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René Schott

G Stacey Staples

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ZEONS, LATTICES OF PARTITIONS, AND FREE PROBABILITY

RENÉ SCHOTT AND G. STACEY STAPLES

Abstract. Central to the theory of free probability is the notion of summing multiplicative functionals on the lattice of non-crossing partitions. In this paper, a graph-theoretic perspective of partitions is investigated in which independent sets in graphs correspond to non-crossing partitions. By associating particular graphs with elements of "zeon" algebras (commutative subalgebras of fermion algebras), multiplicative functions can be summed over segments of lattices of partitions by employing methods of "zeon-Berezin" operator calculus. In particular, properties of the algebra are used to "sieve out" the appropriate segments and sub-lattices. The work concludes with an application to joint moments of quantum random variables.

1. Introduction

The current work builds on the combinatorial approaches to multiple stochastic integrals developed by Rota and Wallstrom [8] and extended to free stochastic processes by Anshelevich [1].

A precursor to the current work was the graph-theoretic approach to multiple stochastic integrals developed by Staples [14]. In that paper, multiple stochastic integrals of classical processes and processes defined within a Clifford algebra of arbitrary signature were recovered from cycles contained in weighted graphs.

Another precursor was the joint paper by the current authors [10] in which Clifford-algebraic methods were applied to partitions and non-overlapping partitions to recover Bell numbers, Stirling numbers of the second kind, and Bessel numbers.

The lattice of non-crossing partitions is essential to the combinatorics of free probability theory, as computing moments of free random variables relies on summing multiplicative functionals on the lattice of non-crossing partitions. In the current work, graph-theoretic perspectives of partitions are investigated in which vertex independent sets in graphs correspond to non-crossing partitions. By associating appropriate graphs with elements of "zeon" algebras (commutative subalgebras of fermion algebras), multiplicative functions can be summed over segments of lattices of partitions by employing methods of "zeon-Berezin" operator calculus. In particular, properties of the algebra are used to "sieve out" the appropriate segments and sub-lattices.

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For essential background on free probability and the necessary combinatorics, the reader is referred to the works of Nica and Speicher [7], and Anshelevich [1].

The work of Franz Lehner on cumulants in non-commutative probability theory is also of interest, as it provides another graph-theoretic perspective on stochastic measures [6].

The rest of the paper is arranged as follows. In Section 2, zeon algebras are defined and the properties of zeon operator calculus necessary for later sections are revealed. Essential terminology from graph theory and the relationships among set partitions, cycle covers, and independent sets are detailed in Section 3, where some familiar counting numbers are recovered with the zeon operator methods.

In Section 4, zeon-algebraic methods are developed to perform computations involving multiplicative functionals on lattices of partitions. These methods combine properties of zeons with the graph-theoretic perspective to define an operator calculus that naturally "sieves out" the appropriate lattice segments for computation.

The work concludes with Section 5, where the methods are applied to free cumulants in a quantum probability space to recover moments of quantum random variables.

The Mathematica examples generated throughout the paper illustrate the usefulness of zeon-Berezin operator calculus methods in performing symbolic computations. The Mathematica procedures underlying the examples are available through the second named author's web page at `http://www.siue.edu/~sstaple`.

2. Zeon Algebras

Zeon algebras are commutative algebras whose generators square to zero. Their combinatorial properties have been applied to the study of graphs in a number of works by the current authors (cf. [11], [12], [9]), although the name "zeons" is attributed to Feinsilver [4].

Definition 2.1. Let $\mathbb{F}$ be a field. For fixed $n \geq 0$, the 2^n-dimensional *zeon algebra* $\mathcal{Z}_n$ is defined as the associative algebra generated by the collection $\{\zeta_i\}$ along with the scalar $\zeta_\emptyset = 1 \in \mathbb{F}$, subject to the following multiplication rules:

$$[\zeta_i, \zeta_j] := \zeta_i\, \zeta_j - \zeta_j\, \zeta_i = 0 \text{ for } 1 \leq i, j \leq n \text{ and} \tag{2.1}$$

$$\zeta_i{}^2 = 0 \tag{2.2}$$

Note that the even subalgebra of the Grassmann algebra over a $2n$-dimensional vector space V contains $\mathcal{Z}_n$. In particular, if $\{\mathbf{e}_i\}$ is an orthogonal basis for V, one can define $\zeta_i := \mathbf{e}_{2i-1} \wedge \mathbf{e}_{2i}$ for $1 \leq i \leq n$, where $\wedge$ denotes the exterior product.

Let $[n] = \{1, 2, \ldots, n\}$ and denote arbitrary, canonically ordered subsets of $[n]$ by capital Roman characters. Let $2^{[n]}$ denote the *power set* of $[n]$. The basis elements of $\mathcal{Z}_n$ can then be indexed by these finite subsets by writing

$$\zeta_I = \prod_{k \in I} \zeta_k. \tag{2.3}$$

Arbitrary elements of $\mathcal{Z}_n$ have the form

$$u = \sum_{I \in 2^{[n]}} u_I \, \zeta_I, \tag{2.4}$$

where $u_I \in \mathbb{F}$ for each I.

It will also be convenient to associate multi-indices with integers. To this end, we adopt the notational convention of letting $\underline{j}$ denote the subset of $[n]$ uniquely associated with the integer $0 \le j < 2^n$ by

$$j = \sum_{k \in \underline{j}} 2^k. \tag{2.5}$$

Definition 2.2. Any nonzero product of k generators in $\mathcal{Z}_n$ is referred to as a *blade of grade* k, or a k-*blade*.

Definition 2.3. For $0 \le k \le n$, the *grade-k part* of $u \in \mathcal{Z}_n$ is defined as the sum of grade-k monomials in the expansion of u. In other words,

$$\langle u \rangle_k = \sum_{\substack{I \in 2^{[n]} \\ |I| = k}} u_I \, \zeta_I. \tag{2.6}$$

The vector space spanned by grade-k blades in $\mathcal{Z}_n$ is denoted by $\langle \mathcal{Z}_n \rangle_k$

Notation. The notation $\langle\langle u \rangle\rangle_k$ is used to denote the sum of the coefficients in the grade-k part of u. That is,

$$\langle\langle u \rangle\rangle_k = \sum_{\substack{I \in 2^{[n]} \\ |I| = k}} u_I. \tag{2.7}$$

Definition 2.4. Given arbitrary $u = \sum_{I \in 2^{[n]}} u_I \, \zeta_I$ and $v = \sum_{I \in 2^{[n]}} v_I \, \zeta_I$, the *zeon inner product* of u and v is defined by

$$\langle u, v \rangle = \sum_{I \in 2^{[n]}} u_I \, v_I. \tag{2.8}$$

Consequently, the expansion of $u \in \mathcal{Z}_n$ can be written

$$u = \sum_{I \in 2^{[n]}} \langle u, \zeta_I \rangle \, \zeta_I. \tag{2.9}$$

This inner-product defines a norm on $\mathcal{Z}_n$ by

$$\|u\| = \langle u, u \rangle^{\frac{1}{2}}. \tag{2.10}$$

This norm is referred to as the *zeon inner-product norm.*

2.1. Zeon operator calculus. The operator calculus here follows naturally from Grassmann-Berezin calculus. Notationally, the work in this section draws largely from [13].

Definition 2.5. In $\mathcal{Z}_n$, the j^{th} *lowering operator* L_j is defined by linear extension of

$$\zeta_I L_j = \begin{cases} \zeta_{I \setminus \{j\}} & \text{if } j \in I, \\ 0 & \text{otherwise,} \end{cases} \tag{2.11}$$

while the j^{th} *raising operator* R_j is similarly defined by linear extension of

$$\zeta_I R_j = \begin{cases} 0 & \text{if } j \in I, \\ \zeta_{I\cup\{j\}} & \text{otherwise} \end{cases} \tag{2.12}$$

for each $j = 1, \dots, n$.

Note that the j^{th} lowering operator is easily regarded as a derivation:

$$\zeta_I L_j = \frac{\partial}{\partial \zeta_j}\zeta_I,$$

although it is not a derivation in the technical sense.

The *zeon canonical raising and lowering operators* $\mathcal{L}$ and $\mathcal{R}$ are defined as the sums of the raising and lowering operators, respectively. In particular,

$$\mathcal{L} := \sum_{j=1}^{n} L_j, \text{ and } \mathcal{R} := \sum_{j=1}^{n} R_j.$$

Note that the action of $\mathcal{L}$ on basis blades of $\mathcal{Z}_n$ has the following combinatorial interpretation: $\mathcal{L}$ maps each blade indexed by set I to a sum of blades indexed by proper subsets of I having cardinality $|I|-1$. The canonical raising operator has a similar interpretation. In particular,

$$\zeta_I \mathcal{L} = \sum_{\substack{J\subsetneq I\\ |J|=|I|-1}} \zeta_J,$$

$$\zeta_I \mathcal{R} = \sum_{\substack{J\supsetneq I\\ |J|=|I|+1}} \zeta_J.$$

In light of the graded structure of $\mathcal{Z}_n = \bigoplus_{k=0}^{n} \langle\mathcal{Z}_n\rangle_k$, these operators induce *level-k lowering and raising operators* $\mathcal{L}^{(k)} : \langle\mathcal{Z}_n\rangle_k \to \langle\mathcal{Z}_n\rangle_{k-1}$ for $1 \le k \le n$ and $\mathcal{R}^{(k)} : \langle\mathcal{Z}_n\rangle_k \to \langle\mathcal{Z}_n\rangle_{k+1}$ for $0 \le k \le n-1$, respectively.

In this context, the zeon canonical raising and lowering operators are correctly regarded as direct sums of level-k raising and lowering operators, i.e.,

$$\mathcal{L} := \bigoplus_{k=1}^{n} \mathcal{L}^{(k)}, \tag{2.13}$$

$$\mathcal{R} := \sum_{k=0}^{n-1} \mathcal{R}^{(k)}. \tag{2.14}$$

Following the formalism of Berezin [2], the following combinatorial integral is defined on the zeons: for any $\{i_1, \dots, i_p\} \subseteq [n]$, the composite map $\frac{\partial}{\partial\zeta_{i_1}} \circ \cdots \circ \frac{\partial}{\partial\zeta_{i_p}}$ is denoted by $\int d\zeta_{i_1} \cdots d\zeta_{i_p}$.

