

2003

On the geometry and topology of moduli spaces of multi-polygonal linkages

Michael Edward Holcomb

Louisiana State University and Agricultural and Mechanical College, holcomb@math.lsu.edu

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ON THE GEOMETRY AND TOPOLOGY OF MODULI SPACES OF
MULTI-POLYGONAL LINKAGES

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by

Michael Holcomb

B.S. Degree with Honors, Oklahoma State University, 1996

M.S., Purdue University, 1998

May 2003

Acknowledgments

This dissertation would not be possible without several contributions. It is a pleasure to thank my thesis advisor Dr. J. William Hoffman, who has been extremely helpful in advising me on my research and other academic considerations. He is very knowledgeable in many areas of mathematics and has willingly shared this expertise and his academic experiences with me. This has helped me grow as a mathematician.

This work was motivated by discussions with Dr. James Madden about linkages. He posed a question about whether spaces of configurations of certain linkages were constructible. This got me interested in studying linkages.

It is a pleasure also to thank Dr. Richardson, as well as the other faculty of the LSU Mathematics Department, for providing me with a pleasant working environment. A special thanks to Mrs. Rouse and Dr. Oxley for their time and energy in the Communicating Mathematics Course, and their encouragement and suggestions, that has helped my teaching improve greatly.

This dissertation is dedicated to my friends and the staff of Christ the King Church, and my parents and my brothers and sister and their families for their support and encouragement.

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Abstract

The geometric, topological, and symplectic properties of moduli spaces (spaces of configurations modulo rotations and translations) of polygonal linkages have been studied by Kapovich, Millson, and Kamiyama, et. al. One can form a polygonal linkage by taking two free linkages and identifying initial and terminal vertices. This can be generalized so that one takes three free linkages and identifies initial and terminal vertices. Then one obtains a linkage which contains multiple polygons, any two of which have shared edges. The geometric and topological properties of moduli spaces of these multi-polygonal linkages are studied. These spaces turn out to be compact algebraic varieties. Multi-Quadrilateral linkages whose moduli spaces are at most one dimensional are classified. The dimensions and some Euler characteristics are computed, and conditions under which these spaces are smooth manifolds are determined. Some conditions are also given for when the moduli spaces are connected and when they are disjoint unions of two moduli spaces of polygonal linkages.

1. Introduction

There is a substantial amount of literature on the topological properties of moduli spaces of special polygonal linkages, particularly equilateral polygonal linkages. A polygonal linkage in \mathbb{R}^n is simply a polygon whose vertices are hinges and whose edges are bars. We are allowed to deform and collapse the polygon without changing the lengths of any bars or any incidences of vertices and edges. This process can change some (or possibly all) of the angles at hinges. An equilateral polygonal linkage is one where each bar has the same length.

The moduli space of a polygonal linkage is the space of realizations of the linkage. The generic dimension of this space depends on the number of edges in the linkage and the ambient space. For example, the dimension of the moduli space of a generic hexagon in \mathbb{R}^3 is 18. But it is convenient to reduce this dimension by fixing one vertex at the origin and one adjacent vertex along one of the coordinate axes. Rigorously speaking, we take the 18-dimensional moduli space, and we mod out by the isometry group $SO(3)$.

Because the moduli spaces of most linkages cannot be visualized, the topological and symplectic properties are studied. Kapovich and Millson studied symplectic geometry in [10] and used a “walls and chambers” approach in [9], where topology in one chamber can change upon crossing a wall into an adjacent chamber. Kamiyama studied Euler characteristics in [2] and [6], homology groups and Poincaré Polynomials in [3] and [4], Chern numbers in [5], symplectic volume in [7], and conditions in which the moduli space is a smooth manifold in [2].

In this thesis we explore combinations of two polygonal linkages which share at least one edge. We build polygonal and multi-polygonal linkages up from free

linkages. The notation for these linkages is set up in Chapters 2 and 3. The moduli spaces are compact, Hausdorff, real triangulable algebraic varieties. In Chapter 4, we investigate multi-quadrilateral linkages and classify the moduli spaces which have dimension less than two.

In Chapter 5 we find topological results for certain classes of multi-polygonal linkages. First we find two classes of multi-polygonal linkages that are smooth manifolds. Next, we observe that moduli spaces of multi-polygonal linkages can be thought of as a fibered product of moduli spaces of polygonal linkages. Then we join two “long enough” edges to a polygonal linkage and determine when the resulting moduli space is either equal to or is close to a disjoint union of copies of the original polygonal moduli space. Finally, we find a class of multi-polygonal linkages whose moduli space is connected.

In Chapter 6 we map the moduli space to S^1 via p , by sending a configuration to a pre-specified angle in that configuration. We make use of some technical theorems in differential geometry and algebraic topology to determine when all fibers of the map p are homeomorphic, and what can cause the fibers to change topological type and at what angle values. We conclude the section with a specific example involving an Euler characteristic computation.

2. Polygonal Linkages and Free Linkages

We define a free linkage in \mathbb{R}^p in terms of the lengths of each edge. Let

$$\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{R}_{>0}^p$$

be a vector of positive lengths. The set

$$\mathcal{F}(p; \mathbf{d}) = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{pn} \mid |x_i - x_{i+1}| = d_i \text{ for } i = 1, \dots, n\}$$

denotes the space of embeddings of linkages with length vector \mathbf{d} . Since we will study linkages in the plane, we will write $\mathcal{F}(\mathbf{d})$ for $\mathcal{F}(2; \mathbf{d})$. When the lengths are understood, we will simply use \mathcal{F} .

Note that we will use \mathcal{F} as a particular embedding, but we will also use \mathcal{F} as a generic object. Sometimes we will say “moving continuously from one point to another point in \mathcal{F} ”, and other times we will say “continuously deforming \mathcal{F} ”. The difference in usage should be clear from the context.

We define $l_{\mathcal{F}}$ to be the length between the initial vertex x_1 and the terminal vertex x_{n+1} in \mathcal{F} .

When length vectors have many repeated edges, we will let $d^{<n>}$ denote repeated lengths. For example, we will write $(2, 2, 1, 1, 1, 1)$ as $(2^{<2>}; 1^{<4>})$. And if \mathbf{d} has n edges of length a , then we simply write $\mathbf{d} = (a^{<n>})$.

We can now form more linkages composed of free linkages. We will use the following notation for two or more ordered free linkages that have the same initial and terminal points.

$$(\mathcal{F}_1, \dots, \mathcal{F}_r), \quad \mathcal{F}_j \in F(p; \mathbf{d}_j)$$

Notice that $(\mathcal{F}_1, \mathcal{F}_2)$ denotes a polygonal linkage. For example, if we specify a priori a free linkage $\mathcal{F}_1(3, 4)$ and a free linkage $\mathcal{F}_2(2, 2, 1, 3)$, and we further force

the initial and terminal points of \mathcal{F}_1 and \mathcal{F}_2 to agree, then we obtain a hexagonal linkage with lengths in the following order: 3,4,3,1,2,2.

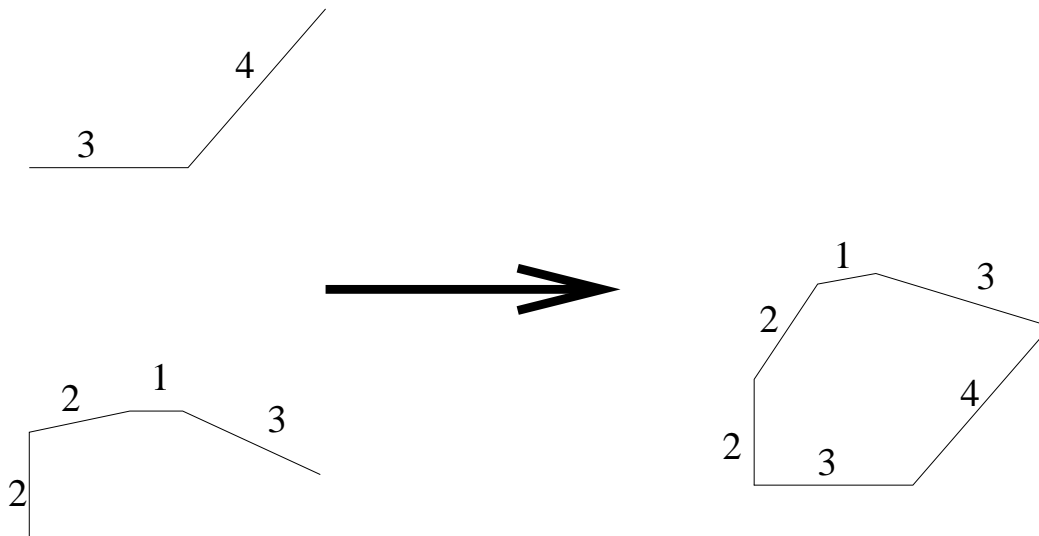


FIGURE 2.1. Forming a Hexagonal Linkage from Two Free Linkages

In this paper we will study multi-polygonal linkages composed of three free linkages embedded in \mathbb{R}^2 . They will be denoted $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$. We call them multi-polygonal because any pair of free linkages forms a polygonal linkage.

We use $C(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ to denote the space of all embeddings of \mathcal{F}_1 , \mathcal{F}_2 , and \mathcal{F}_3 with the initial points identified at the origin and the terminal points identified. This space has the following nice properties.

Proposition 2.0.1. *The space $C(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ is a compact, Hausdorff, real algebraic variety.*

More generally, the space $C(\mathcal{F}_1, \dots, \mathcal{F}_r)$ is defined similarly and has the same nice properties.

For example, if \mathcal{F} has n edges, then $C(\mathcal{F})$ equals a product of n circles.

In addition, we can mod out by the group $SO(2)$ of orientation-preserving isometries of \mathbb{R}^2 . Because we already specified that initial and terminal points of each

free linkage must agree, we can rotate the first edge of only one free linkage (we will do this to \mathcal{F}_1) so that it lies on the positive x -axis.

