Orientations of Graphs Which Have Small Directed Graph Minors.

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ORIENTATIONS OF GRAPHS WHICH HAVE SMALL DIRECTED GRAPH MINORS

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Table of Contents

List of Figures

Abstract

Graphs are characterized by whether or not they have orientations to avoid one or more of the digraphs $\vec{K}_3$, $\vec{S}_3$, and $\vec{P}_3$. $\vec{K}_3$, $\vec{S}_3$ and $\vec{P}_3$ are created by starting with a triangle, a three point star, or a path of length three respectively, and replacing each edge with a pair of arcs in opposite directions. Conditions are described when all orientations of 3-connected and 4-connected graphs must have one or more of the above digraphs as a minor. It is shown that double wheels, and double wheels without an axle, are the only 4-connected graphs with an orientation not having a $\vec{K}_3$-minor. For $\vec{S}_3$, it is shown that the only 4-connected graphs which may be oriented without the minor are K_5 and C_6^2. It is also shown that all 3-connected graphs which do not have a W_5-minor have an orientation without an $\vec{S}_3$-minor, while every orientation of a graph with a W_6-minor has an $\vec{S}_3$-minor. It is demonstrated that K_5, C_6^2, and C_6^2 plus an edge are the only 4-connected graphs with an orientation without a $\vec{P}_3$-minor. Additionally, some restrictions on large 3-connected graphs without a $\vec{P}_3$-minor are given, and it is shown that if a 3-connected graph has a large wheel as a minor and has an orientation without a $\vec{P}_3$-minor, then the graph must be a wheel.

Certain smaller digraphs $\vec{P}_1$, $\vec{P}_2$, and $M = \vec{K}_3\backslash a$ are also considered as possible minors of orientations of graphs. It is shown that a graph has an orientation without a $\vec{P}_1$-minor if and only if it is a forest. It is shown that every orientation of a graph has a $\vec{P}_2$-minor if and only if the graph has T^2 or K_4^+ as a minor. To describe graphs with an orientation without an M-minor, a similar small list

of graphs is given, and it is shown that if none of the given graphs is a minor of a graph, then that graph has an orientation without an M-minor.

1 Introduction

This dissertation considers a relationship between graphs and directed graphs. Much work in graph theory describes situations where certain graphs or characteristics must be or cannot be present in a graph. The best known result of this type is Kuratowski's Theorem, which gives a complete characterization of planar graphs, graphs that can be drawn on the plane without any two edges crossing. This dissertation attempts to characterize graphs by describing graphs for which every possible orientation must contain certain small directed graphs.

While the reader is assumed to be familiar with most terms in graph theory, definitions from *Graph Theory* [6] by Frank Harary are used here. In particular a graph has no loops and no multiple edges. A multigraph may have loops and multiple edges. In addition, an *orientation* of a graph G is a directed graph, or digraph, with underlying graph G. Most of the work in this dissertation involves both graphs and directed graphs. To help keep the situations seperate, *node* and *arc* are used in discussing directed graphs, while *vertex* and *edge* are used in discussing graphs. In a directed graph the arc from a node u to a node v is denoted by $\langle u, v\rangle$, while in a graph the edge joining two vertices u and v is denoted by (u, v). Given any graph G, the directed graph $\vec{G}$ is created by replacing every edge of G with a pair of arcs in opposite directions.

Three directed graphs of primary importance in this dissertation are obtained from three common graphs: K_3, the triangle, S_3, the

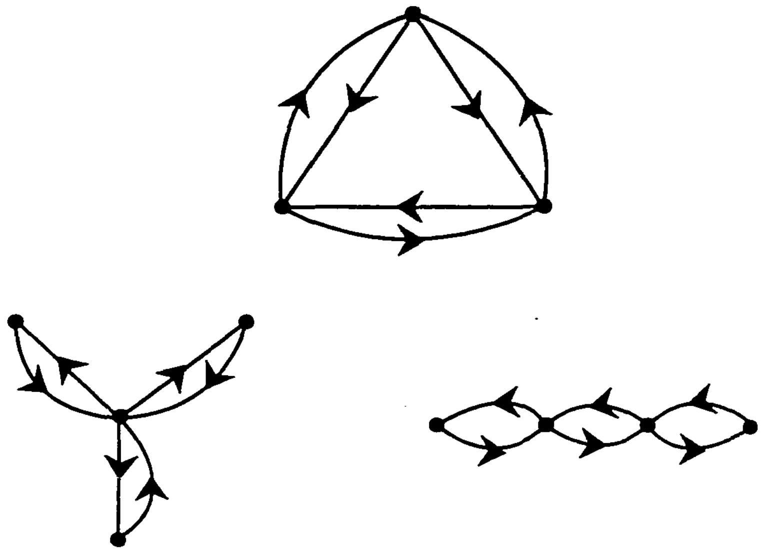

Figure 1: $\vec{K}_3$, $\vec{S}_3$, and $\vec{P}_3$

3-point star, and P_3, the path of length three. The digraphs are $\vec{K}_3$, $\vec{S}_3$, and $\vec{P}_3$, and are shown in Figure 1.

Several graphs are needed to illustrate the central problem of this dissertation. A *theta graph* is obtained by adding an edge to a 4-vertex circuit. The *wheel* W_n consists of an n-vertex circuit, called the *rim*, and an additional vertex, called the *hub*, which is adjacent to every vertex of the rim. A *double wheel,* W_n^2, consists of a circuit of length n, called the *rim*, and two additional vertices, called *hubs*, which are adjacent to each other and to every vertex on the rim. The edge incident with both hubs is called the *axle* and any edge from a hub to a vertex on the rim is called a *spoke.* The *complete bipartite graph,* $K_{k,n}$, consists of $n + k$ vertices, $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_n$, with an edge between a_i and b_j for all $i \in \{1, 2, \ldots, k\}$ and all $j \in \{1, 2, \ldots, n\}$, and no other edges. The *complete graph on n vertices,* K_n, is the graph with n vertices and all possible edges. To create C_n^2, start with a circuit whose vertices, in cyclic order, are $v_1, v_2, \ldots, v_n$

and add an edge between v_i and v_{i+2} for $i \in \{1, 2, \ldots, n-2\}$ and an edge between the pairs of vertices v_n and v_2, and v_{n-1} and v_1. To create the graph A_n, start with the complete bipartite graph $K_{3,n}$, with vertex set $\{a_1, a_2, a_3, b_1, \ldots, b_n\}$, and add edges (a_1, a_2) and (a_2, a_3). To create V_n, start with a circuit whose vertices, in cyclic order, are $v_1, v_2, \ldots, v_{2n-2}$, and add an edge between v_i and v_{2n-i} for $i \in \{2, 3, \ldots, n-1\}$, and an edge between v_1 and v_n.

The relation between graphs most often considered in this dissertation is the *minor* relation. A graph H is a minor of a graph G, written $H <_m G$, if H can be formed by starting with G and using the operations of *deleting a vertex*, written $G \backslash v$, *deleting an edge*, written $G \backslash e$, and *contracting an edge*, written G/e as often as desired. If the contraction of an edge results in a multigraph with either loops or families of parallel edges, the multigraph will be *simplified* by deleting the loops and all but one edge from each family of parallel edges. The new vertex created by the contraction of the edge (u, v) is notated as uv. Note that the minor relation is independent of the order in which the operations are performed. A graph H is a *topological minor of* G, written $H <_{tm} G$ if H is a minor of G that can be created when an edge can be contracted only if it is incident to a vertex of degree 2.

The minor relation can be extended to digraphs. The digraph D is called a minor of the digraph F, written $D <_m F$, if D can be formed by starting with F and performing the same three operations used to obtain a graph minor. Deleting a node or an arc is completely analogous; *contracting the arc* $a = \langle n_1, n_2 \rangle$ produces the digraph D/a with nodes $[N(D) \backslash \{n_1, n_2\}] \bigcup \{n_1 n_2\}$ and arcs found

by removing the arc a and with $i = 1$ or 2, replacing arcs $\langle n_i, x\rangle$ and $\langle x, n_i\rangle$ with $\langle n_1 n_2, x\rangle$ and $\langle x, n_1 n_2\rangle$, respectively.

The main results of this dissertation concern the issue of which graphs have an orientation not containing $\vec{K}_3$, $\vec{S}_3$, and $\vec{P}_3$ as minors. Two results by Pierre Duchet and Viktor Kaneti [2][3] speak to this problem. In [2] they prove a conjecture of Meyniel, that given a directed graph D such that every node of D has both in-degree and out-degree at least two, D contains $\vec{K}_3$ as a minor. Duchet and Kaneti [3] showed that every directed graph with n vertices and at least $5n-8$ arcs contains $\vec{K}_4$ as a minor. (Note: Duchet and Kaneti use the notation K_3^* and K_4^* where this dissertation uses $\vec{K}_3$ and $\vec{K}_4$.)

In considering $\vec{K}_3$, it is first shown that double wheels and double wheels minus the axle can be oriented in a unique manner to not contain a $\vec{K}_3$-minor. The main result on $\vec{K}_3$ shows that this orientation, given in Theorem 4.1.7 and Corollary 4.1.8, is the only way to orient a 4-connected graph to avoid having $\vec{K}_3$ as a minor.

Theorem 4.1.21 *A 4-connected graph G has an orientation without $\vec{K}_3$ as a minor if and only if G is a double wheel, or a double wheel minus the axle.*

This result, like other results throughout this dissertation discussing 4-connected graphs, relies on an extremely useful characterization of 4-connected graphs given in [7].

When the digraph $\vec{S}_3$ is considered, there are some interesting results involving 3-connected graphs. Using a list of all 3-connected graphs without a 5-wheel by Oxley [9], section 4.2 shows that every 3-connected graph not containing the 5-wheel, W_5, can be oriented

so that it does not contain $\vec{S_3}$ as a minor, yet every orientation of a graph containing the 6-wheel, W_6, contains $\vec{S_3}$ as a minor. Moving on to consider 4-connected graphs, the final result on $\vec{S_3}$ shows that every orientation of almost every 4-connected graph has $\vec{S_3}$ as a minor.

Theorem 4.2.11 *The only 4-connected graphs that have an orientation without $\vec{S_3}$ as a minor are K_5 and C_6^2.*

When considering orientations of graphs having the digraph $\vec{P_3}$ as a minor, the first result for 3-connected graphs is Theorem 4.3.1.

Theorem 4.3.1 *There exists n, such that any 3-connected graph which has W_n as a minor, and can be oriented to not have $\vec{P_3}$ as a minor, must be a wheel.*

Oporowski, Oxley, and Thomas [8] showed that every sufficiently large 3-connected graph must contain a subgraph isomorphic to a subdivision of W_k, V_k, or $K_{3,k}$. Combining this result with Theorem 4.3.1 results in Corollary 4.3.2.

Corollary 4.3.2: *For all $k \geq 40$ there is an integer N such that every 3-connected graph with at least N vertices, which can be oriented to not have a $\vec{P_3}$-minor, is either a wheel or has a subgraph isomorphic to a subdivision of $K_{3,k}$.*

In the 4-connected case only a very few 4-connected graphs can be oriented to not have a $\vec{P_3}$-minor.

