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Mohamed Mnif

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NUMERICAL METHODS FOR OPTIMAL INSURANCE DEMAND UNDER MARKED POINT PROCESSES SHOCKS

MOHAMED MNIF

Abstract. This paper deals with numerical solutions of maximizing expected utility from terminal wealth under a non-bankruptcy constraint. The wealth process is subject to shocks produced by a general marked point process. The problem of the agent is to derive the optimal insurance strategy which allows "lowering" the level of the shocks. This optimization problem is related to a suitable dual stochastic control problem in which the delicate boundary constraints disappear. In Mnif [11], the dual value function is characterized as the unique viscosity solution of the corresponding Hamilton Jacobi Bellman Variational Inequality (HJBVI in short). We characterize the optimal insurance strategy by the solution of the variational inequality which could be solved numerically by using an algorithm based on policy iterations.

1. Introduction

The problem of optimal risk control of a financial corporation has recently gained a considerable interest from the academic and practitionner communities. The most typical example is an insurance company, where the insurer receives a rate of premiums, and he is faced with shocks. He prefers to cover only a portion of the shocks and to pay to the reinsurer a certain part of the premiums. In return, the latter is obliged to pay a part of the claim when it happens. This type of contract is needed by legal restrictions. Such a problem could be solved by using the maximizing utility technics. It is useful in the context of an insurance syndicate when the insurers can be seen as individuals endowed with utility functions [1].

In this paper, we study the optimal insurance demand problem of an agent whose wealth is subject to shocks produced by some marked point process. Such a problem was formulated by Bryis [3] in continuous-time with Poisson shocks. Gollier [6] studied a similar problem where shocks are not proportional to wealth. An explicit solution to the problem is provided by Bryis by writing the associated Hamilton-Jacobi-Bellman (HJB in short) equation. In Bryis [3] and Gollier [6], they modeled the insurance premium by an affine function of the insurance strategy $\theta = (\theta_t)_{t\in[0,T]}$ which is the rate of insurance decided to be covered by the agent. If

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[1]Lloyds of London is one example of such syndicated market place.

the agent is subject to some accident at time t which costs an amount Z, then he will pay $\theta_t Z$ and the insurance company reimburses the amount $(1-\theta_t)Z$. They didn't assume any constraint on the insurance strategy, which is not realistic.

In risk theory, Hipp and Plum [7] analysed the trading strategy, in risky assets, which is optimal with respect to the criterion of minimizing the ruin probability. They derived the HJB equation related to this problem and proved the existence of a solution and a verification theorem. When the claims are exponentially distributed, the ruin probability decreases exponentially and the optimal amount invested in risky assets converges to a constant independent of the reserve level. Schmidli [13] studied the optimal proportional reinsurance policy which minimizes the ruin probability in an infinite horizon. He derived the associated HJB equation, proved the existence of a solution and a verification theorem in the diffusion case. He proved that the ruin probability decreases exponentially whereas the optimal proportion to insure is constant. Moreover, he gave some conjecture in the Cramér-Lundberg case. Højgaard and Taksar [8] studied another problem of proportional reinsurance. They considered the issue of reinsurance optimal fraction, that maximizes the return function. They modelled the reserve process as a diffusion process.

In this paper, we model the claims by using a compound Poisson process. The insurance trading strategy is constrained to remain in $[0,1]$. We impose a constraint of non-bankruptcy on the wealth process X_t of the agent for all t. The objective of the agent is to maximize the expected utility of the terminal wealth over all admissible strategies and to determine the optimal policy of insurance.

The latter stochastic control problem with state constraint by duality methods is studied in Mnif [11]. He characterized the dual value function by a PDE approach as the unique solution to the associated HJBVI. In this paper, we determine numerically the optimal strategy of investment and the optimal reserve process. The optimal strategy is usually determined in a feedback form by using the primal approach and solving the associated HJB equation. Thanks to a verification theorem (See Theorem 4.1), the optimal reserve process is related to the derivative of the dual value function with respect to the dual state variable. When the shocks are modeled by a Poisson process, we can obtain an explicit expression of the optimal strategy of insurance in terms of the dual value function . The paper is organized as follows. Section 2 describes the model. In Section 3, we formulate the dual optimization problem and we derive the associated HJBVI for the value function. In Section 4, we prove a verification theorem. We show that if there exists a solution to the HJBVI, then subject to some regularity conditions, it is the value function of the dual problem. The optimal insurance strategy could be completely characterized by the value function of the dual problem. Section 5 is devoted to a numerical analysis of the HJBVI.

2. Problem Formulation

Let $(\Omega,\mathcal{F},P)$ be a complete probability space. We assume that the claims are generated by a compound Poisson process. More precisely, we consider an integer-valued random measure $\mu(dt,dz)$ with compensator $\pi(dz)dt$. We assume that $\pi(dz)=\varrho G(dz)$ where $G(dz)$ is a probability distribution on the bounded set $C \subseteq I\!R_+$ and ϱ is a positive constant. In this case, the integral, with respect to the random measure $\mu(dt, dz)$, is simply a compound Poisson process: we have $\int_0^t \int_C z\mu(du, dz) = \sum_{i=1}^{N_t} Z_i$, where $N = \{N_t, t \geq 0\}$ is a Poisson process with intensity ϱ and $\{Z_i, i \in I\!N\}$ is a sequence of random variables with common distribution G which represent the claim sizes.

Let $T > 0$ be a finite time horizon. We denote by $I\!F = (\mathcal{F}_t)_{0 \leq t \leq T}$ the filtration generated by the random measure $\mu(dt, dz)$. By definition of the intensity $\pi(dz)dt$, the compensated jump process:

$$\tilde{\mu}(dt, dz) := \mu(dt, dz) - \pi(dz)dt$$

is such that $\{\tilde{\mu}([0, t] \times B), 0 \leq t \leq T\}$ is a $(P, I\!F)$ martingale for all $B \in \mathcal{C}$, where $\mathcal{C}$ is the Borel σ-field on C.

An insurance strategy is a predictable process $\theta = (\theta_t)_{0 \leq t \leq T}$ which represents the rate of insurance covered by the agent. We assume that the insurance premium is an affine function of the insurance strategy. Given an initial wealth $x \geq 0$ at time t and an insurance strategy θ, the agent's wealth process at time $s \in [t, T]$ is then given by :

$$X_s^{t,x,\theta} \quad := \quad x + \int_t^s (\alpha - \beta(1 - \theta_u))\, du - \int_t^s \int_C \theta_u z \mu(du, dz). \tag{2.1}$$

We assume that $\alpha \geq \beta \geq 0$ which means that the premium rate received by the agent is lower than the premium rate paid to the insurer. In the literature, this problem is known as the proportional reinsurance one. The agent is an insurer who has to pay a premium to the reinsurer. We impose that the insurance strategy satisfies:

$$\theta_s \in [0, 1] \quad \text{a.s. for all } t \leq s \leq T. \tag{2.2}$$

We also impose the following non-bankruptcy constraint on the wealth process:

$$X_s^{t,x,\theta} \geq 0 \quad \text{a.s. for all } t \leq s \leq T. \tag{2.3}$$

Given an initial wealth $x \geq 0$ at time t, an admissible policy θ is a predictable stochastic process $(\theta_s)_{t \leq s \leq T}$, such that conditions (2.2) and (2.3) are satisfied. We denote by $\mathcal{A}(t, x)$ the set of all admissible policies and $\mathcal{S}(t, x) := \{X^{t,x,\theta}$ such that $\theta \in \mathcal{A}(t, x)\}$.

Our agent has preferences modeled by a utility function U which satisfies the following assumption:

Assumption 2.1. We assume that the agent's utility is described by a CRRA (Constant Relative Risk Aversion) utility function i.e. $U(x) = \frac{x^\eta}{\eta}$, where $\eta \in (0, 1)$.

We denote by I the inverse of U' and we introduce the conjugate function of U defined by

$$\begin{aligned} \tilde{U}(y) \quad &:= \quad \sup_{x>0}\{U(x) - xy\}, \quad y > 0 \\ &= \quad U(I(y)) - yI(y). \end{aligned} \tag{2.4}$$

A straightforward calculus shows that $\tilde{U}(y) = \dfrac{y^{-\gamma}}{\gamma}$ where $\gamma = \frac{\eta}{1-\eta}$ and $\tilde{U}'(y) = -I(y)$ for all $y > 0$.

