Communications on Stochastic Analysis

Volume 4 | Number 3 Article 1

9-1-2010

Covariance identities and mixing of random transformations on the Wiener space

Nicolas Privault

Communications on Stochastic Analysis
Vol. 4, No. 3 (2010) 299-309

COVARIANCE IDENTITIES AND MIXING OF RANDOM TRANSFORMATIONS ON THE WIENER SPACE

NICOLAS PRIVAULT

Abstract. In this paper we derive criteria for the mixing of random transformations of the Wiener space. The proof is based on covariance identities for the Hitsuda–Skorokhod integral.

1. Introduction and Notation

In this paper we derive sufficient conditions for the mixing of random transformations on the Wiener space (W, μ) where $W = \mathcal{C}_0(\mathbf{R}_+, \mathbf{R}^d)$ is the Banach space of continuous functions started at 0. Recall that a measure preserving transformation $T : W \to W$ is said to be mixing (here of order 2) if

$$\lim_{m\to\infty} \mathrm{Cov}\,(F, G \circ T^m) = 0,$$

for all $F, G \in L^2(W)$. The mixing property implies the ergodicity of $T : W \to W$, i.e. the relation

$$F \circ T = F, \qquad \mu - a.s.,$$

holds if and only if F is constant, or equivalently,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F \circ T^k = E[F],$$

for all $F \in L^1(W)$, cf. e.g. [1] for a survey.

As noted above, the mixing and ergodicity properties rely on the invariance of the Wiener measure μ under the transformation $T : (W, \mu) \to (W, \mu)$. It is well known that when $(B_t)_{t\in\mathbf{R}_+}$ is a standard Brownian motion and $(R_t)_{t\in\mathbf{R}_+}$ is an adapted process of isometries of $\mathbf{R}^d$, the process $(\tilde{B}_t)_{t\in\mathbf{R}_+}$ defined by

$$d\tilde{B}_t = R_t dB_t$$

remains a standard Brownian motion. The associated transformation

$$T : W \longrightarrow W, \quad (B_t)_{t\in\mathbf{R}_+} \mapsto (\tilde{B}_t)_{t\in\mathbf{R}_+},$$

called the Lévy transform, preserves the Wiener measure and defines a distribution-preserving mapping

$$R^* : L^p(W) \to L^p(W), \qquad p \geq 1,$$

Received 2009-11-3; Communicated by Hui-Hsiung Kuo.
2000 *Mathematics Subject Classification.* Primary 60H07; Secondary 60G30.
Key words and phrases. Malliavin calculus, Hitsuda–Skorokhod integral, Wiener measure, random isometries, mixing, ergodicity.

that sends any functional F of the form

$$F = f\left(\int_0^\infty h_1(t)dB_t, \dots, \int_0^\infty h_n(t)dB_t\right), \tag{1.1}$$

$h_1, \dots, h_n \in L^2(\mathbf{R}_+; \mathbf{R}^d)$, $f \in \mathcal{C}_b^1(\mathbf{R}^n)$, to

$$R^*F := F \circ T = f\left(\int_0^\infty R_t^\dagger h_1(t) \cdot dB_t, \dots, \int_0^\infty R_t^\dagger h_n(t) \cdot dB_t\right).$$

Next, consider a random and possibly anticipating isometry of $H = L^2(\mathbf{R}_+; \mathbf{R}^d)$ denoted by

$$R : L^2(\mathbf{R}_+; \mathbf{R}^d) \to L^2(\mathbf{R}_+; \mathbf{R}^d)$$

and denote by δ the extension in Hitsuda–Skorokhod sense of the Itô integral. It has been shown in [7] that sending F as in (1.1) above to

$$R^*F := f(\delta(Rh_1), \dots, \delta(Rh_n)),$$

defines a law preserving mapping $R^* : L^p(W) \to L^p(W)$, $p \geq 1$, provided the trace condition

$$\operatorname{trace}(DRh)^n = 0$$

holds for all $h \in L^2(\mathbf{R}_+; \mathbf{R}^d)$, cf. Proposition 2.3 below.

In case $R : L^2(\mathbf{R}_+; \mathbf{R}^d) \to L^2(\mathbf{R}_+; \mathbf{R}^d)$ is deterministic, necessary and sufficient conditions for the mixing and ergodicity of $T : W \to W$ have been given on the spectral type of R using Wiener chaos, cf. [2] Chapter 14, § 2, Theorems 1 and 2, and [9], Theorem 2.

Although the question whether the Lévy transform $T : W \to W$ is ergodic is still open in case the process $(R_t)_{t\in\mathbf{R}_+}$ of isometries is adapted, cf. [3], sufficient conditions for the mixing of R^* have been obtained in the random case using the anticipating Girsanov identity, cf. [8] and Theorems 3 and 4 of [11]. However these conditions are too strong to be satisfied by the Lévy transform.