Given $u \in \mathcal{Z}_n$ and fixed multi-index $I \in 2^{[n]}$, the following shortened notation is defined:

$$\int u\, d\zeta_I := \int u\, d\zeta_{I_1} \cdots d\zeta_{I_{|I|}}. \tag{2.15}$$

Note that for any permutation $\sigma \in S_{|I|}$, commutativity of $\mathcal{Z}_n$ gives

$$\int u \, d\zeta_I = \int u \, d\zeta_{I_{\sigma(1)}} \cdots d\zeta_{I_{\sigma(|I|)}}.$$

The next result is immediate from the properties of $\mathcal{Z}_n$ and the preceding definitions.

Lemma 2.6. *Given $u \in \mathcal{Z}_n$ and fixed multi-index $I \in 2^{[n]}$,*

$$\int u \, d\zeta_I = \sum_{\substack{J \in 2^{[n]} \\ I \subseteq J}} u_J \, \zeta_{J \setminus I}. \tag{2.16}$$

When $I = [n]$, the following special case is obtained.

Definition 2.7. The *Berezin integral* is the linear map $\mathcal{Z}_n \to \mathbb{F}$ defined by

$$\int u \, d\zeta_{\sigma(1)} \cdots d\zeta_{\sigma(n)} = u_{[n]}, \tag{2.17}$$

for any permutation $\sigma \in S_n$. In other words, the Berezin integral is the "top-form" coefficient in the expansion of u.

Definition 2.8. Let $b = \{b_1, \ldots, b_k\} \subseteq [n]$. The *projective Berezin integral* is the linear map $\mathcal{Z}_n \to \mathbb{F}$ defined by

$$\oint u \, d\zeta_{b_1} \cdots d\zeta_{b_k} = \langle u, \zeta_b \rangle. \tag{2.18}$$

In particular, $\oint u \, d\zeta_b$ is the scalar part of $\int u \, d\zeta_b$.

Note that when $b = [n]$, the projective Berezin integral coincides with the usual Berezin integral.

3. Cycle Covers, Independent Sets, and Partitions

A *graph* $G = (V, E)$ is a set of vertices V and a set $E \subseteq V \times V$ of ordered pairs of vertices called *edges*. Two vertices $v_i, v_j \in V$ are said to be *adjacent* if $(v_i, v_j) \in E$. An edge of the form $(v, v) \in E$ is referred to as a *loop* at vertex v. When the relation on V defined by E is symmetric, the graph is said to be *undirected*. A *simple graph* is an undirected graph with no loops.

An *independent set* in a graph G is a set of pairwise nonadjacent vertices. A *clique* in a graph G is a set of pairwise adjacent vertices.

A *walk of length k*, or *k-walk*, in a graph is a sequence of vertices $v_0, v_1, v_2, \ldots, v_k$ with the property that v_i and v_{i+1} are adjacent for each $i = 0, 1, \ldots, k-1$. The vertices v_0 and v_k are referred to as the *initial vertex* and *terminal vertex* of the walk, respectively. A *cycle of length k*, or *k-cycle*, in a graph is a k-walk in which the vertices are pairwise distinct except for the initial and terminal vertices, which coincide.

Given a graph $G = (V, E)$, a *subgraph* of G is a graph $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. Note that G' must be a graph; i.e., vertices appearing within ordered pairs in E' must be elements of V'. A *cycle cover* of a graph G is a set of subgraphs $\{C_1, \ldots, C_k\}$ of G such that (i) each subgraph is a cycle, and (ii) each vertex of G is contained in exactly one of the subgraphs C_j, $(1 \leq j \leq k)$.

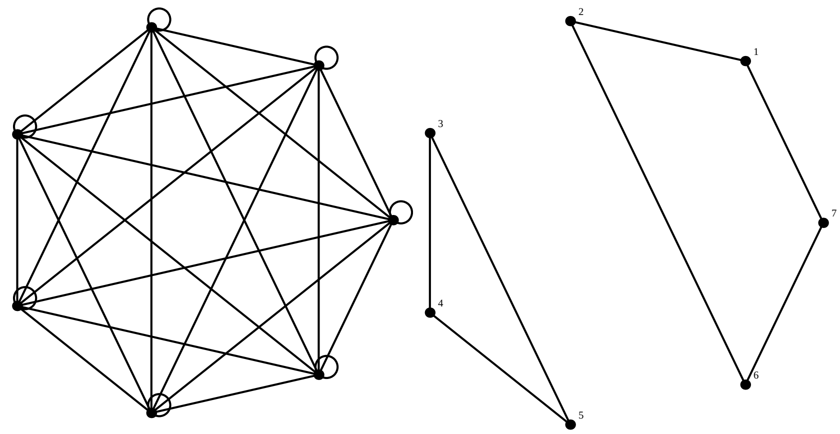

FIGURE 1. K_7° and a cycle cover.

When vertices are ordered and placed on a circle, one can define cycle covers associated with partitions. First, some notation and terminology regarding partitions is recalled.

The partition notation here follows that of Rota and Wallstrom. Given a finite set b, let $\Pi(b)$ denote the set of all partitions of b. In particular, an element $\sigma \in \Pi(b)$ is a collection of nonempty disjoint subsets, called *blocks*, whose union is b. Denote by $|\sigma|$ the number of blocks contained in σ.

The set $\Pi(b)$ is partially ordered by defining $\sigma \leq \pi$ if and only if every block of σ is a subset of some block in π. Accordingly, two partitions of particular interest are defined by

$$\hat{1}_b := b,$$
$$\hat{0}_b := \{b_1\} \cup \cdots \cup \{b_{|b|}\}.$$

When the set being partitioned is clear, one writes simply $\hat{1}$ or $\hat{0}$.

The meet of two partitions $\sigma \wedge \pi$ is defined as the partition whose blocks are the nonempty pairwise intersections of some block of σ with some block of π. The *join* of two partitions, denoted $\sigma \vee \pi$, is the smallest partition containing both σ and π. Note that $\Pi(b)$ is a *lattice*.

A *segment* $[\sigma, \pi]$ of the lattice $\Pi(b)$ is defined by

$$[\sigma, \pi] := \{\rho \in \Pi(b) : \sigma \leq \rho \leq \pi\}. \tag{3.1}$$

The following lemma follows immediately from the definition of a cycle cover.

Lemma 3.1. *Let G be a graph on n vertices. Any cycle cover of G determines a partition of the n-set.*

Recalling the integer-subset correspondence defined in (2.5), define the graph $G = (V, E)$ whose vertices $\{v_1, \ldots, v_{2^n-1}\}$ are in one-to-one correspondence with the nonempty subsets of the n-set with adjacency determined by the following condition:

$$(v_i, v_j) \in E \Leftrightarrow \underline{i} \cap \underline{j} \neq \emptyset. \tag{3.2}$$

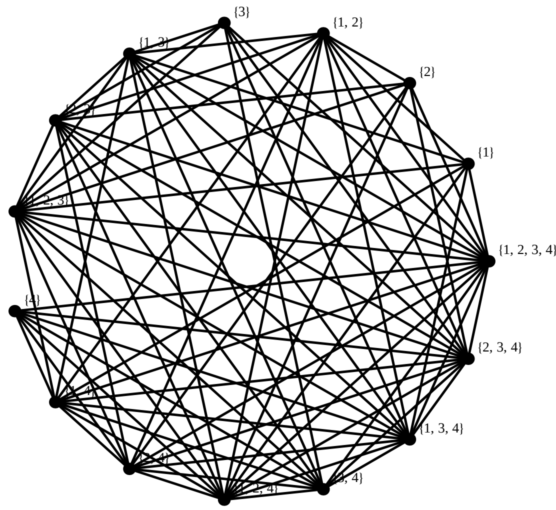

FIGURE 2. Partitions of $\{1, \ldots, 4\}$ among independent sets.

It becomes evident that partitions of the n-set are now associated with independent sets of the graph G. For example, the partitions of $\{1, 2, 3, 4\}$ are found among the independent sets of the graph in Figure 2.

A partition π of the n-set is said to be *crossing* if for some pair of blocks $\gamma, \delta \in \pi$ there exist $i, k \in \gamma$ and $j, \ell \in \delta$ such that $i < j < k < \ell$. For disjoint sets $\gamma, \delta \in 2^{[n]}$, the notation $Cr(\gamma, \delta)$ is defined to indicate that γ and δ form a *crossing* partition of $\gamma \cup \delta$.

The partition π is said to be *overlapping* if there exist blocks $\gamma, \delta \in \pi$ such that $\min \gamma < \min \delta < \max \gamma < \max \delta$. For disjoint sets $\gamma, \delta \in 2^{[n]}$, the notation $Ov(\gamma, \delta)$ is defined to indicate that γ and δ form an *overlapping* partition of $\gamma \cup \delta$.

Notation. Henceforth, the lattice of non-crossing partitions of the n-set will be denoted by $\mathrm{NC}(n)$, while the lattice of non-overlapping partitions of the n-set will be denoted by $\mathrm{NO}(n)$.

Defining the simple graph $G = (V, E)$ whose vertices are in one-to-one correspondence with the nonempty subsets of the n-set with adjacency determined by the following condition:

$$Cr(\underline{i}, \underline{j}) \vee [\underline{i} \cap \underline{j} \neq \emptyset] \Leftrightarrow (v_i, v_j) \in E, \tag{3.3}$$

where $\vee$ represents logical OR, non-crossing partitions of the n-set are now associated with independent sets of the graph G.

Similarly, if one defines the simple graph $H = (V_H, E_H)$ whose vertices are in one-to-one correspondence with the nonempty subsets of the n-set with adjacency determined by the following condition:

$$Ov(\underline{i}, \underline{j}) \vee [\underline{i} \cap \underline{j} \neq \emptyset] \Leftrightarrow (v_i, v_j) \in E_H, \tag{3.4}$$

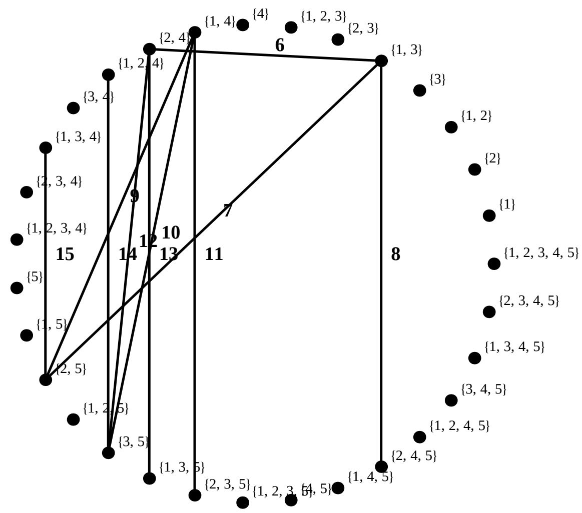

FIGURE 3. Crossing blocks for $n = 5$.

non-overlapping partitions of $[n]$ are associated with independent sets of the graph H.

