbf{v}); \mathbf{w}\} = \frac{1}{(2\pi)^n} \mathcal{K}_{-n-\beta} \{A(\mathbf{v}); -\mathbf{w}\}, \quad A \in \mathcal{D}(\mathbb{S}).$$

**3.2. The operators  $\mathfrak{K}_q$  and  $\mathfrak{L}_q$ .** It is convenient to consider a variant of the operators  $\mathcal{K}_\beta$  in case  $\beta \in \mathbb{Z}$ . Let us start with some notation. If  $q \in \mathbb{N}$  we denote as  $\mathcal{P}_q$  the space of restrictions of homogeneous polynomials of degree  $q$  to  $\mathbb{S}$ , that is  $\mathcal{P}_q = \mathcal{H}_q \oplus \mathcal{H}_{q-2} \oplus \mathcal{H}_{q-4} \oplus \dots$ . Let  $\mathcal{X}$  be a space of functions or generalized functions over  $\mathbb{S}$ , as  $\mathcal{D}(\mathbb{S})$ ,  $L^2(\mathbb{S})$ , or  $\mathcal{D}'(\mathbb{S})$ , that equals the closure in  $\mathcal{X}$  of the sum  $\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots$ <sup>5</sup>. Then  $\mathcal{X}_q$  is the space  $\mathcal{X}$  if  $1 - n \leq q \leq -1$ ; if  $q \geq 0$ ,  $\mathcal{X}_q$  is the sum  $\widehat{\bigoplus}_{\mathcal{X}} \big|_{m \neq q, q-2, \dots} \mathcal{H}_m$ , while if  $q \leq -n$  then  $\mathcal{X}_q = \mathcal{X}_{-n-q}$ . Notice that

$$\mathcal{X}_q \oplus \mathcal{P}_{-n-q} = \mathcal{X}, \quad q \leq -n, \quad \mathcal{X}_q \oplus \mathcal{P}_q = \mathcal{X}, \quad q \geq 0.$$

We define the operators  $\mathfrak{K}_q : \mathcal{D}_q \rightarrow \mathcal{D}_q$  as  $\Pi \mathcal{K}_{q\iota}$ , where  $\iota$  is the canonical injection of  $\mathcal{D}_q$  into  $\mathcal{D}(\mathbb{S})$  and  $\Pi$  the canonical projection of  $\mathcal{D}(\mathbb{S})$  onto  $\mathcal{D}_q$ . We can also consider the  $\mathfrak{K}_q$  as operators from  $\mathcal{D}'_q$  to itself, by duality or employing the expansion (3.5). The Propositions 3.5 and 3.6 immediately give the ensuing.

**Proposition 3.7.** *The operators  $\mathfrak{K}_q$  are isomorphisms of the space  $\mathcal{X}_q$  to itself for<sup>6</sup>  $\mathcal{X} = \mathcal{D}(\mathbb{S})$  or  $\mathcal{D}'(\mathbb{S})$ . Its inverses are given as*

$$(3.12) \quad \mathfrak{K}_q^{-1} \{A(\mathbf{v}); \mathbf{w}\} = \frac{1}{(2\pi)^n} \mathfrak{K}_{-n-q} \{A(\mathbf{v}); -\mathbf{w}\}, \quad A \in \mathcal{X}_q.$$

Observe that for  $\mathcal{X} = \mathcal{D}(\mathbb{S})$  or  $\mathcal{D}'(\mathbb{S})$  we have  $\mathcal{X}_q = \mathcal{K}_q(\mathcal{X})$ , for  $q < 0$ . This is not true for  $q \geq 0$ , but we have  $\mathcal{X}_q = \Pi \mathcal{K}_q(\mathcal{X})$  where  $\Pi$  is the canonical projection of  $\mathcal{X}$  onto  $\mathcal{X}_q$ .

The operators  $\mathfrak{L}_q : \mathcal{P}_q \rightarrow \mathcal{P}_q$  are defined as  $\Pi \mathcal{L}_{q\iota}$ , where  $\iota$  is the canonical injection of  $\mathcal{P}_q$  into  $\mathcal{D}(\mathbb{S})$  and  $\Pi$  the canonical projection of  $\mathcal{D}(\mathbb{S})$  onto  $\mathcal{P}_q$ . They are isomorphisms of the space  $\mathcal{P}_q$ .

<sup>5</sup>Such closures will be denoted as  $\widehat{\bigoplus}_{\mathcal{X}} \big|_{m=0}^{\infty} \mathcal{H}_m$

<sup>6</sup>The results also holds for  $\mathcal{X} = L^2(\mathbb{S})$ , but we will not need this case presently.

## 4. THE FOURIER TRANSFORM OF THICK TEST FUNCTIONS

In this section we will construct a space  $\mathcal{W}(\mathbb{R}^n)$  such that it is possible to define an operator

$$(4.1) \quad \mathcal{F}_{*,t} : \mathcal{S}_*(\mathbb{R}^n) \longrightarrow \mathcal{W}(\mathbb{R}^n) ,$$

the Fourier transform of *test functions*, which has the expected properties of such a transform.

Let us start by observing that if  $\phi$  is a thick test function in  $\mathbb{R}^n$ , then in general it is *not* locally integrable at the origin, so that, in general, it does not give a unique distribution. Therefore, we cannot imbed  $\mathcal{S}_*(\mathbb{R}^n)$  into  $\mathcal{S}'(\mathbb{R}^n)$  and consequently, if  $\phi \in \mathcal{S}_*(\mathbb{R}^n)$  then in general we cannot define  $\mathcal{F}(\phi)$  as a distribution of the space  $\mathcal{S}'(\mathbb{R}^n)$ <sup>7</sup>. On the other hand, any  $\phi \in \mathcal{S}_*(\mathbb{R}^n)$  does have regularizations  $f \in \mathcal{S}'(\mathbb{R}^n)$ ; however  $f$  is not unique, since if  $f_0$  is a regularization, then so are all distributions of the form  $f_0 + g$ , where  $\text{supp } g \subset \{\mathbf{0}\}$ , that is, where  $g$  is a sum of derivatives of the Dirac delta function at the origin. It will be convenient to use the notation  $\mathcal{S}_{*,\text{reg}}(\mathbb{R}^n)$  for the subspace of  $\mathcal{S}'(\mathbb{R}^n)$  whose elements are the regularizations of thick test functions.

Our first task is then to identify those distributions of the form  $\mathcal{F}(f_0)$  where  $f_0$  is a regularization of a thick test function  $\phi \in \mathcal{S}_*(\mathbb{R}^n)$ . It should be clear that if  $\Phi_0 = \mathcal{F}(f_0)$  for one such regularization of  $\phi$ , then so are all distributions of the form  $\Phi_0 + p$  for any polynomial  $p$  and, conversely, if  $\Phi$  is the Fourier transform of a regularization of  $\phi$  then  $\Phi = \Phi_0 + p$  for some polynomial  $p$ . Observe now that if  $\phi \in \mathcal{S}_*(\mathbb{R}^n)$  then  $\phi$  coincides with a test function of the space  $\mathcal{S}(\mathbb{R}^n)$  outside any ball around the origin, while at the origin it has a strong asymptotic expansion of the form  $\phi(\mathbf{x}) \sim \sum_{m=-M}^{\infty} a_m(\mathbf{w}) r^m$ , as  $r \rightarrow 0^+$ , where  $a_m \in \mathcal{D}(\mathbb{S})$ . We can therefore readily obtain the properties of the Fourier transform  $\Phi_0 = \mathcal{F}(f_0)$  of the pseudofunction  $f_0 = \mathcal{P}f(\phi)$ , the finite part regularization of  $\phi$ . Indeed,  $\Phi_0$  is smooth in all of  $\mathbb{R}^n$ , and our analysis of the Section 3 combined with the techniques of [39] or of [14, Chpt. 4] yield the asymptotic expansion of  $\Phi_0(\mathbf{u})$  as  $|\mathbf{u}| \rightarrow \infty$  as follows: if  $\mathbf{u} = s\mathbf{v}$  are polar coordinates then we have the strong expansion

$$(4.2) \quad \begin{aligned} \Phi_0(s\mathbf{v}) \sim & \sum_{m \leq -n} s^{-m-n} (\mathcal{K}_{-m-n} \{a_m(\mathbf{w}); \mathbf{v}\} + \mathcal{L}_{-m-n} \{a_m(\mathbf{w}); \mathbf{v}\} \ln s) \\ & + \sum_{m=-n+1}^{\infty} s^{-m-n} \mathcal{K}_{-m-n} \{a_m(\mathbf{w}); \mathbf{v}\} , \end{aligned}$$

as  $s \rightarrow \infty$ , uniformly with respect to  $\mathbf{v}$ . Consequently we introduce the space  $\mathcal{W}_{\text{pre}}(\mathbb{R}^n)$ .