Let

$$M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) = C(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)/SO(2).$$

These denote moduli spaces, which are sets of equivalence classes of embeddings of $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ in \mathbb{R}^2 modulo the action of the special Euclidean group $SE(2, \mathbb{R}) = \mathbb{R}^2 \rtimes SO(2)$.

We have the following propositions about $M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$.

Proposition 2.0.2. *The space $M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ is a compact, Hausdorff, real algebraic variety.*

Proposition 2.0.3. *The space $M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ is triangulable.*

See Théorème 9.2.1 in [1] for the proof of the second proposition. More generally, the space $M(\mathcal{F}_1, \dots, \mathcal{F}_r)$ is defined similarly and has the same propositions true about them.

We will study moduli spaces of the form $M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$. Points in the moduli space will be represented by P with possible subscripts.

If we rearrange the free linkages by a nontrivial permutation σ , then we obtain a space $M(\mathcal{F}_{\sigma(1)}, \dots, \mathcal{F}_{\sigma(n)})$ which is different from, but homeomorphic to, the space $M(\mathcal{F}_1, \dots, \mathcal{F}_n)$.

3. Multi-polygonal Linkages

We use the following notation with a multi-polygonal linkage $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ in the plane. The length vector for \mathcal{F}_i will equal $(d_{1,i}, \dots, d_{n,i})$. We will denote the j th vertex of the i th free linkage by $x_{j,i}$. The edge $e_{j,i}$ will link $x_{j,i}$ to $x_{j+1,i}$. When we reference the angle between $e_{j,i}$ and the x -axis, we will think of $e_{j,i}$ as the vector $d_{j,i}e^{\sqrt{-1}\theta_{j,i}}$ directed from $x_{j,i}$ to $x_{j+1,i}$.

We identify $x_{1,1}$, $x_{1,2}$, and $x_{1,3}$, the initial points of the free linkages, at the origin, and we identify $x_{n_1,1}$, $x_{n_2,2}$, and $x_{n_3,3}$.

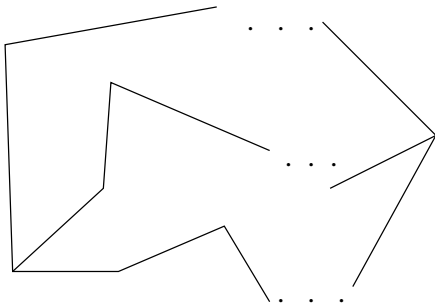


FIGURE 3.1. Generic Multi-polygonal Linkage Formed by Three Free Linkages

4. Multi-Quadrilateral Linkages

If $n_1 = n_2 = n_3 = 2$, then we have multiple joined quadrilateral linkages. To investigate the moduli spaces of these linkages, it will be convenient to investigate the sublinkages which contain $\mathcal{F}_1 \in \mathcal{F}(d_{1,1}, d_{2,1})$. So we will denote the linkage which contains only $\mathcal{F}_2 \in \mathcal{F}(d_{1,2}, d_{2,2})$ and \mathcal{F}_1 by $(\mathcal{F}_1, \mathcal{F}_2)$. Likewise, we will denote the linkage which contains only $\mathcal{F}_3 \in \mathcal{F}(d_{1,3}, d_{2,3})$ and \mathcal{F}_1 by $(\mathcal{F}_1, \mathcal{F}_3)$.

For simple notation, the lengths $a, b, c, d, e,$ and f will refer to $d_{1,1}, d_{2,1}, d_{1,2}, d_{2,2}, d_{1,3},$ and $d_{2,3}$ respectively. We can assume that

$$\min\{a + b, c + d, e + f\} = a + b \quad (4.1)$$

so that \mathcal{F}_1 has sum of lengths of its edges minimal. Then l refers to the common length from $x_{1,1}$ to $x_{3,1}$, from $x_{1,2}$ to $x_{3,2}$, and from $x_{1,3}$ to $x_{3,3}$. Finally, the line containing $x_{1,1}$ and $x_{3,1}$ will be referred to as L .

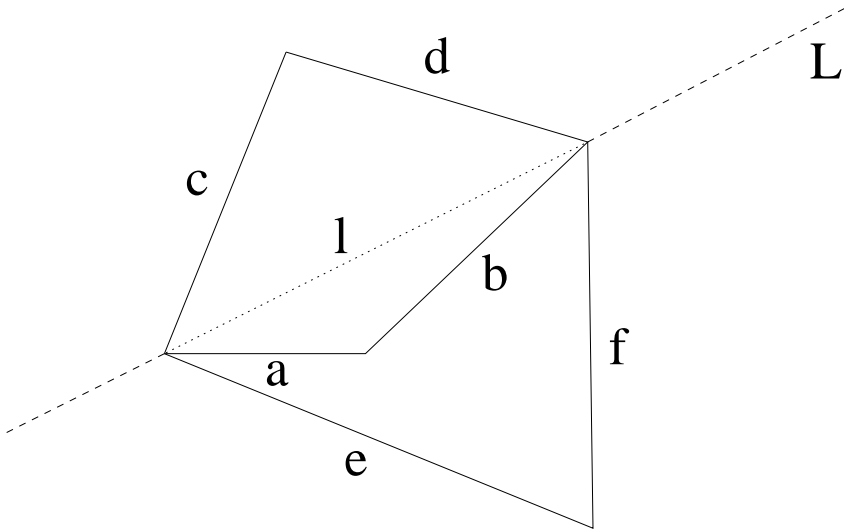


FIGURE 4.1. A generic multi-quadrilateral linkage

Definition 1. A multi-quadrilateral linkage $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ is called generic if $a = b,$ $c = d,$ and $e = f$ are not simultaneously true.

We will only investigate the generic case. The moduli spaces of these generic multi-quadrilateral linkages will have dimension less than two.

For each i , let θ_i be the clockwise angle from $e_{1,i}$ to $e_{2,i}$. We focus mostly on the angle θ_1 . The map $p : M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \rightarrow S^1$ sends a particular configuration $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ to the angle θ_1 . The lengths of the edges in $(\mathcal{F}_1, \mathcal{F}_2)$ can restrict the angle θ_1 . Likewise, the lengths of the edges in $(\mathcal{F}_1, \mathcal{F}_3)$ can restrict θ_1 . So we have restriction maps $p|_{(\mathcal{F}_1, \mathcal{F}_2)}$ whose image is Θ_1 and $p|_{(\mathcal{F}_1, \mathcal{F}_3)}$ whose image is Θ_2 . Each Θ_i is a subset of $[0, 2\pi)$.

Lemma 4.0.4. *The image of the map p is $\Theta_1 \cap \Theta_2$.*

Because of condition (4.1), the image of p is either empty, the angle π , a closed interval containing π , or all of $[0, 2\pi)$.

Definition 2. *The free linkage \mathcal{F}_i is collapsible if there exists at least one configuration of $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ where \mathcal{F}_i lies completely in a straight line such that $\theta_i = \pi$.*

Definition 3. *The free linkage \mathcal{F}_i is foldable if there exists at least one configuration of $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ where \mathcal{F}_i lies completely in a straight line such that $\theta_i = 0$.*

Note that \mathcal{F}_1 is already collapsible because of condition (4.1). Here are the conditions on the lengths that determine if \mathcal{F}_i is collapsible or foldable.

\mathcal{F}_2	collapsible:	$a + b = c + d.$
\mathcal{F}_2	foldable:	$ c - d \geq a - b $ and $ c - d \geq e - f .$
\mathcal{F}_3	collapsible:	$a + b = e + f.$
\mathcal{F}_3	foldable:	$ e - f \geq a - b $ and $ e - f \geq c - d .$

Here is an example. The length constraints

$$a + b < c + d$$

$$a - b < c - d$$

on $(\mathcal{F}_1, \mathcal{F}_2)$ impose the angle constraint $\theta_{min} \leq \theta_1 \leq \theta_{max}$, where

$$\theta_{min} = \cos^{-1} \frac{a^2 + b^2 - (d - c)^2}{2ab}$$

and

$$\theta_{max} = 2\pi - \theta_{min}.$$

Here are the possible subsets $im(p)$ described above.

1. If $im(p)$ is empty, then it is because $a + b < |c - d|$ or $a + b < |e - f|$. In this case, no angles are possible for θ_1 , and the moduli space is empty.
2. If $im(p) = \{\pi\}$, then it is because $a + b = |c - d|$ and $a + b \geq |e - f|$, or because $a + b = |e - f|$ and $a + b \geq |c - d|$. In this case, the only possible angle for θ_1 is π and the moduli space is either one or two points, with one point occurring if the inequality is actually an equality.
3. If $im(p)$ is an interval containing 0, then $im(p)$ is all of $[0, 2\pi)$. Here the moduli space is four circles with possible intersections. (CASE(A))
4. If $im(p)$ is an interval not containing 0, then $im(p)$ is of the form $[\theta_{min}, \theta_{max}] \subset (0, 2\pi)$, and the moduli space is one circle or a figure-8. (CASE (B))

CASE (A): Both 0 and π are possible angles for θ_1 .

In terms of collapsibility and foldability, this case means that \mathcal{F}_1 is collapsible and foldable. We define a vector $\mathbf{v} = (v_1, v_2, v_3, v_4)$ of zeros and ones as follows.

We let

$$v_1 = \begin{cases} 1 & \text{if } \mathcal{F}_2 \text{ is collapsible;} \\ 0 & \text{if } \mathcal{F}_2 \text{ is not collapsible,} \end{cases}$$

$$v_2 = \begin{cases} 1 & \text{if } \mathcal{F}_2 \text{ is foldable;} \\ 0 & \text{if } \mathcal{F}_2 \text{ is not foldable,} \end{cases}$$

$$v_3 = \begin{cases} 1 & \text{if } \mathcal{F}_3 \text{ is collapsible;} \\ 0 & \text{if } \mathcal{F}_3 \text{ is not collapsible,} \end{cases}$$

and

$$v_4 = \begin{cases} 1 & \text{if } \mathcal{F}_3 \text{ is foldable;} \\ 0 & \text{if } \mathcal{F}_3 \text{ is not foldable.} \end{cases}$$

We also define the pairs of \mathbf{v} as (v_1, v_2) and (v_3, v_4) .