Theorem 4.3.9 *The only 4-connected graphs having an orientation without $\vec{P_3}$ as a minor are K_5, C_6^2, and C_6^2 plus an edge.*

While the main results of this dissertation involve the directed graphs already mentioned, Chapters 2 and 3 show some interesting

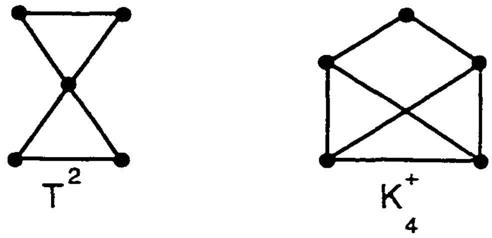

Figure 2: The graphs T^2 and K_4^+.

results by asking the same questions about other directed graphs. It was natural and necessary to begin with simpler digraphs. Chapter 2 considers two digraphs which are minors of $\vec{K}_3$, $\vec{S}_3$, and $\vec{P}_3$: $\vec{P}_1$, the directed 2-circuit, is the digraph with two nodes and two arcs in opposite directions between these two nodes; $\vec{P}_2$ is the digraph formed by replacing both edges of a path of length two with two arcs in opposite directions. The results of Chapter 2 culminate with the following:

Theorem 2.5 *Every orientation of a graph G contains $\vec{P}_2$ as a minor if and only if G has as a minor T^2 or K_4^+ (shown in Figure 2).*

Chapter 3 describes graphs for which every orientation has the digraph $M = \vec{K}_3 \backslash a$ for any arc a as a minor. For the digraph M the results are precise. The main results of this chapter are Theorem 3.7 and Corollary 3.8, which give necessary and sufficient conditions for every orientation of a graph to have M as a minor. In addition, Lemma 3.2 and Theorem 3.3 show that the only 3-connected graph that can be oriented without having M as a minor is a wheel, and that the wheel has a unique orientation, up to reversal of all arcs, without an M-minor.

Chapter 4 is broken into three sections, one for each of the digraphs $\vec{K}_3$, $\vec{S}_3$, and $\vec{P}_3$. The results of this chapter have already been mentioned.

Chapter 5 looks at pairs of the digraphs from Chapter 4. It is shown in Theorem 5.2 that every orientation of a large 3-connected graph has either $\vec{S_3}$ or $\vec{K_3}$ as a minor. For the other possible pairings, an arbitrarily large 3-connected graph may be oriented to avoid having the considered digraphs as a minor. For the pair $\vec{P_3}$ and $\vec{K_3}$, Theorem 5.3 and Corollary 5.4 show that arbitrarily large wheels and A_n, for arbitrarily large n, have orientations for which neither digraph is a minor.

For the pair $\vec{P_3}$ and $\vec{S_3}$, Theorem 5.5 shows that an orientation of A_n exists for which neither digraph is a minor.

Some additional terms needed throughout the body of this dissertation are defined here. A *partially oriented* graph is an ordered triple of disjoint sets (V, E, A), whose members are called *vertices*, *edges*, and *arcs* respectively, such that E is a subset of the set of 2-element subsets of V and A is a subset of $V \times V$. A digraph $D = (N, A)$ is a *completion of the orientation* of a partially oriented graph $G = (V, E, \hat{A})$ if $N = V$, $A = E \bigcup \hat{A}$, and there exists an arc from the node a to the node b in D if and only if there is either an arc from the vertex a to the vertex b in G or there is an edge in G between a and b and no arc $\langle a, b \rangle$. A graph *underlies* a partially oriented graph if the graph underlies every directed graph that completes the orientation of the partially oriented graph. Deletion of edges, arcs, and vertices, and the contraction of edges or arcs, from partially oriented graphs is analogous to such deletions and contractions in graphs and digraphs.

A cubic graph G is *cyclically 4-connected* if, for any set S of fewer than 4 vertices, sufficient to disconnect the graph, at least

one component of $G\backslash S$ does not contain a circuit. A graph G is an *uncontractible n-connected graph* if, for every edge e in G, the graph G/e is not n-connected.

Given a graph, a partially oriented graph, or a digraph with vertex (or node) v, *splitting* v is done by deleting v, adding vertices v_1 and v_2, adding an edge e incident with v_1 and v_2, and adding a minimal set of other edges incident with either v_1 or v_2 so that if e is contracted the result is the original graph. In the case of a digraph, this operation will result in a partially oriented graph.

Given two vertex-disjoint graphs G and H, a *one-sum* of G and H is obtained by identifying a vertex of G with a vertex of H. For a graph G, the *line graph $L(G)$* is a graph H where the vertices of H represent the edges of G, and there is an edge in H between two vertices if and only if those two vertices represent two edges in G which are both incident with a common vertex in G.

2 Orientations of Graphs That Have a $\vec{P}_1$-minor or a $\vec{P}_2$-minor

In building up to the main results about $\vec{K}_3$, $\vec{S}_3$, and $\vec{P}_3$ this dissertation begins with trying to find graphs which may be oriented to avoid having either $\vec{P}_1$ or $\vec{P}_2$ as a minor. This chapter has three main results. Observation 2.1 describes all graphs for which every orientation contains a $\vec{P}_1$-minor. Theorem 2.3 considers the special situation of a *rooted minor.* A *root* of a graph is a vertex distinguished from the other vertices of the graph. $G[a, b]$ is a graph G with roots a and b. An $a - b$ *rooted minor,* $H[a, b]$ of $G[a, b]$, is obtained using the same operations of contracting an edge, deleting an edge, and deleting a vertex as a minor of G, except that neither a nor b may be deleted, and if an edge incident with a or b (but not both) is contracted then the new vertex created is given the label a or b respectively and is a root of the graph obtained. Theorem 2.3 describes every directed graph without a rooted $\vec{P}_1$-minor. The last result of this section, Theorem 2.5, gives a necessary and sufficient condition for every orientation of a graph to have a $\vec{P}_2$-minor.

Observation 2.1: *Every orientation of a graph G has a $\vec{P}_1$ - minor if and only if G is not a forest.*

<u>Proof</u>

It is apparent that a forest cannot be oriented to have a $\vec{P}_1$-minor. If G is not a forest, then G has a triangle as a minor. Checking the only two possible orientations of a triangle shows that every orientation of the triangle, and therefore every orientation of G,

has a $\vec{P}_1$-minor. Therefore, every orientation of a graph G has a $\vec{P}_1$-minor if and only if G is not a forest. □

While the contraction of any arc of a triangle oriented as a directed circuit results in $\vec{P}_1$, if the triangle is not oriented as a circuit there is only one arc which, when contracted, results in $\vec{P}_1$. Therefore, while it is certain every orientation of any graph which is not a forest has a $\vec{P}_1$-minor, it cannot be guaranteed that a chosen pair of vertices will be the vertices of the $\vec{P}_1$-minor. This problem of rooted minors is looked at in the following Lemma.

Lemma 2.2: *Let G be a theta-graph with the two vertices of degree two labeled a and b. Every orientation of G has an $a-b$ rooted $\vec{P}_1$-minor.*

<u>Proof</u>

Label the remaining two vertices c and d, and let D be an orientation of G. Without loss of generality, by reversing the directions of all arcs, if necessary, it may be assumed that $\langle a, c\rangle \in A(D)$. If $\langle d, a\rangle \in A(d)$, then contracting the arcs between c and d and c and b will create the rooted $a-b$ 2-circuit, thus $\langle a, d\rangle \in A(D)$. Consider the two possible orientations of the arc between c and d. By symmetry assume that $\langle c, d\rangle \in A(D)$. Contracting $\langle a, d\rangle$ and the arc between b and c creates an $a-b$ rooted 2-circuit. Therefore, every orientation of $G[a, b]$ has a rooted $a-b$ $\vec{P}_1$-minor. □

Before moving on to the next result, a few additional terms are defined. Given a graph G and an edge e in G, *adding an edge in parallel to e* creates the multigraph that has a pair of edges between the two vertices joined by e in G (and is otherwise identical to G).

Adding an edge in series to e replaces the edge e, incident with vertices a and b, with a path of length 2. The graph created has one additional vertex, v, not in G, such that v is of degree 2 and v is adjacent to a and b. A *series-parallel extension of a graph G* is any multigraph that can be obtained, starting from G, by a series of edges added in parallel and edges added in series. Any series-parallel extension of K_2 is simply called a *series-parallel multigraph.* If a series-parallel multigraph is obtained by a series of operations ending with the addition of an edge in parallel, then the multigraph has a pair of duplicate edges. If the last step was the addition of an edge in series, then the multigraph has a vertex of degree 2.

The idea of series-parallel extension can be extended to digraphs. The only additional consideration is that the direction of an arc be preserved. Thus, two arcs are *parallel arcs* if they have the same in-vertex and same out-vertex. Similarly, in the series extension an arc is replaced with a directed path of length 2.

Theorem 2.3: *A directed graph D, with underlying 2-connected rooted graph $G[a, b]$, does not have an $a - b$ rooted 2-circuit minor if and only if D is a directed series parallel extension of an arc joining a and b.*

Proof

The if part of the Theorem is shown first, by proving the slightly stronger statement that a directed graph D, with underlying rooted graph $G[a, b]$ does not have an $a - b$ rooted 2-circuit minor if D is a directed series parallel extension of an arc joining a and b. If D contains only the arc joining a and b, then D does not have a rooted $a - b$ 2-circuit-minor. Use induction and suppose any directed series

parallel extension of an arc between a and b with fewer arcs than D does not have a rooted $a-b$ 2-circuit-minor.

Since D is a directed series-parallel extension of an $a-b$ arc, either there are two parallel arcs in D, or there is a vertex, v, different from a and b, with out-degree and in-degree one. If D contains parallel arcs, then, since a 2-circuit does not have any families of parallel arcs, deleting one of the arcs cannot affect whether or not D has a rooted $a-b$ 2-circuit-minor. Therefore, by the inductive hypothesis D would not have a rooted $a-b$ 2-circuit.

Now suppose there exists a vertex, v, with out-degree and in-degree one, and let e and f be the two arcs incident with v. Note that $D/e \simeq D/f$. Since the 2-circuit has only two vertices, if D has a rooted $a-b$ 2-circuit-minor, either e or f must be contracted or deleted in the process that leads from D to the $a-b$ rooted 2-circuit. Deleting either e or f results in a digraph in which v is only incident with a single arc. If $D\backslash e$ or $D\backslash f$ has an $a-b$ rooted 2-circuit minor, then so does $D\backslash v$, contradicting the inductive hypothesis. Therefore, D has a rooted $a-b$ 2-circuit-minor if and only if D/e has the same minor. Therefore, by the inductive hypothesis D does not have a rooted $a-b$ 2-circuit-minor.

To show the second part of the Theorem, suppose D is not a directed series parallel extension of an $a-b$ arc. Either G is not a series-parallel extension of an $a-b$ edge, or G is a series-parallel extension of an $a-b$ edge and at least one arc in D is in the opposite direction it would be if D were a directed-series-parallel extension of an arc between a and b. If G is not a series-parallel extension of an $a-b$ edge, then G with the edge (a,b) added contains a K_4-minor

with a and b distinct vertices. Deleting the edge (a, b) from K_4 produces a theta-graph with a and b as the two vertices of degree 2. So by Lemma 2.2 every orientation of G has a rooted $a - b$ 2-circuit-minor.