The agent's objective is to find the value function which is defined as

$$v(t,x) := \sup_{\theta \in \mathcal{A}(t,x)} E(U(X_T^{t,x,\theta})). \tag{2.5}$$

3. Dual Optimization Problem

First we introduce some notations. Let $x \geq 0$ and $t \in [0,T]$. We denote by $\mathcal{P}(\mathcal{S}(t,x))$ the set of all probability measures $Q \sim P$ with the following property: there exists $A \in \mathcal{I}_p$, set of non-decreasing predictable processes with $A_0 = 0$, such that :

$$X - A \text{ is a } Q - \text{local super-martingale for any } X \in \mathcal{S}(t,x). \tag{3.1}$$

The upper variation process of $\mathcal{S}(t,x)$ under $Q \in \mathcal{P}(\mathcal{S}(t,x))$ is the element $\tilde{A}^{\mathcal{S}(t,x)}(Q)$ in $\mathcal{I}_p$ satisfying (3.1) and such that $A - \tilde{A}^{\mathcal{S}(t,x)}(Q) \in \mathcal{I}_p$ for any $A \in \mathcal{I}_p$ satisfying (3.1).

From Lemma 2.1 of Föllmer and Kramkov [5], we can derive $\mathcal{P}(\mathcal{S}(t,x))$ and $\tilde{A}^{\mathcal{S}(t,x)}(Q)$. This result states that $Q \in \mathcal{P}(\mathcal{S}(t,x))$ if and only if there is an upper bound for all the predictable processes arising in the Doob-Meyer decomposition of the special semi-martingale $V \in \mathcal{S}(t,x)$ under Q. In this case, the upper variation process is equal to this upper bound.

It is well-known from the martingale representation theorem for random measures (see e.g. Brémaud [2]) that all probability measures $Q \sim P$ have a density process in the form :

$$Z_s^\rho = \mathcal{E}\left(\int_t^s \int_C (\rho_u(z) - 1)\tilde{\mu}(du, dz)\right), \ s \in [t,T], \tag{3.2}$$

where $\rho \in \mathcal{U}_t = \{(\rho_s(z))_{t \leq s \leq T}$ predictable process : $\rho_s(z) > 0$, a.s., $t \leq s \leq T$,$z \in C$, $\int_t^T \int_C \Big(|\log \rho_s(z)| + \rho_s(z)\pi(dz)\Big)ds < \infty$ and $E[Z_T^\rho] = 1\}$.

By Girsanov's theorem, the predictable compensator of an element $X^\theta \in \mathcal{S}(t,x)$ under $P^\rho = Z_T^\rho.P$ is :

$$A_s^{\rho,\theta} = \int_t^s (\alpha - \beta)du + \int_t^s \theta_u(\beta - \int_C \rho_u(z)\, z\, \pi(dz))du.$$

We deduce from Lemma 2.1 of Föllmer and Kramkov [5] that $\mathcal{P}(\mathcal{S}(t,x)) = \{P^\rho : \rho \in \mathcal{U}_t\}$ and the upper variation process of P^ρ is :

$$\tilde{A}_s^{\mathcal{S}(t,x)}(P^\rho) = \int_t^s (\alpha - \beta)du + \int_t^s (\beta - \int_C \rho_u(z)\, z\, \pi(dz))_+ du.$$

From the non-decreasing property of U, we have

$$v(t,x) = \sup_{H \in \mathcal{C}_+(t,x)} E[U(H)],$$

where $\mathcal{C}_+(t,x) = \{H \in L_+^0(\mathcal{F}_T) : X_T^{t,x,\theta} \geq H \; a.s. \text{ for } \theta \in \mathcal{A}(t,x)\}$. Mnif and Pham [12] gave the following dual characterization of the set $\mathcal{C}_+(t,x)$

$$H \in \mathcal{C}_+(t,x) \Longleftrightarrow J(H) \leq x, \tag{3.3}$$

where $J(H)$ is defined by

$$J(H) := \sup_{Z\in\mathcal{P}^0(t,x),\tau\in\mathcal{T}_t} E\left[Z_T H 1_{\tau=T} - \int_t^\tau Z_u(\alpha-\beta+(\beta-\int_C \rho_u(z)\, z\, \pi(dz))_+)du\right], \tag{3.4}$$

$\mathcal{P}^0(t,x)$ is the subset of elements $P^\rho \in \mathcal{P}(\mathcal{S}(t,x))$ such that $\tilde{A}_T^{\mathcal{S}(t,x)}(P^\rho)$ is bounded and $\mathcal{T}_t$ is the set of all stopping times valued in $[0,T]$.

Following Mnif [11], the dual problem of (2.5) is written as:

$$\tilde{v}(t,y) := \inf_{Y\in\mathcal{Y}^0(t)} E\left[\tilde{U}(yY_T^{\rho,D}) + \int_t^T yY_u^{\rho,D}(\alpha-\beta+(\beta-\int_C \rho_u(z)\, z\, \pi(dz))_+)du\right], \tag{3.5}$$

where

$$\mathcal{Y}^0(t) := \{Y^{\rho,D} = Z^\rho D,\ Z^\rho \in \mathcal{P}^0(t,x),\ D\in\mathcal{D}_t\},$$

and $\mathcal{D}_t$ the set of nonnegative, nonincreasing predictable and càdlàg processes $D = (D_s)_{t\le s\le T}$ with $D_t = 1$. We shall adopt a dynamic programming principle approach to study the dual value function (3.5). We recall the dynamic programming principle for our stochastic control problem: for any stopping time $0\le\tau\le T$, $0\le t\le T$ and $0\le h\le T-t$,

$$\begin{aligned}\tilde{v}(t,y) &= \inf_{Y^{\rho,D}\in\mathcal{Y}^0(t)} E\Big[\tilde{v}\left((t+h)\wedge\tau, Y_{(t+h)\wedge\tau}^{\rho,D}\right) \\ &\quad + \int_t^{(t+h)\wedge\tau} Y_u^{\rho,D}\left(\alpha-\beta+\left(\beta-\int_C \rho_u(z)\, z\, \pi(dz)\right)_+\right)du\Big],\end{aligned} \tag{3.6}$$

where $a\wedge b = \min(a,b)$ (see e.g. Fleming and Soner [4]).

We denote by $\mathcal{L}_t$ the set of adapted processes $(L_s)_{t\le s\le T}$ with possible jump at time $s=t$ and satisfying the equation

$$dL_s = -\frac{dD_s}{D_s}1_{\{D_s>0\}},\ t\le s\le T,\ L_{t^-}=0. \tag{3.7}$$

The Hamilton Jacobi Bellman Variational Inequality arising from the dynamic programming principle (3.6) is written as

$$\min\left\{\frac{\partial\tilde{v}}{\partial t}(t,y) + H\left(t,y,\tilde{v},\frac{\partial\tilde{v}}{\partial y}\right), -\frac{\partial\tilde{v}}{\partial y}(t,y)\right\} = 0, \tag{3.8}$$

$(t,y)\in[0,T)\times(0,\infty)$, with the terminal condition

$$\tilde{v}(T,y) = \tilde{U}(y)\,, y\in(0,\infty), \tag{3.9}$$

where

$$\begin{aligned}&H\left(t,y,\tilde{v},\frac{\partial\tilde{v}}{\partial y}\right) \\ &:= \inf_{\rho\in\Sigma}\left\{A^\rho\left(t,y,\tilde{v},\frac{\partial\tilde{v}}{\partial y}\right) + y\left(\alpha-\beta+(\beta-\int_C \rho(z)\, z\, \pi(dz))_+\right)\right\},\end{aligned}$$

$$A^{\rho}\left(t,y,\tilde{v},\frac{\partial\tilde{v}}{\partial y}\right) := \int_C \left(\tilde{v}(t,\rho(z)y) - \tilde{v}(t,y) - (\rho(z)-1)y\frac{\partial\tilde{v}}{\partial y}(t,y)\right)\pi(dz),$$

and

$$\Sigma := \left\{\rho \text{ positive Borel function s.t. } \int_C \Big(|\log\rho(z)| + \rho(z)\Big)\pi(dz) < \infty\right\}.$$

This divides the time-space solvency region $[0,T)\times(0,\infty)$ into a no-jump region

$$R_1 = \left\{(t,y)\in[0,T]\times(0,\infty),\ \text{s.t. } \frac{\partial\tilde{v}}{\partial t}(t,y) + H\left(t,y,\tilde{v},\frac{\partial\tilde{v}}{\partial y}\right) = 0\right\}$$

and a jump region

$$R_2 = \left\{(t,y)\in[0,T]\times(0,\infty),\ \text{s.t. } \frac{\partial\tilde{v}}{\partial y}(t,y) = 0\right\}.$$