**Definition 4.1.** The space  $\mathcal{W}_{\text{pre}}(\mathbb{R}^n)$  consists of those smooth functions  $\Phi$  defined in  $\mathbb{R}^n$  that admit a strong asymptotic expansion of the form

$$(4.3) \quad \Phi(s\mathbf{v}) \sim \sum_{q=0}^Q (A_q(\mathbf{v}) + P_q(\mathbf{v}) \ln s) s^q + \sum_{q=1}^{\infty} A_{-q}(\mathbf{v}) s^{-q} ,$$

where  $A_q \in \mathcal{K}_q(\mathcal{D}(\mathbb{S}))$  for  $q \leq -n$ ,  $A_q \in \mathcal{D}(\mathbb{S})$  for  $q > -n$ , and where the  $P_q \in \mathcal{P}_q$  for  $q \in \mathbb{N}$ . The topology of  $\mathcal{W}_{\text{pre}}(\mathbb{R}^n)$  is constructed as explained in Subsection 2.1.1.

Our analysis so far yields the ensuing result.

<sup>7</sup>It is possible to consider  $\mathcal{F}(\phi)$  as a distribution of the Lizorkin distributional spaces, but for our purposes a different approach is more convenient.

**Theorem 4.2.** *The Fourier transform is an isomorphism of the vector spaces  $\mathcal{S}_{*,\text{reg}}(\mathbb{R}^n)$  and  $\mathcal{W}_{\text{pre}}(\mathbb{R}^n)$ .*

Notice that we have not defined a topology for the space  $\mathcal{S}_{*,\text{reg}}(\mathbb{R}^n)$  yet; once a topology is introduced, we shall see that the Fourier transform is not only an algebraic isomorphism, but actually an isomorphism of topological vector spaces. First, however, we need to consider the notions of delta part and polynomial part of distributions.

**4.1. Delta parts and polynomial parts.** In general it is not possible to separate the contribution to a distribution from a given point; to talk about the “delta part at  $\mathbf{x}_0$ ” of *all* distributions does not make sense. However, *sometimes*, we can actually separate the delta part [8].

**Definition 4.3.** Let  $f_0 \in \mathcal{D}'(\mathbb{R}^n \setminus \{\mathbf{0}\})$  be a distribution defined in the complement of the origin. Suppose the pseudofunction  $\mathcal{P}f(f_0(\mathbf{x}))$  exists in  $\mathcal{D}'(\mathbb{R}^n)$  (respectively in  $\mathcal{D}'_*(\mathbb{R}^n)$ ). Let  $f \in \mathcal{D}'(\mathbb{R}^n)$  (respectively in  $f \in \mathcal{D}'_*(\mathbb{R}^n)$ ) be any regularization of  $f_0$ . Then the delta part at 0 of  $f$  is the distribution  $f - \mathcal{P}f(f_0(\mathbf{x}))$ , whose support is the origin<sup>8</sup>.

It is easy to construct distributions whose delta part is not defined. Indeed, the function  $\sin r^{-k}$  is locally integrable in  $\mathbb{R}^n$ , and thus it gives a well defined regular distribution in  $\mathcal{D}'(\mathbb{R}^n)$ . If  $k > n$ , then the distributional derivative  $(\bar{\nabla}_i) \sin r^{-k}$  is another well defined distribution, but its delta part at the origin is *not* defined, since  $\mathcal{P}f(\nabla_i \sin r^{-k})$  does not exist.

It must be emphasized that even though when it exists  $\mathcal{P}f(f_0(\mathbf{x}))$  is in a way *the* natural regularization of  $f_0$ , other regularizations appear also very naturally, as we now illustrate. Consider the distribution  $\mathcal{P}f(r^{-k})$  in  $\mathbb{R}^n$ . Then the distributional derivative  $\bar{\nabla}_i \mathcal{P}f(r^{-k})$  is a regularization of  $-kx_i r^{-k-2}$ , the ordinary derivative of  $r^{-k}$ ; however [13, (3.16)] if  $k - n = 2m$  is an even positive integer, then

$$(4.4) \quad \bar{\nabla}_i \mathcal{P}f(r^{-k}) = \mathcal{P}f(-kx_i r^{-k-2}) - \frac{c_{m,n}}{(2m)!k} \nabla_i \Delta^m \delta(\mathbf{x}),$$

where  $c_{m,n}$  is given by (2.1). Therefore,  $(-c_{m,n}/(2m)!k) \nabla_i \Delta^m \delta(\mathbf{x})$  is the delta part of the distribution  $\bar{\nabla}_i \mathcal{P}f(r^{-k})$  in  $\mathcal{D}'(\mathbb{R}^n)$ . In the space  $\mathcal{D}'_*(\mathbb{R}^n)$ , now for any integer  $k \in \mathbb{Z}$ , the delta part of the thick derivative  $\nabla_i^* \mathcal{P}f(r^{-k})$  is given [36, Thm.7.1] as  $Cn_i \delta_*^{[-k-n+1]}$ .<sup>9</sup>

Another illustration of “natural” regularizations that differ from the finite part is the following. In  $\mathbb{R}^n$  for  $n \geq 2$ , and for  $m \in \mathbb{N}$ , the distribution  $\lambda^{n+2m} \mathcal{P}f(|\lambda \mathbf{x}|^{-n-2m})$  is a regularization of  $r^{-n-2m}$  and in  $\mathcal{D}'(\mathbb{R}^n)$  its delta part is  $\ln \lambda c_{m,n} \nabla^{2m} \delta(\mathbf{x}) / (2m)!$ , while in  $\mathcal{D}'_*(\mathbb{R}^n)$  its delta part is  $\ln \lambda C \delta_*^{[2m]}$ , as follows from [36, (5.13), (5.14)]. More generally [11] let  $A$  be a non singular  $n \times n$  matrix, let  $b \in \mathcal{D}(\mathbb{S})$ , and put  $B_\alpha(\mathbf{z}) = b(\mathbf{z}/|\mathbf{z}|) |\mathbf{z}|^\alpha$ , the extension to  $\mathcal{D}'(\mathbb{R}^n \setminus \{\mathbf{0}\})$  that is homogeneous of degree  $\alpha$ . Then in  $\mathcal{D}'_*(\mathbb{R}^n)$  we have  $\mathcal{P}f\{B_\alpha(A\mathbf{z}); \mathbf{x}\} = \mathcal{P}f\{B_\alpha(\mathbf{z}); A\mathbf{x}\}$  if  $\alpha \notin \mathbb{Z}$ , but if  $\alpha = k \in \mathbb{Z}$ ,

$$(4.5) \quad \mathcal{P}f\{B_k(A\mathbf{z}); \mathbf{x}\} = \mathcal{P}f\{B_k(\mathbf{z}); A\mathbf{x}\} - Cb(\mathbf{w}) |\mathbf{A}\mathbf{w}|^k \ln |\mathbf{A}\mathbf{w}| \delta_*^{[-k-n]}(\mathbf{x}),$$

so that the distribution  $\mathcal{P}f\{B_k(\mathbf{z}); A\mathbf{x}\}$  has a non trivial delta part while  $\mathcal{P}f\{B_k(\mathbf{z}); \mathbf{x}\}$  does not.