In this note we recover the latter results using covariance identities for the Hitsuda–Skorokhod integral, and our proofs do not rely on the anticipative Girsanov theorem and the associated $\mathcal{H}C^1$ smoothness hypotheses, cf. [10].

More precisely, the next proposition recovers and extends Theorem 4 of [11] as a consequence of Proposition 3.1 below.

Proposition 1.1. *Let $(R_m)_{m\geq 1}$ be a sequence of random mappings with values in the isometries of $H = L^2(\mathbf{R}_+; \mathbf{R}^d)$, such that $(R_m h)_{m\geq 1}$ is bounded in $\mathbb{D}_{p,2}(H)$ for some $p > 1$ and*

$$\operatorname{trace}(DR_m h)^k = 0, \qquad k \geq 2, \quad h \in H, \quad m \geq 1.$$

Then the law preserving transformation R_m^* that maps any F of the form (1.1) to

$$R_m^*F := f(\delta(R_m h_1), \dots, \delta(R_m h_n))$$

is mixing provided

$$\lim_{m\to\infty} \langle h, R_m h\rangle = 0$$

in probability for all $h \in H$.

We refer to Section 5 of [11] for examples of transformations $T_m : W \to W$ of the Wiener space W, such that

$$\delta(R_m h) = \delta(h) \circ T_m, \qquad m \geq 1, \quad h \in H,$$

and satisfying the hypotheses of Proposition 1.1.

We proceed in two steps to prove Proposition 1.1. In Section 2 we derive covariance identities for the Hitsuda–Skorokhod integral and in Section 3 we prove Proposition 1.1 as an application of those identities.

We close this section with some facts and notation on the Malliavin calculus, cf. e.g. [4], [6], [12]. For any separable Hilbert space X, consider the Malliavin derivative D with values in $H = L^2(\mathbf{R}_+, X \otimes \mathbf{R}^d)$, defined by

$$D_t F = \sum_{i=1}^n h_i(t) \frac{\partial f}{\partial x_i} \left(\int_0^\infty h_1(t) dB_t, \dots, \int_0^\infty h_n(t) dB_t \right), \qquad t \in \mathbf{R}_+,$$

for F of the form (1.1). Let $\mathbb{D}_{p,k}(X)$ denote the completion of the space of smooth X-valued random variables under the norm

$$\|u\|_{\mathbb{D}_{p,k}(X)} = \|u\|_{L^p(W,X)} + \sum_{l=1}^k \|D^l u\|_{L^p(W, X \otimes H^{\otimes l})}, \qquad p > 1,$$

where $X \otimes H$ denotes the completed symmetric tensor product of X and H. For all $p, q > 1$ such that $p^{-1} + q^{-1} = 1$ and all $k \geq 1$, let

$$\delta : \mathbb{D}_{p,k}(X \otimes H) \to \mathbb{D}_{q,k-1}(X)$$

denote the bounded Hitsuda–Skorokhod integral operator adjoint of

$$D : \mathbb{D}_{p,k}(X) \to \mathbb{D}_{q,k-1}(X \otimes H),$$

with

$$E[\langle F, \delta(u) \rangle_X] = E[\langle DF, u \rangle_{X \otimes H}], \qquad F \in \mathbb{D}_{p,k}(X), \quad u \in \mathbb{D}_{q,k}(X \otimes H).$$

Recall the relations

$$D_t \delta(u) = u_t + \delta(D_t^* u), \qquad t \in \mathbf{R}_+, \quad u \in \mathbb{D}_{2,2}(H),$$

and

$$F\delta(u) = \delta(uDF) + \langle u, DF \rangle, \qquad F \in \mathbb{D}_{2,1}, \quad u \in \mathbb{D}_{2,2}(H), \tag{1.2}$$

and that $\delta(u)$ coincides with the Itô integral of $u \in L^2(W; H)$ with respect to Brownian motion, i.e.

$$\delta(u) = \int_0^\infty u_t dB_t,$$

when u is square-integrable and adapted with respect to the Brownian filtration, and in particular when $u \in H$ is deterministic.

2. Covariance Identities

In this section we state several covariance identities in the next lemmas, which will be used to prove Proposition 1.1.