Theorem 4.0.5. *(Case (A)) Suppose that $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ is generic and that 0 and π are both possible angles for θ_1 . Then the moduli space of the multi-quadrilateral linkage $M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ is obtained from four circles, with possible identifications. These can be determined as follows.*

$$\begin{aligned} \text{Number of components} &= 2^{\# \text{ of } (0,0) \text{ pairs in } \mathbf{v}} \\ \text{Points in } p^{-1}(0) &= 2^{2-\# \text{ of } 1\text{'s in even positions of } \mathbf{v}} \\ \text{Points in } p^{-1}(\pi) &= 2^{2-\# \text{ of } 1\text{'s in odd positions of } \mathbf{v}} \end{aligned}$$

Proof. We let $P(\theta_2, \theta_3)$ be any point in $p^{-1}(\theta_1)$, where initially $\theta_1 \neq 0, \pi$ and where θ_2 and θ_3 depend on θ_1 . We denote a point in $p^{-1}(0)$ by $P(\theta_2(0), \theta_3(0))$, and we denote a point in $p^{-1}(\pi)$ by $P(\theta_2(\pi), \theta_3(\pi))$.

First, we find a circle of configurations starting at a point $P(\theta_2, \theta_3)$. Suppose $\theta_2 \in I_2$ and $\theta_3 \in I_3$, where one of the following is true about I_2 and I_3 .

$$\begin{array}{lll}
I_2 = (0, \pi) & \text{and} & I_3 = (0, \pi) \\
I_2 = (0, \pi) & \text{and} & I_3 = (\pi, 2\pi) \\
I_2 = (\pi, 2\pi) & \text{and} & I_3 = (0, \pi) \\
I_2 = (\pi, 2\pi) & \text{and} & I_3 = (\pi, 2\pi)
\end{array}$$

We can perturb θ_1 continuously on the interval $[0, 2\pi)$. As we do so, the angle θ_2 can remain in I_2 if \mathcal{F}_2 is neither collapsible nor foldable. Likewise, the angle θ_3 can remain in I_3 if \mathcal{F}_3 is neither collapsible nor foldable. But if either \mathcal{F}_2 or \mathcal{F}_3 is collapsible or foldable, then θ_2 can remain in $\overline{I_2}$ and θ_3 can remain in $\overline{I_3}$ as θ_1 changes.

Therefore, the path-component containing each configuration in $p^{-1}(\theta_1)$ contains at least that circle of configurations. Whether the component is bigger depends on whether any other circles of configurations meet it when θ_1 equals 0 or π . This in turn depends on how many points lie in $p^{-1}(0)$ and $p^{-1}(\pi)$.

Here are the four points in $p^{-1}(\theta_1)$.

$$P_1 = P(\theta_2, \theta_3)$$

$$P_2 = P(\theta_2, 2\pi - \theta_3)$$

$$P_3 = P(2\pi - \theta_2, \theta_3)$$

$$P_4 = P(2\pi - \theta_2, 2\pi - \theta_3)$$

We also have points $P_{1,0} = P(\theta_2(0), \theta_3(0))$ and $P_{2,0} = P(\theta_2(0), 2\pi - \theta_3(0))$, and $P_{i,0}$ are defined similarly for $i = 3, 4$. The points $P_{i,\pi}$ are similarly defined.

For $i = 2$ or 3 , the linkage \mathcal{F}_i is foldable if and only if $\theta_i(0) = 0$ and is collapsible if and only if $\theta_i(\pi) = \pi$. Therefore, \mathcal{F}_i is foldable if and only if $P_{1,0} = P_{5-i,0}$ and $P_{4,0} = P_{i,0}$. Also, \mathcal{F}_i is collapsible if and only if $P_{1,\pi} = P_{5-i,\pi}$ and $P_{4,\pi} = P_{i,\pi}$.

We have the following equivalences, each of whose proofs are easy.

Lemma 4.0.6. *[Points in $p^{-1}(0)$] The following are equivalent.*

1. \mathcal{F}_i is foldable.
2. $\theta_i(0) = 0$.
3. $P_{1,0} = P_{5-i,0}$.
4. $P_{4,0} = P_{i,0}$.

Lemma 4.0.7. *[Points in $p^{-1}(\pi)$] The following are equivalent.*

1. \mathcal{F}_i is collapsible.
2. $\theta_i(\pi) = \pi$.
3. $P_{1,\pi} = P_{5-i,\pi}$.
4. $P_{4,\pi} = P_{i,\pi}$.

The following lemmas describe when the P_i points lie in the same path components.

Lemma 4.0.8. *The following are equivalent.*

1. P_1 and P_2 are in the same path component.
2. $\theta_3(0) = 0$ or $\theta_3(\pi) = \pi$.
3. \mathcal{F}_3 is collapsible or foldable or both.

4. $\mathbf{v} = (*, *, 0, 1)$ or $\mathbf{v} = (*, *, 1, 0)$ or $\mathbf{v} = (*, *, 1, 1)$.

We now give lemmas with only conditions on P_i and the \mathbf{v} vector.

Lemma 4.0.9. *The following are equivalent.*

1. P_1 and P_3 are in the same path component.
2. $\mathbf{v} = (1, 0, *, *)$ or $\mathbf{v} = (0, 1, *, *)$ or $\mathbf{v} = (1, 1, *, *)$.

Lemma 4.0.10. *The following are equivalent.*

1. P_1 and P_2 are in the same path component.
2. P_3 and P_4 are in the same path component.

Lemma 4.0.11. *The following are equivalent.*

1. P_1 and P_3 are in the same path component.
2. P_2 and P_4 are in the same path component.

So if \mathbf{v} has two $(0, 0)$ pairs, then each P_i lies in a different path component. Also, the points $P_{i,0}$ are all distinct, and the points $P_{i,\pi}$ are all distinct. The resulting moduli space is four disjoint circles.

If \mathbf{v} has one $(0, 0)$ pair, then P_1 lies in the same path component as P_2 or P_3 but not both. In the former case, P_3 and P_4 both lie in a second path component. In the latter case, P_2 and P_4 both lie in a second path component. And if \mathbf{v} has no $(0, 0)$ pairs, then all four points lie in the same path component. These moduli spaces result from four disjoint circles being pinched together in different ways. See Figures 4.3 and 4.4.

□

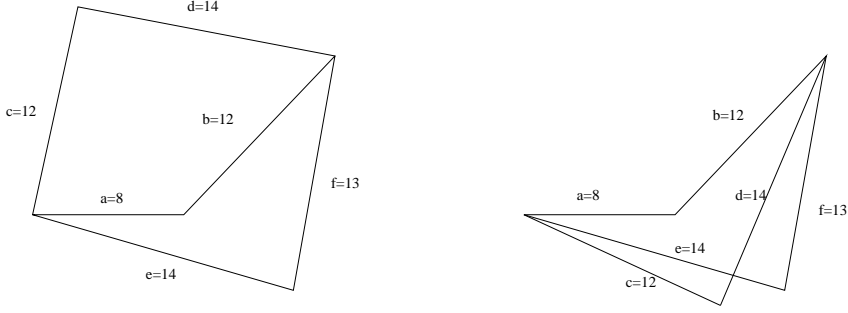


FIGURE 4.2. We cannot change the \mathcal{F}_2 configuration in the first figure (above the dashed line) into the \mathcal{F}_2 configuration in the second figure (below the dashed line) without dismantling the linkage.

We next consider case **(B)**. Suppose $im(p) = [\theta_{min}, \theta_{max}] \subseteq (0, 2\pi)$. We define a new vector $\mathbf{v} = (v_1, v_2, v_3, v_4)$ of zeros and ones as follows. We let

$$v_1 = \begin{cases} 1 & \text{if } \mathcal{F}_2 \text{ is collapsible;} \\ 0 & \text{if } \mathcal{F}_2 \text{ is not collapsible,} \end{cases}$$

$$v_2 = \begin{cases} 1 & \text{if } \mathcal{F}_2 \text{ is foldable at } \theta_{min} \text{ and } \theta_{max}; \\ 0 & \text{if } \mathcal{F}_2 \text{ is not foldable at } \theta_{min} \text{ or } \theta_{max}, \end{cases}$$

$$v_3 = \begin{cases} 1 & \text{if } \mathcal{F}_3 \text{ is collapsible;} \\ 0 & \text{if } \mathcal{F}_3 \text{ is not collapsible,} \end{cases}$$

and

$$v_4 = \begin{cases} 1 & \text{if } \mathcal{F}_3 \text{ is foldable at } \theta_{min} \text{ and } \theta_{max}; \\ 0 & \text{if } \mathcal{F}_3 \text{ is not foldable at } \theta_{min} \text{ or } \theta_{max}. \end{cases}$$

We also define the pairs of (v_1, v_2, v_3, v_4) as (v_1, v_2) and (v_3, v_4) .