To these ends, define the simple graph $G_{cr} = (V_{cr}, E_{cr})$ whose vertices are in one-to-one correspondence with the nonempty subsets of the n-set with adjacency determined by the following condition:

$$Cr(\underline{i}, \underline{j}) \Leftrightarrow (v_i, v_j) \in E_{cr}. \tag{3.5}$$

In other words, a pair of vertices are adjacent in G_{cr} if and only if their associated sets form a crossing partition of their union.

Similarly, define the simple graph $G_{ov} = (V_{ov}, E_{ov})$ whose vertices are in one-to-one correspondence with the nonempty subsets of $[n]$ with adjacency determined by the following condition:

$$Ov(\underline{i}, \underline{j}) \Leftrightarrow (v_i, v_j) \in E_{ov}. \tag{3.6}$$

In other words, a pair of vertices are adjacent in G_{ov} if and only if their associated sets form an overlapping partition of their union.

Finally, define the empty graph $G_\emptyset = (V_\emptyset, E_\emptyset)$ whose vertices are in one-to-one correspondence with the nonempty subsets of $[n]$ with $E_\emptyset = \emptyset$.

Example 3.2. Non-crossing partitions of the set $\{1, 2, 3, 4, 5\}$ are represented among the independent sets of the graph in Figure 3, while non-overlapping partitions are represented among the independent sets of the graph in Figure 4.

Definition 3.3. Let $\Pi(n)$ denote the lattice of partitions of the n-set, and let $\mathcal{E}$ denote a collection of pairs of disjoint subsets of $[n]$. The *$\mathcal{E}$-restricted lattice of partitions*, denoted $\Pi_{\mathcal{E}}(n)$ is defined by

$$\pi \in \Pi_{\mathcal{E}}(n) \Leftrightarrow (\gamma, \delta \in \pi \Rightarrow \{\gamma, \delta\} \notin \mathcal{E}). \tag{3.7}$$

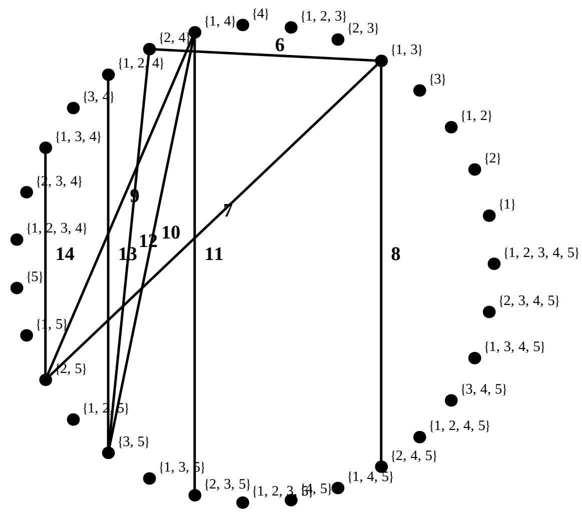

FIGURE 4. Overlapping blocks for $n = 5$.

A pair $\{\gamma, \delta\} \in \mathcal{E}$ is called a *forbidden pair* of blocks.

The collection $\mathcal{E}$ is now used to define a graph.

Definition 3.4. Let $G = (V, E)$ be a simple graph on $2^n - 1$ vertices associated with the nonempty subsets of the n-set, and let adjacency in G be determined by

$$(v_i, v_j) \in E \Leftrightarrow \{\underline{i}, \underline{j}\} \in \mathcal{E}. \tag{3.8}$$

The graph G is called a *partition constraint graph* of the n-set. The *G-restricted lattice of partitions*, whose blocks satisfy the constraints imposed by G, will be denoted $\Pi_G(n)$.

Let $G = (V, E)$ be a simple graph on $|V|$ vertices, and define a labeling $g : E \to \mathcal{Z}_{|V|+|E|}$ of the edges of G with generators of $\mathcal{Z}_{|V|+|E|}$ according to $g(e_j) = \zeta_{|V|+j}$ for $j = 1, 2, \ldots, |E|$.

Each vertex v of G is now associated uniquely within $\mathcal{Z}_{|V|+|E|}$ by the product of generators labeling the edges incident with v. That is, define $\varkappa : V \to \mathcal{Z}_{|V|+|E|}$ by

$$\varkappa(v_j) = \prod_{\substack{\text{edges } e_k \\ \text{incident with } v_j}} \zeta_{g(e_k)}. \tag{3.9}$$

By convention, set $\varkappa(v_j) := 1$ if v_j is an isolated vertex.

Now define the *zeon representative of* G by

$$\Gamma := \sum_{j=1}^{|V|} \zeta_{\underline{j}}\, \varkappa(v_j), \tag{3.10}$$

where $1 \leq j \leq n$ and $\underline{j}$ denotes the subset representation of the integer j.

Finally, define the projection $\varepsilon_{[|V|]} : \mathcal{Z}_{|V|+|E|} \to \mathcal{Z}_{|V|}$ by linear extension of

$$\zeta_I \, \varepsilon_{[|V|]} := \zeta_{I \cap [|V|]}. \tag{3.11}$$

Theorem 3.5. *Let G be a simple graph whose vertices are in one-to-one correspondence with the nonempty subsets of $[n]$, in which each pair of vertices is adjacent if and only if they are associated with a forbidden pair of blocks of a partition of $[n]$. Let Γ denote the zeon representative of G, and let X_k denote the number of k-block partitions of $[n]$, constrained by the forbidden blocks. Then,*

$$\left\langle \int \exp\left(t\,\Gamma\right) \varepsilon_{[n]} \, d\zeta_1 \cdots d\zeta_n, t^k \right\rangle = X_k. \tag{3.12}$$

Proof. It is clear that $\varepsilon_{[n]}\left(\exp(t\,\Gamma)\right)$ is a polynomial in t with coefficients in $\mathcal{Z}_n$. By definition of the exponential function, the coefficient of t^k is $\frac{\Gamma^k}{k!}$. By construction, Γ^k corresponds to a sum of k subsets of vertices in the graph. Each of these subsets represents k blocks which may or may not correspond to a partition of the n-set. By the null-square property of zeons, each of these subsets is pairwise nonadjacent; i.e., each represents an independent set of size k in the graph. Applying the Berezin integral reveals the top-form coefficient in the zeon expansion; i.e., the union of the blocks is $[n]$. $\square$

Example 3.6. A zeon representative of the graph appearing in Figure 3 is used to count k-block non-crossing partitions of the 5-set. Note that the sum is the fifth Catalan number: $C_5 = \frac{1}{6}\binom{10}{5} = 42$.

$$\Gamma = \zeta_{\{1\}} + \zeta_{\{2\}} + \zeta_{\{3\}} + \zeta_{\{4\}} + \zeta_{\{5\}} + \zeta_{\{1,2\}} + \zeta_{\{1,5\}} + \zeta_{\{2,3\}} + \zeta_{\{3,4\}} + \zeta_{\{4,5\}} + \zeta_{\{1,2,3\}} + \zeta_{\{1,2,5\}} + \zeta_{\{1,4,5\}} + \zeta_{\{2,3,4\}} + \zeta_{\{3,4,5\}} + \zeta_{\{1,2,3,4\}} + \zeta_{\{1,2,3,5\}} + \zeta_{\{1,2,4,5\}} + \zeta_{\{1,2,4,14\}} + \zeta_{\{1,3,4,5\}} + \zeta_{\{1,3,4,15\}} + \zeta_{\{1,3,5,13\}} + \zeta_{\{2,3,4,5\}} + \zeta_{\{2,3,5,11\}} + \zeta_{\{2,4,5,8\}} + \zeta_{\{1,2,3,4,5\}} + \zeta_{\{1,3,6,7,8\}} + \zeta_{\{1,4,9,10,11\}} + \zeta_{\{2,4,6,12,13\}} + \zeta_{\{2,5,7,9,15\}} + \zeta_{\{3,5,10,12,14\}}$$

When $k = 1$, $\frac{1}{k!} \int \Gamma^k \epsilon_{[5]} d\zeta_1 \ldots d\zeta_5 = 1$

When $k = 2$, $\frac{1}{k!} \int \Gamma^k \epsilon_{[5]} d\zeta_1 \ldots d\zeta_5 = 10$

When $k = 3$, $\frac{1}{k!} \int \Gamma^k \epsilon_{[5]} d\zeta_1 \ldots d\zeta_5 = 20$

When $k = 4$, $\frac{1}{k!} \int \Gamma^k \epsilon_{[5]} d\zeta_1 \ldots d\zeta_5 = 10$

When $k = 5$, $\frac{1}{k!} \int \Gamma^k \epsilon_{[5]} d\zeta_1 \ldots d\zeta_5 = 1$

Corollary 3.7. *Let G and Γ be defined as in the statement of Theorem 3.5, and let X denote the number of partitions of $[n]$ subject to the forbidden blocks constraint. Then,*

$$\int \exp\left(\Gamma\right) \varepsilon_{[n]} \, d\zeta_1 \cdots d\zeta_n = X. \tag{3.13}$$

Proof. This follows immediately from Theorem 3.5 and the expansion

$$\exp(\Gamma)\varepsilon_{[n]} = \sum_{k=0}^{\infty} \frac{\Gamma^k}{k!}\varepsilon_{[n]} = \sum_{k=0}^{n} \frac{\Gamma^k}{k!}\varepsilon_{[n]}, \tag{3.14}$$

from which

$$\int \exp(\Gamma)\varepsilon_{[n]}\, d\zeta_1 \cdots d\zeta_n = \sum_{k=0}^{n} X_k. \tag{3.15}$$

□

Using this graph-theoretic approach, some familiar numbers can now be recovered: Stirling numbers of the second kind, Bell numbers, Catalan numbers, and Bessel numbers.