If G is a series parallel extension of an $a-b$ edge, then every edge of G is part of an $a - b$ path. Without loss of generality suppose every $a-b$ path in G is oriented in D so that it is possible to reduce that path through contraction and deletion of arcs to the arc $\langle a, b\rangle$. Suppose there exists an $a - b$ path, P, with an arc oriented so that through contraction and deletion the arc $\langle b, a\rangle$ is left. Since G is 2-connected there is another $a-b$ path, internally vertex disjoint from P, which can be reduced to $\langle a, b\rangle$. Therefore, D has an $a - b$ rooted 2-circuit as a minor. If no path exists which can be reduced through contraction and deletion to $\langle b, a\rangle$ then D is a directed-series-parallel extension of the arc $\langle a, b\rangle$. Therefore, the $a - b$ rooted 2-circuit is a minor of D.

Therefore, D, with underlying 2-connected rooted graph $G[a, b]$, does not have an $a - b$ rooted 2-circuit as a minor if and only if D is a directed series parallel extension of an arc joining a and b. □

Now consider graphs that underlie digraphs with a $\vec{P}_2$-minor.

Observation 2.4: *For a connected graph G the following are equivalent.*

(*i*) *No orientation of G has a $\vec{P}_2$-minor.*

(*ii*) *G has at most one cycle.*

(*iii*) *G has no T^2 or theta minor.*

<u>Proof</u>

(i) implies (iii) will be shown first, followed by (ii) implies (i), and then (iii) implies (ii). To see that (i) implies (iii) note that a theta-graph oriented so that the 4-circuit is a directed 4-circuit has a $\vec{P}_2$-minor obtained by contracting the one arc not in the 4-circuit. T^2 oriented so that both triangles are directed 3-circuits has a $\vec{P}_2$-minor obtained by contracting one edge from each triangle. Therefore, a graph G has an orientation with a $\vec{P}_2$-minor if G has a theta-graph or T^2 as a minor, and so (i) implies (iii).

$\vec{P}_2$ has two directed circuits. Therefore the underlying graph of any directed graph with a $\vec{P}_2$-minor must have at least two circuits, and so (ii) implies (i). To see that (iii) implies (ii), note that, as in the above paragraph, any graph without T^2 as a minor contains at most one block containing a circuit, and in any graph without a theta-graph as a minor every block must be either a circuit or K_2. Therefore, (iii) implies (ii). □

Observation 2.1 characterizes all graphs that can and must have a $\vec{P}_1$-minor. An equivalent statement for graphs that can and must have a $\vec{P}_2$-minor cannot be made as cleanly. The first part of characterizing such graphs is shown in Observation 2.4 which describes which graphs can be oriented to have a $\vec{P}_2$-minor. Theorem 2.5 answers the question of which graphs must have a $\vec{P}_2$-minor no matter the orientation chosen.

Theorem 2.5: *Every orientation of a graph G has a $\vec{P}_2$-minor if and only if T^2 or K_4^+ is a minor of G.*

<u>Proof</u>

First, to demonstrate the if part of the Theorem, suppose G has T^2 or K_4^+ as a minor. Since every orientation of a triangle has

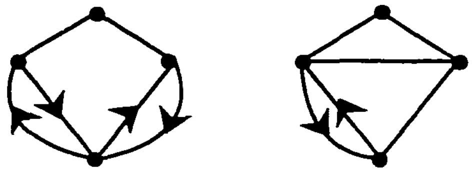

Figure 3: Minors of the possible orientations of K_4^+.

a directed 2-circuit-minor, every orientation of T^2 has a $\vec{P}_2$-minor. Suppose G has K_4^+ as a minor. Let a and b be the two vertices of K_4^+ not adjacent to the vertex of degree 2. For every orientation of K_4^+ either the contraction of the arc between a and b results in a directed graph with the completion of the orientation of the partially oriented graph shown on the left of Figure 3 as a minor, or the contraction of one of the arcs between a node of degree 3 and a or b results in a directed graph with the completion of the orientation of the partially oriented graph shown on the right of Figure 3 as a minor.

In the first case $\vec{P}_2$ is a subgraph of the partially oriented graph shown; therefore, every completion of this orientation has a $\vec{P}_2$-minor. In the second case, no matter how this orientation is completed, contracting one of the edges of the triangle shown with no arcs will result in a graph with a $\vec{P}_2$-subgraph.

Now, to demonstrate the only if, suppose G is a connected graph with neither K_4^+ nor T^2 as a minor. Since G does not have a T^2-minor , G has at most one block which is not K_2, and so without loss of generality suppose G is a block. Every 3-connected graph has a K_4-minor, and additionally every 3-connected graph not equal to K_4 has a W_4-minor. Therefore if G is 3-connected and not equal to K_4 then G has a W_4-minor. W_4 minus a spoke equals K_4^+. Therefore,

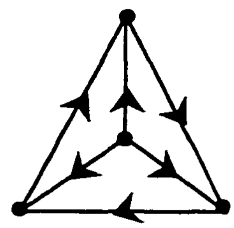

Figure 4: An orientation of K_4 not having a $\vec{P}_2$-minor.

if G is 3-connected either G is K_4 or G has a K_4^+-minor. The orientation of K_4 shown in Figure 4 does not have a $\vec{P}_2$-minor.

Suppose G is not 3-connected. Let $\{x, y\}$ be a 2-vertex-cut of G. Let $C_1, C_2, \ldots, C_n$ be the connected components of $G \backslash \{x, y\}$. For $i \in \{1, 2, \ldots, n\}$ let $\hat{C}_i$ be the induced subgraph of G with vertices $x, y, V(C_i)$ minus the edge (x, y), if it is an edge. Since T^2 is not a minor of G there is at most one i such that $\hat{C}_i$ contains a circuit. Furthermore, because it is assumed that G is a block, each $\hat{C}_i$ is a path if it coes not contain a cycle.

If $n \geq 3$ and there exists i and j, $i \neq j$ such that $|V(\hat{C}_i)|$ and $|V(\hat{C}_j)|$ are both greater than 3, then contracting all the edges from $\hat{C}_k$, k different from both i and j creates a graph with T^2 as a minor. Moreover, if $i \in \{1, 2, \ldots, n\}$ exists such that $\hat{C}_i$ contains a circuit, then T^2 is a minor of G.

Thus, assuming $n \geq 3$, G must be a minor of a graph formed by joining x and y with an edge, joining x and y with a single path of arbitrary length, and joining x and y with an arbitrary number of paths of length 2. Such a graph may be oriented as shown in Figure 5 to avoid having a $\vec{P}_2$-minor.

If $n = 2$, suppose $\hat{C}_1$ contains a block with a cycle. If $\hat{C}_1$ contains more than one block with a cycle then T^2 is a minor of G, so $\hat{C}_1$

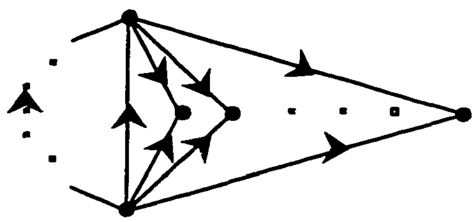

Figure 5: A digraph without a $\vec{P}_2$-minor.

consists of a block with a cycle, a path from x to this block, and another path from y to this block. There exist x' and y' in this block intersect the x-block and y-block paths respectively such that $\{x', y'\}$ is a 2-vertex-cut of G.

If $G\backslash\{x', y'\}$ has more than two components the above analysis for $n \geq 3$ describes the situation. If $G\backslash\{x', y'\}$ has only two components then either the block with a cycle in $\hat{C}_1$ contains a theta-graph with x' and y' as the two vertices of degree 2, or this block contains a cycle with the edge (x', y'). If the block contains the described theta-graph, then K_4^+ is a minor of G. If the block contains the described edge then contracting (x', y') shows that T^2 is a minor of G. Therefore the only case left is where $G\backslash\{x, y\}$ contains no blocks and $n = 2$, but this is simply a circuit. Therefore every orientation of a graph G has a $\vec{P}_2$-minor if and only if T^2 or K_4^+ is a minor of G. $\square$

3 Orientations of Graphs with a $\vec{K}_3\backslash a$−minor

Before moving on to graphs containing three 2-circuits, a specific small directed graph, between the cases considered in Chapter 2 and the cases to be considered in Chapters 4 and 5 is considered. This chapter places conditions on both graphs and digraphs to avoid the directed graph $M = \vec{K}_3\backslash a$ (shown in Figure 6) as a minor.

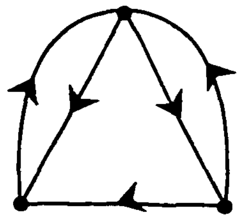

Figure 6: The graph M

Theorems 3.3 and 3.7, and Corollary 3.8 present the main results of this chapter. Theorem 3.3 shows that the only 3-connected graphs with an orientation not having M as a minor are wheels. Additionally, Lemma 3.2 shows that wheels have a unique orientation without an M-minor. Theorem 3.7 and Corollary 3.8 give a complete description of graphs for which every orientation contains an M-minor. In Theorem 3.7 the description is constructive, while in Corollary 3.8 a short list of graphs is given, and it is shown that every orientation of a graph G has an M-minor if and only if one of the graphs on the given list is a minor of G.

It is shown first that a unique orientation, up to reversal of all arcs, of K_4 exists without an M-minor.

Lemma 3.1: *An orientation of K_4 does not have an M-minor if and*

only if one vertex is oriented as a source or sink, and the remaining arcs form a directed circuit.

Proof

First it is shown that an orientation of K_4 does not have an M-minor only if one vertex is oriented as a source or sink, and the remaining arcs form a directed circuit. Suppose D is a directed graph with underlying graph K_4 that does not contain a source or sink. Label the vertices of D a, b, c, d. Without loss of generality suppose $\langle a, b\rangle, \langle c, a\rangle, \langle d, a\rangle \in A(D)$. Since b is not a sink, either $\langle b, c\rangle \in A(D)$ or $\langle b, d\rangle \in A(D)$, and the two cases are interchangeable.

Suppose $\langle b, c\rangle \in A(D)$, if $\langle c, d\rangle \in A(D)$, then contracting $\langle c, a\rangle$ leaves M. So consider the case $\langle d, c\rangle \in A(D)$. By assumption d is not a source, so $\langle b, d\rangle \in A(D)$, but contracting $\langle b, c\rangle$ leaves M. Therefore, any orientation of K_4 without a sink or source has an M-minor.

Now suppose c is a source and $\langle a, b\rangle \in A(D)$. If $\langle d, b\rangle \in A(D)$ then contracting $\langle c, b\rangle$ leaves M, so suppose $\langle b, d\rangle \in A(D)$. If $\langle a, d\rangle \in A(D)$ then contracting $\langle c, d\rangle$ leaves M. Therefore, an orientation of K_4 does not contain M only if one vertex is a sink or source and the rest of the vertices are oriented as a circuit.