Theorem 4.0.12. (case **(B)**) Suppose that $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ is generic and that π , but not 0, is a possible angle for θ_1 . Then the moduli space of the multi-quadrilateral linkage $M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ is formed by two circles if \mathbf{v} has no $(1, 1)$ pairs or two figure-

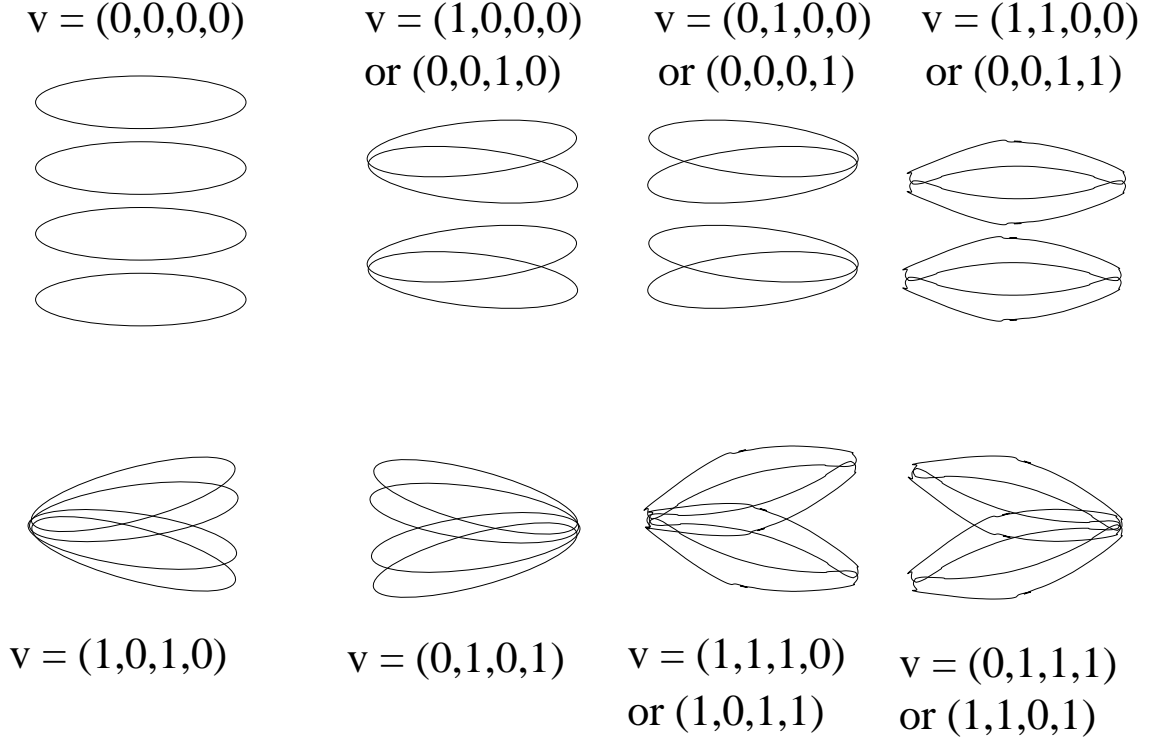


FIGURE 4.3. Case (A) Moduli Spaces of Multi-Quadrilateral Linkages, Part 1

θ_1 's if \mathbf{v} has at least one $(1,1)$ pair. Furthermore, we have the following.

$$\begin{aligned}
 \text{Number of components} &= 2^{\# \text{ of } (0,0) \text{ pairs in } \mathbf{v}} \\
 \text{Points in } p^{-1}(\theta_{min}) = p^{-1}(\theta_{max}) &= 2^{2-\# \text{ of } 1\text{'s in even positions of } \mathbf{v}} \\
 \text{Points in } p^{-1}(\pi) &= 2^{2-\# \text{ of } 1\text{'s in odd positions of } \mathbf{v}}
 \end{aligned}$$

Proof. We start by showing the existence of a circle or figure-8 of configurations. Instead of mapping $M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \rightarrow S^1$, we now map $p : M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \rightarrow [\theta_{min}, \theta_{max}]$. Suppose \mathcal{F}_2 forces θ_1 to lie in that interval. Then there is a set of configurations where the angle θ_3 in \mathcal{F}_3 lies in $[0, \pi]$, and there is a set of configurations where the angle θ_3 lies in $[\pi, 2\pi]$. Assume that $\theta_3 \in [0, \pi]$. Since \mathcal{F}_2 is folded when θ_1 equals θ_{min} or θ_{max} , we see that $p^{-1}(\theta_{min})$ and $p^{-1}(\theta_{max})$ are both one point. If \mathcal{F}_2 is not collapsible, then $p^{-1}(\theta_1)$ is two points for every $\theta_1 \in (\theta_{min}, \theta_{max})$. This gives us a circle of configurations. If \mathcal{F}_2 is collapsible, then $p^{-1}(\theta_1)$ is two points

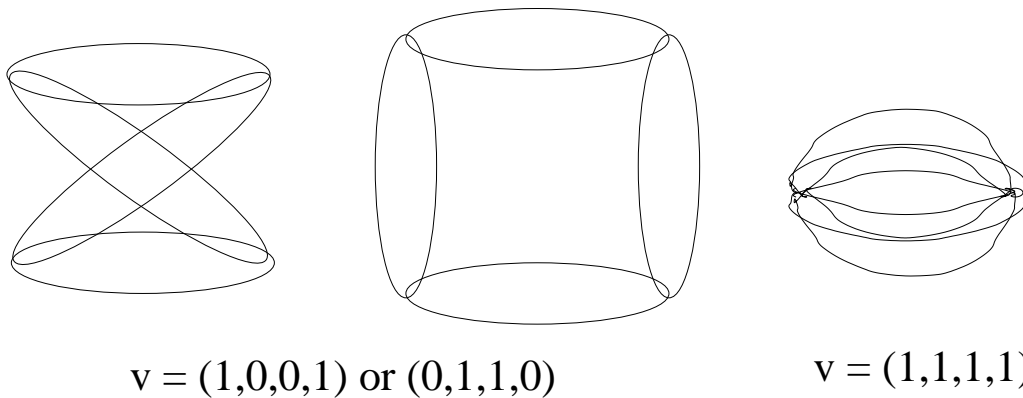


FIGURE 4.4. Case **(A)** Moduli Spaces of Multi-Quadrilateral Linkages, Part 2

for every $\theta_1 \in (\theta_{min}, \theta_{max}) \setminus \{\pi\}$, and $p^{-1}(\pi)$ is one point. This gives us a figure-8 of configurations.

Similarly, we obtain respective configurations when $\theta_3 \in [\pi, 2\pi]$. □

5. General Results

5.1 Dimension and Smoothness

Under certain conditions, the moduli space of a multi-polygonal linkage is a smooth manifold. We utilize the implicit function theorem and compute a Jacobian based on equations whose zero set is the moduli space. Although the moduli spaces are algebraic varieties, we use equations based on the angles $\theta_{i,j}$ between the edges $e_{i,j}$ and the positive x -axis.

We think of each $e_{i,j}$ as a complex number $d_{i,j}e^{\sqrt{-1}\theta_{i,j}}$. The sum of real parts of the complex numbers obtained from edges of \mathcal{F}_1 must equal the sum of real parts of complex numbers obtained from edges of \mathcal{F}_2 and of \mathcal{F}_3 . Likewise, we can say a similar statement about the imaginary parts.

The following equations describe the moduli space.

$$a + \cos \theta_{2,1} + \dots + \cos \theta_{n_1,1} = \cos \theta_{1,2} + \dots + \cos \theta_{n_2,2}$$

$$\sin \theta_{2,1} + \dots + \sin \theta_{n_1,1} = \sin \theta_{1,2} + \dots + \sin \theta_{n_2,2}$$

$$a + \cos \theta_{2,1} + \dots + \cos \theta_{n_1,1} = \cos \theta_{1,3} + \dots + \cos \theta_{n_3,3}$$

$$\sin \theta_{2,1} + \dots + \sin \theta_{n_1,1} = \sin \theta_{1,3} + \dots + \sin \theta_{n_3,3}$$

Now we create functions whose zero set is the moduli space.

$$f_1 = a + \cos \theta_{2,1} + \dots + \cos \theta_{n_1,1} - \cos \theta_{1,2} - \dots - \cos \theta_{n_2,2}$$

$$f_2 = \sin \theta_{2,1} + \dots + \sin \theta_{n_1,1} - \sin \theta_{1,2} - \dots - \sin \theta_{n_2,2}$$

$$f_3 = a + \cos \theta_{2,1} + \dots + \cos \theta_{n_1,1} - \cos \theta_{1,3} - \dots - \cos \theta_{n_3,3}$$

$$f_4 = \sin \theta_{2,1} + \dots + \sin \theta_{n_1,1} - \sin \theta_{1,3} - \dots - \sin \theta_{n_3,3}$$

Theorem 5.1.1. *The moduli space $M(a, 1^{<n_1-1>}; 1^{<n_2>}; 1^{<n_3>})$ is a smooth manifold of dimension $n_1 + n_2 + n_3 - 5$ when the following conditions are met.*

1. *The numbers n_2 and n_3 do not have the same parity*
2. *The length a lies in the interval $(0, b) \setminus \mathbb{Z}$, where $b = \min\{n_2, n_3\}$*

Proof. The condition (2) above assures us that the linkage is actually possible since $a < b$ and that the first free linkage has non-integer length between $x_{1,1}$ and $x_{n_1+1,1}$ if it lies completely in a straight line. We assume n_2 is odd and n_3 is even. The proof when n_2 is even and n_3 is odd goes the same way.

Here is a heuristic dimension count. We have $n_1 - 1$ degrees of freedom for the linkage with the $\theta_{i,1}$ angles, as the first edge lies on the positive x -axis. But we only have $n_2 - 2$ degrees of freedom for the linkage with the $\theta_{i,2}$ angles. This is because, once we have chosen the first $n_2 - 2$ angles, we must have the last two edges connect up to the terminal point of the first linkage. We cannot do this if $|x_{n_2-2,2} - x_{n_2,2}| > 2$, and we can do this in only one or two ways if $|x_{n_2-2,2} - x_{n_2,2}| \leq 2$. A similar argument shows that we only have $n_3 - 2$ degrees of freedom for the linkage with the $\theta_{i,3}$ angles. The total number of free angles is $n_1 + n_2 + n_3 - 5$, which equals the difference in the number of equations and unknowns above.