In Mnif [11], The dual value function is characterized as the unique viscosity solution of the associated HJBVI (3.8)- (3.9) in the set of functions $D_\gamma([0,T]\times(0,\infty))$ defined as follows:

$$D_\gamma([0,T]\times(0,\infty)) := \Big\{ f:[0,T]\times(0,\infty)\to I\!R \text{ such that },$$
$$\sup_{y>0}\frac{|f(t,y)|}{y+y^{-\gamma}} < \infty \text{ and } \sup_{x>0,y>0}\frac{|f(t,x)-f(t,y)|}{|x-y|(1+x^{-(\gamma+1)}+y^{-(\gamma+1)})} < \infty\Big\}.$$

4. Verification Theorem

The main result of this section is the following verification theorem. It characterizes the optimal wealth process. When we model the jump by a Poisson process, the optimal insurance strategy is expressed in terms of the HJBVI solution. Our stochastic control problem is unusual, in the sense that, the control ρ is an unbounded predictable process and L, given by (3.7), is also unbounded. For technical reasons, we need to add the following integrability conditions that will be checked later in the case of the Poisson process (See Example 4.9 and Remark 4.7).

Assumption 4.1. For all $t\in[0,T]$ and $(\rho,D)\in\mathcal{U}_t\times\mathcal{D}_t$. We assume that :
(i) for all $\gamma^{'}\geq 2\gamma$, we have $E[\exp(\gamma^{'}L_T)]<\infty$,
(ii) there exist two Borel functions $C_{1\rho}$, $C_{2\rho}$ such that

$$C_{1\rho}(z) \leq \rho_s(z) \leq C_{2\rho}(z)\ ds\otimes\pi(dz) \text{ a.e., } (s,z)\in[t,T]\times C,$$

$\int_C C_{1\rho}(z)^{-\gamma^{'}}\pi(dz)<\infty$ and $\int_C C_{2\rho}(z)\pi(dz)<\infty$.

The following lemma states the growth condition of the dual value function $\tilde{v}$.

Lemma 4.2. *The dual value function $\tilde{v}$ is locally bounded and satisfies*

$$\sup_{y>0}\frac{|\tilde{v}(t,y)|}{y+\tilde{U}(y)} < \infty. \tag{4.1}$$

Proof. See Appendix. □

Theorem 4.3. *Suppose that there exists a solution to the HJBVI (3.8), denoted by* $\hat{v}$ *with terminal condition*

$$\hat{v}(T, y) = \tilde{U}(y) \quad \text{for all } y \in (0, \infty),$$

such that $\hat{v}$ *is continuously differentiable with respect to* t *and* y, $\frac{\partial \hat{v}}{\partial y}$ *is continuously differentiable with respect to* t *and* y *in the no jump region* R_1 *and* $\hat{v}$ *satisfies the growth condition* (4.1). *Suppose that Assumption 4.1 holds. Suppose further that there exist a Borel function* $\hat{\rho} \in \mathcal{U}_t$, *a process* $\hat{D} \in \mathcal{D}_t$, $t \in [0, T]$ *and a positive real* $\hat{y}$ *such that with probability* 1 *we have*

$$(s, \hat{y}\hat{Y}_s) \in R_1 \quad ds \otimes dP \ \text{a.s.} \ \ s \in [t, T], \tag{4.2}$$

$$\int_t^T \frac{\partial \hat{v}}{\partial y}(s, \hat{y}\hat{Y}_{s^-})\hat{Y}_{s^-} d\hat{L}_s = 0, \tag{4.3}$$

$$\frac{\partial \hat{v}}{\partial y}(t, \hat{y}\hat{Y}_t) + x = 0, \tag{4.4}$$

where $\hat{Y} := Z^{\hat{\rho}}\hat{D} = \hat{Z}\hat{D}$. *Then* $\hat{v}$ *is the value function of the dual problem,* $(\hat{D}, \hat{\rho})$ *is the solution of the dual problem. The optimal wealth process is given by:*

$$X_s^* = -\frac{\partial \hat{v}}{\partial y}(s, \hat{y}\hat{Y}_s)\, ds \otimes dP \ \text{a.s.} \ \ s \in [t, T]. \tag{4.5}$$

Proof. See Appendix □

Remark 4.4. Hypothesis (4.2) means that $((s, \hat{y}\hat{Y}_s))_{s\in[t,T]}$ remains almost surely in the no jump region. The process might have jumps in the region R_2 but it immediately reaches the region R_1.

Remark 4.5. Hypothesis(4.3) means that the process $\hat{D}$ regulates the process $\hat{Y}$ and it only decreases when the wealth process hits zero.

Remark 4.6. If all the shocks have the same size denoted by δ, then the optimal insurance strategy is given by

$$\theta_s^* = \frac{\frac{\partial \hat{v}}{\partial y}(s, \hat{\rho}_s\hat{y}\hat{Y}_{s^-}) - \frac{\partial \hat{v}}{\partial y}(s, \hat{y}\hat{Y}_{s^-})}{\delta} \quad \text{a.e. in } s \in [t, T]. \tag{4.6}$$

From definition of $\hat{L}$ (see assumption 4.3), $\hat{L}$ decreases only on the set $\{\frac{\partial \hat{v}}{\partial y}(s, \hat{y}\hat{Y}_s) = 0\}$ or on this set, we have $\frac{\partial^2 \hat{v}}{\partial y^2}(s, \hat{y}\hat{Y}_s) = 0$ and so $\dfrac{\partial^2 \hat{v}}{\partial y^2}(s, \hat{y}\hat{Y}_s)\hat{D}_s d\hat{L}_s = 0$. By Itô's lemma we obtain

$$\begin{aligned} dX_s^* &= \frac{\partial^2 \hat{v}}{\partial y^2}(s, \hat{y}\hat{Y}_s)\hat{Y}_s d\hat{L}_s + \varrho(\hat{\rho}_s - 1)\hat{y}\hat{Y}_s \frac{\partial^2 \hat{v}}{\partial y^2}(s, \hat{y}\hat{Y}_s) ds \\ &\quad -\frac{\partial^2 \hat{v}}{\partial s \partial y}(s, \hat{y}\hat{Y}_s) ds - (\frac{\partial \hat{v}}{\partial y}(s, \hat{\rho}_s\hat{y}\hat{Y}_{s^-}) - \frac{\partial \hat{v}}{\partial y}(s^-, \hat{y}\hat{Y}_{s^-}))dN_s \\ &= \varrho(\hat{\rho}_s - 1)\hat{y}\hat{Y}_s \frac{\partial^2 \hat{v}}{\partial y^2}(s, \hat{Y}_s) ds - \frac{\partial^2 \hat{v}}{\partial s \partial y}(s, \hat{y}\hat{Y}_s) ds \\ &\quad -\theta_s^* \delta dN_s. \end{aligned} \tag{4.7}$$

Using Hypothesis (4.2), the regularity on the function $\hat{v}$ and Itô's lemma, we have

$$\begin{aligned}
&\frac{\partial^2 \hat{v}}{\partial y \partial s}(s, \hat{y}\hat{Y}_{s^-}) + \varrho(\hat{\rho}_s \frac{\partial \hat{v}}{\partial y}(s, \hat{\rho}_s \hat{y}\hat{Y}_{s^-}) - \frac{\partial \hat{v}}{\partial y}(s, \hat{y}\hat{Y}_{s^-})) \\
&-\varrho(\hat{\rho}_s - 1)\frac{\partial \hat{v}}{\partial y}(s, \hat{y}\hat{Y}_{s^-}) - \varrho(\hat{\rho}_s - 1)\hat{y}\hat{Y}_{s^-}\frac{\partial^2 \hat{v}}{\partial y^2}(s, \hat{y}\hat{Y}_{s^-}) \\
&+(\alpha - \beta + (\beta - \varrho\delta\hat{\rho}_s)_+) = 0.
\end{aligned} \tag{4.8}$$

Plugging (4.8) into (4.7) and using (4.4), we obtain

$$\begin{aligned}
X^*_s &= x + \int_t^s (\alpha - \beta + (\beta - \varrho\delta\hat{\rho}_u)_+)du - \int_t^s \theta^*_u \delta dN_u \\
&\quad + \int_t^s \varrho\delta\hat{\rho}_u\theta^*_u du,
\end{aligned}$$

and so θ^* is the optimal insurance strategy.