Let $\left\{ {n \atop k} \right\}$ denote the *Stirling number of the second kind* defined as the number of partitions of the n-set into k nonempty blocks. Let B_n denote the n^{th} *Bell number*, defined as the total number of partitions of the n-set into nonempty blocks.

Let $C(n,k)$ denote the number of non-crossing partitions of $[n]$ into k blocks, and let C_n denote the n^{th} *Catalan number*. It is known that C_n gives the total number of non-crossing partitions of the n-set.

Let $\mathcal{B}(n,k)$ denote the number of non-overlapping partitions of $[n]$ into k blocks, and let $\mathcal{B}_n$ denote the n^{th} *Bessel number*. It is known that $\mathcal{B}_n$ gives the total number of non-overlapping partitions of the n-set [5].

Considering the zeon representatives of the graphs $G_\emptyset$, G_{cr}, and G_{ov} as defined previously, the next result is an immediate consequence of the previous corollary.

Corollary 3.8. *Let $\Gamma_\emptyset$ denote the zeon representative of $G_\emptyset$, let Γ_{cr} denote the zeon representative of G_{cr}, and let Γ_{ov} denote the zeon representative of G_{ov}. Then,*

$$\left\langle \int \exp\left(t\,\Gamma_\emptyset\right) \varepsilon_{[n]}\, d\zeta_1 \cdots d\zeta_n, t^k \right\rangle = \left\{ {n \atop k} \right\}, \tag{3.16}$$

$$\int \exp\left(\Gamma_\emptyset\right) \varepsilon_{[n]}\, d\zeta_1 \cdots d\zeta_n = B_n, \tag{3.17}$$

$$\left\langle \int \exp\left(t\,\Gamma_{cr}\right) \varepsilon_{[n]}\, d\zeta_1 \cdots d\zeta_n, t^k \right\rangle = C(n,k), \tag{3.18}$$

$$\int \exp\left(\Gamma_{cr}\right) \varepsilon_{[n]}\, d\zeta_1 \cdots d\zeta_n = C_n, \tag{3.19}$$

$$\left\langle \int \exp\left(t\,\Gamma_{ov}\right) \varepsilon_{[n]}\, d\zeta_1 \cdots d\zeta_n, t^k \right\rangle = \mathcal{B}(n,k), \tag{3.20}$$

$$\int \exp\left(\Gamma_{ov}\right) \varepsilon_{[n]}\, d\zeta_1 \cdots d\zeta_n = \mathcal{B}_n. \tag{3.21}$$

4. Computations on Lattices of Partitions

In this section, functions are defined on $\Pi([n])$, the lattice of partitions of the n-set. Given a partition $\pi \in \Pi([n])$, the notation $|\pi|$ will be used to denote the number of *blocks* (i.e., the number of pairwise disjoint subsets) in the partition π. Blocks $b \in \pi$ will also be allowed to serve as multi-indices in the zeon algebra. Note that when b is a block; i.e., subset, $|b|$ denotes the cardinality of b.

The general strategy is to first define a complex-valued function $f : 2^{[n]} \to \mathbb{C}$ satisfying $f(\emptyset) := 1$, and then to extend this to a complex-valued function $h : \Pi([n]) \to \mathbb{C}$ by taking products over the blocks of each partition.

Let $f : 2^{[n]} \to \mathbb{C}$ be a function on the power set of $[n]$ with $f(\emptyset) = 1$. Define the function $h_f : \Pi([n]) \to \mathbb{C}$ by

$$h_f(\pi) = \prod_{b\in\pi} f(b) \tag{4.1}$$

for each $\pi \in \Pi([n])$.

With f as defined above, let $G = (V, E)$ be a partition constraint graph of the n-set. Now define the *weighted zeon representative of* G by

$$\Upsilon := \sum_{j=1}^{|V|} f(\underline{j})\, \zeta_{\underline{j}}\, \varkappa(v_j), \tag{4.2}$$

where $\varkappa$ is defined as in (3.9).

Note that Υ is a weighted sum over the vertices of the graph G, which are in one-to-one correspondence with nonempty subsets of the n-set, i.e. blocks of partitions. For $1 \le j \le 2^n - 1$, vertex v_j is associated with block $\underline{j}$ having corresponding weight $f(\underline{j})$. Moreover, by construction each vertex v_j is associated with the blade $\zeta_{\underline{j}}\varkappa(v_j) = \zeta_{\underline{j}\cup E_j}$, where E_j represents the set of edges incident with vertex v_j in G. Recalling that a pair of vertices in G is adjacent (incident with a common edge) if and only if the vertices represent a forbidden pair of blocks in the lattice of partitions, it is evident that products of terms of Υ will be zero when their corresponding blocks are either not pairwise disjoint or contain forbidden pairs of blocks.

Theorem 4.1. *Let $0 < k \le n$, and let Υ be the weighted zeon representative of a partition constraint graph G. Then,*

$$\int \Upsilon^k \varepsilon_{[n]}\, d\zeta_1 \cdots d\zeta_n = \sum_{\substack{\pi\in\Pi_G(n)\\ |\pi|=k}} k!\, h_f(\pi). \tag{4.3}$$

Proof. By the null-square property of zeons, the multinomial theorem gives

$$\begin{aligned}\Upsilon^k &= \sum_{\substack{\ell_1+\cdots+\ell_{|V|}=k\\ (\ell_1,\ldots,\ell_{|V|})\in\{0,1\}^{|V|}}} \binom{k}{\ell_1,\ldots,\ell_{|V|}} \prod_{m=1}^{|V|} f(\underline{m})^{\ell_m} {\zeta_{\underline{m}}}^{\ell_m} \varkappa(v_m)^{\ell_m}\\ &= \sum_{\{I\subseteq V: |I|=k\}} k! \prod_{v_\ell\in I} f(\underline{\ell})\zeta_{\underline{\ell}}\varkappa(v_\ell). \end{aligned} \tag{4.4}$$

Note that by construction of $\varkappa(v_\ell)$, the product $\prod_{v_\ell \in I} f(\underline{\ell})\zeta_{\underline{\ell}}\varkappa(v_\ell)$ is zero if the blocks represented by vertex subset I are either not pairwise disjoint or include forbidden pairs. Hence, the sum is over vertex subsets representing blocks of restricted partitions. In order to sieve out those subsets representing partitions of the n-set, the zeon-Berezin integral is applied after projection onto $\mathcal{Z}_n$. That is,

$$\begin{aligned}\int \Upsilon^k \varepsilon_{[n]}\, d\zeta_1 \cdots d\zeta_n &= \sum_{\{I: \bigcup_{v_\ell \in I} \underline{\ell} = [n] \text{ and } |I| = k\}} k! \prod_{v_\ell \in I} f(\underline{\ell}) \\ &= \sum_{\substack{\pi \in \Pi_G(n) \\ |\pi| = k}} k! \prod_{b \in \pi} f(b). \end{aligned} \tag{4.5}$$

□

Example 4.2. In Figure 5, the weighted zeon representative Υ of G_{cr} is used to compute

$$\sum_{\substack{\pi \in NC(5) \\ |\pi| = k}} k!\, h_f(\pi) = \sum_{\substack{\pi \in NC(5) \\ |\pi| = k}} k! \prod_{b \in \pi} f(b)$$

for a given scalar-valued function $f : 2^{[5]} \to \mathbb{F}$. Note that $f(b)$ is written as f_b for convenience.

Corollary 4.3.

$$\int e^{\Upsilon} \varepsilon_{[n]}\, d\zeta_1 \cdots d\zeta_n = \sum_{\pi \in \Pi_G(n)} h_f(\pi). \tag{4.6}$$

Proof. By definition of the exponential function and the null square property of zeons,

$$e^{\Upsilon} = \sum_{k=0}^{\infty} \frac{1}{k!} \Upsilon^k = \sum_{k=0}^{n} \frac{1}{k!} \Upsilon^k. \tag{4.7}$$

Whence, applying Theorem 4.1,

$$\begin{aligned}\int e^{\Upsilon} \varepsilon_{[n]}\, d\zeta_1 \cdots d\zeta_n &= \sum_{k=0}^{n} \frac{1}{k!} \int \Upsilon^k \varepsilon_{[n]}\, d\zeta_1 \cdots d\zeta_n = \sum_{k=0}^{n} \sum_{\substack{\pi \in \Pi_G(n) \\ |\pi| = k}} h_f(\pi) \\ &= \sum_{\pi \in \Pi_G(n)} h_f(\pi). \end{aligned} \tag{4.8}$$

□

Remark 4.4. Note that $\exp(t\,\Upsilon)$ is a polynomial in t with zeon coefficients. In particular, the Berezin integral of the coefficient of t^k represents a sum over k-block partitions of the n-set. That is,

$$\left\langle \int e^{t\,\Upsilon} \varepsilon_{[n]}\, d\zeta_1 \cdots d\zeta_n, t^k \right\rangle = \sum_{\substack{\pi \in \Pi_G(n) \\ |\pi| = k}} h_f(\pi). \tag{4.9}$$

```
In[36]:= Print["ϒ = ", ϒ]
         For[k = 1, k ≤ 5, k++,
          Print["When k = ", k, ",  ∫ϒ^k ϵ_[5] dζ_1...dζ_5 = ", ZeonBerezin[Proj[Expand[ϒ^k], 5], ζ_{1,2,3,4,5}]]]
```