The matrix of the Jacobian can be written in block form

$$\begin{pmatrix} A & B_1 & 0 \\ 0 & B_2 & C \end{pmatrix}.$$

The entries of A come from taking partials of the first two equations with respect to the $\theta_{i,2}$ angles. The entries of B_1 come from taking partials of the first two equations with respect to the $\theta_{i,1}$ variables. The entries of B_2 come from taking partials of the last two equations with respect to the $\theta_{i,1}$ variables. The entries

of C come from taking partials of the last two equations with respect to the $\theta_{i,3}$ variables.

$$A = \begin{pmatrix} \sin \theta_{1,2} & \cdots & \sin \theta_{n_2,2} \\ -\cos \theta_{1,2} & \cdots & -\cos \theta_{n_2,2} \end{pmatrix}$$

$$B_1 = \begin{pmatrix} -\sin \theta_{2,1} & \cdots & -\sin \theta_{n_1,1} \\ \cos \theta_{2,1} & \cdots & \cos \theta_{n_1,1} \end{pmatrix}$$

$$B_2 = \begin{pmatrix} -\sin \theta_{2,1} & \cdots & -\sin \theta_{n_1,1} \\ \cos \theta_{2,1} & \cdots & \cos \theta_{n_1,1} \end{pmatrix}$$

$$C = \begin{pmatrix} \sin \theta_{1,3} & \cdots & \sin \theta_{n_3,3} \\ -\cos \theta_{1,3} & \cdots & -\cos \theta_{n_3,3} \end{pmatrix}$$

We now state some lemmas whose proofs are easy.

Lemma 5.1.2. *The following are equivalent.*

1. *The rank of A is 2.*
2. *The edges of \mathcal{F}_2 do not lie in a single straight line.*
3. *The Jacobian is column equivalent to*

$$\begin{pmatrix} A & 0 & 0 \\ 0 & B_2 & C \end{pmatrix}.$$

Lemma 5.1.3. *The following are equivalent.*

1. *The rank of C is 2.*
2. *The edges of \mathcal{F}_3 do not lie in a single straight line.*

3. The Jacobian is column equivalent to

$$\begin{pmatrix} A & B_1 & 0 \\ 0 & 0 & C \end{pmatrix}.$$

Lemma 5.1.4. *The following are equivalent.*

1. The rank of A is one and the rank of $[A|B_1]$ is 2.
2. All the edges of \mathcal{F}_2 lie in a straight line which is different from the x -axis.

Lemma 5.1.5. *The following are equivalent.*

1. The rank of C is one and the rank of $[B_2|C]$ is 2.
2. All the edges of \mathcal{F}_3 lie in a straight line which is different from the x -axis.

Using these lemmas, we can show that if a is not an integer, if n_2 is odd, and if n_3 is even, then the Jacobian has full rank.

If the edges of \mathcal{F}_2 in a configuration of the linkage do not lie in a straight line, and if the edges of \mathcal{F}_3 in that configuration do not lie in a straight line, then both A and C have rank 2. Therefore the Jacobian has full rank.

If all of the edges of \mathcal{F}_2 lie on the x -axis, then A has rank 1 and the distance between the initial and terminal points is odd. Since \mathcal{F}_3 has an even number of edges, it cannot possibly lie in a straight line. This forces C to have rank 2. Therefore the Jacobian is column equivalent to

$$\begin{pmatrix} A & B_1 & 0 \\ 0 & 0 & C \end{pmatrix}.$$

Because \mathcal{F}_1 has an edge a which is not integer, the remaining edges of \mathcal{F}_1 cannot all lie in the same straight line as \mathcal{F}_2 . Therefore B_1 has a column which is not a scalar multiple of any column in A . Therefore $[A|B_1]$ has rank 2, and the Jacobian

has full rank. A similar argument holds if we assume all of the edges of \mathcal{F}_3 lie on the x -axis.

If all the edges of \mathcal{F}_2 lie in a straight line which is not the x -axis, then the rank of $[A|B_1]$ is 2. Since the distance between the initial and terminal points is odd, the edges of \mathcal{F}_3 cannot all lie in that straight line. Therefore C has rank 2, and the Jacobian is column equivalent to

$$\begin{pmatrix} A & B_1 & 0 \\ 0 & 0 & C \end{pmatrix}.$$

Therefore the Jacobian has full rank. A similar argument holds if we assume all of the edges of \mathcal{F}_3 lies in a straight line which is not the x -axis.

□

We will also need these smoothness results later.

Theorem 5.1.6. *The moduli space $M(a, b; 1^{<n_2>}; 1^{<n_3>})$ is a smooth manifold of dimension $n_2 + n_3 - 3$ when $a + b < \min\{n_2, n_3\}$ and one of the following conditions are met.*

1. *The numbers n_2 and n_3 do not have the same parity, $a + b \notin \mathbb{Z}$, and $|a - b| \notin \mathbb{Z}$.*
2. *The numbers n_2 and n_3 are both odd, and $[|a - b|, a + b]$ has empty intersection with the odd integers.*
3. *The numbers n_2 and n_3 are both even, and $[|a - b|, a + b]$ has empty intersection with the even integers (note that this condition disallows $a = b$).*

Proof. We build a matrix similar to that in the proof of Theorem 5.1.1. Again we start by writing functions whose zero set is the moduli space.

$$f_1 = a + b \cos \theta_{2,1} - \cos \theta_{1,2} - \dots - \cos \theta_{n_2,2}$$

$$f_2 = b \sin \theta_{2,1} - \sin \theta_{1,2} - \dots - \sin \theta_{n_2,2}$$

$$f_3 = a + b \cos \theta_{2,1} - \cos \theta_{1,3} - \dots - \cos \theta_{n_3,3}$$

$$f_4 = b \sin \theta_{2,1} - \sin \theta_{1,3} - \dots - \sin \theta_{n_3,3}$$

The dimension count goes similarly as in the proof of Theorem 5.1.1.

The Jacobian has the block form

$$\begin{pmatrix} A & B_1 & 0 \\ 0 & B_2 & C \end{pmatrix}$$

where A and C are the same as before. Now B_1 is just the single column

$$(-b \sin \theta_{2,1}, b \cos \theta_{2,1})^T$$

and B_2 is the single column

$$(-b \sin \theta_{2,1}, b \cos \theta_{2,1})^T.$$

So we want to show that, under the conditions of 5.1.6, the Jacobian has full rank.

We will use Lemmas 5.1.2, 5.1.3, 5.1.4, and 5.1.5

Suppose that the numbers n_2 and n_3 do not have the same parity, that $a+b \notin \mathbb{Z}$, and that $|a-b| \notin \mathbb{Z}$. Then the edges of \mathcal{F}_2 and of \mathcal{F}_3 cannot possibly all lie on the x -axis. So if the edges of either \mathcal{F}_2 or of \mathcal{F}_3 all lie in a single straight line, then that line is different from the x -axis. If neither \mathcal{F}_2 nor \mathcal{F}_3 have all edges in a straight line, then the Jacobian has full rank.

So suppose the edges of \mathcal{F}_2 lie in a straight line which is different from the x -axis. The distance between the initial and terminal points has the same parity as n_2 and

different parity than n_3 . So the edges of \mathcal{F}_3 do not all lie in the same straight line. So C has rank 2, $[A|B_1]$ has rank 2, and the Jacobian is column equivalent to

$$\begin{pmatrix} A & B_1 & 0 \\ 0 & 0 & C \end{pmatrix}.$$

Thus the Jacobian has full rank. A similar argument follows if the edges of \mathcal{F}_3 lie in a straight line which is different from the x -axis.

Now suppose that the numbers n_2 and n_3 are both odd, and $[|a - b|, a + b]$ has empty intersection with the odd integers. Now regardless of the angle between a and b , the distance between the initial and terminal points will not equal an odd integer. So neither \mathcal{F}_2 nor \mathcal{F}_3 can lie completely in a straight line. This means that both A and C have rank 2, and that the Jacobian has full rank.

Similarly, suppose that the numbers n_2 and n_3 are both even, and $[|a - b|, a + b]$ has empty intersection with the even integers. Now regardless of the angle between a and b , the distance between the initial and terminal points will not equal an even integer. So neither \mathcal{F}_2 nor \mathcal{F}_3 can lie completely in a straight line. This means that both A and C have rank 2, and that the Jacobian has full rank.

□

5.2 Moduli Space as a Fibered Product

Definition 4. *Suppose Y_1 , Y_2 , and Z are spaces and suppose $q_1 : Y_1 \rightarrow Z$, and $q_2 : Y_2 \rightarrow Z$ are maps. The subspace of $Y_1 \times Y_2$ defined by*

$$\{(y_1, y_2) \in Y_1 \times Y_2 \mid q_1(y_1) = q_2(y_2)\}$$

is called the fibered product of Y_1 and Y_2 relative to Z , and is denoted $Y_1 \times_Z Y_2$.

Now suppose $Y_1 = M(\mathcal{F}_1, \mathcal{F}_2)$ and $Y_2 = M(\mathcal{F}_1, \mathcal{F}_3)$ and $Z = M(\mathcal{F}_1)$. The first two spaces are moduli spaces of polygonal linkages and the last space is the moduli space of one free linkage. We have the following homeomorphism result.

Theorem 5.2.1. *The space $M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ is homeomorphic to $M(\mathcal{F}_1, \mathcal{F}_2) \times_{M(\mathcal{F}_1)} M(\mathcal{F}_1, \mathcal{F}_3)$.*

Note: For this theorem we could possibly have an empty moduli space. So $\emptyset \times_Z X = \emptyset$ for every space X .

Proof. We will construct a continuous function

$$f : M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \rightarrow M(\mathcal{F}_1, \mathcal{F}_2) \times_{M(\mathcal{F}_1)} M(\mathcal{F}_1, \mathcal{F}_3)$$

and a continuous inverse function f^{-1} .