It remains to be shown that an orientation of K_4 does not have an M-minor if one vertex is oriented as a source or sink, and the remaining arcs form a directed circuit. Note that every arc adjacent to the source or sink is interchangeable, and that the remaining arcs form another interchangeable set. It is enough therefore to consider

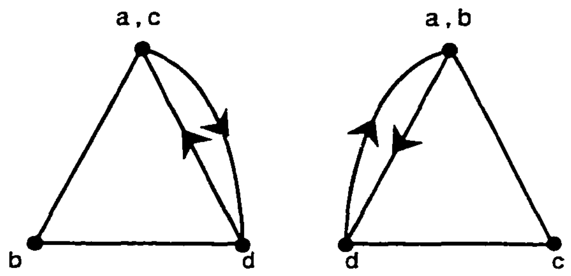

Figure 7: D contract $\langle c, a\rangle$ or $\langle a, b\rangle$

the orientation and labeling from above with the contraction of either $\langle c, a\rangle$ or $\langle a, b\rangle$, as shown in Figure 7. In either case M is clearly not a minor of the resulting digraph. Therefore, M is not a minor of this orientation of K_4. Therefore, an orientation of K_4 does not have an M-minor if and only if a vertex is oriented as a source or sink, and the remaining arcs form a directed circuit. □

Extending this to an orientation of all wheels gives the following result.

Lemma 3.2: *For $n \geq 3$ the only orientation of the wheel W_n without M as a minor is the orientation of the hub as a source or sink and the rim as a directed circuit.*

Proof

If $n = 3$, then $W_n = W_3 = K_4$, and since any vertex of W_3 can be considered the hub the claim is true. In addition it is necessary to show that the Lemma is true for W_4 before using induction. Let D be a directed graph with underlying graph W_4. If the hub is not oriented as a source or sink there is an arc on the rim which, when contracted, leaves a digraph that is a completion of the partially oriented graph shown in Figure 8. When the orientation of this graph is completed the triangle drawn beneath the directed 2-circuit

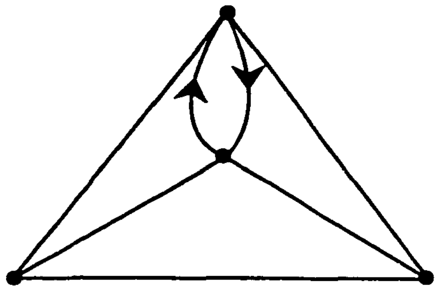

Figure 8: D contract a rim arc.

will provide the second directed 2-circuit needed to obtain M. If the hub is a source or sink but the rim is not oriented as a circuit it is possible to contract an arc on the rim and obtain an orientation of K_4 with a source or sink, but with the remaining vertices not oriented as a directed circuit. By Lemma 3.1 this orientation also has an M-minor. Now use induction and suppose that for some $k \geq 4$ and all n, $4 \leq n \leq k$, the claim is true. Let $n = k + 1$ and consider the graph W_n.

The contraction of any edge along the rim of W_n leaves the graph W_k, for which the inductive hypothesis holds. Therefore, since $n \geq 5$, it is obvious that any orientation of W_n besides one where the hub is either a source or sink and the rim is a directed circuit will have an M-minor. Furthermore, given W_n with this special orientation, the contraction of any arc on the rim leaves a graph that does not have an M-minor.

Since $n \geq 5$, $|V(W_n)| \geq 6$. Contracting any arc along the rim would not allow the creation of M, so if W_n has an M-minor it is necessary to contract at least two spokes. This leaves two cases to consider: the contraction of consecutive spokes and the contraction of non-consecutive spokes.

In the case of two consecutive spokes being contracted, the resulting directed graph, after deletion of the loop incident with the hub, could be obtained by contracting a single spoke of W_k; therefore, by the inductive hypothesis, this directed graph does not have an M-minor.

In the case of two non-consecutive spokes the resulting graph is not 2-connected. Since M is 2-connected it is going to be contained in one of the blocks. Therefore this second case contains M if and only if deleting one or more vertices along the rim leaves a block with an M-minor. However, deleting these vertices creates a graph which is a minor of W_k and therefore does not have an M-minor. Therefore, for $n \geq 3$ the only orientation of the wheel, W_n that does not have an M-minor is the orientation of the hub as a source or sink and the rim as a directed circuit. □

One additional operation on a graph that needs to be defined is *splitting a vertex*, which is the replacement of a vertex v by two adjacent vertices v' and v'' such that each vertex formerly joined by an edge to v is joined by an edge to exactly one of v' and v''. With the addition of this operation a characterization of 3-connected graphs given by Tutte [12] can be stated.

Theorem: *A graph G is 3-connected if and only if G is a wheel or can be obtained from a wheel by a sequence of operations of the following two types:*

1. The addition of a new edge

2. Splitting a vertex of degree at least 4 so that the two new vertices v' and v'' each have degree greater than or equal to 3.

Using this characterization the following can be shown.

Theorem 3.3: *Let D be a directed graph with underlying graph G. If G is 3-connected and M is not a minor of D, then D is a wheel oriented as described in Lemma 3.2.*

Proof

Let D be an orientation of a 3-connected graph G. If G is a wheel, then by Lemma 3.2 D does not have an M-minor if and only if D is oriented in the special way described in Lemma 3.2. If G is not a wheel, then since G is 3-connected, G contains a wheel-minor.

No edges can be added to K_4 since all possible edges are already present. Additionally, no vertex of K_4 has degree greater than three. Thus, by the above mentioned characterization of 3-connected graphs by Tutte, if G is not a wheel W_4 is a minor of G. Suppose G is not a wheel. G has a minor H obtained by adding an edge to W_n, $n \geq 4$, or by splitting a vertex of degree greater than 3 in W_n, $n \geq 4$. Moreover, by contracting and deleting edges of H it is possible to obtain a minor of G obtained by adding an edge or splitting a vertex of degree greater than 3 in W_4

First consider the case in which G has W_4 plus an edge as a minor. By Lemma 3.2, the only orientation of W_4 which does not have M as a minor has the rim oriented as a directed circuit. With the rim oriented as a directed circuit, however, the partially oriented graph obtained by the contraction of the added edge has M as a minor, as shown in Figure 9.

The hub is the only vertex of degree greater than 3 in W_n, $n \geq 4$. Therefore, if G does not have W_4 plus an edge as a minor, then G has a graph obtained by splitting the hub of W_4 as a minor. Taking

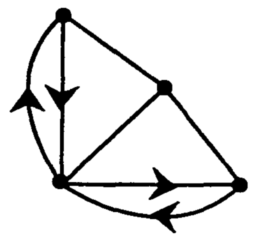

Figure 9: (W_4 plus e) contract e

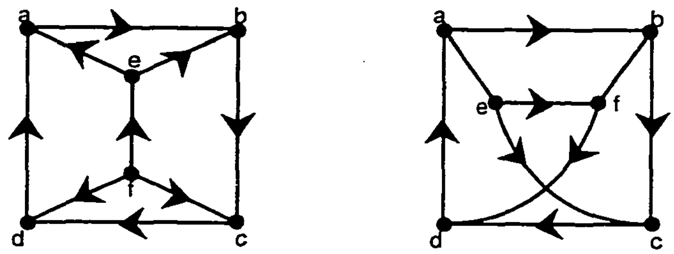

Figure 10: Two splittings of the hub of W_4

into account the result of Lemma 3.2, there are two cases, both shown in Figure 10. In either case M may be obtained as shown in Figure 11.

Therefore, given that D is a directed graph with underlying graph G, if G is 3-connected and M is not a minor of D, then D is a wheel oriented as described in Lemma 3.2. □

This describes the 3-connected case completely. In considering the not-3-connected situation begin with the following.

Lemma 3.4: *Any orientation of a graph G, made by joining two vertices with an arbitrary number of internally vertex disjoint paths*

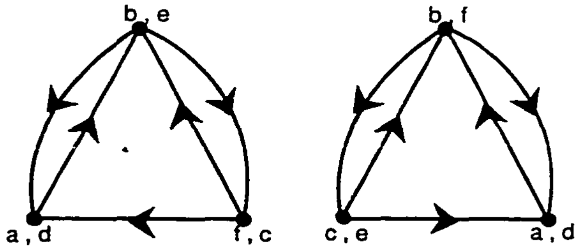

Figure 11: M contained in either splitting of the hub of W_4

of arbitrary length, does not have an M-minor.

<u>Proof</u>

Let G be a graph consisting of two vertices, x and y, and an arbitrary number of internally vertex disjoint paths connecting x and y. It is enough to show that the underlying multigraph of M is not a minor of G. Every vertex besides x and y has degree 2 in G, but no vertex in the underlying multigraph of M has degree 2. Therefore, every vertex of G besides x and y must be deleted or identified using contractions with x or y. This, however, leaves only two vertices and the underlying multigraph of M has three vertices. Therefore, no orientation of a graph G, made by joining two vertices with an arbitrary number of internally vertex disjoint paths of arbitrary length, has an M-minor. □

Lemma 3.5: *Given a directed graph D that does not have an M-minor, the directed multigraph D^+ obtained by adding an arc in parallel with an arc of D or subdividing an arc in D does not have an M-minor.*

<u>Proof</u>

Let D be a directed graph without M as a minor. First consider the case where an arc is added in parallel to an arc already in D. Since there are no parallel arcs in M, this additional arc does not affect whether or not D has an M-minor, and so M is not a minor of this version of D^+.

Consider the case where an arc is subdivided. Let the new vertex, of degree 2, be labeled v. M has no vertex of degree 2. Therefore, v must either be deleted or joined by contraction to one of its neighbors. In either case, however, the directed graph created is a minor

of D. Therefore, given a directed graph D that does not have M as a minor, the directed graph D^+ made by adding an arc in parallel or subdividing an arc in D does not have M as a minor. □

Corollary 3.6: *Given a directed graph D that does not have an M-minor, any directed series parallel extension of D does not have an M-minor.*

Proof

A directed series parallel extension is obtained from D by adding arcs in parallel and by subdividing arcs. Therefore, by Lemma 3.5 any directed series parallel extension of D does not have an M-minor. □

Theorem 3.7: *A directed graph D with underlying graph G does not have an M-minor if and only if every block of G is a directed series parallel extension of either a wheel oriented as described in Lemma 3.2 or an arbitrary orientation of a graph as described in Lemma 3.4.*

Proof

First, to prove the if part of the Theorem, note that the underlying graph of M is 2-connected. Thus, M is either a minor of an orientation of a single block of G or M is not a minor of D. Therefore, assume that G is a block. Lemmas 3.2 and 3.4, and Corollary 3.6 show that if D is a directed series parallel extension of either a wheel oriented as described in Lemma 3.2 or an arbitrary orientation of a graph as described in Lemma 3.4, then D does not have M as a minor.

To prove the other half of the Theorem, suppose that D is a directed graph (with 2-connected underlying graph) that does not

have an M-minor. Let D^- be a directed graph so that D is a directed series-parallel extension of D^- and so that D is not a directed series-parallel extension of any proper minor of D^-. By Corollary 3.6, D has an M-minor if and only if D^- has an M-minor. Now suppose that G^-, the underlying graph of D^-, has W_n , $n \geq 3$ as a minor. If G^- is not a wheel then G^- has a 2-connected minor obtained from a wheel by the addition of an edge or by splitting a vertex. If G^- has a wheel plus an edge as a minor, then either G^- has a 3-connected graph besides a wheel as a minor, and by Theorem 3.3 every orientation of G^- has an M-minor, or D^- has a minor obtained from an orientation of a wheel by adding an arc between two adjacent nodes in the opposite direction of the already existent arc, and so, by the uniqueness of the orientation given in Lemma 3.2, D^- has an M-minor.