Remark 4.7. If all the shocks have the same size denoted by δ, then the set $\mathcal{U}_t$ is given by $\mathcal{U}_t = \{(\rho_s)_{t \le s \le T}$ predictable process : $\rho_s > 0$, a.s., $t \le s \le T$ and $E[Z^\rho_T] = 1\}$. In this case Assumption 4.1(ii) is automatically checked.

Remark 4.8. Theorem 5.1 of Mnif and Pham [12] could be viewed as a dual verification theorem which caracterizes the solution of the primal approach. The theorem 4.3 brings a new information by using PDE arguments which concern the wealth process and the optimal strategy in the case of the Poisson process.

Example 4.9. If all the shocks have the same size denoted by δ and if $\alpha = \beta = \pi\delta$ (cheap reinsurance), then the Hamiltonian H has the following expression

$$\begin{aligned}
&H\left(t, y, \tilde{v}, \frac{\partial \tilde{v}}{\partial y}\right) = \\
&\inf_{\rho > 0}\left\{\pi\left(\tilde{v}(t, \rho y) - \tilde{v}(t, y) - (\rho - 1)y\frac{\partial \tilde{v}}{\partial y}(t, y)\right) + y\beta(1 - \rho)_+\right\}
\end{aligned}$$

As it is seen in Lemma 4.1 in Mnif [11], the dual value function is convex in y and so

$$\pi\left(\tilde{v}(t, \rho y) - \tilde{v}(t, y) - (\rho - 1)y\frac{\partial \tilde{v}}{\partial y}(t, y)\right) + y\beta(1 - \rho)_+ \ge 0$$

and the equality is obtained when $\rho = 1$. In this case $H\left(t, y, \tilde{v}, \frac{\partial \tilde{v}}{\partial y}\right) = 0$. The solution of the HJBVI (3.8) with terminal condition (3.9) is given by

$$\tilde{v}(t, y) = \tilde{U}(y),$$

and the solution of the dual problem is given by $\hat{\rho} \equiv 1$ and $\hat{D} \equiv 1$. From the Verification Theorem the optimal wealth process is given by $X^* \equiv x$, the insurance strategy $\theta^* \equiv 0$ and so Assumption 4.1 is checked.

5. Numerical Study

Here we restrict ourselves to the case where the integer valued random measure $\mu(dt, dz)$ is a Poisson process with the constant intensity π. All the claims have the same size denoted by δ. Our purpose is to solve the following variational inequality:

$$\min\left\{\frac{\partial\tilde{v}}{\partial t}(t,y)+\inf_{\rho>0}\left\{A^{\rho}(t,y,\tilde{v},\frac{\partial\tilde{v}}{\partial y})+y\left(\alpha-\beta+(\beta-\rho\delta\pi)_{+}\right)\right\},\right. \tag{5.1}$$

$$\left.-\frac{\partial\tilde{v}}{\partial y}(t,y)\right\}=0, \tag{5.2}$$

for all $(t, y) \in [0, T) \times (0, \infty)$, with the terminal condition $\tilde{v}(T, y) = \tilde{U}(y)$, where

$$A^{\rho}(t,y,\tilde{v},\frac{\partial\tilde{v}}{\partial y})=\pi\left(\tilde{v}(t,\rho y)-\tilde{v}(t,y)-(\rho-1)y\frac{\partial\tilde{v}}{\partial y}(t,y)\right).$$

It is more appropriate to study numerically the function

$$J(t,y):=e^{-rt}\tilde{v}(t,y), \tag{5.3}$$

where r is a positive constant. We will explain in Remark 5.2 the advantage of the introduction of the function J. We proceed with another technical change of variable which brings $[0, T] \times (0, \infty)$ into $[0, T] \times (0, 1)$, namely

$$\begin{cases}\tilde{y}=\frac{y}{1+y}\\ \bar{v}(t,\tilde{y})=J(t,y).\end{cases}$$

The function $\bar{v}$ satisfies

$$\min\left\{\frac{\partial\bar{v}}{\partial t}(t,\tilde{y})+\inf_{\rho>0}\left\{\bar{A}^{\rho}(t,\tilde{y},\bar{v},D\bar{v})+\frac{\tilde{y}}{(1-\tilde{y})}(\alpha-\beta+(\beta-\rho\delta\pi)_{+})\right\},\right.$$
$$\left.-(1-\tilde{y})^{2}D\bar{v}(t,\tilde{y})\right\}=0 \tag{5.4}$$

for all $(t, \tilde{y}) \in [0, T) \times (0, 1)$, where

$$\bar{A}^{\rho}(t,\tilde{y},\bar{v},D\bar{v}):=$$
$$\pi\left(\bar{v}(t,\frac{\rho\tilde{y}}{1+\tilde{y}(\rho-1)})-\bar{v}(t,\tilde{y})-(\rho-1)(1-\tilde{y})\tilde{y}D\bar{v}(t,\tilde{y})\right)-r\bar{v}(t,\tilde{y})$$

and $D\bar{v}$ is the derivative of $\bar{v}$ with respect to the state variable. The terminal condition is given by

$$\bar{v}(T,\tilde{y})=\frac{e^{-rT}\tilde{y}^{-\gamma}}{\gamma(1-\tilde{y})^{-\gamma}} \tag{5.5}$$

for all $\tilde{y} \in (0, 1)$.

In Mnif [11], we have proved that the dual value function (3.5), within a change of variables, is the unique viscosity solution of variational inequality (5.4). This solution can be approximated by the following numerical method:

(i) approximate variational inequality (5.4) by using a consistent finite difference approximation which satisfies the discrete maximum principle (DMP) (see Lapeyre, Sulem and Talay [10]),

(ii) solve the discrete equation by means of the Howard algorithm (policy iteration) (see Howard [9]). Finally a reverse change of variables is performed in order to display results of variational inequality (5.1).

5.1. Finite difference approximation. Let $h := (h_t, h_{\tilde{y}})$ be the finite difference step in the time coordinate and the finite difference step in the state coordinate. The step h_t is defined by $h_t := \frac{T}{N}, (N \in I\!N^*)$. Let $M \in I\!N^*$ be the number of discretization steps in the state coordinate ($h_{\tilde{y}}$ is not uniform for all elements of the grid). Let $(t_i, \tilde{y}_j), 0 \le i \le N,\ 1 \le j \le M-1$ be the points of the grid $\Omega_{N,M}$.We choose a fully implicit θ-scheme. We consider an approximation scheme of (5.4) of the following form:

$$S(h, t, \tilde{y}, \bar{v}^h(t, \tilde{y}), \bar{v}^h) = 0,\ (t, \tilde{y}) \in \Omega_{N,M}, \tag{5.6}$$

where

$$\begin{aligned} & S(h, t, \tilde{y}, \bar{v}^h(t, \tilde{y}), \bar{v}^h) := \min\Big\{ \frac{\bar{v}^h(t + h_t, \tilde{y}) - \bar{v}^h(t, \tilde{y})}{h_t} - r\bar{v}^h(t, \tilde{y}) \\ + \quad & \inf_{\rho > 0} \Big\{ \pi\Big(\bar{v}^h(t, Pr(\frac{\rho\tilde{y}}{1 + \tilde{y}(\rho - 1)})) - \bar{v}^h(t, \tilde{y}) + ((1-\rho)(1-\tilde{y})\tilde{y})_+ D_+ \bar{v}^h(t, \tilde{y}) \\ - \quad & ((1-\rho)(1-\tilde{y})\tilde{y})_- D_- \bar{v}^h(t, \tilde{y}) \Big) + \frac{\tilde{y}}{(1-\tilde{y})}(\alpha - \beta + (\beta - \rho\delta\pi)_+) \Big\} \\ - \quad & (1-\tilde{y})^2 D\bar{v}^h(t, \tilde{y}) \Big\}; \end{aligned}$$

$$D_+ \bar{v}^h(t, \tilde{y}) := \frac{\bar{v}^h(t, \tilde{y} + h_{\tilde{y}}) - \bar{v}^h(t, \tilde{y})}{h_{\tilde{y}}},\ D_- \bar{v}^h(t, \tilde{y}) := \frac{\bar{v}^h(t, \tilde{y}) - \bar{v}^h(t, \tilde{y} - h_{\tilde{y}})}{h_{\tilde{y}}},$$

$$\begin{aligned} ((1-\rho)(1-\tilde{y})\tilde{y})_+ &= \max\left((1-\rho)(1-\tilde{y})\tilde{y}, 0\right), \\ ((1-\rho)(1-\tilde{y})\tilde{y})_- &= \max\left(-(1-\rho)(1-\tilde{y})\tilde{y}, 0\right), \end{aligned}$$

and $(t, Pr(\frac{\rho\tilde{y}}{1+\tilde{y}(\rho-1)}))$ is the projection of $(t, \frac{\rho\tilde{y}}{1+\tilde{y}(\rho-1)})$ on the grid. We take $\bar{v}^h(t_i, \tilde{y}_M) = \bar{v}^h(t_i, \tilde{y}_{M-2})$ for all $0 \le i \le N-1$. For terminal condition, we set

$$\bar{v}^h(T, \tilde{y}_j) \quad = \quad \frac{e^{-rT}\tilde{y}_j^{-\gamma}}{\gamma(1-\tilde{y}_j)^{-\gamma}} \ \text{ for all } 1 \le j \le M-1.$$