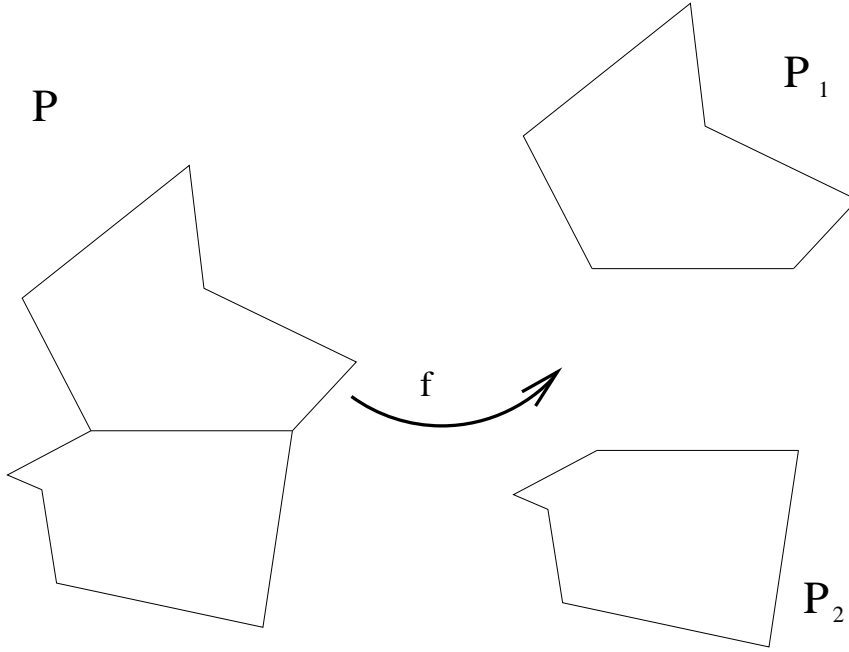


FIGURE 5.1. The effect of the f map when \mathcal{F}_1 has one edge

Suppose P is an embedding of $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$. We use the symbol $P|_{\setminus \mathcal{F}}$ to denote the embedding of $(\mathcal{F}_1, \mathcal{F}_2)$ after removing all edges of the \mathcal{F} linkage from that embedding.

Also suppose that P_1 and P_2 are embeddings of $(\mathcal{F}_1, \mathcal{F}_2)$ and $(\mathcal{F}_1, \mathcal{F}_3)$ respectively such that \mathcal{F}_1 is embedded the same way for each. We use the notation $P_1 \# P_2$ to represent the embedding of $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ where the common linkage is identified.

We define $f(P) = (P_1|_{\setminus \mathcal{F}_3}, P_2|_{\setminus \mathcal{F}_2})$ and $f^{-1}(P_1, P_2) = P_1 \# P_2$. The composition $f(f^{-1})$ takes a pair of embeddings of polygonal linkages with identical configurations of \mathcal{F}_1 , identifies the edges of \mathcal{F}_1 , and then maps this embedding to a 2-tuple of embeddings, one with the \mathcal{F}_2 removed and one with the \mathcal{F}_3 removed. It is clear to see that this operation is the identity on $M(\mathcal{F}_1, \mathcal{F}_2) \times_{M(\mathcal{F}_1)} M(\mathcal{F}_1, \mathcal{F}_3)$.

The composition $f^{-1}(f)$ takes an embedding of the multi-polygonal linkage, then maps this embedding to a 2-tuple of embeddings, one with the \mathcal{F}_2 removed and one with the \mathcal{F}_3 removed. This pair of embeddings has the same configuration of the \mathcal{F}_1 free linkage, so it lies in the fibered product. So the pair is then mapped to one multi-polygonal linkage embedding where the common edges are identified. This is clearly the identity on $M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$.

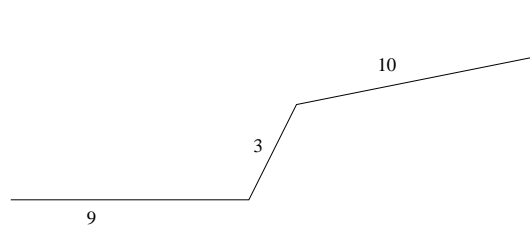
□

Notice that if \mathcal{F}_1 is formed by one edge, then $M(\mathcal{F}_1)$ is just a point, and the fibered product $M(\mathcal{F}_1, \mathcal{F}_2) \times_{M(\mathcal{F}_1)} M(\mathcal{F}_1, \mathcal{F}_3)$ becomes the usual cross product $M(\mathcal{F}_1, \mathcal{F}_2) \times M(\mathcal{F}_1, \mathcal{F}_3)$.

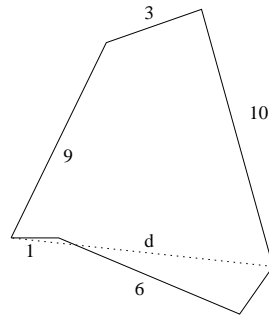
5.3 Disjoint Union of Polygonal Moduli Spaces

Definition 5. *The length range of a free linkage with vertices $\{x_i\}_{i=1}^n$ is the closed interval $[l_1, l_2]$, where l_1 is the minimum value of $|x_{n+1} - x_1|$ over all points in the moduli space of \mathcal{F} , and $l_2 = \sum_{i=1}^n |x_{i+1} - x_i|$.*

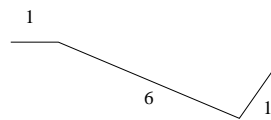
Proposition 5.3.1. *Suppose the free linkage \mathcal{F}_1 has length range $[l_{11}, l_{12}]$ and suppose the free linkage \mathcal{F}_2 has length range $[l_{21}, l_{22}]$. Furthermore, suppose \mathcal{F}_3 consists of two edges of lengths a and b , where $|a - b| < \max\{l_{11}, l_{21}\}$ and $a + b > \min\{l_{12}, l_{22}\}$. Then $M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ equals the disjoint union of two copies of $M(\mathcal{F}_1, \mathcal{F}_2)$.*



Minimum distance from initial to terminal point: 0
 Maximum distance from initial to terminal point: 22



Here is a point in the moduli space of the polygonal linkage formed by two free linkages.
 Here $4 < d < 8$.

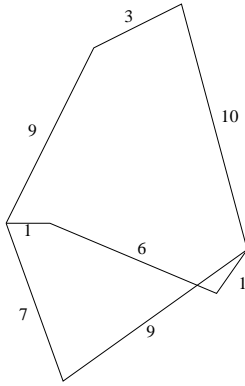
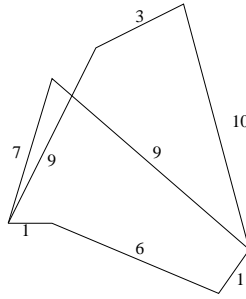


Minimum distance from initial to terminal point: 4
 Maximum distance from initial to terminal point: 8

FIGURE 5.2. Constraints on Lengths with First Two Free Linkages

Proof. Let P be any point in $M(\mathcal{F}_1, \mathcal{F}_2)$. We will construct two points in $M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ by attaching \mathcal{F}_3 . The conditions $|a - b| < \max\{l_{11}, l_{21}\}$ and $a + b > \min\{l_{12}, l_{22}\}$ ensure that we can attach the linkage so that the initial point agrees with the identified initial points of \mathcal{F}_1 and \mathcal{F}_2 and the terminal point agrees with the identified terminal points of \mathcal{F}_1 and \mathcal{F}_2 . There are two ways to make this attachment, one in which the angle between a and b is in the interval $(0, \pi)$, and one in which the angle is in the interval $(\pi, 2\pi)$. Now because of the conditions above, there are no configurations where the angle between a and b is 0 or π . Since the angle between a and b has to change continuously as we deform any linkage in the moduli space, there is no path that we can take to get from one point to the other. Refer to figures 5.3 and 5.3.

□

P_1  P_2 

The third free linkage has lengths that sum to greater than 8 and whose difference is less than 4. There is no path of configurations to go from the first point to the second.

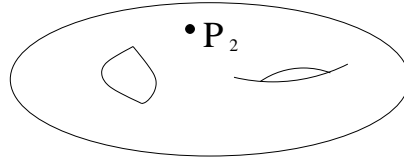
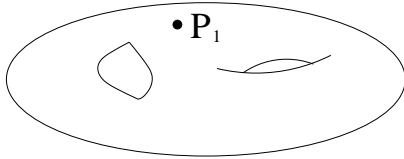


FIGURE 5.3. Illustration of Points in Two Disjoint Copies of Moduli Space of Polygonal Linkage

5.4 Disjoint Union Modulo a Subspace

Now if $|a - b|$ can equal $\max\{l_{11}, l_{21}\}$ or if $a + b$ can equal $\min\{l_{12}, l_{22}\}$ (or both), then there is a path between the points described above. This is because the angle between a and b can equal 0 or π (or both). With this observation we have the following propositions.

Proposition 5.4.1. *Suppose we have the same conditions as in 5.3.1 except that $|a - b| = \max\{l_{11}, l_{21}\} > 0$. Then the moduli space $M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ is a disjoint union of two copies of $M(\mathcal{F}_1, \mathcal{F}_2)$, modulo the subspace M_0' of $M(\mathcal{F}_1, \mathcal{F}_2)$ where the angle between a and b is 0.*

Similarly we have the following.

Proposition 5.4.2. *Suppose we have the same conditions as in 5.3.1 except that $a + b = \min\{l_{12}, l_{22}\}$. Then the moduli space $M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ is a disjoint union of*

two copies of $M(\mathcal{F}_1, \mathcal{F}_2)$, modulo the subspace $M_{\pi'}$ of $M(\mathcal{F}_1, \mathcal{F}_2)$ where the angle between a and b is π .