Before that we describe the application of covariance identities to mixing in case $R : H \to H$ is deterministic. By polarization of the Gaussian moment identity

$$E\left[\left(\int_0^\infty h(t)dB_t\right)^{2k}\right] = (2k)!!\|h\|^{2k}, \qquad h \in L^2(\mathbf{R}_+;\mathbf{R}^d), \quad k \in \mathbf{N},$$

where !! denotes the double factorial, we find that for any family of sequences

$$(k_{1,m})_{m\geq 1}, \ldots, (k_{n,m})_{m\geq 1},$$

such that

$$k_{l,m} < k_{l+1,m}, \qquad m, l \geq 1,$$

the joint Gaussian moment

$$E\left[\delta(R^{k_{1,m}}h_1)^{l_1}\cdots\delta(R^{k_{n,m}}h_n)^{l_n}\right]$$

$l_1, \ldots, l_n \geq 1$, $h_1, \ldots, h_n \in H$, can be written as a linear combination and product of terms of the form

$$\langle R^{k_{a,m}}h_a, R^{k_{b,m}}h_b\rangle = \langle R^{k_{a,m}-k_{b,m}}h_a, h_b\rangle,$$

$1 \leq a \leq b \leq n$, which tend to zero whenever $a \neq b$ and $k_{b,m} - k_{a,m}$ tends to $+\infty$ as m goes to infinity. It follows that

$$\lim_{m\to\infty} E\left[\delta(R^{k_{1,m}}h_1)^{l_1}\cdots\delta(R^{k_{n,m}}h_n)^{l_n}\right] = E[\delta(h_1)^{l_1}]\cdots E[\delta(h_n)^{l_n}],$$

when $k_{b,m} - k_{a,m}$ tends to $+\infty$ as m goes to infinity, $1 \leq a < b \leq n$, provided

$$\lim_{n\to\infty}\langle R^n h, h\rangle = 0, \qquad h \in H, \tag{2.1}$$

showing that R^* is mixing of order n for all $n \geq 2$.

This type of argument will be applied in Section 3 to prove mixing of order 2 in the anticipating case using the covariance identities for the Hitsuda–Skorokhod integral stated in the following lemmas.

For $u \in \mathbb{D}_{2,1}(H)$ we identify $Du = (D_t u_s)_{s,t\in\mathbf{R}_+}$ to the random operator $Du : H \to H$ almost surely defined by

$$(Du)v(s) = \int_0^\infty (D_t u_s)v_t dt, \quad s \in \mathbf{R}_+, \quad v \in L^2(W;H),$$

where the product of $D_t u_s \in X \otimes H$ with $v_t \in H$ is defined in X via

$$(a \otimes b)c = a\langle b, c\rangle, \qquad a \otimes b \in X \otimes H, \quad c \in H.$$

The adjoint D^*u of Du on $H \otimes H$ is defined as

$$(D^*u)v(s) = \int_0^\infty (D_s^\dagger u_t)v_t dt, \qquad s \in \mathbf{R}_+, \quad v \in L^2(W;H),$$

where $D_s^\dagger u_t$ denotes the transpose matrix of $D_s u_t$ in $\mathbf{R}^d \otimes \mathbf{R}^d$.

Lemma 2.1. *For any $n \geq 1$ and $u, v \in \mathbb{D}_{n+1,2}(H)$ we have*

$$\begin{aligned} &\operatorname{Cov}(\delta(v), \delta(u)^n) \qquad (2.2) \\ &= \sum_{k=1}^{n} \frac{n!}{(n-k)!} E\left[\delta(u)^{n-k}\left(\langle u, (Du)^{k-1} v\rangle + \langle D^* u, D((Du)^{k-1} v)\rangle\right)\right]. \end{aligned}$$

Proof. We use the same argument as in the proof of Theorem 2.1 of [5] which deals with the case $u = v$. We have