<sup>8</sup>Notice that this delta part is in fact a *spherical* delta part.

<sup>9</sup>Naturally, when  $k - n = 2m \geq 0$ , the projection of the thick delta part is precisely the distributional delta part, and this agrees with [36, (7.7)].

When  $f_0$  is a smooth function defined in  $\mathbb{R}^n \setminus \{\mathbf{0}\}$  such that the Hadamard regularization exists at the origin, and  $f \in \mathcal{D}'(\mathbb{R}^n)$  is a regularization of  $f_0$ , then we call  $f_0$  the *ordinary part* of  $f$ . Thus, for instance,  $-kx_i r^{-k-2}$  is the ordinary part of  $(\nabla_i) \mathcal{P}f(r^{-k})$ .

In a similar fashion, one may consider the *polynomial part* of distributions. Not all distributions have a well defined polynomial part, but all the elements of  $\mathcal{W}_{\text{pre}}(\mathbb{R}^n)$  do. Let us start with the case of a distribution that is homogeneous of degree  $q \geq 0$  in  $\mathbb{R}^n \setminus \{\mathbf{0}\}$ , that is  $F_q(\mathbf{u}) = A_q(\mathbf{v}) s^q$ ,  $\mathbf{u} = s\mathbf{v}$  being polar coordinates and  $A_q \in \mathcal{D}'(\mathbb{S})$ . Then we can write  $A_q$  in terms of spherical harmonics as  $A_q(\mathbf{v}) = \sum_{m=0}^{\infty} Y_{m,q}(\mathbf{v})$ , where  $Y_{m,q} \in \mathcal{H}_m$ . Therefore

$$(4.6) \quad F_q(\mathbf{u}) = E_q(\mathbf{u}) + \tilde{F}_q(\mathbf{u}),$$

where  $E_q = \Pi_{\text{pol}}(F_q)$  is the homogeneous polynomial of degree  $q$  given as

$$(4.7) \quad E_q(\mathbf{u}) = \Pi_{\text{pol}}(F_q) = \left( \sum_{k \leq q/2} Y_{q-2k,q}(\mathbf{v}) \right) s^q,$$

and  $\tilde{F}_q = F_q - E_q$  is the polynomial free part of  $F_q$ .

In the general case when  $F$  has the asymptotic expansion of the form

$$(4.8) \quad F(s\mathbf{v}) \sim \sum_{q=0}^Q (A_q(\mathbf{v}) + P_q(\mathbf{v}) \ln s) s^q + \sum_{q=1}^{\infty} A_{-q}(\mathbf{v}) s^{-q},$$

then the *polynomial part* of  $F$  is the polynomial

$$(4.9) \quad \Pi_{\text{pol}}(F) = \sum_{q=0}^Q \Pi_{\text{pol}}(A_q(\mathbf{v}) s^q).$$

The *polynomial free part* of  $F$  is  $F - \Pi_{\text{pol}}(F)$ .

It is possible to define the polynomial part of other distributions, not just those with an asymptotic expansion of the form (4.8), but this construction is enough for our purposes, since it gives the polynomial part in  $\mathcal{W}_{\text{pre}}(\mathbb{R}^n)$ . It should also be noticed that the polynomial part we constructed is a *radial* polynomial part, since we have employed polar coordinates.

The polynomial part allows us to understand why  $\mathcal{K}_q(\mathcal{D}(\mathbb{S})) = \mathcal{D}_q$  and  $\mathcal{K}_q(\mathcal{D}'(\mathbb{S})) = \mathcal{D}'_q$  for  $q < -n$ . In fact we have the following.

**Lemma 4.4.** *Let  $A \in \mathcal{D}(\mathbb{S})$ . If  $m \in \mathbb{N}$ , then  $A \in \mathcal{K}_{-(n+m)}(\mathcal{D}(\mathbb{S}))$  if and only if the function  $A(\mathbf{v}) s^m$  is polynomial free. Similarly, if  $A \in \mathcal{D}'(\mathbb{S})$ , then  $A \in \mathcal{K}_{-(n+m)}(\mathcal{D}'(\mathbb{S}))$  if and only if the distribution  $A(\mathbf{v}) s^m$  is polynomial free.*

*Proof.* Follows at once from the Propositions 3.5 and 3.6. □

**4.2. The space  $\mathcal{W}(\mathbb{R}^n)$ .** We can now consider the topology of the spaces  $\mathcal{W}_{\text{pre}}(\mathbb{R}^n)$  and  $\mathcal{S}_{*,\text{reg}}(\mathbb{R}^n)$ , as well as define the space  $\mathcal{W}(\mathbb{R}^n)$ .

The space  $\mathcal{S}_{*,\text{reg}}(\mathbb{R}^n)$  admits the representation

$$(4.10) \quad \mathcal{S}_{*,\text{reg}}(\mathbb{R}^n) = \mathcal{S}_{*,\text{ord}}(\mathbb{R}^n) \oplus \mathcal{D}'_{\{\mathbf{0}\}}(\mathbb{R}^n),$$

where  $\mathcal{D}'_{\{\mathbf{0}\}}(\mathbb{R}^n)$  is the space of distributions with support at the origin and where  $\mathcal{S}_{*,\text{ord}}(\mathbb{R}^n)$  is the space of ordinary parts of regularizations of thick test functions. Clearly the  $\mathcal{P}f$  operator is an isomorphism of  $\mathcal{S}_*(\mathbb{R}^n)$  onto  $\mathcal{S}_{*,\text{ord}}(\mathbb{R}^n)$ . We define the topology of  $\mathcal{S}_{*,\text{ord}}(\mathbb{R}^n)$  by asking  $\mathcal{P}f$  to be a topological isomorphism. The space  $\mathcal{D}'_{\{\mathbf{0}\}}(\mathbb{R}^n)$  has a topology as a

closed subspace of  $\mathcal{S}'(\mathbb{R}^n)$ . The topology of  $\mathcal{S}_{*,\text{reg}}(\mathbb{R}^n)$  is the direct sum topology. Notice that the topology of  $\mathcal{S}_{*,\text{reg}}(\mathbb{R}^n)$  is stronger but not equal to the subspace topology inherited from  $\mathcal{S}'(\mathbb{R}^n)$ . We can now complete the Theorem 4.2: *The Fourier transform is a topological isomorphism of the spaces  $\mathcal{S}_{*,\text{reg}}(\mathbb{R}^n)$  and  $\mathcal{W}_{\text{pre}}(\mathbb{R}^n)$ .*

We now define the space  $\mathcal{W}(\mathbb{R}^n)$ .

**Definition 4.5.** The space  $\mathcal{W}(\mathbb{R}^n)$  is formed by the polynomial free elements of  $\mathcal{W}_{\text{pre}}(\mathbb{R}^n)$ , with the subspace topology. Explicitly,  $\Phi \in \mathcal{W}$  if it is smooth in  $\mathbb{R}^n$  and at infinity it has an asymptotic expansion

$$(4.11) \quad \Phi(s\mathbf{v}) \sim \sum_{q=0}^Q (A_q(\mathbf{v}) + P_q(\mathbf{v}) \ln s) s^q + \sum_{q=1}^{\infty} A_{-q}(\mathbf{v}) s^{-q},$$

where  $A_q \in \mathcal{D}_q$  for  $q \in \mathbb{Z}$  and the  $P_q \in \mathcal{P}_q$  are homogeneous polynomials of degree  $q$ .