Since the moduli space is the union of two algebraic subvarieties, which are in turn triangulable, we have the following Mayer-Vietoris sequence.

$$\dots \rightarrow H_i(M_{\theta'}) \rightarrow H_i(M(\mathcal{F}_1, \mathcal{F}_2)) \oplus H_i(M(\mathcal{F}_1, \mathcal{F}_2)) \rightarrow H_i(M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)) \rightarrow \dots$$

This gives us the following Euler characteristics.

$$\chi(M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)) = 2\chi(M(\mathcal{F}_1, \mathcal{F}_2)) - \chi(M_{\theta'})$$

5.5 Connectedness

We recall Theorem 1 from [9] regarding polygonal linkages. Kapovich and Millson use M_r as the notation for the moduli space of a polygonal linkage with a vector $r = (r_1, \dots, r_n)$ of edge lengths. They also allow some, but not all, edges to have a possible length of 0 and normalize their polygons so that the perimeter is 1. That theorem generalizes to the following form.

Theorem 5.5.1. *The space M_r is not connected if and only if there are three different sides $e_i, e_j,$ and e_k (with lengths r_i, r_j and r_k respectively) in the normalized polygon so that*

$$r_i + r_j > \frac{1}{2} \sum_{i'=1}^n r_{i'}, \quad r_j + r_k > \frac{1}{2} \sum_{i'=1}^n r_{i'}, \quad r_i + r_k > \frac{1}{2} \sum_{i'=1}^n r_{i'}$$

For the theorem below, we have the following proposition.

Proposition 5.5.2. *Let $M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ be the moduli space of a multi-polygonal linkage, and let $p : M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \rightarrow S^1$ be an onto map which sends a configuration with angle $\theta_{1,1}$ to the angle $\theta_{1,1}$. Suppose the following conditions hold.*

1. $M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ contains a subspace homeomorphic to S^1 which intersects each fiber.
2. For every $\theta \in \text{im}(p)$, $p^{-1}(\theta)$ is connected.

Then $M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ is connected.

We use this theorem to give a nice class of multi-polygonal linkages with connected moduli spaces. For this theorem, let

$$(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) = (\mathcal{F}(a, b), \mathcal{F}_2(r_1, \dots, r_{n_2}), \mathcal{F}_3(r_1', \dots, r_{n_3}'))$$

be a multi-polygonal linkage where \mathcal{F}_2 has length range containing $[|a - b|, a + b]$ and \mathcal{F}_3 has length range containing $[|a - b|, a + b]$. Let P' be the perimeter of the polygonal linkage $(\mathcal{F}(|a - b|), \mathcal{F}_2)$ and let P'' be the perimeter of the polygonal linkage $(\mathcal{F}(|a - b|), \mathcal{F}_3)$

Theorem 5.5.3. *Let $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ be a multi-polygonal linkage as described above. Let r_i and r_j be the two longest edges in \mathcal{F}_2 and let r_i' and r_j' be the two longest edges in \mathcal{F}_3 . Suppose that $r_1 + r_j \leq P'/2$ and that $r_i' + r_j' \leq P''/2$. Then the moduli space $M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ is connected.*

Proof. The length range conditions above assure that the space $M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ contains at least one point in each fiber, so that we can change the angle $\theta = \theta_{1,1}$ between a and b and obtain a copy of S^1 that intersects each fiber. We do this by starting with a configuration P at $\theta = 0$. Then we find two paths z_1 (by perturbing P so that θ increases from 0 to π) and z_2 (by perturbing P so that θ decreases from 2π to π) from P to a common configuration with angle $\theta = \pi$. The composition of the paths z_1 and z_2^{-1} gives us a subspace homeomorphic to S^1 which intersects every fiber.

Now we show that each fiber is connected. Let θ be any angle between a and b , and let d be the distance from the initial point to the terminal point of \mathcal{F}_1 . Let $P_{d,2}$ be the perimeter of the polygonal linkage $(\mathcal{F}(d), \mathcal{F}_2)$. Let $P_{d,3}$ be the perimeter of the polygonal linkage $(\mathcal{F}(d), \mathcal{F}_3)$. Since $d \geq |a - b|$, we have that $P_{d,2} \geq P'$ and that $P_{d,3} \geq P''$. Therefore

$$r_i + r_j \leq P_{d,2}/2$$

and

$$r_i' + r_j' \leq P_{d,3}/2.$$

The new polygonal linkages therefore do not have three long edges, regardless of how long d is. Since $M(d)$ is just a point, by Section 5.2 the fiber is now a cross product of moduli spaces of these polygonal linkages. Each of these moduli spaces is connected. Therefore the fiber is connected. Since θ was arbitrary, every fiber of the map is connected, and so is the moduli space $M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$. \square

6. Specific Results

6.1 Bundle Structure

Now we investigate moduli spaces of the form $M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ where $\mathcal{F}_1 = \mathcal{F}(a, b)$ is now the 2-bar linkage, $\mathcal{F}_2 = \mathcal{F}(1^{<n_1>})$, and $\mathcal{F}_3 = \mathcal{F}(1^{<n_2>})$. Let θ be the angle from the x -axis to b , and let $\theta_1 = \pi - \theta$ be the angle from b to a . Consider the map $p : M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \rightarrow S^1$ which maps a configuration to its angle θ_1 . When we form a triangle with lengths a and b , with an angle θ_1 between them, we have a unique number $d(\theta_1)$ that is the length of the third side.

So we fix θ_1 and we consider $p^{-1}(\theta_1) \subset M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$. We have a homeomorphism $\phi : p^{-1}(\theta_1) \rightarrow M(\mathcal{F}(d(\theta_1)), \mathcal{F}_2, \mathcal{F}_3)$ which just rotates the length $d(\theta_1)$ segment to the positive x -axis. The following observation will be used later.

Proposition 6.1.1. *The space $p^{-1}(\theta_1)$ is an algebraic subvariety of $M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$.*

We combine this with the homeomorphism between $M(\mathcal{F}(d(\theta_1)), \mathcal{F}_2, \mathcal{F}_3)$ and $M(\mathcal{F}(d(\theta_1)), \mathcal{F}_2) \times M(\mathcal{F}(d(\theta_1)), \mathcal{F}_3)$ established in section 5.2.

But here $M(\mathcal{F}(d(\theta_1)), \mathcal{F}_2)$ and $M(\mathcal{F}(d(\theta_1)), \mathcal{F}_3)$ are moduli spaces of quasi-equilateral linkages, so we know something about the moduli spaces from Theorems in [8].

Theorem 6.1.2. *Suppose the multi-polygonal linkage $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ satisfies any one of the following three conditions.*

1. *The interval $[|a - b|, a + b]$ does not intersect the odd integers, and \mathcal{F}_2 and \mathcal{F}_3 both have an odd number of edges.*

2. The interval $[|a - b|, a + b]$ does not intersect the even integers, and \mathcal{F}_2 and \mathcal{F}_3 both have an even number of edges.
3. The interval $[|a - b|, a + b]$ does not intersect the integers, and \mathcal{F}_2 and \mathcal{F}_3 have opposite parity of number of edges.

then the map $p : M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \rightarrow S^1$ is a locally trivial fibration, and therefore $\chi(M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)) = 0$.

We need the following theorems.

Theorem 6.1.3. *Let f_1, \dots, f_r be C^∞ functions on \mathbb{R}^n with $r \leq n$ and coordinates x_1, \dots, x_n . Let a be a point with $f_1(a) = \dots = f_r(a) = 0$. Let $K \subseteq \{1, \dots, n\}$ be a subset of size r . Suppose that the $r \times r$ matrix*

$$\left(\frac{\partial f_i}{\partial x_k} \right),$$

with $i = 1, \dots, r$ and $k \in K$, has rank r . Then

1. *The set $M \subseteq \mathbb{R}^n$ defined by $f_1 = \dots = f_r = 0$ is a smooth manifold of dimension $n - r$ in a neighborhood of a .*
2. *For any $j \notin K$, the projection $M \rightarrow \mathbb{R}$ given by x_j is a submersion in a neighborhood of a .*

Proof. The implicit function theorem shows in fact that $\{x_j\}_{j \notin K}$ is a system of local coordinates for a manifold structure on M in a neighborhood of a . \square

Theorem 6.1.4. *Let $f : X \rightarrow \mathbb{R}$ be a smooth map of smooth manifolds. Assume that f is a proper map (that is, $f^{-1}(K)$ is compact for any compact set K). Assume f is a submersion everywhere on X . Then X is a locally trivial fibration of smooth manifolds.*

Proof. We apply the proof of Theorem 3.1 in [11] to the function f , which is a Morse function without critical points.

□

Recall that a smooth map $f : X \rightarrow Y$ is a submersion at x if and only if the map $T_x(X) \rightarrow T_{f(x)}(Y)$ of tangent spaces is onto.

Proof of theorem 6.1.2. Let $x \in X = M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ and let $Y = S^1$. Then x represents a configuration of a multi-polygonal linkage which is described as the zero set of the following equations, where θ_1 is the angle from b to a in x .

$$f_1 = a + b \cos \theta - \cos \theta_{1,2} - \dots - \cos \theta_{n_2,2}$$

$$f_2 = b \sin \theta - \sin \theta_{1,2} - \dots - \sin \theta_{n_2,2}$$

$$f_3 = a + b \cos \theta - \cos \theta_{1,3} - \dots - \cos \theta_{n_3,3}$$

$$f_4 = b \sin \theta - \sin \theta_{1,3} - \dots - \sin \theta_{n_3,3}$$

$$f_5 = \theta + \theta_1 - \pi$$

We need to show that the Jacobian of this set of equations has full rank. The Jacobian has the form

$$\begin{pmatrix} A & B_1 & 0 \\ 0 & B_2 & C \\ 0 & 1 & 0 \end{pmatrix}$$

where A , B_1 , B_2 , and C are defined as before.

If the conditions of $[|a - b|, a + b]$ not intersecting integers of the proper parity above are satisfied, then neither \mathcal{F}_2 nor \mathcal{F}_3 can lie in a single straight line. This will force both A and C to have rank 2, and the last row will contribute a 5th

linearly independent row. Therefore the Jacobian has full rank. The first result then follows from Theorems 6.1.3 and 6.1.4.