$$\begin{aligned} &E[\delta(v)\delta(u)^n] = E[\langle v, D\delta(u)^n\rangle] \\ &= nE[\delta(u)^{n-1}\langle v, D\delta(u)\rangle] \\ &= nE[\delta(u)^{n-1}\langle v, u\rangle] + nE[\delta(u)^{n-1}\langle v, \delta(D^* u)\rangle] \\ &= nE[\delta(u)^{n-1}\langle v, u\rangle] + \sum_{k=1}^{n} \frac{n!}{(n-k)!} E[\delta(u)^{n-k}\langle (Du)^{k-1} v, \delta(D^* u)\rangle] \\ &\quad - \sum_{k=1}^{n-1} \frac{n!}{(n-k-1)!} E[\delta(u)^{n-k-1}\langle (Du)^{k} v, \delta(D^* u)\rangle] \\ &= nE[\delta(u)^{n-1}\langle v, u\rangle] + \sum_{k=1}^{n} \frac{n!}{(n-k)!} E[\langle D(\delta(u)^{n-k}(Du)^{k-1} v), D^* u\rangle] \\ &\quad - \sum_{k=1}^{n-1} \frac{n!}{(n-k-1)!} E[\delta(u)^{n-k-1}\langle (Du)^{k-1} v \otimes \delta(D^* u), Du\rangle] \\ &= nE[\delta(u)^{n-1}\langle v, u\rangle] + \sum_{k=1}^{n} \frac{n!}{(n-k)!} E[\delta(u)^{n-k}\langle D((Du)^{k-1} v), D^* u\rangle] \\ &\quad + \sum_{k=1}^{n-1} \frac{n!}{(n-k-1)!} E[\delta(u)^{n-k-1}\langle (Du)^{k-1} v \otimes (D\delta(u) - \delta(D^* u)), D^* u\rangle] \\ &= nE[\delta(u)^{n-1}\langle v, u\rangle] + \sum_{k=1}^{n} \frac{n!}{(n-k)!} E[\delta(u)^{n-k}\langle D((Du)^{k-1} v), D^* u\rangle] \\ &\quad + \sum_{k=2}^{n} \frac{n!}{(n-k)!} E[\delta(u)^{n-k}\langle (Du)^{k} v \otimes u, D^* u\rangle] \\ &= nE[\delta(u)^{n-1}\langle v, u\rangle] + \sum_{k=1}^{n} \frac{n!}{(n-k)!} E[\delta(u)^{n-k}\langle D((Du)^{k-1} v), D^* u\rangle] \\ &\quad + \sum_{k=2}^{n} \frac{n!}{(n-k)!} E[\delta(u)^{n-k}\langle u, (Du)^{k-1} v\rangle] \\ &= \sum_{k=1}^{n} \frac{n!}{(n-k)!} E\left[\delta(u)^{n-k}\left(\langle D^* u, D((Du)^{k-1} v)\rangle + \langle u, (Du)^{k-1} v\rangle\right)\right]. \end{aligned}$$

□

For $k \geq 2$ the trace of $(Du)^k$ is defined by

$$\begin{aligned} \text{trace}\,(Du)^k &= \langle Du, (D^*u)^{k-1}\rangle_{H\otimes H} \\ &= \int_0^\infty \cdots \int_0^\infty \langle D^\dagger_{t_{k-1}} u_{t_k}, D_{t_{k-2}} u_{t_{k-1}} \cdots D_{t_1} u_{t_2} D_{t_k} u_{t_1}\rangle_{\mathrm{R}^d\otimes\mathrm{R}^d} dt_1 \cdots dt_k. \end{aligned}$$

The next result is a consequence of Lemma 2.1.

Lemma 2.2. *Let $u \in I\!D_{n+1,2}(H)$ such that $\|u\|_H$ is deterministic and*

$$\text{trace}\,(Du)^k = 0, \qquad k \geq 2.$$

Then for any $n \geq 1$ and $v \in I\!D_{n+1,2}(H)$ we have

$$\begin{aligned} \text{Cov}\,(\delta(v), \delta(u)^n) &= nE\left[\delta(u)^{n-1}\langle u, v\rangle\right] \\ &+ \sum_{k=1}^n \frac{n!}{(n-k)!} E\left[\delta(u)^{n-k}\,\text{trace}\,((Du)^k Dv)\right]. \end{aligned} \tag{2.3}$$

Proof. For all $1 \leq k \leq n$ we have

$$(Du)^{k-1} v \in I\!D_{(n+1)/k,1}(H), \qquad \delta(u) \in I\!D_{(n+1)/(n-k+1),1}(\mathbf{R}),$$

and

$$\begin{aligned} &\langle D^*u, D((Du)^{k-1}v)\rangle \\ &= \int_0^\infty \cdots \int_0^\infty \langle D^\dagger_{t_{k-1}} u_{t_k}, D_{t_k}(D_{t_{k-2}} u_{t_{k-1}} \cdots D_{t_0} u_{t_1} v_{t_0})\rangle dt_0 \cdots dt_k \\ &= \int_0^\infty \cdots \int_0^\infty \langle D^\dagger_{t_{k-1}} u_{t_k}, D_{t_{k-2}} u_{t_{k-1}} \cdots D_{t_0} u_{t_1} D_{t_k} v_{t_0}\rangle dt_0 \cdots dt_k \\ &\quad + \int_0^\infty \cdots \int_0^\infty \langle D^\dagger_{t_{k-1}} u_{t_k}, D_{t_k}(D_{t_{k-2}} u_{t_{k-1}} \cdots D_{t_0} u_{t_1}) v_{t_0}\rangle dt_0 \cdots dt_k \\ &= \text{trace}\,((Du)^k Dv) + \sum_{i=0}^{k-2} \int_0^\infty \cdots \int_0^\infty \\ &\langle D^\dagger_{t_{k-1}} u_{t_k}, D_{t_k} u_{t_{k+1}} \cdots D_{t_{i+1}} u_{t_{i+2}} (D_{t_i} D_{t_k} u_{t_{i+1}}) D_{t_{i-1}} u_{t_i} \cdots D_{t_0} u_{t_1} v_{t_0}\rangle dt_0 \cdots dt_k \\ &= \text{trace}\,((Du)^k Dv) + \sum_{i=0}^{k-2} \frac{1}{k-i} \int_0^\infty \cdots \int_0^\infty \\ &\langle D_{t_i} \langle D^\dagger_{t_{k-1}} u_{t_k}, D_{t_k} u_{t_{k+1}} \cdots D_{t_{i+1}} u_{t_{i+2}} D_{t_k} u_{t_{i+1}}\rangle, D_{t_{i-1}} u_{t_i} \cdots D_{t_0} u_{t_1} v_{t_0}\rangle dt_0 \cdots dt_k \\ &= \text{trace}\,((Du)^k Dv) + \sum_{i=0}^{k-2} \frac{1}{k-i} \langle (Du)^i v, D\,\text{trace}\,(Du)^{k-i}\rangle \\ &= \text{trace}\,((Du)^k Dv). \end{aligned}$$