If G^- has a 2-connected minor that is not a wheel, obtained from a wheel by splitting a vertex, then either G^- has a 3-connected minor that is not a wheel, and so D has an M-minor, or G^- has a wheel with an edge, e subdivided as a minor. Let x and y be the vertices of G^- incident with e and let z be the vertex added by the subdivision of e. If $G\backslash\{x,y\}$ has a component C containing the vertex z, but containing no other vertex of G^-, then, since the vertex z is a vertex of G^-, $C\bigcup\{x,y\}$ is not a directed series parallel extension of an $\{x,y\}$ arc, and so D^- has minors obtained by orienting the edge e in both possible directions. Therefore, by the uniqueness shown in Lemma 3.2, D^- has an M-minor.

If $G\backslash\{x,y\}$ does not have a component containing z and no other vertex of G^- then G has a minor obtained by adding an edge to the

wheel subdivide an edge incident with z, but not incident with x or y. The graph obtained this way is a 3-connected graph which is not a wheel, and so by Theorem 3.3 every orientation of G would have an M-minor.

Therefore, if G^- has W_n, $n \geq 3$ as a minor, and G^- is not a wheel, then D has an M-minor. Therefore, if G has W_n , $n \geq 3$ as a minor and D does not have M as a minor, D is a series parallel extension of a wheel oriented as described in Lemma 3.2.

Now suppose that G^- does not contain W_3 as a minor. Which implies that G^- is a series parallel graph. Choose x and y so that G^- is a series parallel extension of an $x - y$ edge. Choose $\{x, y\}$ so that the number of components of $G^- \backslash \{x, y\}$ is as large as possible.

Suppose that G^- is not a number of $x - y$ paths. Let $C_1, C_2, \ldots, C_n$ be the connected components of $G^- \backslash \{x, y\}$ labeled so that $C_1 \cup \{x, y\}$ is not a path. Note that if $n = 1$ then either G^- is 3-connected, or G^- is a path, which implies that $C_1 \cup \{x, y\}$ contains a circuit. Since C_1 is connected, if this circuit contains both x and y, then $K_4 = W_3$ is a minor of G^-. Therefore, at least one edge connects a circuit in $C_1 \cup \{x, y\}$ to either x or y. If $n \geq 3$, then the components C_2 and C_3 can be used to create a 2-circuit rooted to x and y, which, along with the 2-circuit that can be created from the $C_1 \cup \{x, y\}$ and the extra edge that must exist in $C_1 \cup \{x, y\}$, will create the minor M. Therefore, if $C_1 \cup \{x, y\}$ is not a path then $n = 2$.

In the case $C_1 \cup \{x, y\}$ is not a path and $n = 2$, let B be a 2-connected subgraph of $C_1 \cup \{x, y\}$. There are vertices a and b that are the only attachments of the block B to the rest of $C_1 \cup \{x, y\}$.

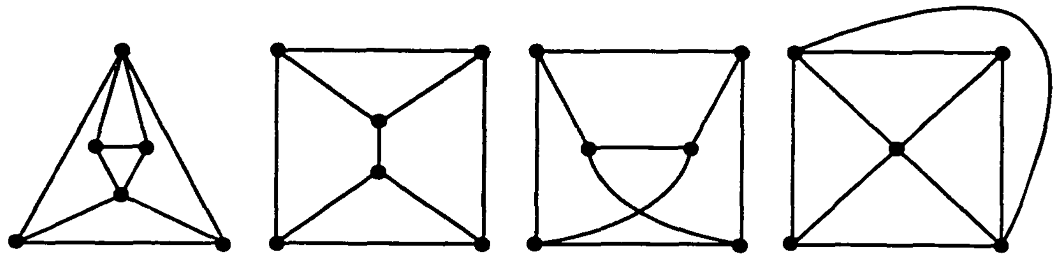

Figure 12: Minimal graphs which must have M as a minor

Consider the 2-cut {a,b}. Since $C_1 \cup \{x, y\}$ is a series parallel extension of an $x-y$ path, and since $G^{-} \backslash \{x, y\}$ has only two components, G^{-} is a series parallel extension of some number of $a-b$ paths. Moreover, $G^{-} \backslash \{a, b\}$ has at least three components, the one containing C_2, and two from the circuit contained in the block a and b attach to the rest of G^{-}. (If the block B were contained in a single component of $G^{-} \backslash \{a, b\}$, then K_4 would be a minor of G^{-}, contradicting the choice of the original x and y.) Therefore, a directed graph D with underlying graph G does not have M as a minor if and only if every block of G is a directed series parallel extension of either a wheel oriented as described in Lemma 3.2 or an arbitrary orientation of a graph as described in Lemma 3.4. □

An immediate corollary of Theorem 3.7 is a description, in terms of minors, of the graphs for which any orientation has an M-minor.

Corollary 3.8: *Every orientation of a graph G has M as a minor if and only if G has one of the graphs shown in Figure 12 as a minor.*

<u>Proof</u>

The four graphs shown in Figure 12 are, on the far left, a non-series parallel extension of K_4 followed by a trio of 3-connected graphs that are not wheels. Theorem 3.7 shows that all four must have an M-minor no matter how they are oriented.

By Tutte's characterization of 3-connected graphs, every 3-connected graph that is not a wheel contains one of the last three graphs shown in Figure 12 as a minor. Thus, if G does not have any of the last three graphs shown in Figure 12 as a minor, then G does not contain any 3-connected minors except, possibly, for wheels. If G contains a wheel as a minor and G is not a series parallel extension of a wheel then G has the first graph shown in Figure 12 as a minor. Thus if G has none of the graphs given in Figure 12 as a minor, then either G does not have a wheel as a minor, or G is a series parallel extension of a wheel (perhaps being a wheel itself). Therefore, G is either a series parallel graph or a series parallel extension of a wheel, and so can be oriented to not have an M-minor, or G has one of the given graphs as a minor. $\square$

4 Orienting Graphs to Avoid $\vec{K}_3$, $\vec{S}_3$, or $\vec{P}_3$

This chapter considers three digraphs, $\vec{K}_3$, $\vec{S}_3$, and $\vec{P}_3$ (shown in Figure 1). Each is treated individually in sections 4.1, 4.2 and 4.3, respectively. Due to the increased complexity of the directed graphs being considered, added restrictions are necessary. In all three cases the majority of the work done considers 4-connected graphs, as these provide enough structure to achieve interesting results.

In Section 4.1 the primary results are Observation 4.1.2, Theorem 4.1.7, and Theorem 4.1.21. Observation 4.1.2 gives an orientation of K_5 without a $\vec{K}_3$-minor, and shows that this orientation is unique. Observation 4.1.2 is extended in Theorem 4.1.7, where it is shown that every double wheel has a unique orientation without a $\vec{K}_3$-minor. Finally, Theorem 4.1.21 shows that the double wheels, oriented as given in Theorem 4.1.7, are the maximal 4-connected graphs without a $\vec{K}_3$-minor.

In Section 4.2 the primary results are given in Lemma 4.2.1, Theorem 4.2.4, and Corollary 4.2.11. Lemma 4.2.1 demonstrates a class of 3-connected graphs which can be oriented to not have an $\vec{S}_3$-minor. Theorem 4.2.4 again considers the 3-connected case, and shows that only a very restricted class of large 3-connected graphs can be oriented to not have a $\vec{S}_3$-minor. Finally, Corollary 4.2.11 completes the 4-connected case by listing the only two 4-connected graphs with an orientation without an $\vec{S}_3$-minor.

The primary results for Section 4.3 are presented in Theorem 4.3.1 and Corollary 4.3.9. Theorem 4.3.1 provides restrictions on

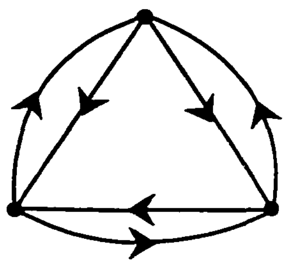

Figure 13: $\vec{K}_3$

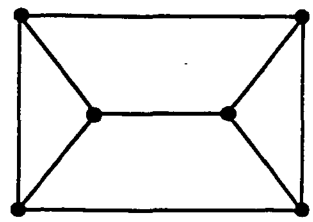

Figure 14: $(K_5\backslash e)^*$

3-connected graphs with an orientation without a $\vec{P}_3$-minor. While Corollary 4.3.9 lists 4-connected graphs that can be oriented to not have a $\vec{P}_3$-minor.

4.1 Orienting Graphs to Avoid a $\vec{K}_3$-minor

First the digraph $\vec{K}_3$ is considered (shown in Figure 13).

The goal of this section is to show that double wheels are the maximal 4-connected graphs that may be oriented to not have a $\vec{K}_3$-minor. The first step toward this goal is as follows:

Lemma 4.1.1: *Every orientation of $L(K_{3,3})$ or $L((K_5\backslash e)^*)$ has $\vec{K}_3$ as a minor.*

Proof

Note that because both $L(K_{3,3})$ and $L((K_5\backslash e)^*)$ have the graph S, shown in Figure 16, as a minor, it is enough to show that every orientation of the graph S has a $\vec{K}_3$-minor.

Let D be an orientation of S and suppose that D is a completion of the orientation of either of the partially oriented graphs shown in

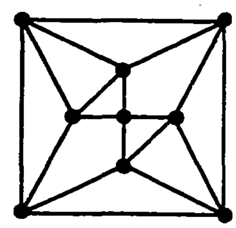

Figure 15: $L((K_5\backslash e)^*)$

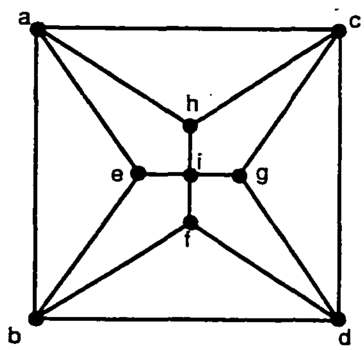

Figure 16: The graph S. A minor of both $L(K_{3,3})$ and $L((K_5\backslash e)^*)$

Figure 17. In the first case, no matter which edge of the triangle (a, b, e) can be contracted to obtain $\vec{P}_1$, the resulting graph has $\vec{K}_3$ as a minor.

In the second case shown, to avoid either an $a-c$ rooted 2-circuit or a $b-d$ rooted 2-circuit (either of which clearly makes it possible to obtain a $\vec{K}_3$-minor), it is necessary to orient so that the contraction

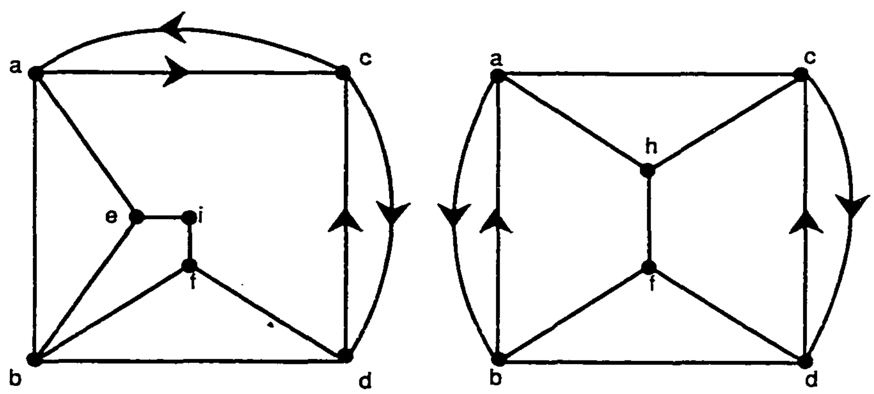

Figure 17: Two partial orientations of minors of S

of the edges (b, d) and (a, c) leaves a partially oriented graph, which obviously has a $\vec{K}_3$-minor.