$$\begin{aligned}\Upsilon = {} & f_{\{1\}}\zeta_{\{1\}} + f_{\{2\}}\zeta_{\{2\}} + f_{\{3\}}\zeta_{\{3\}} + f_{\{4\}}\zeta_{\{4\}} + f_{\{5\}}\zeta_{\{5\}} + f_{\{1,2\}}\zeta_{\{1,2\}} + \\ & f_{\{1,5\}}\zeta_{\{1,5\}} + f_{\{2,3\}}\zeta_{\{2,3\}} + f_{\{3,4\}}\zeta_{\{3,4\}} + f_{\{4,5\}}\zeta_{\{4,5\}} + f_{\{1,2,3\}}\zeta_{\{1,2,3\}} + f_{\{1,2,5\}}\zeta_{\{1,2,5\}} + \\ & f_{\{1,4,5\}}\zeta_{\{1,4,5\}} + f_{\{2,3,4\}}\zeta_{\{2,3,4\}} + f_{\{3,4,5\}}\zeta_{\{3,4,5\}} + f_{\{1,2,3,4\}}\zeta_{\{1,2,3,4\}} + f_{\{1,2,3,5\}}\zeta_{\{1,2,3,5\}} + \\ & f_{\{1,2,4,5\}}\zeta_{\{1,2,4,5\}} + f_{\{1,2,4\}}\zeta_{\{1,2,4,14\}} + f_{\{1,3,4,5\}}\zeta_{\{1,3,4,5\}} + f_{\{1,3,4\}}\zeta_{\{1,3,4,15\}} + f_{\{1,3,5\}}\zeta_{\{1,3,5,13\}} + \\ & f_{\{2,3,4,5\}}\zeta_{\{2,3,4,5\}} + f_{\{2,3,5\}}\zeta_{\{2,3,5,11\}} + f_{\{2,4,5\}}\zeta_{\{2,4,5,8\}} + f_{\{1,2,3,4,5\}}\zeta_{\{1,2,3,4,5\}} + \\ & f_{\{1,3\}}\zeta_{\{1,3,6,7,8\}} + f_{\{1,4\}}\zeta_{\{1,4,9,10,11\}} + f_{\{2,4\}}\zeta_{\{2,4,6,12,13\}} + f_{\{2,5\}}\zeta_{\{2,5,7,9,15\}} + f_{\{3,5\}}\zeta_{\{3,5,10,12,14\}}\end{aligned}$$

When k = 1, $\int \Upsilon^k \epsilon_{[5]} d\zeta_1 \ldots d\zeta_5 = f_{\{1,2,3,4,5\}}$

When k = 2, $\int \Upsilon^k \epsilon_{[5]} d\zeta_1 \ldots d\zeta_5 = 2 f_{\{4,5\}} f_{\{1,2,3\}} + 2 f_{\{3,4\}} f_{\{1,2,5\}} + 2 f_{\{2,3\}} f_{\{1,4,5\}} + 2 f_{\{1,5\}} f_{\{2,3,4\}} + 2 f_{\{1,2\}} f_{\{3,4,5\}} + 2 f_{\{5\}} f_{\{1,2,3,4\}} + 2 f_{\{4\}} f_{\{1,2,3,5\}} + 2 f_{\{3\}} f_{\{1,2,4,5\}} + 2 f_{\{2\}} f_{\{1,3,4,5\}} + 2 f_{\{1\}} f_{\{2,3,4,5\}}$

When k = 3, $\int \Upsilon^k \epsilon_{[5]} d\zeta_1 \ldots d\zeta_5 =$
$6 f_{\{5\}} f_{\{1,4\}} f_{\{2,3\}} + 6 f_{\{4\}} f_{\{1,5\}} f_{\{2,3\}} + 6 f_{\{3\}} f_{\{1,5\}} f_{\{2,4\}} + 6 f_{\{5\}} f_{\{1,2\}} f_{\{3,4\}} + 6 f_{\{2\}} f_{\{1,5\}} f_{\{3,4\}} + 6 f_{\{1\}} f_{\{2,5\}} f_{\{3,4\}} + 6 f_{\{4\}} f_{\{1,2\}} f_{\{3,5\}} + 6 f_{\{3\}} f_{\{1,2\}} f_{\{4,5\}} + 6 f_{\{2\}} f_{\{1,3\}} f_{\{4,5\}} + 6 f_{\{1\}} f_{\{2,3\}} f_{\{4,5\}} + 6 f_{\{4\}} f_{\{5\}} f_{\{1,2,3\}} + 6 f_{\{3\}} f_{\{5\}} f_{\{1,2,4\}} + 6 f_{\{3\}} f_{\{4\}} f_{\{1,2,5\}} + 6 f_{\{2\}} f_{\{5\}} f_{\{1,3,4\}} + 6 f_{\{2\}} f_{\{4\}} f_{\{1,3,5\}} + 6 f_{\{2\}} f_{\{3\}} f_{\{1,4,5\}} + 6 f_{\{1\}} f_{\{5\}} f_{\{2,3,4\}} + 6 f_{\{1\}} f_{\{4\}} f_{\{2,3,5\}} + 6 f_{\{1\}} f_{\{3\}} f_{\{2,4,5\}} + 6 f_{\{1\}} f_{\{2\}} f_{\{3,4,5\}}$

When k = 4, $\int \Upsilon^k \epsilon_{[5]} d\zeta_1 \ldots d\zeta_5 = 24 f_{\{3\}} f_{\{4\}} f_{\{5\}} f_{\{1,2\}} + 24 f_{\{2\}} f_{\{4\}} f_{\{5\}} f_{\{1,3\}} + 24 f_{\{2\}} f_{\{3\}} f_{\{5\}} f_{\{1,4\}} + 24 f_{\{2\}} f_{\{3\}} f_{\{4\}} f_{\{1,5\}} + 24 f_{\{1\}} f_{\{4\}} f_{\{5\}} f_{\{2,3\}} + 24 f_{\{1\}} f_{\{3\}} f_{\{5\}} f_{\{2,4\}} + 24 f_{\{1\}} f_{\{3\}} f_{\{4\}} f_{\{2,5\}} + 24 f_{\{1\}} f_{\{2\}} f_{\{5\}} f_{\{3,4\}} + 24 f_{\{1\}} f_{\{2\}} f_{\{4\}} f_{\{3,5\}} + 24 f_{\{1\}} f_{\{2\}} f_{\{3\}} f_{\{4,5\}}$

When k = 5, $\int \Upsilon^k \epsilon_{[5]} d\zeta_1 \ldots d\zeta_5 = 120 f_{\{1\}} f_{\{2\}} f_{\{3\}} f_{\{4\}} f_{\{5\}}$

FIGURE 5. Summing a function over $NC(5)$.

Corollary 4.5. *Let $\Upsilon_\emptyset$, Υ_{cr}, and Υ_{ov} be the weighted zeon representatives of graphs $G_\emptyset$, G_{cr}, and G_{ov} as used in Corollary 3.8. Then,*

$$\int {\Upsilon_\emptyset}^k \varepsilon_{[n]}\, d\zeta_1 \cdots d\zeta_n = \sum_{\substack{\pi\in\Pi([n])\\ |\pi|=k}} k!\, h_f(\pi), \tag{4.10}$$

$$\int {\Upsilon_{cr}}^k \varepsilon_{[n]}\, d\zeta_1 \cdots d\zeta_n = \sum_{\substack{\pi\in\mathrm{NC}(n)\\ |\pi|=k}} k!\, h_f(\pi), \text{ and} \tag{4.11}$$

$$\int {\Upsilon_{ov}}^k \varepsilon_{[n]}\, d\zeta_1 \cdots d\zeta_n = \sum_{\substack{\pi\in\mathrm{NO}(n)\\ |\pi|=k}} k!\, h_f(\pi). \tag{4.12}$$

Proof. By construction of the graphs $G_\emptyset$, G_{cr}, and G_{ov}, Theorem 4.1 results in summing over k-block partitions with no restrictions, no crossing blocks, and no overlapping blocks, respectively. □

4.1. Computations on Lattice Segments. The graph-theoretic approach detailed above does not easily lend itself to summing values over segments of a lattice of partitions. In this section, methods are developed using zeon-Berezin operator calculus to sum functions over lattice segments. The initial results are established on the lattice of partitions $\Pi([n])$.

Let $\mathcal{L}$ denote the canonical lowering operator on $\mathcal{Z}_n$ and define the operator Φ on $\mathcal{Z}_n$ by linear extension of

$$\zeta_b \Phi := f(b)\,\zeta_b \tag{4.13}$$

for each subset $b \subseteq [n]$.

Let t denote a scalar variable. For each block $b \subseteq [n]$, define the linear operator $\mathcal{D}_b$ by

$$\mathcal{D}_b := \frac{1}{|b|!} \bigoplus_{k=0}^{|b|-1} \frac{\partial^k}{\partial t^k}. \tag{4.14}$$

Theorem 4.6. *For fixed partition $\pi \in \Pi([n])$ and nonzero scalar parameter t, the following holds:*

$$\prod_{b\in\pi} \mathcal{D}_b \left[\oint \left(t + \sum_{j=0}^{|b|-1} \frac{1}{j!} \zeta_b \mathcal{L}^j \Phi \right)^{|b|} d\zeta_{b_1} \cdots d\zeta_{b_{|b|}} \right]_{t=0} = \sum_{\sigma\le\pi} h_f(\sigma). \tag{4.15}$$

Consequently,

$$\mathcal{D}_{[n]} \left[\oint \left(t + \sum_{j=0}^{n-1} \frac{1}{j!} \zeta_{[n]} \mathcal{L}^j \Phi \right)^{n} d\zeta_1 \cdots d\zeta_n \right]_{t=0} - \prod_{b\in\pi} \mathcal{D}_b \left[\oint \left(t + \sum_{j=0}^{|b|-1} \frac{1}{j!} \zeta_b \mathcal{L}^j \Phi \right)^{|b|} d\zeta_{b_1} \cdots d\zeta_{b_{|b|}} \right]_{t=0} = \sum_{\sigma>\pi} h_f(\sigma). \tag{4.16}$$

Proof. Begin by noting that for any block b, $\zeta_b\mathcal{L}$ is a sum of blades representing all proper subsets of b having cardinality $|b|-1$.