Initial conditions are usually needed to approximate the derivatives which leads to a linear system to solve (see for example Lapeyre, Sulem and Talay [10]). In our case, we approximate the first derivative with respect to the state variable when $\tilde{y} = \tilde{y}_1$ by using the forward finite difference discretization only since $\bar{v}(t, 0) = \tilde{v}(t, 0) = \infty$, $t \in [0, T)$ (see inequality (6.5)). For the approximation of $D\bar{v}(t, \tilde{y})$ when $\tilde{y} = \tilde{y}_{M-1}$, we use the backward finite difference discretization only since $\bar{v}(t, 1) = \tilde{v}(t, \infty)$, $t \in [0, T)$ is not defined. Taking into account these conditions, the approximation (5.6) leads to a system of $N \times (M-1)$ equations with $N \times (M-1)$

unknowns $\{\bar{v}^h(t_i,\tilde{y}_j)\ , 0 \le i \le N-1, 1 \le j \le M-1\}$:

$$\min\Big\{\bar{v}^h(t_{i+1},\tilde{y}_j) - \bar{v}^h(t_i,\tilde{y}_j) + \min_{\rho\in\mathcal{M}^\rho}\left\{h_t\bar{A}^{\rho,t_i}\bar{v}^h(t_i,\tilde{y}_j) + h_t l^\rho(\tilde{y}_j)\right\},$$
$$\bar{B}\bar{v}^h(t_i,\tilde{y}_j)\Big\} = 0, \tag{5.7}$$

for all $0 \le i \le N-1, 1 \le j \le M-1$, with terminal condition:

$$\bar{v}^h(T,\tilde{y}_j) = \frac{e^{-rT}\tilde{y}_j^{-\gamma}}{\gamma(1-\tilde{y}_j)^{-\gamma}} \text{ for all } 1 \le j \le M-1,$$

where $\mathcal{M}^\rho = \{(\rho_{ij})_{0\le i\le N-1\,,\,1\le j\le M-1}, \rho_{ij} > 0\}$, $\bar{A}^{\rho,t_i}$ is the $(M-1)\times(M-1)$ matrix associated to the approximation of the operator $\bar{A}^\rho$ at time t_i, l^ρ is $(M-1)$ vector such that

$$l^\rho(\tilde{y}_j) = \frac{\tilde{y}_j}{1-\tilde{y}_j}(\alpha - \beta + (\beta - \rho\delta\pi)_+), \text{ for all } 1 \le j \le M-1$$

and $\bar{B}$ is a $(M-1)\times(M-1)$ matrix associated to the second term of our variational inequality, which verifies

$$\begin{cases} \bar{B}(j,j) = -\frac{1}{\tilde{y}_j-\tilde{y}_{j-1}} \text{ for all } 2 \le j \le M-1 \\ \bar{B}(j,j-1) = \frac{1}{\tilde{y}_j-\tilde{y}_{j-1}} \text{ for all } 2 \le j \le M-1 \\ \bar{B}(i,j) = 0 \text{ if not }. \end{cases}$$

Let $\mathcal{A}_p$ denote the set of control functions $\rho\ : \Omega_{N,M} \longrightarrow \mathcal{M}^\rho$. The system of equations (5.7) can be written as a system of N stationary inequalities:

$$\min\left\{\bar{v}^{h,t_{i+1}} - \bar{v}^{h,t_i} + \min_{\rho\in\mathcal{A}_p}\left\{h_t\bar{A}^{\rho,t_i}\bar{v}^{h,t_i} + h_t l^\rho\right\}, \bar{B}\bar{v}^{h,t_i}\right\} = 0, \tag{5.8}$$

for all $i = 0...N-1$, with terminal condition:

$$\bar{v}^{h,T} = \left(\frac{e^{-rT}\tilde{y}_j^{-\gamma}}{\gamma(1-\tilde{y}_j)^{-\gamma}}\right)_{j=1..M-1},$$

where $\bar{v}^{h,t_i}$ a vector which approximates $(\bar{v}(t_i,\tilde{y}_j))_{j=1...M-1}$.

The convergence of the numerical scheme in not proved in our situation as in the case of Tourin and Zariphopoulou [14] (They studied numerical schemes for investment consumption models with transaction costs). The system of N stationary inequalities (5.8) can be solved by Howard algorithms. We describe this algorithm below.

Remark 5.1. Barles and Souganidis [1] proved that a numerical scheme consistent monotone and stable converges to the unique viscosity solution of the HJB since a comparison theorem holds for the limiting equation in class of bounded functions. In our case, the dual value function is not bounded and since we used the discoutinuous viscosity solutions, it is not obvious that the semi-relaxed limits of our sequences are in the space $D_\gamma([0,T]\times(0,\infty))$.

Remark 5.2. The introduction of the function J (see equality (5.3)), insures that the matrix $\bar{A}^{\rho,t_i}, i = 0...N-1$ is diagonally dominant.

5.2. The Howard algorithm. To solve Equation (5.8), we use the Howard algorithm (see Lapeyre Sulem and Talay [10]), also named policy iteration.

It consists in computing two sequences $(\rho^{t_i,n})_{n\in\mathbb{N}}$ and $(\bar{v}^{h,t_i,n})_{n\in\mathbb{N}}, i=0...N-1$, (starting from $\bar{v}^{h,t_i,1}, i=0...N-1$) defined by:

- Step $2n-1$. To $\bar{v}^{h,t_i,n}$ is associated to another strategy $\rho^{t_i,n}$
$$\rho^{t_i,n}\in\arg\min_{\rho\in\mathcal{A}_p}\left\{\bar{A}^{\rho,t_i}\bar{v}^{h,t_i,n}+l^{\rho,n}\right\},\ i=0...N-1.$$
- Step $2n$. To the strategy $\rho^{t_i,n}$, we compute a partition $(D_1^n\cup D_2^n)$ such that
$$\begin{aligned}&\bar{v}^{h,t_{i+1},n}+(h_t\bar{A}^{\rho^{t_i,n},t_i}-I)\bar{v}^{h,t_i,n}+h_tl^{\rho^{t_i,n}}\\ \le\ &\bar{B}\bar{v}^{h,t_i,n},\ i=0...N-1,\ \text{ on } D_1^n,\end{aligned}$$
$$\begin{aligned}&\bar{v}^{h,t_{i+1},n}+(h_t\bar{A}^{\rho^{t_i,n},ih_t}-I)\bar{v}^{h,t_i,n}+h_tl^{\rho^{ih_t,n}}\\ \ge\ &\bar{B}\bar{v}^{h,t_i,n},\ i=0...N-1,\ \text{ on } D_2^n.\end{aligned}$$
The solution $\bar{v}^{h,t_i,n+1}$ is obtained by solving two linear systems:
$$\begin{aligned}&\bar{v}^{h,t_{i+1},n+1}+(h_t\bar{A}^{\rho^{t_i,n},t_i}-I)\bar{v}^{h,t_i,n+1}\\ &+h_tl^{\rho^{t_i,n}}=0,\ i=0...N-1,\ \text{ on } D_1^n,\end{aligned}$$
and
$$\bar{B}\bar{v}^{h,t_i,n+1}=0,\ i=0...N-1,\ \text{ on } D_2^n.$$
- If $|\bar{v}^{h,t_i,n+1}-\bar{v}^{h,t_i,n}|\le\epsilon$, $i=0...N-1$, stop, otherwise, go to step $2n+1$.