To avoid both of the two partially oriented graphs already considered, it is necessary that three of the triangles (a, b, e), (a, c, h), (b, d, f), and (c, d, g) be oriented as directed series parallel extensions of the arc on the edges (a, b), (a, c), (b, d), and (c, d), respectively. Without loss of generality, assume that the triangles (a, b, e), (a, c, h), and (b, d, f) are so oriented. Consider that if $\langle a, b\rangle, \langle c, a\rangle \in A(D)$, then contracting (h, i), (e, i), (f, i), (b, d), and (c, d) results in $\vec{K}_3$. Similarly, if $\langle a, b\rangle, \langle b, d\rangle \in A(D)$, then $\vec{K}_3$ is a minor of the orientation. Therefore, it may be assumed that $\langle a, c\rangle, \langle d, b\rangle \in A(d)$, but in this case contraction of (a, b), (c, d), (f, i), and (h, i) demonstrates the existence of a $\vec{K}_3$-minor.

Therefore, every orientation of S has a $\vec{K}_3$-minor, and every orientation of $L(K_{3,3})$ or $L((K_5 \backslash e)^*)$ has a $\vec{K}_3$-minor. □

K_5 is the next graph considered.

Observation 4.1.2: *A unique orientation (up to reversal of all arcs) of K_5 exists that does not have $\vec{K}_3$ as a minor.*

Proof

First note that a unique orientation of K_5 exists with no node having 2 out-arcs and 2 in-arcs. The sum of all the out-degrees of the vertices of an orientation of K_5 is 10. In addition, at most one vertex can have out-degree 4 and at most one can have out-degree 0. If there is a vertex of out-degree 4, then there must be one of out-degree 3 as well (since $4 + 1 + 1 + 1 + 1 = 8$) and therefore, the out-degrees must be 4,3,1,1,1. If there is no vertex of out-degree 4 then the only possible out-degrees are 3,3,3,1,0, which are exactly

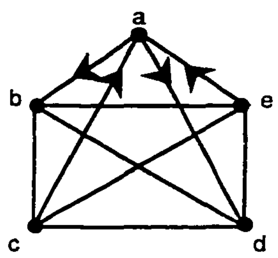

Figure 18: A labeling of K_5

the in-degrees in the first case, and as such are the out-degrees of the orientation with all arcs reversed.

To show that the orientation with no node of in-degree and out-degree two does not have a $\vec{K}_3$-minor notice that every vertex of $\vec{K}_3$ has in-degree and out-degree 2. Since $\vec{K}_3$ has three vertices, one of the vertices of K_5 is not joined with any others by contraction in decreasing the number of vertices to 3, but this lone vertex does not have in-degree 2 and out-degree 2. Therefore, this orientation of K_5 does not have $\vec{K}_3$ as a minor.

To show the uniqueness of this orientation it is enough to show that any orientation of K_5 with a vertex of out-degree 2 has $\vec{K}_3$ as a minor. Therefore, let D be an orientation of K_5 with a vertex of out-degree 2, and label it as indicated in Figure 18.

Contracting (b, c) and (d, e) shows that unless both $\langle b, e\rangle, \langle c, d\rangle \in A(D)$ or $\langle e, b\rangle, \langle d, c\rangle \in A(D)$ then $\vec{K}_3$ is a minor of D. However, contracting (b, d) and (c, e) shows that unless both $\langle b, e\rangle, \langle d, c\rangle \in A(D)$ or $\langle e, b\rangle, \langle c, d\rangle \in A(D)$ then $\vec{K}_3$ is a minor of D. Clearly both cases cannot be true, and therefore D must have $\vec{K}_3$ as a minor. Therefore, there is a unique orientation of K_5 that does not have $\vec{K}_3$ as a minor. □

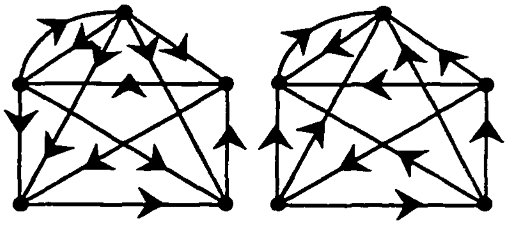

Figure 19: Two directed graphs not having $\vec{K}_3$ as a minor

Some immediate corollaries of this follow.

Corollary 4.1.3: *The directed graph D obtained by adding an arc to create a 2-circuit to an orientation of K_5 has $\vec{K}_3$ as a minor unless it is one of the directed graphs shown in Figure 19.* (Note: the two directed graphs shown in Figure 19 are the reversals of each other)

Proof

By Observation 4.1.2 there is a unique orientation (up to reversal) of K_5 that does not have $\vec{K}_3$ as a minor. Use the orientation with a vertex of out-degree 4. If any arc besides the one between the vertex of out-degree 4 and the vertex of out-degree 3 is reversed, then there would be a vertex of out-degree 2. Therefore, $\vec{K}_3$ is a minor of the directed graph D with a 2-circuit and underlying graph K_5 unless it is one of the two given. Additionally, the two orientations given do not have a $\vec{K}_3$-minor for the same reason as given in Observation 4.1.2; that is, after performing two contractions there will be at least one vertex of either in-degree 1 or out-degree 1. □

Corollary 4.1.4: *Every completion of the orientation of the partially oriented graph shown in Figure 20 has a $\vec{K}_3$-minor.*

Proof

The contraction of (a, b) or (b, c) creates a directed graph with underlying graph K_5 and a 2-circuit. By the uniqueness shown in

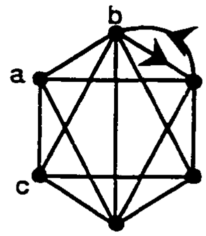

Figure 20: A graph which must have $\vec{K}_3$ as a minor

Corollary 4.1.3, if the contraction of (a, b) does not have $\vec{K}_3$ as a minor then the contraction of (b, c) must have $\vec{K}_3$ as a minor. □

Lemma 4.1.5: *Every orientation of $C^2_{2n+1}, n \geq 3$ has $\vec{K}_3$ as a minor.*

Proof

Note first that C^2_{2n+1} is a minor of C^2_{2n+3}. To see this, take C^2_{2n+3} and label the vertices 1 to $2n+3$ around the circuit in the obvious way. Contract edges (1,3) and (2,4). The resulting graph is C^2_{2n+1}. Therefore, it is enough to show that every orientation of C^2_7 has $\vec{K}_3$ as a minor.

Label the vertices of C^2_7 1 to 7 in the obvious way. Let D be an orientation of C^2_7. If none of the contractions of an edge $(i, i+2\text{mod}(7))$ creates an $i - i + 1\text{mod}(7)$ rooted 2-circuit, then for each of the triangles $(i, i+1, i+2), i \in \{1, 3, 5\}$, the contraction of either $(i, i+1)$ or $(i+1, i+2)$ creates an i-$i+2$ rooted 2-circuit. Therefore, $\vec{K}_3$ is a minor of this orientation.

If for some i the contraction $(i, i+2\text{mod}(7))$ creates an $i, i+1\text{mod}(7)$ rooted 2-circuit, then, by Corollary 4.1.4, $\vec{K}_3$ is a minor of D. Therefore, every orientation of C^2_7 has $\vec{K}_3$ as a minor, and every orientation of C^2_{2n+1}, $n \geq 3$, has $\vec{K}_3$ as a minor.□

Lemma 4.1.6: *Every orientation of C^2_{2n}, $n \geq 4$ has $\vec{K}_3$ as a minor.*

Proof

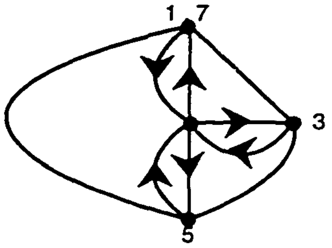

Figure 21:

Note first that C^2_{2n} is a minor of $C^2_{2(n+1)}$. Take $C^2_{2(n+1)}$ and number the vertices 1 to $2n+2$ around the circuit in the obvious way. Contract edges (1,3) and (2,4). The resulting graph is C^2_{2n}. Therefore, it is enough to show that every orientation of C^2_8 has $\vec{K}_3$ as a minor.

Label the vertices of C^2_8 1 to 8 in the obvious way. Let D be an orientation of C^2_8 and suppose that the contraction of edges (2,8) and (4,6) results in a 1-2 rooted 2-circuit and a 4-5 rooted 2-circuit, respectively. It is clear from the resulting graph that if there is a 2-4 rooted 2-circuit in the triangle (2,3,4) then $\vec{K}_3$ is a minor of D. If there is not a 2-4 rooted 2-circuit in the triangle (2,3,4), then the contraction of (2,4) must result in a partially oriented graph with the partially oriented graph shown in Figure 21 as a minor. For at least one of the edges in the circuit (17,3,5), contraction will result in $\vec{K}_3$. Therefore, at least one of the contractions of the edges (2,8) and (4,6) does not result in a 1-2 rooted 2-circuit or a 4-5 rooted 2-circuit, respectively.

Next suppose that neither of the contractions of the edges (2,8) and (4,6) results in a 1-2 rooted 2-circuit nor in a 4-5 rooted 2-circuit, respectively. In this case the partially oriented graph in Figure 22 is a minor of D. Clearly, if there is a 2-4 rooted 2-circuit or a 6-8

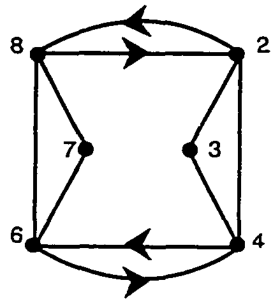

Figure 22:

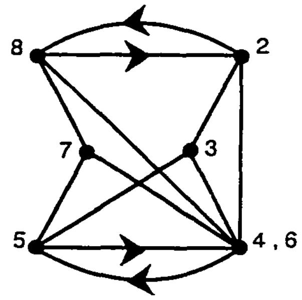

Figure 23: A partial orientation of C_8^2

rooted 2-circuit, then $\vec{K}_3$ is a minor of D, which implies that the contraction of (2,4) and (6,8) results in a 2-3 rooted 2-circuit and a 6-7 rooted 2-circuit, respectively. Thus, by relabeling the vertices of D, the previous case shows that $\vec{K}_3$ is a minor of D.

The only case that remains is the one in which D has a completion of the partially oriented graph shown in Figure 28 as a minor. Considering the triangles (2,3,4) and (6,7,8) and the same argument as for the previous two cases shows that $\vec{K}_3$ is a minor of D. Therefore, every orientation of C_8^2 has $\vec{K}_3$ as a minor, and every orientation of C_{2n}^2, $n \geq 4$ has $\vec{K}_3$ as a minor. □

Now a class of graphs that can be oriented to not have a $\vec{K}_3$-minor is demonstrated.