The space  $\mathcal{W}(\mathbb{R}^n)$  is exactly the space needed to define the Fourier transform of thick test functions; the condition  $A_q \in \mathcal{D}_q$  in the expansion (4.11), which is equivalent to the fact that  $\Phi$  is polynomial free, will play a very important role in the behavior of the Fourier transform of thick distributions. Notice in fact that

$$(4.12) \quad \mathcal{W}_{\text{pre}}(\mathbb{R}^n) = \mathcal{W}(\mathbb{R}^n) \oplus \mathcal{P}(\mathbb{R}^n),$$

as topological vector spaces. Therefore the space  $\mathcal{W}(\mathbb{R}^n)$  can also be constructed as a quotient space. Namely, if we define the equivalence relation  $F \sim G$  if  $F - G$  is a polynomial, then  $\mathcal{W}(\mathbb{R}^n)$  is canonically isomorphic to  $\mathcal{W}_{\text{pre}}(\mathbb{R}^n) / \sim$ . Similarly, if we consider the equivalence relation  $f \sim g$  when  $\text{supp}(f - g) \subset \{\mathbf{0}\}$  in  $\mathcal{S}_{*,\text{reg}}(\mathbb{R}^n)$ , then  $\mathcal{S}_*(\mathbb{R}^n) \simeq \mathcal{S}_{*,\text{ord}}(\mathbb{R}^n) \simeq \mathcal{S}_{*,\text{reg}}(\mathbb{R}^n) / \sim$ .

When  $\phi \in \mathcal{S}_*(\mathbb{R}^n)$  we shall denote by  $\mathcal{F}_{*,t}(\phi)$  the element  $\Pi_{\mathcal{W}_{\text{pre}},\mathcal{W}}(\mathcal{F}(\mathcal{P}f(\phi)))^{10}$  of  $\mathcal{W}(\mathbb{R}^n)$ , and call it the thick Fourier transform of  $\phi$ . We can also define a Fourier transform in  $\mathcal{W}(\mathbb{R}^n)$ ,  $\mathcal{F}_t^* : \mathcal{W}(\mathbb{R}^n) \rightarrow \mathcal{S}_*(\mathbb{R}^n)$ , as

$$(4.13) \quad \mathcal{F}_t^* \{\Phi(\mathbf{u}); \mathbf{x}\} = (2\pi)^n \mathcal{F}_{*,t}^{-1} \{\Phi(\mathbf{u}); -\mathbf{x}\}.$$

We immediately obtain the following important result.

**Theorem 4.6.** *The thick Fourier transform  $\mathcal{F}_{*,t}$  is a topological isomorphism of  $\mathcal{S}_*(\mathbb{R}^n)$  onto  $\mathcal{W}(\mathbb{R}^n)$ . The thick Fourier transform  $\mathcal{F}_t^*$  is a topological isomorphism of  $\mathcal{W}(\mathbb{R}^n)$  onto  $\mathcal{S}_*(\mathbb{R}^n)$ .*

## 5. THE SPACE $\mathcal{W}'(\mathbb{R}_c^n)$

In this section we shall consider the distributions of the space  $\mathcal{W}'(\mathbb{R}^n)$ . The first thing we would like to point out is that the functions of  $\mathcal{W}(\mathbb{R}^n)$  are smooth functions in  $\mathbb{R}^n$  with a special type of thick behavior at infinity; therefore the elements of  $\mathcal{W}'(\mathbb{R}^n)$  are actually distributions over the space  $\mathbb{R}_c^n = \mathbb{R}^n \cup \{\infty\}$ , the one point compactification of  $\mathbb{R}^n$ . From now on we shall also employ the more informative notation  $\mathcal{W}'(\mathbb{R}_c^n)$  when we want to call attention to the dimension  $n$  and the simpler notation  $\mathcal{W}'$  when no explicit mention of  $n$  is needed. The elements of  $\mathcal{W}'$  shall be called *sl-thick distributions*, since the thick test functions of  $\mathcal{W}$  have a special type of logarithmic expansion at infinity.

Several distributions defined on  $\mathbb{R}^n$  admit canonical extensions to  $\mathcal{W}'(\mathbb{R}_c^n)$ . Indeed, if  $\mathcal{W}(\mathbb{R}^n) \subset \mathcal{A}(\mathbb{R}^n)$  continuously and with dense image, where  $\mathcal{A}(\mathbb{R}^n)$  is a space of

<sup>10</sup>In general  $\mathcal{F}(\mathcal{P}f(\phi))$  does not belong to  $\mathcal{W}$ .

test functions, then  $\mathcal{A}'(\mathbb{R}^n)$  is canonically imbedded into  $\mathcal{W}'(\mathbb{R}_c^n)$ . The simplest case is when  $\mathcal{A}(\mathbb{R}^n) = \mathcal{E}(\mathbb{R}^n)$ , the space of all smooth functions in  $\mathbb{R}^n$ , which gives that each distribution of compact support,  $f \in \mathcal{E}'(\mathbb{R}^n)$  admits a canonical extension to  $\mathcal{W}'(\mathbb{R}_c^n)$ , namely one whose support in  $\mathbb{R}_c^n$  is precisely the original support of  $f$ ,

$$(5.1) \quad \langle f, \Phi \rangle_{\mathcal{W}' \times \mathcal{W}} = \langle f, \Phi \rangle_{\mathcal{E}' \times \mathcal{E}} .$$

Actually we can also take  $\mathcal{A}(\mathbb{R}^n) = \mathcal{K}(\mathbb{R}^n)$ , so that any distribution  $f \in \mathcal{K}'(\mathbb{R}^n)$  admits a canonical extension to  $\mathcal{W}'(\mathbb{R}_c^n)$ , given by the Cesàro evaluation,

$$(5.2) \quad \langle f, \Phi \rangle_{\mathcal{W}' \times \mathcal{W}} = \langle f, \Phi \rangle \quad (\text{C})$$

since  $\langle f, \Phi \rangle$  (C) exists whenever  $\Phi \in \mathcal{K}(\mathbb{R}^n)$  [14] and  $\mathcal{W}(\mathbb{R}^n) \subset \mathcal{K}(\mathbb{R}^n)$ . We shall employ the same notation for both the distribution of  $\mathcal{K}'(\mathbb{R}^n)$  and its canonical extension to  $\mathcal{W}'(\mathbb{R}_c^n)$ . On the other hand,  $\mathcal{W}(\mathbb{R}^n)$  is not contained in  $\mathcal{S}(\mathbb{R}^n)$ , and this means that tempered distributions do not have *canonical* extensions in  $\mathcal{W}'(\mathbb{R}_c^n)$ . In fact, it is not hard to see that actually all elements of  $\mathcal{S}'(\mathbb{R}^n)$  have *many* extensions to  $\mathcal{W}'(\mathbb{R}_c^n)$ , but it is not possible to construct a continuous extension procedure, similarly to the situation explained in [7].

Another important class of *sl*-thick distributions are the thick deltas at infinity.

**Definition 5.1.** If  $G \in \mathcal{D}'_q$  then we define  $G(\mathbf{v}) \delta_\infty^{[q]}$ , the thick delta at infinity of order  $q$  as

$$(5.3) \quad \langle G(\mathbf{v}) \delta_\infty^{[q]}, \Phi \rangle_{\mathcal{W}' \times \mathcal{W}} = \frac{1}{C} \langle G, A_q \rangle_{\mathcal{D}'_q \times \mathcal{D}_q} ,$$

if  $\Phi \in \mathcal{W}$  has the asymptotic expansion (4.11). Similarly, if  $H \in \mathcal{P}'_q = \mathcal{P}_q$  then we define  $H(\mathbf{v}) \delta_{\ln, \infty}^{[q]}$  the thick logarithmic delta of order  $q$  at infinity as

$$(5.4) \quad \langle H(\mathbf{v}) \delta_{\ln, \infty}^{[q]}, \Phi \rangle_{\mathcal{W}' \times \mathcal{W}} = \frac{1}{C} \langle H, P_q \rangle_{\mathcal{P}'_q \times \mathcal{P}_q} .$$

Sometimes one may construct extensions of a tempered distribution  $g$  by considering a finite part at infinity<sup>11</sup>, a construction we shall now denote as  $\mathcal{P}f_{\mathcal{W}}(g)$ , or later simply as  $\mathcal{P}f(g)$  if there is no danger of confusion. Consider for example the distribution  $\mathcal{P}f(s^\lambda)$ ,  $s = |\mathbf{u}|$ , of  $\mathcal{S}'(\mathbb{R}^n)$ : this tempered distribution yields the *sl*-thick distribution  $\mathcal{P}f_{\mathcal{W}}(s^\lambda)$  obtained from the generally divergent integral  $\int_{\mathbb{R}^n} s^\lambda \Phi(\mathbf{u}) d\mathbf{u}$ ,  $\Phi \in \mathcal{W}$ , by taking the radial finite part at  $\mathbf{0}$ , or at  $\infty$ , or at both. Using the ideas of the Example 2.5 we can see the structure of  $\mathcal{P}f_{\mathcal{W}}(s^\lambda)$ .