To show that the Euler characteristic of $M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ is zero, we use the fact that $\chi(M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)) = \chi(F)\chi(S^1)$, where F is the fiber over every point. Since $\chi(S^1) = 0$, we see that $\chi(M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)) = 0$ also. \square

6.2 When Fibers Change Topological Type

In general, the fibers over different angles will not all be homeomorphic. There can be degenerate fibers where the topology changes, and the fibers of points in open intervals between the degenerate fibers will be homeomorphic to one another.

The theorem 5.1.6 will now be used in a more general setting. We recall the following variant of the Poincaré Duality Theorem from Spanier. See [12, pages 297]

Theorem 6.2.1. *Let (X, A) be a compact relative n -manifold such that $X - A$ is orientable over R . For all q and R -modules G there is an isomorphism*

$$H_q(X - A; G) = \bar{H}^{n-q}(X, A; G), \quad (6.2)$$

where \bar{H} is the homology theory used in [12].

Here is a consequence of that theorem.

Proposition 6.2.2. *Let (X, A) be a relative n -manifold. Suppose that (X, A) is orientable over a ring R . Then $\chi(X) = \chi(A) + (-1)^n \chi(X - A)$.*

Proof. From the long exact homology sequence with coefficients in a ring R , we have

$$\cdots \rightarrow H_i(A) \rightarrow H_i(X) \rightarrow H_i(X, A) \rightarrow \cdots$$

This sequence gives us the following well-known property of Euler characteristics.

$$\chi(X) = \chi(A) + \chi(X, A)$$

On the other hand, Theorem 6.2.1 states that there is a canonical isomorphism

$$H_i(X - A) \simeq \bar{H}^{n-i}(X, A) \cong H^{n-i}(X, A)$$

with the second isomorphism holding because of tautness. The tautness properties hold because (X, A) admits a triangulation. By the universal coefficient theorem, $rk(H^j(X, A)) = rk(H_j(X, A))$.

Since any manifold is $\mathbb{Z}/2$ -orientable, we have the following for $G = \mathbb{Z}/2$ (or G can be any $\mathbb{Z}/2$ -module).

$$H_q(X - A; \mathbb{Z}/2) = H^{n-q}(X, A; \mathbb{Z}/2) = H^{n-q}(X, A; \mathbb{Z}/2)$$

Now we have the following Euler characteristic computation.

$$\begin{aligned} \chi(X) &= \chi(A) + \chi(X, A) \\ &= \chi(A) + \sum_{i=0}^n (-1)^i rk(H_i(X, A)) \\ &= \chi(A) + \sum_{i=0}^n (-1)^i rk(H^i(X, A)) \\ &= \chi(A) + \sum_{i=0}^n (-1)^{n-i} rk(H^{n-i}(X, A)) \\ &= \chi(A) + \sum_{i=0}^n (-1)^{n-i} rk(H_i(X - A)) \\ &= \chi(A) + (-1)^n \sum_{i=0}^n (-1)^i rk(H_i(X - A)) \\ &= \chi(A) + (-1)^n \chi(X - A) \end{aligned}$$

□

Theorem 6.2.3. *Let $p : X \rightarrow S^1$ be a proper, onto map, and suppose there exists a finite set $T = \{\theta_i\} \subset S^1$ such that $p|_{X-p^{-1}(T)}$ is a smooth map of maximal rank. Then the map p is a locally trivial fibration on $p^{-1}(I)$ for each open interval I of $S^1 - T$.*

Proof. The proof uses Theorem 6.1.4. □

Let $X = M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ and let A be the disjoint union of the singular fibers $\{F_i\}$. There are finitely many such fibers. These occur as $p^{-1}(\theta_i)$ for θ_i angles causing the distance between the initial and terminal points to be integers of the proper parity. Then $\chi(A) = \sum \chi(F_i)$.

Then $X - A$ is the disjoint union of the inverse images of the open intervals in $S^1 - \cup\{\theta_i\}$. Since all fibers are homeomorphic on a fixed interval, we can contract these inverse images down to inverse images of a single point in each interval. So B is homotopic to $\sqcup F_j'$ where the F_j' fibers are in one-to-one correspondence with the open intervals of $S^1 - \cup\{\theta_i\}$. Then $\chi(B) = \sum \chi(F_j')$.

As a result we obtain $\chi(M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)) = \sum \chi(F_i) + (-1)^n \sum \chi(F_i')$.

6.3 Example Where Fibers Change Topological Type

For example, consider the linkage $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) = (\mathcal{F}_1(1.1, 0.9), \mathcal{F}_2(1, 1, 1), \mathcal{F}_3(1, 1))$ whose picture is given in figure 6.1.

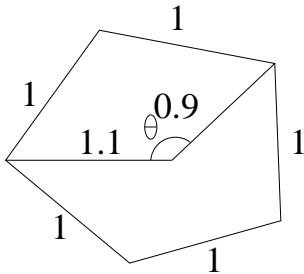


FIGURE 6.1. Sample multi-polygonal linkages with three degenerate fibers

When $\theta = \cos^{-1}(51/99)$ and $-\cos^{-1}(51/99)$, the distance between the initial and terminal edges equals 1. The fibers over these two angles is the cross product of the moduli space of an equilateral quadrilateral linkage and that of a triangle. The former moduli space is formed by three circles, any two of which meet in one point. The latter moduli space is two points. So the moduli space of each fiber is homeomorphic to a disjoint union of two copies of the space in Figure 6.3.

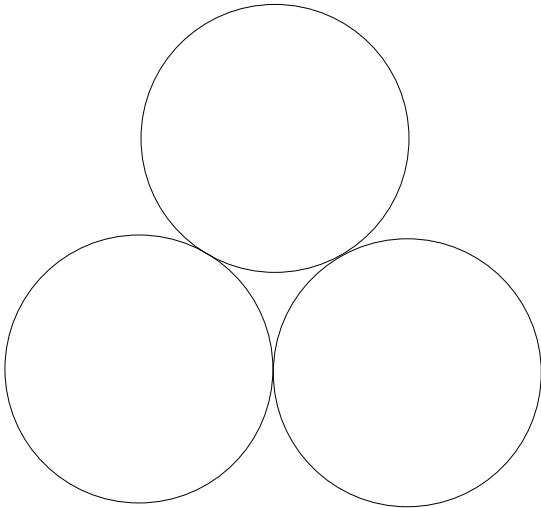


FIGURE 6.2. The moduli space of an equilateral quadrilateral linkage

On the interval $(-\cos^{-1}(51/99), \cos^{-1}(51/99))$ the fiber over any angle is homeomorphic to the cross product of the moduli space of $(\mathcal{F}_1(d), \mathcal{F}_2)$ (where $d < 1$) and that of a triangle. The former moduli space is two circles and the latter moduli space is two points. So the fiber over any angle in that interval is a disjoint union of four circles.

On the intervals $(-\pi, -\cos^{-1}(51/99))$ and $(\cos^{-1}(51/99), \pi)$, the fiber over any angle is homeomorphic to the cross product of the moduli space of $(\mathcal{F}_1(d), \mathcal{F}_2)$ (where $1 < d < 2$) and that of a triangle. The former moduli space is one circle and the latter moduli space is two points. So the fiber over any angle in that interval is a disjoint union of two circles.

Finally the fiber of π is homeomorphic to the cross product of the moduli space of $(\mathcal{F}_1(2), \mathcal{F}_2)$ and that of a triangle. The former moduli space is one circle and the latter moduli space is one point because \mathcal{F}_1 has collapsed into a straight line. So the fiber over π is one circle.

So $\chi(M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3))$ is the sum of Euler characteristics of four circles, 2 circles, 2 circles, one circle, and four copies of the space in Figure 6.3. All except the last space have Euler characteristic 0. The last space has Euler characteristic -3 for each copy and a total Euler characteristic of -12. Therefore $\chi(M(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)) = -12$.

See the figure below.

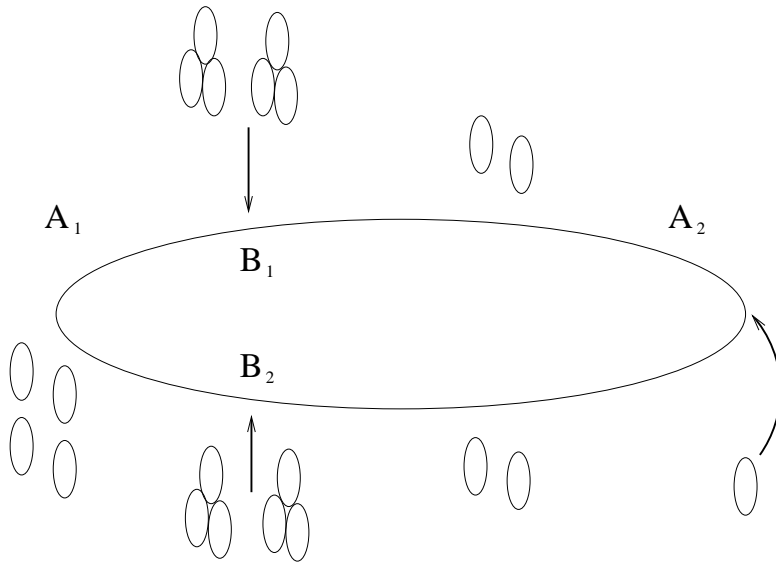


FIGURE 6.3. The fibers of the mapp : $M(\mathcal{F}_1(1.1, 0.9), \mathcal{F}_2(1, 1, 1), \mathcal{F}_3(1, 1)) \rightarrow S^1$

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Vita

Michael Holcomb was born on January 5, 1974, in Oklahoma City, Oklahoma. He finished his undergraduate studies at Oklahoma State University May 1996. He earned a master of science degree in mathematics from Purdue University in May 1998. In August 1999 he came to Louisiana State University to pursue graduate studies in mathematics. He is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in May 2003.