Theorem 4.1.7: *The double wheels, W_n^2, have a unique orientation*

(up to reversal of all arcs) without $\vec{K}_3$ as a minor.

Proof

The first claim is that the orientation with every spoke oriented towards the rim, the rim oriented in a directed circuit, and the axle oriented arbitrarily does not have $\vec{K}_3$ as a minor. If $n = 3$, then $W_3^2 = K_5$, and by Observation 4.1.2 the orientation as given is exactly the unique orientation of K_5 without $\vec{K}_3$ as a minor. Using induction, suppose there exists $k \geq 3$ such that for all $n \leq k$ the orientation of W_n^2 given does not have $\vec{K}_3$ as a minor, and let $n = k + 1$.

By the inductive hypothesis, the contraction of any edge of the rim results in an oriented graph without $\vec{K}_3$ as a minor. Only two possibilities remain to be considered, the contraction of the axle or the contraction of a spoke.

First consider the contraction of a spoke. Assuming the axle is oriented appropriately, the graph in Figure 24 is obtained by contracting a spoke (and clearly the other orientation of the axle results in a subdigraph of this one). In this graph there is a vertex of in-degree 2 and out-degree 1; therefore, if $\vec{K}_3$ is a minor of this graph it is necessary to contract one of the arcs incident with this vertex. However, because the contraction of all these arcs results in the same digraph, the contraction of a rim arc can be assumed, and therefore, by the inductive hypothesis, $\vec{K}_3$ is not a minor of this digraph. Therefore, W_n^2 contract a spoke does not have $\vec{K}_3$ as a minor.

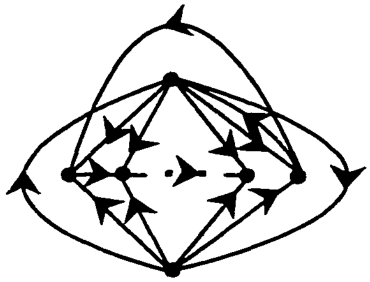

Figure 24: W_n^2 contract a spoke

The remaining case is contracting the axle, but because that leaves a graph with more than three vertices, some additional contraction must be used. Therefore, contracting the axle will not produce a digraph with $\vec{K}_3$ as a minor, and thus an orientation of W_n^2 exists without $\vec{K}_3$ as a minor.

To show the uniqueness of this orientation consider the two cases $n = 3$ and $n = 4$. The case $n = 3$ is the graph K_5 and the uniqueness of the orientation is shown in Observation 4.1.2. If $n \geq 4$ and W_n^2 is oriented in some manner besides that given, then the orientation of W_n^2 has an orientation of W_4^2 oriented in a manner different from that given. Either a contraction of a rim arc will result in K_5 with a 2-circuit, or there is a spoke that can be contracted to result in two 2-circuits between a hub and the rim, or one hub is a source and the other a sink.

In the first case, Corollary 4.1.3 shows a unique orientation without a $\vec{K}_3$-minor. There is a second rim arc, however, whose contraction will also result in K_5 with a 2-circuit. W_4^2 cannot be oriented so that both these contractions result in orientations without a $\vec{K}_3$-minor.

In the second case, if a rim arc does not exist that can be contracted to create a directed 2-circuit, then each hub is either a source

or sink and the rim is not oriented as a circuit. This implies that there is a spoke which when contracted results in two directed 2-circuits. Every completion of the orientation of such a partially oriented graph clearly must have a $\vec{K}_3$-minor.

The final case is when one hub is a source, the other a sink, and the rim is a directed circuit. In this case, contracting a rim arc results in an orientation of K_5, which is shown in Observation 4.1.2 to have a $\vec{K}_3$-minor. Therefore, there is a unique orientation of W_n^2 without a $\vec{K}_3$-minor. □

Corollary 4.1.8: *For $n \geq 5$ there is a unique orientation of $W_n^2 \backslash a$, where a is the axle, without $\vec{K}_3$ as a minor.*

Proof

The existence of such an orientation is immediate from Theorem 4.1.7, so it is enough to consider the question of uniqueness. It is enough to consider the case $n = 5$. Let $G = W_5^2$ and let D be an orientation of G. Suppose one hub is neither a source nor a sink. Label the hubs x and y, and label the rim 1 to 5 in the obvious way such that the contraction of (1,2) creates an $x - 1$ rooted 2-circuit. If this contraction results in a $y - 1$ rooted 2-circuit as well, then $\vec{K}_3$ is clearly a minor of the orientation. If any of the contractions (2,3), (1,5), or $(y, 3)$ create a y-1 rooted 2-circuit, then $\vec{K}_3$ is clearly a minor of the orientation.

Suppose $\langle y, 1\rangle \in A(D)$. The above analysis shows that it can then be assumed that $\langle y, 2\rangle, \langle y, 5\rangle, \langle 3, 2\rangle \in A(D)$ The contraction of $\langle y, 12\rangle$ shows that it can be assumed that $\langle 1, 5\rangle \in A(D)$. Now the contraction of $\langle y, 5\rangle$, however, clearly must have $\vec{K}_3$ as a minor.

Therefore, if either hub is neither a source nor a sink then $\vec{K}_3$ is a minor of D.

If x is a source (or sink) and the rim is not an oriented circuit, then there is a spoke which, when contracted, produces two directed 2-circuits between x and the rim. This clearly has $\vec{K}_3$ as a minor.

In the final case, x is a source, y is a sink, and the rim is a directed circuit. Assuming the rim is oriented 5 to 4 to 3 to 2 to 1 to 5, contracting $\langle x, 1\rangle, \langle 3, y\rangle$ results in a digraph clearly having a $\vec{K}_3$-minor.

Therefore, there is a unique orientation of $W_n^2 \backslash a$, $n \geq 5$ without $\vec{K}_3$ as a minor. $\square$

Using the theorem from [7], which states that the only uncontractible 4-connected graphs are C_n^2 for $n \geq 5$ and the line graphs of the cubic cyclically 4-connected graphs, the promised results on 4-connected graphs can be obtained. First, however, several preliminary results regarding cubic cyclically 4-connected graphs and line graphs need to be stated.

Lemma 4.1.9: *A cubic cyclically 4-connected graph is 3-connected.*

Proof

Let G be a cubic cyclically 4-connected graph, and suppose on the contrary that $\{x, y\} \in V(G)$ is a 2-vertex-cut of G. Since G is cubic, there are edges e_1, e_2, e_3, incident with $\{x, y\}$ such that $G \backslash \{e_1, e_2, e_3\}$ has at least two components. Additionally it may be assumed that e_1, e_2, e_3 are all edges joining different components of $G \backslash \{e_1, e_2, e_3\}$. (If not true, then consider only the edges actually necessary to separate G into the different components.) Since G is

cyclically 4-connected at least one of these components must not have a circuit, and so must have at least two vertices of degree 1. However, since none of the removed edges connected two vertices in the same component, at least four edges must be removed. Because this is a contradiction, a cubic cyclically 4-connected graph is 3-connected. □

Lemma 4.1.10: *Given graphs H and G. If H is a topological minor of G, then $L(H)$ is a minor of $L(G)$.*

Proof

It is enough to show that if e is an edge of G, then $L(G\backslash e)$ is a minor of $L(G)$, and if v is a vertex of degree 2 incident with the edge f, then $L(G/f)$ is a minor of $L(G)$.

First let $e \in E(G)$, and let x be the vertex of $L(G)$ associated with e. Then $L(G\backslash e) = L(G)\backslash x$.

Consider the second case, and let v be a vertex of degree 2 incident with the edge f in G, and let d be the other edge adjacent with v in G. In this case, let β be the edge in $L(G)$ joining the vertices associated with f and d. Then $L(G/f) = L(G)/\beta$ □

Lemma 4.1.11: *Every 3-connected graph has K_4 as a topological minor.*

Proof

Let G be a 3-connected graph, and let v be a vertex of G. Let C be a circuit in $G\backslash v$. Since G is 3-connected there are paths P_1, P_2, and P_3, vertex disjoint except for v, that connect v to C. Delete every edge and vertex that is not part of C or one of the three paths, what remains is a subdivision of K_4

Therefore, K_4 is a topological minor of every 3-connected graph.

□

Two obvious corollaries that follow from Lemma 4.1.11 are useful.

Corollary 4.1.12: *Every cubic cyclic 4-connected graph has K_4 as a topological minor.*

Corollary 4.1.13: *The line graph of a cubic cyclically 4-connected graph has $L(K_4)$ as a minor.*

Lemma 4.1.14: *The only cubic 3-connected graph that does not have either $K_{3,3}$ or $(K_5\backslash e)^*$ as a topological minor is K_4.*

Proof

Let G be a cubic 3-connected graph not equal to K_4. G must have at least six vertices. Let v be a vertex of G and consider the graph $G\backslash v$. $G\backslash v$ is a 2-connected graph. Exactly three vertices of $G\backslash v$ are of degree 2, and every vertex of $G\backslash v$ not adjacent to v in G is of degree 3.

Suppose that $(K_5\backslash e)^*$ is not a topological minor of G. This implies that there is not a circuit in $G\backslash v$ containing the neighbors of v, with a path that separates one of the neighbors of v from the other two neighbors of v along the circuit. If a circuit that contained all the neighbors of v existed however, then such a bridge must exist, or there would be a 2-cut in G. Therefore, the neighbors of v are not contained in a circuit in $G\backslash v$. However, the same argument used in Lemma 4.1.11 shows that $K_{3,3}$ is a topological minor of G.

Therefore, either $K_{3,3}$ or $(K_5\backslash e)^*$ is a topological minor of any cubic 3-connected graph not equal to K_4. □

Combining the results from Lemma 4.1.5, Lemma 4.1.6, and Lemma 4.1.14 gives the following observation.

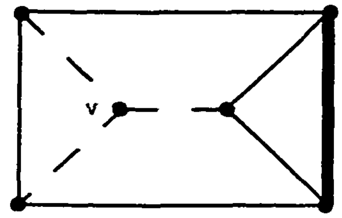

Figure 25: A graph with $(K_5\backslash e)^*$ as a minor

Observation 4.1.15: *The only uncontractible 4-connected graphs that have an orientation without a $\vec{K}_3$-minor are $L(K_4), C_6^2$, and C_5^2.*

$L(K_4) = C_6^2 = W_4^2\backslash a$, where a is the axle, and C_5^2 is better known as $K_5 = W_3^2$ so restating Observation 4.1.15 gives the following.

Observation 4.1.16: *The only uncontractible 4-connected graphs that have an orientation avoiding $\vec{K}_3$ are W_3^2 and $W_4^2\backslash a$, where a is the axle.*

Therefore, every 4-connected graph that can be oriented to not have $\vec{K}_3$ as a minor can be created by adding edges and splitting vertices starting from W_3^2 and $W_4^2\backslash a$, where a is the axle.

Starting with the smaller of these two graphs, it is clear that no edges can be added to K_5; therefore, the following observation should be noted.