Hence (2.2) shows that

$$\text{Cov}\,(\delta(v), \delta(u)^n) = \sum_{k=1}^n \frac{n!}{(n-k)!} E\left[\delta(u)^{n-k}\left(\langle u, (Du)^{k-1} v\rangle + \text{trace}\,((Du)^k Dv)\right)\right]. \tag{2.4}$$

On the other hand the relation

$$D\langle u,u\rangle = 2(D^*u)u$$

shows that

$$\begin{aligned} 2\langle (Du)^{k-1}v,u\rangle &= 2\langle v,(D^*u)^{k-1}u\rangle \\ &= \langle v,(D^*u)^{k-2}D\langle u,u\rangle\rangle \\ &= 0, \qquad k\geq 2, \end{aligned}$$

hence the conclusion from (2.4). □

Note that Lemma 2.2 recovers the following consequence of Corollary 2.2 in [5].

Proposition 2.3. *Let $u\in I\!D_{p,2}(H)$ for some $p>1$, such that $\|u\|_H$ is deterministic and* $\mathrm{trace}\,(Du)^{k+1}=0$, $k\geq 1$. *Then $\delta(u)$ has a centered Gaussian distribution with variance $\|u\|_H^2$.*

Proof. When $u=v\in I\!D_{n+1,2}(H)$ we have

$$\begin{aligned} E[\delta(u)^{n+1}] &= nE\left[\delta(u)^{n-1}\|u\|_H^2\right]+\sum_{k=1}^{n}\frac{n!}{(n-k)!}E\left[\delta(u)^{n-k}\,\mathrm{trace}\,(Du)^{k+1}\right] \\ &= n\|u\|_H^2E\left[\delta(u)^{n-1}\right], \qquad n\geq 1. \end{aligned}$$

The conclusion follows by density of $I\!D_{n+1,2}(H)$ in $I\!D_{p,2}(H)$, $p\leq n+1$, and induction on $n\geq 1$. □

Proposition 2.3 above also recovers Theorem 2.1-*b*) of [7] by taking u of the form $u=Rh$, $h\in H$, where R is a random mapping with values in the isometries of H, such that $Rh\in I\!D_{p,2}(H)$ and $\mathrm{trace}\,(DRh)^{k+1}=0$, $k\geq 1$,

We will also need the following covariance identity. Let

$$k!! = \prod_{i=0}^{[k/2]-1}(k-2i),$$

denote the double factorial of $k\in\mathbf{N}$, where $[k/2]$ denotes the integer part of $k/2$.

Lemma 2.4. *For all $k,n\geq 0$ and $h\in H$, $u\in I\!D_{n+1,2}(H)$, we have*

$$\mathrm{Cov}\,(\delta(h)^{k+1},\delta(u)^n) = \sum_{i=0}^{[k/2]}\frac{k!!}{(k-2i)!!}\langle h,h\rangle^i\mathrm{Cov}\,(\delta(h\delta(h)^{k-2i}),\delta(u)^n). \tag{2.5}$$

Proof. We will show by induction on $k\geq 0$ that

$$\begin{aligned} E[\delta(h)^{k+1}\delta(u)^n] &= E\left[\delta(h)^{k+1}\right]E\left[\delta(u)^n\right] \qquad (2.6)\\ &\quad +\sum_{i=0}^{[k/2]}\frac{k!!}{(k-2i)!!}\langle h,h\rangle^iE\left[\delta(h\delta(h)^{k-2i})\delta(u)^n\right]. \end{aligned}$$