Using the definition of $\mathcal{L}$, the following is easily established:

$$\zeta_b \mathcal{L}^j = j! \sum_{\substack{b'\subseteq b \\ |b'|=|b|-j}} \zeta_{b'}. \tag{4.17}$$

This implies

$$\sum_{j=0}^{|b|-1} \left(\frac{1}{j!} \zeta_b \mathcal{L}^j\right)\Phi = \sum_{\emptyset\neq b'\subseteq b} f(b')\,\zeta_{b'}. \tag{4.18}$$

Whence, applying the multinomial theorem gives

$$\left(t + \sum_{j=0}^{|b|-1} \frac{1}{j!} \zeta_b \mathcal{L}^j \Phi \right)^{|b|} = \sum_{k_\emptyset+\cdots+k_b=|b|} \binom{|b|}{k_\emptyset, \ldots, k_b} t^{k_\emptyset} \prod_{\emptyset\neq b'\subseteq b} (f(b'))^{k_{b'}} \zeta_{b'}^{k_{b'}}. \tag{4.19}$$

By the null-square property of zeons, the only nonzero terms correspond to the case $k_{b'} = 0$ or $k_{b'} = 1$ when $b' \neq \emptyset$. Hence, nonzero terms correspond to products on pairwise disjoint subsets of the block b, with exponent $k_\emptyset$ remaining to be determined. Considering the Berezin integral, only collections of sub-blocks whose union is b remain; i.e., each of these terms represents a partition of b. It follows that $k_\emptyset = |b| - |\sigma|$, where $|b|$ denotes the cardinality of the block b, and $|\sigma|$ denotes the number of blocks in the resulting partition σ of b. In summary,

$$\oint \left(\sum_{k_\emptyset + \cdots + k_b = |b|} \binom{|b|}{k_\emptyset, \ldots, k_b} t^{k_\emptyset} \prod_{b' \subseteq b} (f(b'))^{k_{b'}} \, \zeta_{b'}^{k_{b'}} \right) d\zeta_{b_1} \cdots d\zeta_{b_{|b|}}$$
$$= \sum_{\sigma \in \Pi(b)} \frac{t^{|b|-|\sigma|}|b|!}{(|b|-|\sigma|)!} \prod_{b' \in \sigma} f(b') = \sum_{\sigma \leq \hat{1}_b} \frac{t^{|b|-|\sigma|}|b|!}{(|b|-|\sigma|)!} h_f(\sigma). \tag{4.20}$$

An immediate consequence is that when $0 \leq k < |b|$,

$$\frac{\partial^k}{\partial t^k} \left[\oint \left(t + \sum_{j=0}^{|b|-1} \frac{1}{j!} \zeta_b \mathcal{L}^j \Phi \right)^{|b|} d\zeta_{b_1} \cdots d\zeta_{b_{|b|}} \right]_{t=0} = |b|! \sum_{\substack{\sigma \leq \hat{1}_b \\ |\sigma| = |b|-k}} h_f(\sigma). \tag{4.21}$$

Applying the operator $\mathcal{D}_b := \frac{1}{|b|!} \sum_{k=0}^{|b|-1} \frac{\partial^k}{\partial t^k}$ gives

$$\mathcal{D}_b \left[\oint \left(t + \sum_{j=0}^{|b|-1} \frac{1}{j!} \zeta_b \mathcal{L}^j \Phi \right)^{|b|} d\zeta_{b_1} \cdots d\zeta_{b_{|b|}} \right]_{t=0} = \sum_{\sigma \leq \hat{1}_b} h_f(\sigma). \tag{4.22}$$

Since the blocks of any partition are pairwise disjoint, taking the product over all blocks b in the fixed partition π results in

$$\prod_{b \in \pi} \mathcal{D}_b \left[\oint \left(t + \sum_{j=0}^{|b|-1} \frac{1}{j!} \zeta_b \mathcal{L}^j \Phi \right)^{|b|} d\zeta_{b_1} \cdots d\zeta_{b_{|b|}} \right]_{t=0} = \sum_{\sigma \leq \pi} h_f(\sigma). \tag{4.23}$$

Letting $\pi = \hat{1}_{[n]}$, an immediate consequence of (4.23) is that

$$\mathcal{D}_{[n]} \left[\oint \left(t + \sum_{j=0}^{n-1} \frac{1}{j!} \zeta_{[n]} \mathcal{L}^j \Phi \right)^{n} d\zeta_1 \cdots d\zeta_n \right]_{t=0} = \sum_{\sigma \leq \hat{1}_{[n]}} h_f(\sigma). \tag{4.24}$$

The proof is concluded by combining Equations (4.23) and (4.24) and observing that for fixed partition π,

$$\sum_{\hat{0} \leq \sigma \leq \hat{1}} h_f(\sigma) = \sum_{\sigma \leq \pi} h_f(\sigma) + \sum_{\sigma > \pi} h_f(\sigma) \tag{4.25}$$

□

Zeon operator calculus methods also lead to some special case partition summations.

Definition 4.7. Fix partitions $\pi, \sigma \in \Pi([n])$. We say π is a *proper refinement* of σ and write $\pi \prec \sigma$ if and only if every block of π is a proper subset of some block of σ. That is,

$$\pi \prec \sigma \Leftrightarrow (\forall \gamma \in \pi)(\exists \delta \in \sigma)[\gamma \subsetneq \delta]. \tag{4.26}$$

Note that by changing the limits of summation on the left-hand side of (4.15) in Theorem 4.6, the following corollary is immediately obtained for partitions free of singleton blocks.

Corollary 4.8. *Let $\pi \in \Pi([n])$ contain no singleton blocks, and let t be a nonzero scalar parameter t. Then,*

$$\prod_{b\in\pi} \mathcal{D}_b \left[\oint \left(t + \sum_{j=1}^{|b|-1} \frac{1}{j!} \zeta_b \mathcal{L}^j \Phi \right)^{|b|} d\zeta_{b_1} \cdots d\zeta_{b_{|b|}} \right]_{t=0} = \sum_{\sigma \prec \pi} h_f(\sigma). \tag{4.27}$$

Proof. For each block $b \in \pi$, letting the summation begin with $j = 1$ allows only proper subsets of b to be considered. □

Turning now to the canonical raising operator, another special case partition summation is possible.

Definition 4.9. Let $b \subseteq [n]$ be nonempty. A partition $\pi \in \Pi([n])$ is said to be *b-admissible* if $\exists \gamma \in \pi$ with $b \subseteq \gamma$. That is, b is a block in π or b is a block in some refinement of π. For fixed b, the collection of b-admissible partitions of $[n]$ is denoted by $\Pi_{ad(b)}([n])$.

Notation. For any subset $b \subseteq [n]$, let $\overline{b}$ denote the *complement* of b; i.e, $\overline{b} := [n] \setminus b$.

Proposition 4.10. *Let $b \subseteq [n]$ be a fixed nonempty block, let $\overline{b}$ denote the complement of b, and let t be a nonzero scalar parameter. Then,*

$$\mathcal{D}_{\overline{b}} \left[\oint \left(\sum_{j=0}^{n-|b|} \frac{1}{j!} \zeta_b \mathcal{R}^j \Phi \right) \left(t + \sum_{j=0}^{n-|b|-1} \frac{1}{j!} \zeta_{\overline{b}} \mathcal{L}^j \Phi \right)^{n-|b|} d\zeta_1 \cdots d\zeta_n \right]_{t=0} = \sum_{\sigma \in \Pi_{ad(b)}([n])} h_f(\sigma). \tag{4.28}$$

Proof. First note that the definition of $\mathcal{R}$ implies

$$\zeta_b \mathcal{R}^j = j! \sum_{\substack{b' \supseteq b \\ |b'| = |b|+j}} \zeta_{b'}, \tag{4.29}$$

since there are $j!$ "paths" by which ζ_b can be "raised" to $\zeta_{b'}$; i.e., the additional j index elements of $\zeta_{b'}$ can be appended to the multi-index of ζ_b in any order.

Working in $\mathcal{Z}_n$, it is apparent that $\zeta_b \mathcal{R}^j = 0$ for all $j > n - |b|$, so

$$\sum_{j=0}^{n-|b|} (\frac{1}{j!} \zeta_b \mathcal{R}^j) \Phi = \sum_{\beta \supseteq b} f(\beta)\, \zeta_\beta. \tag{4.30}$$

Using properties of canonical lowering as in the proof of Theorem 4.6,

$$\sum_{j=0}^{n-|b|-1} \frac{1}{j!}\zeta_{\overline{b}}\mathcal{L}^j\Phi = \sum_{\emptyset\neq\overline{b}'\subseteq\overline{b}} f(\overline{b}')\,\zeta_{\overline{b}'}. \tag{4.31}$$

Thus,

$$\begin{aligned}
&\left(\sum_{j=0}^{n-|b|}(\frac{1}{j!}\zeta_b\mathcal{R}^j)\Phi\right)\left(t+\sum_{j=0}^{n-|b|-1}\frac{1}{j!}\zeta_{\overline{b}}\mathcal{L}^j\Phi\right)^{n-|b|}\\
&\quad=\left(\sum_{\beta\supseteq b} f(\beta)\,\zeta_\beta\right)\left(\sum_{k_\emptyset+\cdots+k_{\overline{b}}=|\overline{b}|}\binom{|\overline{b}|}{k_\emptyset,\ldots,k_{\overline{b}}}t^{k_\emptyset}\prod_{\overline{b}'\subseteq\overline{b}}(f(\overline{b}'))^{k_{\overline{b}'}}\,{\zeta_{\overline{b}'}}^{k_{\overline{b}'}}\right)\\
&\quad=\left(\sum_{\beta\supseteq b} f(\beta)\,\zeta_\beta\right)\left(\sum_{k_\emptyset+\cdots+k_{\overline{b}}=|\overline{b}|}\frac{|\overline{b}|!}{k_\emptyset!}t^{k_\emptyset}\prod_{\overline{b}'\subseteq\overline{b}}(f(\overline{b}'))^{k_{\overline{b}'}}\,{\zeta_{\overline{b}'}}^{k_{\overline{b}'}}\right),
\end{aligned}\tag{4.32}$$

where k_γ is either 0 or 1 for every nonempty block γ, and the number of nontrivial blocks in the product is given by $|\overline{b}|-k_\emptyset$.

Now expanding the product and applying the Berezin integral,

$$\begin{aligned}
&\oint\left(\sum_{\beta\supseteq b} f(\beta)\,\zeta_\beta\right)d\zeta_\beta\left(\sum_{k_\emptyset+\cdots+k_{\overline{b}}=|\overline{b}|}\frac{|\overline{b}|!}{k_\emptyset!}t^{k_\emptyset}\prod_{\overline{b}'\subseteq\overline{b}}(f(\overline{b}'))^{k_{\overline{b}'}}\,{\zeta_{\overline{b}'}}^{k_{\overline{b}'}}\right)d\zeta_{\overline{\beta}_1}\cdots d\zeta_{\overline{\beta}_{|\overline{\beta}|}}\\
&\quad=\sum_{\substack{\beta\supseteq b\\ \sigma\leq\hat{1}_{\overline{\beta}}}}\frac{|\overline{b}|!}{(|\overline{b}|-|\sigma|)!}t^{|\overline{b}|-|\sigma|}f(\beta)h_f(\sigma),
\end{aligned}\tag{4.33}$$

where $|\sigma| = |\overline{b}| - k_\emptyset$. The proof is concluded by applying $\mathcal{D}_{\overline{b}}$ and evaluating at $t=0$. □

4.2. Computations on restricted lattice segments. In order to extend this approach to non-crossing or non-overlapping partitions, the ideas underlying the weighted zeon representative of a graph are used to define the restriction mapping below.