**Lemma 5.2.** *The parametric *sl*-thick distribution  $\mathcal{P}f_{\mathcal{W}}(s^\lambda)$  is a meromorphic function of  $\lambda$ , analytic in the region  $(\mathbb{C} \setminus \mathbb{Z}) \cup \{0, 2, 4, \dots\}$ , with simple poles at  $\lambda = m$ ,  $m \in \{-n-1, -n-3, -n-5, \dots\} \cup \{-1, -2, \dots, 1-n\} \cup \{1, 3, 5, \dots\}$ , the residues at these poles being*

$$(5.5) \quad \text{Res}_{\lambda=m} \mathcal{P}f_{\mathcal{W}}(s^\lambda) = -C \delta_\infty^{[-n-m]}(\mathbf{u}) ,$$

and double poles at  $\lambda = m$ ,  $m = -n - 2q \in \{-n, -n - 2, -n - 4, \dots\}$  with singular part

$$(5.6) \quad \frac{C \delta_{\ln, \infty}^{[2q]}(\mathbf{u})}{(\lambda - m)^2} + \frac{c_{q,n} \nabla^{2q} \delta(\mathbf{u})}{(2q)! (\lambda - m)} .$$

The finite part of  $\mathcal{P}f_{\mathcal{W}}(s^\lambda)$  at any pole  $\lambda = m$  is precisely  $\mathcal{P}f_{\mathcal{W}}(s^m)$ .

<sup>11</sup>Clearly the finite part at infinity does *not exist* for all  $g \in \mathcal{S}'(\mathbb{R}^n)$ .

Many of the constructions that we have discussed can also be done in the space  $\mathcal{W}'_{\text{pre}}$ . Notice, however, that several distributions of  $\mathcal{W}'_{\text{pre}}$  could vanish in  $\mathcal{W}$  so that their projection to  $\mathcal{W}'$  could be zero. For instance, the plain thick delta  $\delta_{\infty}^{[0]}$  is not zero in  $\mathcal{W}'_{\text{pre}}$  but it is zero in  $\mathcal{W}'$ . If one considers the finite part  $\mathcal{P}f_{\mathcal{W}'_{\text{pre}}}(s^{\lambda})$  then it would not be analytic at  $\lambda = 0, 2, 4, \dots$ ; for instance, it has a simple pole at  $\lambda = 0$  with residue  $-C\delta_{\infty}^{[0]}$ .

One of the consequences of the fact that  $\mathcal{W}'$  is a space over the compact space  $\mathbb{R}_c^n$  is that several of the usual operations on  $sl$ -thick distributions could have additional terms at infinity. This is the case for the linear changes of variables and for the multiplications by polynomials. Curiously, however, derivatives in  $\mathcal{W}'$  can be defined in the standard way by duality, since the derivative operators send  $\mathcal{W}$  to  $\mathcal{W}$ ,

$$(5.7) \quad \langle \nabla_j(F), \Phi \rangle = -\langle F, \nabla_j(\Phi) \rangle, \quad F \in \mathcal{W}', \Phi \in \mathcal{W}.$$

**5.1. Linear changes of variables in  $\mathcal{W}'$ .** Let  $A$  be a non-singular  $n \times n$  matrix. If  $\Phi \in \mathcal{W}$  then the function  $\Phi_A$  given by  $\Phi_A(\mathbf{u}) = \Phi(A\mathbf{u})$  does not belong to  $\mathcal{W}$ , in general, but it belongs to  $\mathcal{W}'_{\text{pre}}$ . Therefore we define the function of  $\mathcal{W}$  obtained by the change of variables,  $\tau_A^{\mathcal{W}}(\Phi)$  as

$$(5.8) \quad \tau_A^{\mathcal{W}}(\Phi) = \Pi_{\mathcal{W}'_{\text{pre}}, \mathcal{W}}(\Phi_A).$$

We can then define the change of variables in  $sl$ -thick distributions by duality.

**Definition 5.3.** Let  $A$  be a non-singular  $n \times n$  matrix. If  $F \in \mathcal{W}'$  then the distribution  $\tau_A^{\mathcal{W}'}(F)$ , the  $sl$ -thick version of  $F(A\mathbf{u})$ , is defined as

$$(5.9) \quad \langle \tau_A^{\mathcal{W}'}(F), \Phi \rangle = \frac{1}{|\det(A)|} \langle F, \tau_{A^{-1}}^{\mathcal{W}}(\Phi) \rangle.$$

It is important to observe that if  $F \in \mathcal{K}'(\mathbb{R}^n)$  then it has a canonical extension to  $\mathcal{W}'(\mathbb{R}_c^n)$ , and the restriction of  $\tau_A^{\mathcal{W}'}(F)(\mathbf{x})$  to  $\mathbb{R}^n$  is precisely  $F(A\mathbf{x})$  but in general  $\tau_A^{\mathcal{W}'}(F)(\mathbf{x})$  is *not* the canonical extension of  $F(A\mathbf{x})$ . A simple example is provided by the delta function at the origin,  $\delta(\mathbf{x})$ , and the change  $A = tI$  for  $t \neq 0$ . We have  $\delta(t\mathbf{x}) = |t|^{-n} \delta(\mathbf{x})$ , of course, but

$$(5.10) \quad \tau_{tI}^{\mathcal{W}'}(\delta)(\mathbf{x}) = |t|^{-n} \delta(\mathbf{x}) - |t|^{-n} \ln t \delta_{\infty, \ln}^{[0]}(\mathbf{x}).$$

Interestingly, if  $A$  is an orthogonal matrix, in particular if it is a rotation, and  $F \in \mathcal{K}'(\mathbb{R}^n)$  then the canonical extension of  $F(A\mathbf{x})$  is precisely  $\tau_A^{\mathcal{W}'}(F)(\mathbf{x})$ . Therefore we give the following definitions.

**Definition 5.4.** A  $sl$ -thick distribution  $F \in \mathcal{W}'$  is called radial if  $\tau_A^{\mathcal{W}'}(F) = F$  for all orthogonal matrices  $A$ . We say that  $F$  is homogeneous of order  $\lambda$  if

$$(5.11) \quad \tau_{tI}^{\mathcal{W}'}(F)(\mathbf{x}) = t^{\lambda} F(\mathbf{x}), \quad t > 0.$$

Notice that a distribution  $F \in \mathcal{K}'(\mathbb{R}^n)$  is radial if and only if its canonical extension is, but (5.10) shows that a corresponding result does not hold for homogeneous distributions. On the other hand, a distribution of the form  $G(\mathbf{v})\delta_{\infty}^{[q]}$  is radial if and only if  $G$  is constant, where we observe that the plain thick delta at infinity  $\delta_{\infty}^{[q]}$  is a non zero  $sl$ -thick distribution for  $q \neq 0, 2, 4, \dots$  and  $q \neq -n, -n-2, -n-4, \dots$ . Furthermore, since the plain thick logarithmic deltas at infinity  $\delta_{\ln, \infty}^{[1]}, \delta_{\ln, \infty}^{[3]}, \delta_{\ln, \infty}^{[5]}, \dots$  vanish, the distributions  $c\delta_{\ln, \infty}^{[q]}$  for  $q = 0, 2, 4, \dots$  and  $c$  constant are the radial distributions of the form  $G(\mathbf{v})\delta_{\ln, \infty}^{[q]}$ .