Clearly this identity holds when $k=-1$ and when $k=0$. On the other hand by (1.2) we have

$$\begin{aligned} \delta(h)^{k+2} &= \delta(h\delta(h)^{k+1})+\langle h,D\delta(h)^{k+1}\rangle \\ &= (k+1)\langle h,h\rangle\delta(h)^k+\delta(h\delta(h)^{k+1}), \qquad (2.7) \end{aligned}$$

hence, assuming that the identity (2.6) holds up to the rank $k \geq 0$ we have, using (2.7),

$$\begin{aligned}
&E[\delta(h)^{k+2}\delta(u)^n] = (k+1)\langle h,h\rangle E\left[\delta(h)^k\delta(u)^n\right] + E\left[\delta(h\delta(h)^{k+1})\delta(u)^n\right] \\
&= (k+1)\langle h,h\rangle E\left[\delta(h)^k\right] E\left[\delta(u)^n\right] + E\left[\delta(h\delta(h)^{k+1})\delta(u)^n\right] \\
&\quad +(k+1)\langle h,h\rangle \sum_{i=0}^{[(k-1)/2]} \frac{(k-1)!!}{(k-1-2i)!!}\langle h,h\rangle^i E\left[\delta(h\delta(h)^{k-1-2i})\delta(u)^n\right] \\
&= E\left[\delta(h)^{k+2}\right] E\left[\delta(u)^n\right] + E\left[\delta(h\delta(h)^{k+1})\delta(u)^n\right] \\
&\quad + \sum_{i=1}^{[(k+1)/2]} \frac{(k+1)!!}{(k+1-2i)!!}\langle h,h\rangle^i E\left[\delta(h\delta(h)^{k+1-2i})\delta(u)^n\right] \\
&= E\left[\delta(h)^{k+2}\right] E\left[\delta(u)^n\right] \\
&\quad + \sum_{i=0}^{[(k+1)/2]} \frac{(k+1)!!}{(k+1-2i)!!}\langle h,h\rangle^i E\left[\delta(h\delta(h)^{k+1-2i})\delta(u)^n\right].
\end{aligned}$$

□

Finally we will need the covariance identity stated in Lemma 2.5 below, which is proved using Lemmas 2.2 and 2.4.

Lemma 2.5. *Let $u \in \mathbb{D}_{p,2}(H)$ for some $p > 1$, assume that $\|u\|_H$ is deterministic and*

$$\operatorname{trace}(Du)^k = 0, \qquad k \geq 2.$$

Then for any $k, n \geq 1$ and $h \in H$ we have

$$\operatorname{Cov}(\delta(h)^{k+1}, \delta(u)^n) = n \sum_{0\leq 2i\leq k} \frac{k!!}{(k-i)!!}\langle h,h\rangle^i E\left[\delta(h)^{k-2i}\delta(u)^{n-1}\langle h,u\rangle\right] \tag{2.8}$$

$$+ \sum_{0\leq 2i<k}\sum_{l=1}^{n} \frac{n!}{(n-l)!}\frac{k!!}{(k-i)!!}\langle h,h\rangle^i E\left[\langle u,h\rangle\delta\left(\delta(u)^{n-l}\delta(h)^{k-2i-1}(Du)^{l-1}h\right)\right].$$

Proof. Applying Lemmas 2.2 and 2.4 to $u \in \mathbb{D}_{n+1,2}(H)$ and to $v := h\delta(h)^{k-2i}$ we have

$$\begin{aligned}
&\operatorname{Cov}(\delta(h)^{k+1}, \delta(u)^n) = \sum_{0\leq 2i\leq k} \frac{k!!}{(k-2i)!!}\langle h,h\rangle^i \operatorname{Cov}(\delta(h\delta(h)^{k-2i}), \delta(u)^n) \\
&= n \sum_{0\leq 2i\leq k} \frac{k!!}{(k-2i)!!}\langle h,h\rangle^i E\left[\langle h,u\rangle\delta(h)^{k-2i}\delta(u)^{n-1}\right] \\
&+ \sum_{0\leq 2i<k} \frac{k!!}{(k-2i)!!}\langle h,h\rangle^i \sum_{l=1}^{n}\frac{n!}{(n-l)!} E\left[\delta(u)^{n-l}\delta(h)^{k-2i-1}\operatorname{trace}((Du)^l h\otimes h)\right].
\end{aligned}$$