Definition 4.11. Let G be a simple graph on 2^n-1 vertices associated with nonempty subsets of the n-set. The *G-restriction mapping* $\Psi:\mathcal{Z}_n\to\mathcal{Z}_{n+|E|}$ is defined by linear extension of

$$\zeta_{\underline{j}}\Psi := \zeta_{\underline{j}}\varkappa(v_j). \tag{4.34}$$

Theorem 4.12. *For fixed G-restricted partition π, the following holds:*

$$\prod_{b\in\pi}\mathcal{D}_b\left[\oint\left(t+\sum_{j=0}^{|b|-1}\frac{1}{j!}\zeta_b\mathcal{L}^j\,\Phi\,\Psi\right)^{|b|}\varepsilon_{[n]}\,d\zeta_{b_1}\cdots d\zeta_{b_{|b|}}\right]_{t=0}=\sum_{\substack{\sigma\leq\pi\\ \sigma\in\Pi_G(n)}}h_f(\sigma), \tag{4.35}$$

where Φ *and* Ψ *are as defined in* (4.13) *and* (4.34)*. Consequently,*

$$\begin{aligned}
&\mathcal{D}_{[n]}\left[\oint\left(t+\sum_{j=0}^{n-1}\frac{1}{j!}\zeta_{[n]}\mathcal{L}^j\,\Phi\,\Psi\right)^n \varepsilon_{[n]}\,d\zeta_1\cdots d\zeta_n\right]_{t=0}\\
&\quad-\prod_{b\in\pi}\mathcal{D}_b\left[\oint\left(t+\sum_{j=0}^{|b|-1}\frac{1}{j!}\zeta_b\mathcal{L}^j\,\Phi\,\Psi\right)^{|b|} \varepsilon_{[n]}\,d\zeta_{b_1}\cdots d\zeta_{b_{|b|}}\right]_{t=0}\\
&=\sum_{\substack{\sigma>\pi\\ \sigma\in\Pi_G(n)}} h_f(\sigma).
\end{aligned}\tag{4.36}$$

Proof. Recall that for any block b and integer $0\le j\le |b|$,

$$\zeta_b\,\mathcal{L}^j\Phi = j!\sum_{\substack{b'\subseteq b\\ |b'|=|b|-j}} f(b')\,\zeta_{b'}.\tag{4.37}$$

Hence,

$$\frac{1}{j!}\zeta_b\,\mathcal{L}^j\Phi\Psi=\sum_{\substack{\underline{\ell}\subseteq b\\ |\underline{\ell}|=|b|-j}} f(\underline{\ell})\,\zeta_{\underline{\ell}}\varkappa(v_\ell).\tag{4.38}$$

Whereby, substitution and application of the multinomial theorem gives

$$\begin{aligned}
\left(t+\sum_{j=0}^{|b|-1}\frac{1}{j!}\zeta_b\,\mathcal{L}^j\Phi\Psi\right)^{|b|}&=\left(t+\sum_{\emptyset\neq\underline{\ell}\subseteq b} f(\underline{\ell})\,\zeta_{\underline{\ell}}\varkappa(v_\ell)\right)^{|b|}\\
&=\sum_{k_\emptyset+\cdots+k_b=|b|}\binom{|b|}{k_\emptyset,\ldots,k_b}t^{k_\emptyset}\prod_{\emptyset\neq\underline{\ell}\subseteq b} f(\underline{\ell})^{k_{\underline{\ell}}}{\zeta_{\underline{\ell}}}^{k_{\underline{\ell}}}\varkappa(v_\ell)^{k_{\underline{\ell}}}.
\end{aligned}\tag{4.39}$$

By the null-square property of zeons, all integers $k_{\underline{\ell}}$ are either 0 or 1 when $\ell>0$. Construction of G guarantees that the only nonzero terms in the sum correspond to disjoint tuples of subsets of b containing no forbidden pair of blocks. Specifically, a nonzero term of the form

$$t^{k_\emptyset}\prod_{\emptyset\neq\underline{\ell}\subseteq b} f(\underline{\ell})^{k_{\underline{\ell}}}{\zeta_{\underline{\ell}}}^{k_{\underline{\ell}}}\varkappa(v_\ell)^{k_{\underline{\ell}}}$$

corresponds to a disjoint $(|b|-k_\emptyset)$-tuple of subsets of b with no forbidden pair of blocks.

Note that the corresponding multinomial coefficient on the term above is

$$\binom{|b|}{k_\emptyset,\ldots,k_b}=\frac{|b|!}{k_\emptyset!}.\tag{4.40}$$

Projecting onto $\mathcal{Z}_n$ and taking the projective Berezin integral, one now obtains

$$\oint \left(t + \sum_{j=0}^{|b|-1} \frac{1}{j!} \zeta_b \, \mathcal{L}^j \Phi\Psi \right)^{|b|} \varepsilon_{[n]} \, d\zeta_{b_1} \, \cdots \, d\zeta_{b_{|b|}}$$

$$= \oint \left(\sum_{\substack{k_\emptyset, \ldots, k_b \in \{0,1\} \\ k_\emptyset + \cdots + k_b = |b|}} \frac{|b|!}{k_\emptyset !} t^{k_\emptyset} \prod_{\substack{\emptyset \neq \underline{\ell} \subseteq b \\ \text{no forbidden pairs}}} f(\underline{\ell})^{k_{\underline{\ell}}} \zeta_{\underline{\ell}}{}^{k_{\underline{\ell}}} \right) d\zeta_{b_1} \cdots d\zeta_{b_{|b|}}. \quad (4.41)$$

Applying $\mathcal{D}_b$ and evaluating at $t = 0$ gives

$$\mathcal{D}_b \left[\oint \left(t + \sum_{j=0}^{|b|-1} \frac{1}{j!} \zeta_b \, \mathcal{L}^j \Phi\Psi \right)^{|b|} \varepsilon_{[n]} \, d\zeta_{b_1} \, \cdots \, d\zeta_{b_{|b|}} \right]_{t=0} = \sum_{\substack{\sigma \leq b \\ \sigma \in \Pi_G(n)}} h_f(\sigma). \quad (4.42)$$

Whence, taking the product over disjoint blocks of arbitrary partition π completes the first part of the proof.

Substitution in (4.42) gives

$$\mathcal{D}_{[n]} \left[\oint \left(t + \sum_{j=0}^{n-1} \frac{1}{j!} \zeta_{[n]} \, \mathcal{L}^j \Phi\Psi \right)^{n} \varepsilon_{[n]} \, d\zeta_1 \, \cdots \, d\zeta_n \right]_{t=0} = \sum_{\substack{\sigma \leq \hat{1}_{[n]} \\ \sigma \in \Pi_G(n)}} h_f(\sigma), \quad (4.43)$$

from which (4.36) is deduced. □

Example 4.13. Given a scalar-valued function $f : 2^{[5]} \to \mathbb{F}$ as in Example 4.2, the lowering operator and restriction mapping are used to sum over non-crossing partitions of the 5-set with Mathematica.

```
In[59]:= Sum[D[ZeonBerezin[Expand[
           Proj[Expand[(t + Sum[Expand[(1 / j !) ψ5[f[PwrL[ζb, j]], A]], {j, 0, Length[b] - 1}])^5 / 5!] /.
            {t^5 → 0}, 5]], ζb], {t, k}] /. {t → 0}, {k, 0, 4}]
```

Out[59]=

$$\begin{aligned}
&f_{\{1\}} f_{\{2\}} f_{\{3\}} f_{\{4\}} f_{\{5\}} + f_{\{3\}} f_{\{4\}} f_{\{5\}} f_{\{1,2\}} + f_{\{2\}} f_{\{4\}} f_{\{5\}} f_{\{1,3\}} + f_{\{2\}} f_{\{3\}} f_{\{5\}} f_{\{1,4\}} + f_{\{2\}} f_{\{3\}} f_{\{4\}} f_{\{1,5\}} + \\
&f_{\{1\}} f_{\{4\}} f_{\{5\}} f_{\{2,3\}} + f_{\{5\}} f_{\{1,4\}} f_{\{2,3\}} + f_{\{4\}} f_{\{1,5\}} f_{\{2,3\}} + f_{\{1\}} f_{\{3\}} f_{\{5\}} f_{\{2,4\}} + f_{\{3\}} f_{\{1,5\}} f_{\{2,4\}} + \\
&f_{\{1\}} f_{\{3\}} f_{\{4\}} f_{\{2,5\}} + f_{\{1\}} f_{\{2\}} f_{\{5\}} f_{\{3,4\}} + f_{\{5\}} f_{\{1,2\}} f_{\{3,4\}} + f_{\{2\}} f_{\{1,5\}} f_{\{3,4\}} + f_{\{1\}} f_{\{2,5\}} f_{\{3,4\}} + \\
&f_{\{1\}} f_{\{2\}} f_{\{4\}} f_{\{3,5\}} + f_{\{4\}} f_{\{1,2\}} f_{\{3,5\}} + f_{\{1\}} f_{\{2\}} f_{\{3\}} f_{\{4,5\}} + f_{\{3\}} f_{\{1,2\}} f_{\{4,5\}} + f_{\{2\}} f_{\{1,3\}} f_{\{4,5\}} + \\
&f_{\{1\}} f_{\{2,3\}} f_{\{4,5\}} + f_{\{4\}} f_{\{5\}} f_{\{1,2,3\}} + f_{\{4,5\}} f_{\{1,2,3\}} + f_{\{3\}} f_{\{5\}} f_{\{1,2,4\}} + f_{\{3\}} f_{\{4\}} f_{\{1,2,5\}} + f_{\{3,4\}} f_{\{1,2,5\}} + \\
&f_{\{2\}} f_{\{5\}} f_{\{1,3,4\}} + f_{\{2\}} f_{\{4\}} f_{\{1,3,5\}} + f_{\{2\}} f_{\{3\}} f_{\{1,4,5\}} + f_{\{2,3\}} f_{\{1,4,5\}} + f_{\{1\}} f_{\{5\}} f_{\{2,3,4\}} + \\
&f_{\{1,5\}} f_{\{2,3,4\}} + f_{\{1\}} f_{\{4\}} f_{\{2,3,5\}} + f_{\{1\}} f_{\{3\}} f_{\{2,4,5\}} + f_{\{1\}} f_{\{2\}} f_{\{3,4,5\}} + f_{\{1,2\}} f_{\{3,4,5\}} + \\
&f_{\{5\}} f_{\{1,2,3,4\}} + f_{\{4\}} f_{\{1,2,3,5\}} + f_{\{3\}} f_{\{1,2,4,5\}} + f_{\{2\}} f_{\{1,3,4,5\}} + f_{\{1\}} f_{\{2,3,4,5\}} + f_{\{1,2,3,4,5\}}
\end{aligned}$$

Recalling the definition of proper refinements from (4.26), note that by changing the limits of summation in Theorem 4.12, the following corollary is immediately obtained for partitions free of singleton blocks.