It is useful to know the  $sl$ -thick radial homogeneous distributions.

**Proposition 5.5.** *Let  $\lambda \in \mathbb{C}$ . Then the set of  $sl$ -thick radial homogeneous distributions of order  $\lambda$  form a vector space of dimension 1, generated by the distribution*

$$(5.12) \quad \mathcal{P}f_{\mathcal{W}}(s^\lambda) \text{ for } \lambda \in (\mathbb{C} \setminus \mathbb{Z}) \cup \{0, 2, 4, \dots\},$$

$$(5.13) \quad \delta_\infty^{[-n-m]} \text{ for } \lambda = m, \\ m \in \{-n-1, -n-3, -n-5, \dots\} \cup \{1, 2, \dots, n-1\} \cup \{1, 3, 5, \dots\},$$

$$(5.14) \quad \delta_{\text{in}, \infty}^{[-n-m]} \text{ for } \lambda = m, \quad m \in \{-n, -n-2, -n-4, \dots\}.$$

5.1.1. *Multiplication by polynomials in  $\mathcal{W}'$ .* In general if  $\Phi \in \mathcal{W}$  then  $u_j \Phi(\mathbf{u})$  is in  $\mathcal{W}_{\text{pre}}$  but it does not belong to  $\mathcal{W}$ . Therefore we define the multiplication operator

$$(5.15) \quad M_{u_j}^{\mathcal{W}} : \mathcal{W} \longrightarrow \mathcal{W}, \quad M_{u_j}(\Phi) = \Pi_{\mathcal{W}_{\text{pre}}, \mathcal{W}}(u_j \Phi),$$

and by duality the operator  $M_{u_j}^{\mathcal{W}'} : \mathcal{W}' \longrightarrow \mathcal{W}'$  as

$$(5.16) \quad \langle M_{u_j}^{\mathcal{W}'}(F), \Phi \rangle = \langle F, M_{u_j}^{\mathcal{W}}(\Phi) \rangle.$$

The multiplication operators  $M_p^{\mathcal{W}}$  and  $M_p^{\mathcal{W}'}$ , where  $p$  is a polynomial, can be defined in a similar way.

**Example 5.6.** Sometimes  $M_{u_j}^{\mathcal{W}'}(F)$  resembles a standard multiplication, as in the product formula

$$(5.17) \quad M_{u_j}^{\mathcal{W}'}(\mathcal{P}f_{\mathcal{W}}(s^\lambda)) = \mathcal{P}f_{\mathcal{W}}(u_j s^\lambda),$$

but sometimes extra terms at infinity appear, as in the formula

$$(5.18) \quad M_{u_j}^{\mathcal{W}'}(\delta(\mathbf{u})) = -\omega_j \delta_\infty^{[-1]}(\mathbf{u}).$$

## 6. THE FOURIER TRANSFORM OF THICK DISTRIBUTIONS

The Fourier transform of thick tempered distributions  $f \in \mathcal{S}'_*(\mathbb{R}^n)$ ,  $\mathcal{F}_*(f) \in \mathcal{W}'(\mathbb{R}_c^n)$  can now be defined in the usual way,

$$(6.1) \quad \langle \mathcal{F}_* \{f(\mathbf{x}); \mathbf{u}\}, \Phi(\mathbf{u}) \rangle = \langle f(\mathbf{x}), \mathcal{F}_t^* \{\Phi(\mathbf{u}); \mathbf{x}\} \rangle, \quad \Phi \in \mathcal{W}(\mathbb{R}_c^n).$$

Similarly, the Fourier transform of distributions  $G \in \mathcal{W}'(\mathbb{R}_c^n)$ ,  $\mathcal{F}^*(G) \in \mathcal{S}'_*(\mathbb{R}^n)$  is defined as

$$(6.2) \quad \langle \mathcal{F}^* \{G(\mathbf{u}); \mathbf{x}\}, \phi(\mathbf{x}) \rangle = \langle G(\mathbf{u}), \mathcal{F}_{*,t} \{\phi(\mathbf{x}); \mathbf{u}\} \rangle, \quad \phi \in \mathcal{S}_*(\mathbb{R}^n).$$

**Theorem 6.1.** *The thick Fourier transform  $\mathcal{F}_*$  is a topological isomorphism of  $\mathcal{S}'_*(\mathbb{R}^n)$  onto  $\mathcal{W}'(\mathbb{R}_c^n)$ . The thick Fourier transform  $\mathcal{F}^*$  is a topological isomorphism of  $\mathcal{W}'(\mathbb{R}_c^n)$  onto  $\mathcal{S}'_*(\mathbb{R}^n)$ .*

The properties of the Fourier transform of thick distributions are similar to those of the transform in  $\mathcal{S}'(\mathbb{R}^n)$  but one must remember that the operations in  $\mathcal{W}'(\mathbb{R}_c^n)$  may or may not be the standard ones. We have,

$$(6.3) \quad \mathcal{F}_* \{f(A\mathbf{x}); \mathbf{u}\} = \frac{1}{|\det A|} \tau_{A^{-1}}^{\mathcal{W}'}(\mathcal{F}_* \{f(\mathbf{x}); \mathbf{u}\}),$$

for  $A$  a non-singular matrix, and in particular, if  $t \neq 0$

$$(6.4) \quad \mathcal{F}_* \{f(t\mathbf{x}); \mathbf{u}\} = t^{-n} \tau_{t^{-n}I}^{\mathcal{W}'} (\mathcal{F}_* \{f(\mathbf{x}); \mathbf{u}\}) .$$

It follows that  $\mathcal{F}_*$  and  $\mathcal{F}^*$  send radial thick distributions to radial thick distributions, and homogeneous distributions of degree  $\lambda$  to homogeneous distributions of degree  $-n - \lambda$ . We also have the usual interchange of multiplication and differentiation,

$$(6.5) \quad \mathcal{F}_* \{x_j f(\mathbf{x}); \mathbf{u}\} = -i \nabla_{u_j} \mathcal{F}_* \{f(\mathbf{x}); \mathbf{u}\} ,$$

$$(6.6) \quad \mathcal{F}_* \{\nabla_{x_j} f(\mathbf{x}); \mathbf{u}\} = -i M_{u_j}^{\mathcal{W}'} \mathcal{F}_* \{f(\mathbf{x}); \mathbf{u}\} ,$$

where the modified multiplication operator  $M_{u_j}^{\mathcal{W}'}$  is given by (5.16). The formulas for the inverse transforms are a variant of the usual ones,

$$(6.7) \quad (\mathcal{F}^*)^{-1} \{f(\mathbf{x}); \mathbf{u}\} = \frac{1}{(2\pi)^n} \mathcal{F}_* \{f(\mathbf{x}); -\mathbf{u}\} ,$$

$$(6.8) \quad (\mathcal{F}_*)^{-1} \{F(\mathbf{u}); \mathbf{x}\} = \frac{1}{(2\pi)^n} \mathcal{F}^* \{F(\mathbf{u}); -\mathbf{x}\} .$$

Another important property is that the Fourier transforms  $\mathcal{F}_*$  or  $\mathcal{F}^*$  of extensions of distributions of  $\mathcal{S}'(\mathbb{R}^n)$  to  $\mathcal{S}'_*(\mathbb{R}^n)$  or  $\mathcal{W}'(\mathbb{R}_c^n)$  are extensions of the Fourier transform, that is

$$(6.9) \quad \Pi_{\mathcal{W}', \mathcal{S}'} \mathcal{F}_* \{f(\mathbf{x}); \mathbf{u}\} = \mathcal{F} \{\Pi_{\mathcal{S}'_*} f(\mathbf{x}); \mathbf{u}\} ,$$

$$(6.10) \quad \Pi_{\mathcal{S}'_*} \mathcal{F}^* \{F(\mathbf{u}); \mathbf{x}\} = \mathcal{F} \{\Pi_{\mathcal{W}', \mathcal{S}'} F(\mathbf{u}); \mathbf{x}\} .$$

We are now ready to give the Fourier transform of several thick distributions.