Finally we note that

$$\begin{aligned}
&E\left[\delta(u)^{n-l}\delta(h)^{2k-i-1}\operatorname{trace}((Du)^l h\otimes h)\right] \\
&= E\Big[\delta(u)^{n-l}\delta(h)^{2k-i-1}
\end{aligned}$$

$$\int_0^\infty \cdots \int_0^\infty \langle D^\dagger_{t_{l-1}} u_{t_l}, D_{t_{l-2}} u_{t_{l-1}} \cdots D_{t_0} u_{t_1} h_{t_0} \otimes h_{t_l} \rangle dt_0 \cdots dt_l \Big]$$

$$= E\Big[\delta(u)^{n-l} \delta(h)^{2k-i-1}$$

$$\int_0^\infty \cdots \int_0^\infty \langle D^\dagger_{t_{l-1}} \langle u_{t_l}, h_{t_l} \rangle, D_{t_{l-2}} u_{t_{l-1}} \cdots D_{t_0} u_{t_1} h_{t_0} \rangle dt_0 \cdots dt_l \Big]$$

$$= E\Big[\delta(u)^{n-l} \delta(h)^{2k-i-1}$$

$$\int_0^\infty \langle D_{t_{l-1}} \int_0^\infty \langle u_{t_l}, h_{t_l} \rangle dt_l, (Du)^{l-1} h(t_{l-1}) \rangle dt_{l-1} \Big]$$

$$= E\left[\delta(u)^{n-l} \delta(h)^{2k-i-1} \langle D\langle u, h \rangle, (Du)^{l-1} h \rangle \right]$$

$$= E\left[\langle u, h \rangle \delta\left(\delta(u)^{n-l} \delta(h)^{2k-i-1} (Du)^{l-1} h \right) \right],$$

which proves (2.8) for $u \in I\!D_{n+1,2}(H)$.

Next, for any $p, q > 1$ such that $p^{-1} + q^{-1} = 1$ we have

$$\begin{aligned} E\left[|\delta(h)^{k-2i} \delta(u)^{n-1} \langle h, u \rangle| \right] &\leq \|\delta(h)^{k-2i}\|_{2q} \|\delta(u)^{n-1}\|_{2q} \|\langle h, u \rangle\|_p \\ &\leq \|\delta(h)^{k-2i}\|_{2q} \|\delta(u)^{n-1}\|_{2q} \|h\|_H \|u\|_p, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} & E\left[|\langle u, h \rangle \delta\left(\delta(u)^{n-l} \delta(h)^{k-2i-1} (Du)^{l-1} h \right)| \right] \\ & \quad \leq \|\delta\left(\delta(u)^{n-l} \delta(h)^{k-2i-1} (Du)^{l-1} h \right)\|_q \|\langle h, u \rangle\|_p \\ & \quad \leq \|\delta(u)^{n-l} \delta(h)^{k-2i-1} (Du)^{l-1} h\|_{p,1} \|h\|_H \|u\|_p. \end{aligned} \tag{2.10}$$

Hence we can extend Relation (2.8) from $u \in I\!D_{n+1,2}(H)$ to $u \in I\!D_{p,2}(H)$ by density using (2.9), (2.10), and the fact that $\delta(u) \simeq \mathcal{N}(0, \|u\|_H^2)$ from Proposition 2.3. □

3. Mixing

The goal of this section is to prove the following result, from which Proposition 1.1 follows by density.

Proposition 3.1. *Let $(u_m)_{m\geq 1}$ be a bounded sequence in $I\!D_{p,2}(H)$ for some $p > 1$, such that for all $m \geq 1$, $\|u_m\|_H$ is deterministic and*

$$\operatorname{trace}(Du_m)^k = 0, \qquad k \geq 2.$$

Then for all $h \in H$ such that

$$\lim_{m\to\infty} \langle h, u_m \rangle = 0$$

in probability we have

$$\lim_{m\to\infty} \operatorname{Cov}(\delta(h)^{k+1}, \delta(u_m)^n) = 0,$$

for all $k, n \geq 1$.

Proof. From Lemma 2.5 we have

$\operatorname{Cov}(\delta(h)^{k+1}, \delta(u_m)^n)$

$$= \quad n \sum_{0\le 2i\le k} \frac{k!!}{(k-i)!!} \langle h,h\rangle^i E\left[\delta(h)^{k-2i}\delta(u_m)^{n-1}\langle h,u_m\rangle\right]$$

$$+ \sum_{0\le 2i<k} \sum_{l=1}^{n} \frac{n!}{(n-l)!} \frac{k!!}{(k-i)!!} \langle h,h\rangle^i E\left[\langle u_m,h\rangle \delta\left(\delta(u_m)^{n-l}\delta(h)^{k-2i-1}(Du_m)^{l-1}h\right)\right].$$