Corollary 4.14. *Let $\pi \in \Pi_G([n])$ contain no singleton blocks, and let t be a nonzero scalar parameter t. Then,*

$$\prod_{b\in\pi} \mathcal{D}_b \left[\oint \left(t + \sum_{j=1}^{|b|-1} \frac{1}{j!} \zeta_b \mathcal{L}^j \Phi\Psi \right)^{|b|} d\zeta_{b_1} \cdots d\zeta_{b_{|b|}} \right]_{t=0} = \sum_{\substack{\sigma\in\Pi_G([n]) \\ \sigma\prec\pi}} h_f(\sigma). \quad (4.44)$$

5. Free Cumulants

As in Nica and Speicher [7], let $\mathcal{A}$ be a unital algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ a unital linear functional; i.e., $(\mathcal{A}, \varphi)$ is a quantum probability space. A sequence of multilinear functionals $(\varphi_n)_{n\in N}$ is obtained on $\mathcal{A}$ by defining $\varphi_n(a_1, \ldots, a_n) := \varphi(a_1 \cdots a_n)$.

These functionals are extended to the corresponding multiplicative functionals on non-crossing partitions by

$$\varphi_\pi[a_1, \ldots, a_n] := \prod_{b\in\pi} \varphi(b)[a_1, \ldots, a_n], \quad (5.1)$$

where writing $b = \{i_1, \ldots, i_{|b|}\}$ with $i_1 < \cdots < i_{|b|}$, $\varphi(b)[a_1, \ldots, a_n]$ is defined by

$$\varphi(b)[a_1, \ldots, a_n] := \varphi_{|b|}(a_{i_1}, \ldots, a_{i_{|b|}}). \quad (5.2)$$

Let μ denote the Möbius function on the lattice of non-crossing partitions. For each $n \in \mathbb{N}$, the corresponding *free cumulants* $(k_\pi)_{\pi\in NC(n)}$ are multilinear functionals $k_\pi : \mathcal{A}^n \to \mathbb{C}$ satisfying the moment-cumulant formulas:

$$\kappa_{\hat{1}_{[n]}}(a_1, \ldots, a_n) := \sum_{\sigma\in\mathrm{NC}(n)} \varphi_\sigma[a_1, \ldots, a_n]\mu(\sigma, \hat{1}_{[n]}), \quad (5.3)$$

$$\varphi_n(a_1, \cdots, a_n) = \sum_{\pi\in\mathrm{NC}(n)} \kappa_\pi[a_1, \ldots, a_n]. \quad (5.4)$$

Moreover, free cumulant functionals are multiplicative:

$$\kappa_\pi[a_{i_1}, \ldots, a_{i_n}] := \prod_{b\in\pi} \kappa(b)[a_1, \ldots, a_n], \quad (5.5)$$

where, similar to (5.1), writing $b = \{r_1, \ldots, r_{|b|}\}$ with $r_1 < \cdots < r_{|b|}$,

$$\kappa(b)[a_{r_1}, \ldots, a_{r_n}] := \kappa_{\hat{1}_b}(a_{r_1}, \ldots, a_{r_{|b|}}). \quad (5.6)$$

This is simplified by recursively defining the multidimensional R-transform by

$$\varphi_\pi(a_{i_1}, \ldots, a_{i_n}) = \sum_{\substack{\sigma\in\mathrm{NC}(n) \\ \sigma\leq\pi}} R_\sigma(a_{i_1}, \ldots, a_{i_n}), \quad (5.7)$$

where again writing $b = \{r_1, \ldots, r_{|b|}\}$ with $r_1 < \cdots < r_{|b|}$,

$$R_\sigma(a_{i_1}, \ldots, a_{i_n}) = \prod_{b\in\sigma} R(a_{i_{r_1}}, \ldots, a_{i_{r_{|b|}}}). \quad (5.8)$$

Definition 5.1. Let G_{cr} be the simple graph on 2^n-1 vertices as defined in Corollary 3.8. Let $(a_i)_{i\in I}$ be random variables in quantum probability space $(\mathcal{A},\varphi)$. Define the *moment zeon representative of* G_{cr} by

$$\Xi = \sum_{j=1}^{|V|} R(a_{\underline{j}_1}\cdots a_{\underline{j}_{|\underline{j}|}})\,\zeta_{\underline{j}}\varkappa(v_j), \tag{5.9}$$

where $a_\ell \in \mathcal{A}$ for all indices ℓ and $R(a_{\underline{j}_1}\cdots a_{\underline{j}_{|\underline{j}|}})$ is the R-transform of the block $\underline{j}\subseteq[n]$.

Proposition 5.2 (Moments by independent sets)**.**

$$\int e^{\Xi}\varepsilon_{[n]}\,d\zeta_1\cdots d\zeta_n = \varphi(a_{i_1}\cdots a_{i_n}). \tag{5.10}$$

Proof. The result is a corollary of Theorem 4.1. In particular, expanding the exponential function gives

$$e^{\Xi} = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{j=1}^{|V|} R(a_{\underline{j}_1}\cdots a_{\underline{j}_{|\underline{j}|}})\,\zeta_{\underline{j}}\varkappa(v_j)\right)^k. \tag{5.11}$$

Proceeding as in the proof of Theorem 4.1 and keeping in mind equations (5.7) and (5.8), it follows that

$$\begin{aligned}\int e^{\Xi}\varepsilon_{[n]}\,d\zeta_1\cdots d\zeta_n &= \sum_{k=1}^{n}\sum_{\substack{\sigma\in\mathrm{NC}(n)\\|\sigma|=k}} R_\sigma(a_{i_1},\ldots,a_{i_n})\\ &= \sum_{\sigma\in\mathrm{NC}(n)} R_\sigma(a_{i_1},\ldots,a_{i_n}) = \varphi_{\hat{1}_{[n]}}(a_{i_1},\ldots,a_{i_n}).\end{aligned} \tag{5.12}$$

□

Finally, to recover the multiplicative functional $\varphi_\pi(a_{i_1},\ldots,a_{i_n})$ using zeon-Berezin methods, let the operator Θ be defined on $\mathcal{Z}_n$ by linear extension of

$$\zeta_b\,\Theta := R(a_{b_1},\ldots,a_{b_{|b|}})\,\zeta_b. \tag{5.13}$$

Proposition 5.3 (Moments by lowering)**.** *Let $G := G_{cr}$, let Ψ be the G restriction mapping defined in* (4.34)*, and let Θ be defined as in* (5.13)*. Applying the canonical lowering operator yields the following:*

$$\prod_{b\in\pi}\mathcal{D}_b\left[\oint\left(t+\sum_{j=0}^{|b|-1}\frac{1}{j!}\zeta_b\mathcal{L}^j\Theta\Psi\right)^{|b|} d\zeta_{b_1}\cdots d\zeta_{b_{|b|}}\right]_{t=0} = \varphi_\pi(a_{i_1},\ldots,a_{i_n}). \tag{5.14}$$

Proof. Applying results from Theorem 4.12, and again considering equations (5.7) and (5.8),

$$\prod_{b\in\pi}\mathcal{D}_b\left[\oint\left(t+\sum_{j=0}^{|b|-1}\frac{1}{j!}\zeta_b\mathcal{L}^j\Theta\Psi\right)^{|b|}d\zeta_{b_1}\cdots d\zeta_{b_{|b|}}\right]_{t=0} = \sum_{\substack{\sigma\in\mathrm{NC}(n)\\ \sigma\leq\pi}} R_\sigma(a_{i_1},\dots,a_{i_n}) = \varphi_\pi(a_{i_1},\dots,a_{i_n}). \quad (5.15)$$

□

Corollary 5.4. *Setting* $\pi=\hat{1}_{[n]}$ *in Proposition 5.3 gives*

$$\mathcal{D}_{[n]}\left[\int\left(t+\sum_{j=0}^{n-1}\frac{1}{j!}\zeta_{[n]}\mathcal{L}^j\Theta\Psi\right)^{n}d\zeta_1\cdots d\zeta_n\right]_{t=0}=\varphi(a_{i_1},\dots,a_{i_n}). \quad (5.16)$$

6. Conclusion

In earlier work, the authors showed that the operator-theoretic tools of quantum probability can be used to reveal information about combinatorial structures such as random graphs and partitions. The current work shows that this approach can be extended to perform computations over combinatorial structures, notably lattices of partitions.

While the lattice of non-crossing partitions is most important in free probability theory, the zeon-Berezin calculus approach can be applied to lattices of partitions satisfying any restriction criteria. The result is an operator-theoretic combinatorial approach to integration of functions defined on arbitrary graphs and partitions.

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René Schott: IECN and LORIA Université Henri Poincaré-Nancy I, BP 239, 54506 Vandoeuvre-lès-Nancy, France
E-mail address: `schott@loria.fr`
URL: `http://www.loria.fr/~schott`

G. Stacey Staples: Department of Mathematics and Statistics, Southern Illinois University Edwardsville, Edwardsville, IL 62026-1653, USA
E-mail address: `sstaple@siue.edu`
URL: `http://www.siue.edu/~sstaple`