The convergence of the Howard algorithm is obtained heuristically. We have no theoretical result for the convergence. The matrix arising after the discretization of the HJBVI does not satisfy the discrete maximum principle which is a sufficient condition for the convergence of such algorithm.

5.3. Algorithm for the optimal strategy. After the numerical resolution of the Variational Inequality (5.4), we compute the optimal strategy of insurance and the wealth process. From the Verification Theorem, we need to evaluate $\hat{y}$ and to construct the process $(\hat{Y}_{t_i})_{0\le i\le N-1}$.

The optimal insurance strategy and the wealth process are given by the formulas (4.6) and (4.5). We describe the algorithm below.

First step: Given an initial wealth x,

- we compute $\tilde{y}_{j_0}$ s.t $(0,\tilde{y}_{j_0})\in\Omega_{N,M}$ and $\hat{X}(0,\tilde{y}_{j_0})=x$, where $\hat{X}(t_i,\tilde{y}_j)=-(1-\tilde{y}_j)^2\left(\frac{\bar{v}(t_i,\tilde{y}_j)-\bar{v}(t_i,\tilde{y}_{j-1})}{\tilde{y}_j-\tilde{y}_{j-1}}\right)$, $0\le i\le N-1$ and $1\le j\le M$
- we compute $\hat{y}=\frac{\tilde{y}_{j_0}}{1-\tilde{y}_{j_0}}$.

Second step: Let $\hat{Z}_0=\hat{D}_0=1$. For $i=1$ to $N-1$, we construct the process $\hat{Y}_{t_i}=\hat{y}\hat{Z}_{t_i}\hat{D}_{t_i}$ as follows:

- We compute $\frac{\hat{Y}_{t_{i-1}}}{1+\hat{Y}_{t_{i-1}}}$ and we select the nearest point of the grid to $(t_i,\frac{\hat{Y}_{t_{i-1}}}{1+\hat{Y}_{t_{i-1}}})$. This point will be denoted by $(t_i,\tilde{y}_{j_i})$.

- We determine the optimal control ρ which is obtained by the Howard Algorithm at point $(t_i, \tilde{y}_{j_i})$. We denote this control by $\hat{\rho}_{j_i}$.
- We evaluate $\hat{Z}_{t_i} = \hat{Z}_{t_{i-1}} \exp(-\pi h(\hat{\rho}_{j_i} - 1))(1 + (\hat{\rho}_{j_i} - 1)1_{\{\triangle\mu(t_i)=1\}})$. We take $D_{t_i} = D_{t_{i-1}}$.
- We compute $\frac{\hat{\rho}_{j_i}\hat{Y}_{t_{i-1}}}{1+\hat{\rho}_{j_i}\hat{Y}_{t_{i-1}}}$ (resp $\frac{\hat{Y}_{t_i}}{1+\hat{Y}_{t_i}}$) and we select the point of the grid which is the nearest to $(t_i \frac{\hat{\rho}_{j_i}\hat{Y}_{t_{i-1}}}{1+\hat{\rho}_{j_i}\hat{Y}_{t_{i-1}}})$ (resp $\frac{\hat{y}\hat{Y}_{t_i}}{1+\hat{y}\hat{Y}_{t_i}}$) . This point will be denoted by $(t_i, \tilde{y}_{j_i'})$ (resp $(t_i, \tilde{y}_{j_i''})$).
- We make the following instruction: while $\hat{X}(t_i, \tilde{y}_{j_i''}) < 0$, we decrease the process D_{t_i}. We denote by $(t_i, \tilde{y}_{j_i''})$ the new point of the grid.
- The optimal insurance strategy and the optimal wealth process are given by

$$\theta^*_{t_i} = \frac{-\hat{X}(t_i, \tilde{y}_j^{'}) + \hat{X}(t_i, \tilde{y}_j)}{\delta}, \tag{5.9}$$

$$X^*_{t_i} = \hat{X}(t_i, \tilde{y}_j^{''}). \tag{5.10}$$

Numerical resolution of the associated HJBVI is postponed in future research.

6. Appendix

6.1. Proof of Lemma 4.2. Since the controls $\rho_s = 1$ and $D_s = 1$, $s \in [t, T]$ lie in $\mathcal{U}_t \times \mathcal{D}_t$, we have

$$\tilde{v}(t, y) \leq \tilde{U}(y) + Ky, \tag{6.1}$$

where K is a constant.

Let $(Z^n := Z^{\rho^n}, D^n)$ be a minimizing sequence of $\tilde{v}(t, y)$. From the definition of these minimizing sequences, there exist ϵ_n and $n_0 \in I\!N$ such that $\epsilon_n \longrightarrow 0$ when $n \longrightarrow \infty$ and for all $n \geq n_0$, we have

$$\begin{aligned} \tilde{v}(t, y) \quad \geq \quad & E\left[\tilde{U}(yZ^n_T D^n_T)\right] \\ & +yE\left[\int_t^T Z^n_u D^n_u(\alpha - \beta + (\beta - \int_C \rho^n_u(z)\, z\, \pi(dz))_+)du\right] - \epsilon_n. \end{aligned} \tag{6.2}$$

Since $\epsilon_n \longrightarrow 0$ when $n \longrightarrow \infty$, there exists $n_1 \in I\!N$ such that for all $n \geq n_1$, we have $\epsilon_n \leq \tilde{U}(y) + y$. We recall That $\tilde{U}(y) \geq U(0^+) \geq 0$ and so $\tilde{U}(y) + y > 0$ since $y > 0$. Using the boundedness of D^n, Jensen's inequality and the martingale property of Z^n, we have:

$$\begin{aligned} E\left[\tilde{U}(yZ^n_T D^n_T)\right] \quad & \geq \quad \tilde{U}(yE\left[Z^n_T\right]) \\ & \geq \quad \tilde{U}(y). \end{aligned} \tag{6.3}$$

For the second term of the r.h.s of inequality (6.2), since $D_s^n \leq 1$ for all $s \in [t,T]$, using Fubini's theorem and the martingale property of Z^n, we have

$$\begin{aligned}
& E\left[\int_t^T yZ_u^n D_u^n(\alpha-\beta+(\beta-\int_C \rho_u(z)\, z\, \pi(dz))_+)du\right] \\
\geq\ & y(\alpha-\beta)E\left[\int_t^T Z_u^n D_u^n du\right] \\
\geq\ & y(\alpha-\beta)\int_t^T E\left[Z_u^n\right] du \\
\geq\ & K'y, \qquad (6.4)
\end{aligned}$$

where K' is a constant independent of y. Inequalities (6.3) and (6.4) imply that

$$\tilde{v}(t,y) \geq \tilde{U}(y) + K'y. \qquad (6.5)$$

From inequalities (6.1) and (6.5), we deduce that

$$\sup_{y>0} \frac{|\tilde{v}(t,y)|}{y+\tilde{U}(y)} < \infty \qquad (6.6)$$

6.2. Proof of Theorem 4.3. The proof of the theorem is broken in three steps. Let $t \in [0,T]$ and $y \in (0,\infty)$.
First step: We show that

$$\begin{aligned}
& \hat{v}(t,y) \qquad (6.7) \\
\leq\ & \inf_{Y^{\rho,D}\in\mathcal{Y}^0(t)} E\left[\tilde{U}(yY_T^{\rho,D}) + \int_t^T yY_u^{\rho,D}(\alpha-\beta+(\beta-\int_C \rho_u(z)\, z\, \pi(dz))_+)du\right].
\end{aligned}$$

Let $Y^{\rho,D} \in \mathcal{Y}^0(t)$. Let

$$\tau_n = \inf\{u \geq t \text{ such that } \left|\int_C \hat{v}(u,\rho_u(z)yY_{u^-}^{\rho,D}) - \hat{v}(u,yY_{u^-}^{\rho,D})\pi(dz)\right| > n\} \wedge T.$$