**Example 6.2.** Let us compute the Fourier transform  $\mathcal{F}_* \{\delta_*^{[0]}(\mathbf{x}); \mathbf{u}\}$  of the plain thick delta function. Since  $\delta_*^{[0]}(\mathbf{x})$  is radial and homogenous of degree  $-n$ , its transform is radial and homogeneous of degree 0. Also, the projection of  $\delta_*^{[0]}(\mathbf{x})$  onto  $\mathcal{S}'$  is the standard delta function  $\delta(\mathbf{x})$ , whose transform is the constant function 1. From the Proposition 5.5 it follows that the only radial, homogeneous of degree 0  $sl$ -thick distribution whose projection to  $\mathcal{S}'$  is the constant distribution 1 is precisely  $\mathcal{P}f_{\mathcal{W}}(1)$ . Hence

$$(6.11) \quad \mathcal{F}_* \{\delta_*^{[0]}(\mathbf{x}); \mathbf{u}\} = \mathcal{P}f_{\mathcal{W}}(1) .$$

A similar argument yields

$$(6.12) \quad \mathcal{F}_* \{\delta_*^{[2m]}(\mathbf{x}); \mathbf{u}\} = \frac{(-1)^m \Gamma(m+1/2) \Gamma(n/2)}{\Gamma(m+n/2) \Gamma(1/2) (2m)!} \mathcal{P}f_{\mathcal{W}}(s^{2m}) ,$$

and by inversion,

$$(6.13) \quad \mathcal{F}^* \{\mathcal{P}f_{\mathcal{W}}(s^{2m}); \mathbf{x}\} = \frac{(-1)^m \Gamma(m+n/2) \Gamma(1/2) (2m)!}{(2\pi)^n \Gamma(m+1/2) \Gamma(n/2)} \delta_*^{[2m]}(\mathbf{x}) .$$

**Example 6.3.** The ensuing formulas, reminiscent of (3.1), also follow along the same lines,

$$(6.14) \quad \mathcal{F}_* \{\mathcal{P}f(r^\lambda); \mathbf{u}\} = \frac{\pi^{n/2} 2^{\lambda+n} \Gamma(\frac{\lambda+n}{2})}{\Gamma(-\frac{\lambda}{2})} \mathcal{P}f_{\mathcal{W}}(s^{-\lambda-n}) ,$$

$$(6.15) \quad \mathcal{F}^* \{ \mathcal{P}f_{\mathcal{W}}(s^\lambda); \mathbf{x} \} = \frac{\pi^{n/2} 2^{\lambda+n} \Gamma\left(\frac{\lambda+n}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)} \mathcal{P}f(r^{-\lambda-n}),$$

whenever  $\lambda \in \mathbb{C} \setminus \mathbb{Z}$ . Interestingly,  $\mathcal{P}f_{\mathcal{W}}(s^\lambda)$  is analytic at  $0, 2, 4, \dots$  so that (6.13) can be recovered by taking the limit as  $\lambda \rightarrow 2m$  in the right side of (6.15).

**Example 6.4.** Formulas (6.14) and (6.15) are equalities of meromorphic functions and thus by considering the residues, finite parts, or singular parts at the poles of both sides we obtain the Fourier transform of several thick distributions. Let start with  $m \in \{-n-1, -n-3, -n-5, \dots\} \cup \{-1, -2, \dots, 1-n\} \cup \{1, 3, 5, \dots\}$ , so that  $\lambda = m$  is a simple pole of the function in (6.15). From the Lemma 5.2 the residue of the left side is  $\mathcal{F}^* \left\{ -C \delta_\infty^{[-n-m]}(\mathbf{u}); \mathbf{x} \right\}$ , while if we recall [36, (4.13)] that  $\text{Res}_{\mu=k} \mathcal{P}f(r^\mu) = C \delta_*^{[-k-n]}(\mathbf{x})$ , we obtain the residue of the right side as  $Cg(m) \delta_*^{[m]}(\mathbf{x})$  where

$$(6.16) \quad g(\lambda) = \frac{\pi^{n/2} 2^{\lambda+n} \Gamma\left(\frac{\lambda+n}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)}.$$

Therefore

$$(6.17) \quad \mathcal{F}^* \left\{ \delta_\infty^{[-n-m]}(\mathbf{u}); \mathbf{x} \right\} = -g(m) \delta_*^{[m]}(\mathbf{x}),$$

and by inversion,

$$(6.18) \quad \mathcal{F}_* \left\{ \delta_*^{[m]}(\mathbf{x}); \mathbf{u} \right\} = -g(-n-m) \delta_\infty^{[-n-m]}(\mathbf{u}),$$

since  $g(m)g(-n-m) = (2\pi)^n$ . Similarly, consideration of the finite parts of both sides of (6.15) yields

$$(6.19) \quad \mathcal{F}^* \left\{ \mathcal{P}f_{\mathcal{W}}(s^m); \mathbf{x} \right\} = g(m) \left\{ \mathcal{P}f(r^{-m-n}) + \chi_m \delta_*^{[m]}(\mathbf{x}) \right\},$$

and

$$(6.20) \quad \mathcal{F}_* \left\{ \mathcal{P}f(r^{-m-n}); \mathbf{u} \right\} = g(-n-m) \left\{ \mathcal{P}f_{\mathcal{W}}(s^m) + \chi_{-m-n} \delta_\infty^{[-n-m]}(\mathbf{u}) \right\},$$

where

$$(6.21) \quad \chi_m = \chi_{-m-n} = \frac{C}{2} \left( 2 \ln 2 + \psi\left(\frac{m+n}{2}\right) + \psi\left(\frac{-m}{2}\right) \right).$$

Studying the coefficients of order  $-2$  at the poles of order  $2$ ,  $m = -n - 2q$  for  $q \in \mathbb{N}$  gives

$$(6.22) \quad \mathcal{F}^* \left\{ \delta_{\ln, \infty}^{[2q]}(\mathbf{u}); \mathbf{x} \right\} = \frac{(-1)^n 2^{1-2q} \pi^{n/2}}{q! \Gamma\left(\frac{n+2q}{2}\right)} \delta_*^{[-n-2q]}(\mathbf{x}),$$

and

$$(6.23) \quad \mathcal{F}_* \left\{ \delta_*^{[-n-2q]}(\mathbf{x}); \mathbf{u} \right\} = (-1)^n 2^{n+2q-1} \pi^{n/2} q! \Gamma\left(\frac{n+2q}{2}\right) \delta_{\ln, \infty}^{[2q]}(\mathbf{u}).$$

We have considered the transform of plain thick deltas so far, now we compute the Fourier transform of general thick deltas.

**Example 6.5.** Let  $\phi \in \mathcal{S}_*$ , with expansion  $\sum_{j=m}^{\infty} a_j r^j$  at zero and let  $\Phi = \mathcal{F}_{*,t}(\phi) \in \mathcal{W}$ , with expansion  $\sum_{q=0}^{n-m} (A_q(\mathbf{v}) + P_q(\mathbf{v}) \ln s) s^q + \sum_{q=1}^{\infty} A_{-q}(\mathbf{v}) s^{-q}$  at infinity. Then  $A_q = \mathfrak{K}_q(a_{-n-q})$ , therefore if  $G \in \mathcal{D}'_q$  then

$$\langle G \delta_\infty^{[q]}, \Phi \rangle = \frac{1}{C} \langle G, A_q \rangle = \frac{1}{C} \langle G, \mathfrak{K}_q(a_{-n-q}) \rangle = \frac{1}{C} \langle \mathfrak{K}_q(G), a_{-n-q} \rangle = \langle \mathfrak{K}_q(G) \delta_*^{[-n-q]}, \phi \rangle,$$

or

$$(6.24) \quad \mathcal{F}^* \{G(\mathbf{v}) \delta_\infty^{[q]}(\mathbf{u}); \mathbf{x}\} = \mathfrak{K}_q \{G(\mathbf{v}); \mathbf{w}\} \delta_*^{[-n-q]}(\mathbf{x}),$$

giving the transform of *all* thick deltas at infinity  $G\delta_\infty^{[q]}$ , for arbitrary  $q \in \mathbb{Z}$ , since  $G$  needs to be  $\mathcal{D}'_q$ . Similarly, for  $q \in \mathbb{N}$

$$(6.25) \quad \mathcal{F}^* \left\{ H(\mathbf{v}) \delta_{\ln, \infty}^{[q]}(\mathbf{u}); \mathbf{x} \right\} = \mathfrak{L}_q \{H(\mathbf{v}); \mathbf{w}\} \delta_*^{[-n-q]}(\mathbf{x}).$$