Now for any $p,q>1$ such that $p^{-1}+q^{-1}=1$, the bounds (2.9) and (2.10) show that

$$E\left[\delta(h)^{k-2i}\delta(u_m)^{n-1}\langle h,u_m\rangle\right]$$

and

$$E\left[\langle u_m,h\rangle \delta\left(\delta(u_m)^{n-l}\delta(h)^{k-2i-1}(Du_m)^{l-1}h\right)\right]$$

tend to zero as m goes to infinity since $\langle h,u_m\rangle$ is bounded by

$$|\langle h,u_m\rangle| \le \|h\|\|u_m\|, \qquad m\ge 1,$$

and tends to zero in probability on the one hand, and

$$\|\delta(u_m)^{n-l}\delta(h)^{k-2i-1}(Du_m)^{l-1}h\|_{p,1}$$

is bounded in $m\ge 1$, $l=1,\dots,n$, on the other hand. □

Proof of Proposition 1.1. By Proposition 3.1, for all $h,f\in H$ we have

$$\lim_{m\to\infty} \mathrm{Cov}\,(\delta(h)^{k+1}, \delta(R_m f)^n) = 0,$$

$k,m\ge 1$, and by density of the polynomial functionals in $L^2(W)$ this shows that

$$\lim_{m\to\infty} \mathrm{Cov}\,(F, R_m^* G) = 0,$$

for all $F,G\in L^2(\Omega)$. Indeed, recall that in order for mixing to hold it suffices to prove the property on a dense subset of $L^2(W)$ since if $\|F-\tilde F\|_2<\varepsilon$ and $\|G-\tilde G\|_2<\varepsilon$, $\varepsilon>0$, then

$$\begin{aligned}|\mathrm{Cov}\,(F,R_m^*G)| &= |\mathrm{Cov}\,(F-\tilde F, R_m^*G) + \mathrm{Cov}\,(\tilde F, R_m^*(G-\tilde G)) + \mathrm{Cov}\,(\tilde F, R_m^*\tilde G)| \\ &\le \|F-\tilde F\|_2 \mathrm{Var}[R_m^*G]^{1/2} + \mathrm{Var}[\tilde F]^{1/2}\|R_m^*(G-\tilde G)\|_2 + |\mathrm{Cov}\,(\tilde F, R_m^*\tilde G)| \\ &\le \varepsilon(\mathrm{Var}[G]^{1/2} + \mathrm{Var}[\tilde F]^{1/2}) + |\mathrm{Cov}\,(\tilde F, R_m^*\tilde G)|.\end{aligned}$$

□

Acknowledgment. The work described in this paper was fully supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. CityU 102309).

References

1. Blume, F.: Ergodic theory, in *Handbook of measure theory, Vol. II* (2002) 1185–1235, North-Holland, Amsterdam.
2. Cornfeld, I. P., Fomin, S. V., and Sinaĭ, Ya. G.: *Ergodic theory*, volume 245 of *Grundlehren der Mathematischen Wissenschaften*, Springer-Verlag, New York, 1982.
3. Dubins, L. E., Émery, M., and Yor, M.: On the Lévy transformation of Brownian motions and continuous martingales, in *Séminaire de Probabilités XXVII*, volume 1557 of *Lecture Notes in Math.* (1993) 122–132, Springer, Berlin.
4. Nualart, D.: *The Malliavin calculus and related topics*, Probability and its Applications, Springer-Verlag, Berlin, second edition, 2006.

5. Privault, N.: Moment identities for Skorohod integrals on the Wiener space and applications, *Electron. Commun. Probab.* **14** (2009) 116–121 (electronic).
6. Privault, N.: *Stochastic Analysis in Discrete and Continuous Settings*, volume 1982 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 2009.
7. Üstünel, A. S. and Zakai, M.: Random rotations of the Wiener path, *Probab. Theory Relat. Fields* **103** (1995) 409–429.
8. Üstünel, A. S. and Zakai, M.: Ergodicité des rotations sur l'espace de Wiener, *C. R. Acad. Sci. Paris Sér. I Math.* **330** (2000) 725–728.
9. Üstünel, A. S. and Zakai, M.: Some measure-preserving point transformations on the Wiener space and their ergodicity, arXiv:math/0002198v2, 2000.
10. Üstünel, A. S. and Zakai, M.: *Transformation of measure on Wiener space*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2000.
11. Üstünel, A. S. and Zakai, M.: Some ergodic theorems for random rotations on Wiener space, arXiv:math/0110162v1, 2001.
12. Watanabe, S.: *Lectures on Stochastic Differential Equations and Malliavin Calculus*, Tata Institute of Fundamental Research, 1984.

Nicolas Privault: Department of Mathematics, City University of Hong Kong, Tat Chee Avenue, Kowloon Tong, Hong Kong

E-mail address: nprivaul@cityu.edu.hk

URL: http://math.cityu.edu.hk/~nprivaul