Applying the generalized Itô's formula, we have

$$\begin{aligned}
& \hat{v}(T\wedge\tau_n, yY_{T\wedge\tau_n}^{\rho,D}) + \int_t^{T\wedge\tau_n} yY_u^{\rho,D}(\alpha-\beta+(\beta-\int_C \rho_u(z)\, z\, \pi(dz))_+)du \\
=\ & \hat{v}(t,y) + \int_t^{T\wedge\tau_n} \frac{\partial \hat{v}}{\partial u}(u, yY_{u^-}^{\rho,D})du - \int_t^{T\wedge\tau_n} \frac{\partial \hat{v}}{\partial y}(u,yY_{u^-}^{\rho,D})yY_{u^-}^{\rho,D} dL_u \\
& - \int_t^{T\wedge\tau_n}\int_C \frac{\partial \hat{v}}{\partial y}(u,yY_{u^-}^{\rho,D})yY_{u^-}^{\rho,D}(\rho_u(z)-1)\pi(dz)du \\
& + \sum_{t\leq u\leq T\wedge\tau_n} \Big(\hat{v}(u,yY_u^{\rho,D}) - \hat{v}(u,yY_{u^-}^{\rho,D})\Big) \\
& + \int_t^{T\wedge\tau_n} yY_u^{\rho,D}(\alpha-\beta+(\beta-\int_C \rho_u(z)\, z\, \pi(dz))_+)du
\end{aligned}$$

and so we have

$$\begin{aligned}
&\hat{v}(T\wedge\tau_n, yY^{\rho,D}_{T\wedge\tau_n}) + \int_t^{T\wedge\tau_n} yY^{\rho,D}_u(\alpha-\beta+(\beta-\int_C \rho_u(z)\,z\,\pi(dz))_+)du \\
= \quad & \hat{v}(t,y) + \int_t^{T\wedge\tau_n}\left(\frac{\partial \hat{v}}{\partial u}(u, yY^{\rho,D}_u) + A^{\rho}(u, yY^{\rho,D}_u, \hat{v}, \frac{\partial \hat{v}}{\partial y})\right) ds \qquad (6.8)\\
& + \int_t^{T\wedge\tau_n} yY^{\rho,D}_u(\alpha-\beta+(\beta-\int_C \rho_u(z)\,z\,\pi(dz))_+)du \\
& - \int_t^{T\wedge\tau_n} \frac{\partial \hat{v}}{\partial y}(u, yY^{\rho,D}_{u^-})yY^{\rho,D}_{u^-}dL_u \\
& + \int_t^{T\wedge\tau_n}\int_C \hat{v}(u, \rho_u(z)yY^{\rho,D}_{u^-}) - \hat{v}(u, yY^{\rho,D}_{u^-})\tilde{\mu}(du,dz).
\end{aligned}$$

Since $\hat{v}$ is a classical solution of the variational inequality (3.8), we have

$$\begin{aligned}
&\frac{\partial \hat{v}}{\partial u}(u, yY^{\rho,D}_u) + A^{\rho}(u, yY^{\rho,D}_u, \hat{v}, \frac{\partial \hat{v}}{\partial y}) \\
&+yY^{\rho,D}_u(\alpha-\beta+(\beta-\int_C \rho_u(z)\,z\,\pi(dz))_+) \geq 0 \\
&\text{and } -\frac{\partial \hat{v}}{\partial y}(u, yY^{\rho,D}_{u^-})Y^{\rho,D}_{u^-}dL_u \geq 0 \text{ a.e. in } u\in[t,T].
\end{aligned}$$

Taking expectation in (6.8), we have

$$\begin{aligned}
\hat{v}(t,y) \quad \leq \quad & E\Big[\hat{v}(T\wedge\tau_n, yY^{\rho,D}_{T\wedge\tau_n}) \\
& + \int_t^{T\wedge\tau_n} Y^{\rho,D}_u(\alpha-\beta+(\beta-\int_C \rho_u(z)\,z\,\pi(dz))_+)du\Big],
\end{aligned}$$

for all $Y^{\rho,D}\in\mathcal{Y}^0(t)$. It remains to show that

$$\text{the family } \left(\tilde{v}(T\wedge\tau_n, yY^{\rho,D}_{T\wedge\tau_n})\right)_n \text{ is uniformly integrable under } P. \qquad (6.9)$$

We consider the function $g(z)=z^p$, $p>1$ will be chosen later, $z\geq 0$. By using Itô's formula and since the function U is a power utility function, we have

$$\begin{aligned}
g\left(\tilde{U}(Y^{\rho,D}_T)\right) \quad = \quad & g(\tilde{U}(y)) + \int_t^T \gamma p g\left(\tilde{U}(yY^{\rho,D}_u)\right) dL_u \qquad (6.10)\\
+ \quad & \int_t^T\int_C g\left(\tilde{U}(yY^{\rho,D}_u)\right)\left(\rho_u(z)^{-\gamma p}-1\right)\tilde{\mu}(du,dz) \\
+ \quad & \int_t^T\int_C g\left(\tilde{U}(yY^{\rho,D}_u)\right)\left(\rho_u(z)^{-\gamma p}-1+\gamma p(\rho_u(z)-1)\right)\pi(dz)du.
\end{aligned}$$

The solution of (6.10) is given by the Doléans-Dade exponential formula

$$\begin{aligned}
& g\left(\tilde{U}(yY_T^{\rho,D})\right) \\
= \quad & g(\tilde{U}(y))Z_{1T}^{\rho}\exp\left(\gamma pL_T + \int_t^T\int_C \left(\rho_u(z)^{-\gamma p} - 1 + \gamma p(\rho_u(z)-1)\right)\pi(dz)du\right) \\
\leq \quad & \frac{1}{2}g(\tilde{U}(y))\Big((Z_{1T}^{\rho})^2 \\
& + \exp\left(2\gamma pL_T + 2\int_t^T\int_C \left(\rho_u(z)^{-\gamma p} - 1 + \gamma p(\rho_u(z)-1)\right)\pi(dz)du\right)\Big),
\end{aligned}$$

where $(Z_{1u}^{\rho})_{u\in[t,T]}$ is a local martingale defined by

$$Z_{1u}^{\rho} = \mathcal{E}\left(\int_t^u\int_C \left(\rho_u(z)^{-\gamma p} - 1\right)\tilde{\mu}(du,dz)\right).$$

We choose $p = \frac{\gamma^{'}}{2\gamma}$ where $\gamma^{'}$ is defined in Assumption 4.1(i). From Assumption 4.1(ii) and by Jensen inequality, we have

$$\int_0^T\int_C \rho_s(z)^{-\gamma p}\pi(dz)ds \leq \int_0^T \Big(\int_C \rho_s(z)^{-2\gamma p}\pi(dz)\Big)^{\frac{1}{2}}ds$$

and so by Assumption 4.1 there exists a positive constant C_1 such that :

$$E\Big[\exp\left(2\gamma pL_T + 2\int_0^T\int_C \left(\rho_s(z)^{-\gamma p} - 1 + \gamma p(\rho_s(z)-1)\right)\pi(dz)ds\right)\Big] \leq C_1 \tag{6.11}$$

From the definition of $(Z_{1s}^{\rho})_{s\in[t,T]}$, we have

$$Z_{1s}^{\rho} = 1 + \int_t^s\int_C Z_{1u^-}^{\rho}\left(\rho_u(z)^{-\gamma p} - 1\right)\tilde{\mu}(du,dz)$$

Taking expectation under P and using Assumption 4.1(ii), we obtain

$$\begin{aligned}
E\Big[(Z_{1s}^{\rho})^2\Big] \quad & \leq \quad 2\Big(1 + \int_t^s\int_C |Z_{1u}^{\rho}|^2\left(\rho_u(z)^{-\gamma p} - 1\right)^2\pi(dz)du\Big) \\
& \leq \quad 2\Big(1 + E\int_t^s |Z_{1u}^{\rho}|^2 du\Big).
\end{aligned}$$

By Fubini's theorem and Gronwall's lemma , we have

$$E\Big[(Z_{1s}^{\rho})^2\Big] \leq C_1 \tag{6.12}$$

From inequalities (6.11) and (6.12), we obtain that

$$E\left[g\left(\tilde{U}(yY_{T\wedge\tau_n}^{\rho,D})\right)\right] \leq C_1 g(\tilde{U}(y)),$$

and so

$$\sup_{n\in\mathbb{N}} E\left[g\left(\tilde{U}(yY_{T\wedge\tau_n}^{\rho,D})\right)\right] < \infty. \tag{6.13}$$

Similarly, one can prove that $\sup_{n\in\mathbb{N}} E\left[g\left(yY_{T\wedge\tau_n}^{\rho,D}\right)\right] < \infty$. Since $\dfrac{g(x)}{x} \longrightarrow \infty$ when x goes to infinity and from the growth condition (4.1) , the property (6.9) holds.

Sending $n \to \infty$, we have $\tau_n \longrightarrow \infty$ P $a.s.$ By dominated convergence theorem, we have (6.7).