**Example 6.6.** We now consider the transform of the general thick deltas  $f(\mathbf{w}) \delta_*^{[m]}(\mathbf{x})$ . Let  $m = -n - q$ . Different formulas arise depending on  $m$  and  $q$ . If  $1 - n \leq m, q \leq -1$  then  $\mathcal{D}'_q = \mathcal{D}'_m = \mathcal{D}'$  so that inversion of (6.24), remembering (3.12), gives

$$(6.26) \quad \mathcal{F}_* \{f(\mathbf{w}) \delta_*^{[m]}(\mathbf{x}); \mathbf{u}\} = \mathfrak{K}_m \{f(\mathbf{w}); \mathbf{v}\} \delta_\infty^{[-n-m]}(\mathbf{u}).$$

If  $m \geq 0$ , that is  $q \leq -n$ , we decompose  $f \in \mathcal{D}'(\mathbb{S})$  as  $f = p_m + f_m$  where  $f_m \in \mathcal{D}'_q = \mathcal{D}'_m$  and  $p_m \in \mathcal{P}_m$ . We now notice that  $\mathcal{F}_* \left( p \delta_*^{[m]} \right)$  is the finite part regularization  $\mathcal{P}f_{\mathcal{W}}(P_m(\mathbf{u}))$  of a homogeneous polynomial of degree  $m$ , namely  $P_m = \mathcal{F} \left( \Pi_{S'_*, S'} \left( f \delta_*^{[m]} \right) \right)$ , obtaining

$$(6.27) \quad \mathcal{F}_* \{f(\mathbf{w}) \delta_*^{[m]}(\mathbf{x}); \mathbf{u}\} = \mathcal{P}f_{\mathcal{W}}(P_m(\mathbf{u})) + \mathfrak{K}_m \{f_m(\mathbf{w}); \mathbf{v}\} \delta_\infty^{[-n-m]}(\mathbf{u}).$$

In particular, when  $m = 0$ , since  $\mathcal{F}_* \left( \delta_*^{[0]} \right) = \mathcal{P}f_{\mathcal{W}}(1)$ , we obtain

$$(6.28) \quad \mathcal{F}_* \{f(\mathbf{w}) \delta_*^{[0]}(\mathbf{x}); \mathbf{u}\} = M \mathcal{P}f_{\mathcal{W}}(1) + \mathfrak{K}_0 \{f(\mathbf{w}) - M; \mathbf{v}\} \delta_\infty^{[-n]}(\mathbf{u}),$$

where  $M$  is the constant  $M = (1/C) \langle f(\mathbf{w}), 1 \rangle$ .

Finally, if  $m \leq -n$ , i.e.  $q \geq 0$ , the decomposition  $f = p_m + f_m$  where  $f_m \in \mathcal{D}'_q = \mathcal{D}'_m$  and  $p_m \in \mathcal{P}_{-n-m} = \mathcal{P}_q$  yields

$$(6.29) \quad \begin{aligned} \mathcal{F}_* \{f(\mathbf{w}) \delta_*^{[m]}(\mathbf{x}); \mathbf{u}\} &= (2\pi)^n \mathfrak{L}_{-n-m}^{-1} \{p_m(\mathbf{w}); -\mathbf{v}\} \delta_{\ln, \infty}^{[-n-m]}(\mathbf{u}) \\ &\quad + \mathfrak{K}_m \{f_m(\mathbf{w}); \mathbf{v}\} \delta_\infty^{[-n-m]}(\mathbf{u}). \end{aligned}$$

We will consider the thick Fourier transform of several other distributions in forthcoming papers.

## APPENDIX A. GUIDE TO NOTATION

### Constants

$c_{n,m}$ ,	equation (2.1)
$C$ , surface area of the unit sphere,	equation (2.1)

### Spaces

$\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n), \mathcal{D}_*(\mathbb{R}^n)$ ,	paragraph before Definition 2.1
$\mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n), \mathcal{D}'_*(\mathbb{R}^n)$ ,	Definition 2.1
$\mathcal{A}_{*,\mathbf{a}}(\mathbb{R}^n)$ , in particular $\mathcal{S}_{*,\mathbf{a}}(\mathbb{R}^n)$ ,	Definition 2.2
$\mathcal{A}'_{*,\mathbf{a}}(\mathbb{R}^n)$ , in particular $\mathcal{S}'_{*,\mathbf{a}}(\mathbb{R}^n)$ ,	Definition 2.2
$\mathcal{X}_q$ , in particular $\mathcal{D}_q$ and $\mathcal{D}'_q$ ,	Subsection 3.2
$\mathcal{P}_q$ , polynomials of degree $q$ on the sphere,	Subsection 3.2
$\mathcal{S}_{*,\text{reg}}(\mathbb{R}^n)$ ,	paragraph after equation (4.1)
$\mathcal{W}_{\text{pre}}(\mathbb{R}^n)$ ,	Definition 4.1
$\mathcal{W}(\mathbb{R}^n)$ ,	Definition 4.5

### Operators

$\mathcal{K}_\beta \{a(\mathbf{w}); \mathbf{v}\} = \langle K_\beta(\mathbf{w}, \mathbf{v}), a(\mathbf{w}) \rangle_{\mathbf{w}}$ ,	equation (3.5)
$\mathcal{L}_q \{a(\mathbf{w}); \mathbf{v}\} = \langle L_q(\mathbf{w}, \mathbf{v}), a(\mathbf{w}) \rangle_{\mathbf{w}}$ ,	equation (3.6)
$\mathfrak{R}_q \{a(\mathbf{w}); \mathbf{v}\}$ ,	Subsection 3.2
$\mathfrak{L}_q \{a(\mathbf{w}); \mathbf{v}\}$ ,	Subsection 3.2
delta part,	Subsection 4.1
ordinary part,	Subsection 4.1
polynomial part,	Subsection 4.1
polynomial free part,	Subsection 4.1
$\mathcal{F}_t^* \{\Phi(\mathbf{u}); \mathbf{x}\}$ and $\mathcal{F}_{*,t} \{\Phi(\mathbf{u}); \mathbf{x}\}$ , Fourier transform of test functions,	equation (4.13)
$\tau_A^{\mathcal{W}}$ and $\tau_A^{\mathcal{W}'}$ ,	Subsection 5.1
$M_{u_j}^{\mathcal{W}}$ and $M_{u_j}^{\mathcal{W}'}$ ,	Subsection 5.1.1
$\mathcal{F}^* \{f(\mathbf{x}); \mathbf{u}\}$ , Fourier transform of thick distributions,	equation (6.1)
$\mathcal{F}_* \{g(\mathbf{u}); \mathbf{x}\}$ , Fourier transform of $sl$ -thick distribution,	equation (6.2)
<b>Distributions</b>	
$g(\mathbf{w}) \delta_*^{[q]}(\mathbf{x} - \mathbf{a})$ ,	equation (2.3)
$\mathcal{P}f \{g(\mathbf{z}); \mathbf{x}\} = \mathcal{P}f(g)$ ,	Definition 2.4
$G(\mathbf{v}) \delta_\infty^{[q]}$ ,	Definition 5.1
$H(\mathbf{v}) \delta_{\ln, \infty}^{[q]}$ ,	Definition 5.1
$\mathcal{P}f_{\mathcal{W}}(g)$ ,	paragraph after Definition 5.1

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