Second step: We show that $\hat{v}$ is the dual value function and $(\hat{\rho}, \hat{D})$ is the solution of the dual problem i.e:

$$\hat{v}(t,y) \tag{6.14}$$
$$= E\left[\tilde{U}(y\hat{Y}_T^t) + \int_t^T y\hat{Y}_u^t(\alpha - \beta + (\beta - \int_C \hat{\rho}_u(z)\, z\, \pi(dz))_+)du \big| \hat{y}\hat{Y}_t = y\right],$$

where $Y_s^t := \frac{\hat{Y}_s}{\hat{Y}_t}$, $s \in [t,T]$. We consider the processes $\hat{\rho}$ and $\hat{D}$ and the positive number $\hat{y}$ such that (4.2) and (4.3) hold. Then, we have

$$\frac{\partial \hat{v}}{\partial u}(u, \hat{y}\hat{Y}_u) + A^\rho(u, \hat{y}\hat{Y}_u, \hat{v}, \frac{\partial \hat{v}}{\partial y}) + \hat{y}\hat{Y}_u(\alpha - \beta + (\beta - \int_C \rho_u(z)\, z\, \pi(dz))_+) = 0$$
$$\text{and } -\frac{\partial \hat{v}}{\partial y}(u, \hat{y}\hat{Y}_{u^-})\hat{Y}_{u^-} dL_u = 0 \text{ a.e. in } u \in [t,T].$$

Let

$$\hat{\tau}_n = \inf\{u \geq t \text{ such that } \Big| \int_C \hat{v}(s, \hat{y}\hat{\rho}_s(z)\hat{Y}_s) - \hat{v}(s, \hat{y}\hat{Y}_s)\pi(dz)\Big| \geq n\}.$$

Taking expectation in (6.8), we have

$$\hat{v}(t,y) \tag{6.15}$$
$$= E\Big[\hat{v}(T \wedge \hat{\tau}_n, y\hat{Y}_{T\wedge\hat{\tau}_n}^t)$$
$$+ \int_t^{T\wedge\hat{\tau}_n} \hat{y}\hat{Y}_s^t(\alpha - \beta + (\beta - \int_C \hat{\rho}_s(z)\, z\, \pi(dz))_+)ds \big| \hat{y}\hat{Y}_t = y\Big].$$

Since the family $\left(\hat{v}(T \wedge \hat{\tau}_n, y\hat{Y}_{T\wedge\hat{\tau}_n}^t)\right)_n$ is uniformly integrable under P, equation (6.15) implies (6.14) and so $(\hat{\rho}, \hat{D})$ is the solution of the dual problem.

Third step: We show that X^* defined by $X_s^* := -\frac{\partial \hat{v}}{\partial y}(s, \hat{y}\hat{Y}_s)$, $s \in [t,T]$ is the solution of the primal problem. Following the same arguments as in Lemma 6.6 of Mnif and Pham [12], we have from (6.14):

$$\frac{\partial \hat{v}}{\partial y}(t,y) \tag{6.16}$$
$$= -E\left[\hat{Y}_T^t I(y\hat{Y}_T^t) - \int_t^T \hat{Y}_u^t(\alpha - \beta + (\beta - \int_C \hat{\rho}_u(z)\, z\, \pi(dz))_+)du\right],$$

$J(I(y\hat{Y}_T^t)) = -\frac{\partial \hat{v}}{\partial y}(t,y)$ and in particular $I(y\hat{Y}_T^t) \in \mathcal{C}_+(t, -\frac{\partial \hat{v}}{\partial y}(t,y))$ (see characterization 3.3). Moreover, from definition of $\tilde{U}$ and (2.4), we have for all $H \in \mathcal{C}_+(t,x)$:

$$\begin{aligned} U(H) &\leq \tilde{U}(y\hat{Y}_T^t) + y\hat{Y}_T^t H \\ &= U(I(y\hat{Y}_T^t)) - y\hat{Y}_T^t I(y\hat{Y}_T^t) + y\hat{Y}_T^t H. \end{aligned}$$

Hence, by taking expectation, we obtain :

$$\begin{aligned} E[U(H)] &\leq E[U(I(y\hat{Y}_T^t))] + y\left(J(H) + \frac{\partial \hat{v}}{\partial y}(t,y)\right) \\ &\leq E[U(I(y\hat{Y}_T^t))], \end{aligned}$$

where we used expression of $\frac{\partial \hat{v}}{\partial y}(t,y)$ given in equation (6.16), expression of $J(H)$ in Lemma 3.2 in Mnif [11], and the fact that $J(H) \leq x = -\frac{\partial \hat{v}}{\partial y}(t,y)$ (see equality (4.4)). From characterization 3.3, there exists $\theta^* \in \mathcal{A}(t,x)$ such that :

$$I(y\hat{Y}_T^t) \leq X_T^{t,x,\theta^*}, \quad a.s. \tag{6.17}$$

Since $\hat{Y}_. X_.^{t,x,\theta^*} - \int_t^. \hat{Y}_u(\alpha - \beta + (\beta - \int_C \hat{\rho}_u(z)\, z\, \pi(dz))_+)du$ is a supermartingale under P (see Lemma 3.1 in Mnif [11]), we have :

$$E\left[\hat{Y}_T^t X_T^{t,x,\theta^*} - \int_t^T \hat{Y}_u(\alpha - \beta + (\beta - \int_C \hat{\rho}_u(z)\, z\, \pi(dz))_+)du\right] \leq x. \tag{6.18}$$

From equation (6.16), and by (6.17), we actually have

$$\hat{Y}_T^t X_T^{t,x,\theta^*} = \hat{Y}_T^t I(y\hat{Y}_T^t) \text{ a.s.}$$

and equality in (6.18). Therefore $\hat{Y}_. X_.^{t,x,\theta^*} - \int_t^. \hat{Y}_u(\alpha - \beta + (\beta - \int_C \rho_u(z)\, z\, \pi(dz))_+)du$ is a martingale under P, and so relation $X_s^* = -\frac{\partial \hat{v}}{\partial y}(s, \hat{y}\hat{Y}_s) = X_s^{t,x,\theta^*}$ holds for all $s \in [t,T]$.

References

1. Barles, G. and Souganidis, P. E.: Convergence of approximation schemes for fully nonlinear second order equations. Asymptotic Analysis, 4, 271–283, (1991).
2. Brémaud, P.: *Point Processes and Queues*, Springer Verlag, New York (1981).
3. Briys, E.: Insurance and consumption: the continuous-time case.J. Risk, 53, 718–723, (1986).
4. Fleming, W. H. and Soner, H. M.: Controlled Markov Processes and Viscosity solutions, Springer Verlag, New York (1993).
5. Föllmer, H. and Kramkov, D.: Optional Decomposition under Constraints, Probab. Theory Related Fields, 109, PP. 1–25, (1997).
6. Gollier, C.: Insurance and precautionary capital accumulation in a continuous -time model. J. of Risk and Insurance, 61, 78–95, (1994).
7. Hipp, C. and Plum, M.: Optimal investment for insurers, Insurance Math. Econom., 27, PP. 215–228,(2000).
8. Højgaard, B. and Taksar, M.: Optimal proportional reinsurance policies for diffusion models with transaction costs, Insurance Math. Econom. 22,41–51, (1998).
9. Howard, R. A.: *Dynamic Programming and Markov Processes*, MIT Press, Cambridge, Massachusetts, (1960).
10. Lapeyre, B., Sulem, A., and Talay, D.: Understanding Numerical Analysis for Option Pricing, editor: L. C.G.Rogers and D. Talay, Cambridge University Press, Cambridge, to appear.
11. Mnif, M.: Optimal insurance demand under marked point processes shocks: a dynamic programming duality approach, submitted (2012).
12. Mnif, M. and Pham, H.: Stochastic optimization under constraints, Stochastic Process. Appl., 93, 149–180, (2001).
13. Schmidli, H.: Optimal Proportional Reinsurance Policies in a Dynamic Setting, Scand. Actuar. J., 55–68, (2001).

14. Tourin, A. and Zariphopoulou, T.: Viscosity solutions and Numerical schemes for investment consumption models with transaction costs, Numerical Methods in Finance, edited by L.C.G. Rogers and D. Talay, 1997.

Mohamed Mnif: LAMSIN, ENIT, University of Tunis El Manar, Tunis, Tunisia
E-mail address: mohamed.mnif@enit.rnu.tn