Observation 4.1.17: *The only 4-connected graph obtained by splitting a vertex from K_5 is W_4^2.*

<u>Proof</u>

Split a vertex of K_5. The resulting graph has 6 edges not adjacent to the splitting vertices, and seven edges adjacent to the splitting vertices. The total of 13 edges means that this splitting of K_5 can be obtained by deleting two edges from K_6. If two edges of K_6 adjacent to a single vertex were deleted the result would be 3-connected, since

there would be a vertex of degree three. Therefore, a 4-connected graph obtained by splitting a vertex of K_5 equals K_6 delete two non-incident edges. ${W_4}^2$ has 13 edges, including the axle, and is 4-connected, and so must also be obtained from K_6 by deleting two edges that are not incident with each other. Therefore, the only 4-connected graph that can be obtained from K_5 by splitting a vertex is W_4^2. □

Any graph obtained by splitting a vertex of W_4^2 has a graph obtained by splitting a vertex of $W_4^2\backslash a$, where a is the axle. Moving on to consider 4-connected graphs with a $W_4^2\backslash a$-minor, first is the issue of adding edges.

Observation 4.1.18: *Every orientation of $K_6\backslash e$ has $\vec{K}_3$ as a minor.*

Proof

Label the vertices of degree 4 in $K_6\backslash e$ a and b. Contracting any edge incident with a or b in $K_6\backslash e$ results in K_5 which, by Observation 4.1.2, can only be oriented to not have a $\vec{K}_3$-minor if there is no vertex of out-degree and in-degree at least 2. Since $(K_6\backslash e)\backslash\{a, b\} = K_4$, there must be at least one vertex of in-degree at least 2 and one of out-degree at least 2 in $(K_6\backslash e)\backslash\{a, b\}$. If a has out-degree at least 3, then contracting the edge joining a to the vertex of in-degree at least 2 in $(K_6\backslash e)\backslash\{a, b\}$ results in K_5 oriented with a vertex of in-degree and out-degree at least 2.

If a has in-degree at least 3, then by a similar argument $\vec{K}_3$ is a minor of this orientation. Similarly, if b has in-degree or out-degree not equal to 2, then the orientation must have a $\vec{K}_3$-minor. However, since both a and b have in-degree and out-degree 2, contracting any

edge incident with a leaves K_5 with a vertex (b) of in-degree and out-degree 2. Therefore, every orientation of $K_6 \backslash e$ has $\vec{K}_3$ as a minor. □

Corollary 4.1.19: *Let $G = W_5^2 + e$. Every orientation of G has $\vec{K}_3$ as a minor.*

Proof

Let $G = W_5^2 + e$. Without loss of generality, label the vertices of the rim 1 to 5 and suppose the added edge joins vertices 1 and 3. The contraction of either edge (3,4) or (4,5) results in W_4^2, and so, by Theorem 4.1.7, has a unique orientation without $\vec{K}_3$ as a minor. The two different contractions, however, cannot both be accommodated by a single orientation of G. Therefore, every orientation of G has $\vec{K}_3$ as a minor.□

Now consider the splitting of a vertex of $W_n^2 \backslash a$, where a is the axle.

Lemma 4.1.20: *A 4-connected graph, G, obtained by splitting a vertex of $W_n^2 \backslash a$, where a is the axle, $n \geq 4$, has an orientation without $\vec{K}_3$ as a minor if and only if G is a double wheel or a double wheel minus the axle.*

Proof

First note that every vertex of $W_4^2 \backslash a$, where a is the axle, is interchangeable. Therefore, every subdivision of a vertex in W_4^2 results in W_5^2 , $W_5^2 \backslash a$ plus an edge besides the axle, or a graph with a C_7^2-minor. Therefore, subdividing a vertex of $W_4^2 \backslash a$ results in either W_5^2 or in a graph with a $\vec{K}_3$-minor for every possible orientation.

Suppose that G is a 4-connected graph obtained by splitting a vertex of $W_n^2 \backslash a$, $n \geq 5$. A graph obtained by splitting a vertex on

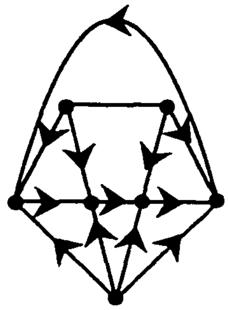

Figure 26: A minor of W_n^2 with the hub subdivided

the rim of W_n^2 has a graph obtained by splitting a vertex of W_4^2; therefore, it is enough to consider splitting the hub of W_n^2.

Any subdivision of a hub has the digraph shown in Figure 26, with the orientation uniquely chosen as shown in Corollary 4.1.8.

Therefore, a 4-connected graph, G, obtained by splitting a vertex of $W_n^2\backslash a$, $n \geq 4$, has an orientation without $\vec{K}_3$ as a minor if and only if G is a double wheel or a double wheel minus the axle. □

The above leads to the final result of this section.

Theorem 4.1.21: *A 4-connected graph G has an orientation avoiding $\vec{K}_3$ as a minor if and only if G is a minor of a double wheel.*

Proof

Theorem 4.1.7 shows that if G is a minor of a double wheel, an orientation of G without a $\vec{K}_3$-minor exists. If G is uncontractible, then, by Observation 4.1.10, G is a minor of a double wheel or has a $\vec{K}_3$-minor. Therefore, G must be obtained by subdividing vertices and adding edges to K_5, or W_4^2. Such a graph is a double wheel, or has $W_5^2\backslash a$ plus an edge besides the axle, $K_6\backslash e$, or W_n^2 $n \geq 4$ subdivide a vertex as a minor. All of these have been shown to have a $\vec{K}_3$-minor. Therefore, a 4-connected graph G has an orientation avoiding $\vec{K}_3$ as a minor if and only if G is a minor of a double wheel.□

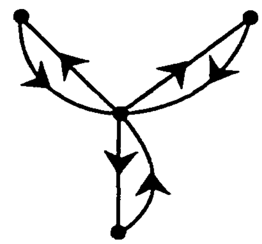

Figure 27: $\vec{S}_3$

4.2 Orienting Graphs to Avoid an $\vec{S}_3$-minor

The second digraph considered is $\vec{S}_3$, shown in Figure 27. Considering 3-connected graphs without an $\vec{S}_3$-minor gives the following.

Lemma 4.2.1: *Every 3-connected graph with no W_5-minor has an orientation without $\vec{S}_3$ as a minor.*

Proof

Oxley provides a complete list of 3-connected graphs without a W_5-minor [9]. Every 3-connected graph without a W_5-minor is a minor of either A_k , H_6 , Q_3 , $K_{2,2,2}$, or H_7.

Claim 4.2.1a *The orientation of A_k with the vertices in the k-set oriented as sinks does not have $\vec{S}_3$ as a minor.*

Proof

$\vec{S}_3$ has four vertices, so if this orientation of A_k were to have $\vec{S}_3$ as a minor, one of the vertices oriented as a sink would not be identified by contraction with any other vertex of A_k. However, every vertex of S_3 has out-degree at least 1; therefore, this orientation of A_k does not have $\vec{S}_3$ as a minor.□

Claim 4.2.1b *The orientation of H_6 given in Figure 28 does not have $\vec{S}_3$ as a minor.*

Proof

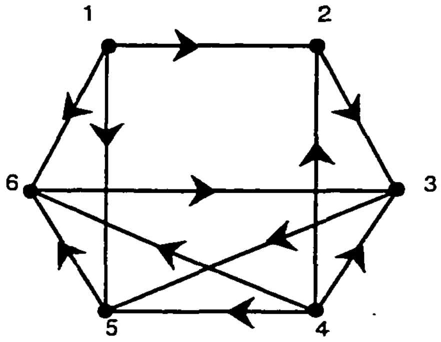

Figure 28: An orientation of H_6 without $\vec{S}_3$ as a minor

Note that the vertex ab made by contracting an arc connecting a and b in a digraph obeys the following inequalities:

$id(ab) \leq id(a) + id(b) - 1$ and $od(ab) \leq od(a) + od(b) - 1$.

Note also that in the orientation of H_6 given, both the vertex 1 and the vertex 4 are sources, and $\vec{S}_3$ has a vertex with in-degree 3 and out-degree 3. Since there are no sources in $\vec{S}_3$, some other vertex must be identified with 4, and some vertex with 1. Moreover, since $\vec{S}_3$ has four vertices, only these two contractions can occur. If an arc incident with either 1 or 4 is contracted the resulting vertex has in-degree less than or equal to two, since no vertex has in-degree greater than three. Therefore, if 1 and 4 are identified with two separate vertices, there is not a vertex of in-degree and out-degree greater than or equal to three. If 1, 4, and another vertex are joined by two contractions the only vertex of out-degree three is the one made by identifying 1,4, and the other vertex, but this vertex has in-degree less than or equal to two. Therefore, this orientation of H_6 does not have $\vec{S}_3$ as a minor.□

Claim 4.2.1c *The orientation of Q_3 given in Figure 29 does not have $\vec{S}_3$ as a minor.*

<u>Proof</u>

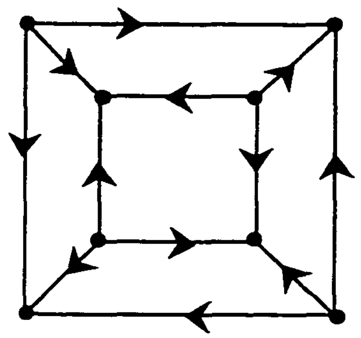

Figure 29: An orientation of Q_3 without $\vec{S}_3$ as a minor

Since Q_3 is cubic, at least four vertices must be joined by contraction to create a vertex of degree six. Moreover, since every vertex is either a sink or a source in the given orientation, every vertex must be identified with another by contraction. Therefore, to create $\vec{S}_3$ from this orientation of Q_3 it would be necessary to identify four vertices to create the center of the star which has total degree six, and to identify three pairs of vertices to create the three vertices of in-degree and out-degree one. Thus, to end with the four vertices of $\vec{S}_3$ it would be necessary to begin with at least ten vertices. Q_3 has only eight vertices. Therefore, the given orientation of Q_3 does not have $\vec{S}_3$ as a minor.□

Claim 4.2.1d *The orientation of $K_{2,2,2}$ given in Figure 30 does not have $\vec{S}_3$ as a minor.*

Proof

Since the vertices 1 and 6 are sources they must be identified by contraction to another vertex, and since $K_{2,2,2}$ has only six vertices these are the only two contractions that can occur. The identification of 1 with any vertex adjacent to 1 results in a vertex of out-degree four and in-degree two. The same is true for vertex 6. Therefore, if 1 and 6 are identified with different vertices there is no

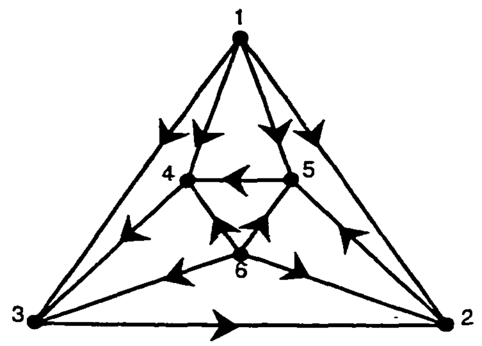

Figure 30: An orientation of $K_{2,2,2}$ without $\vec{S}_3$ as a minor

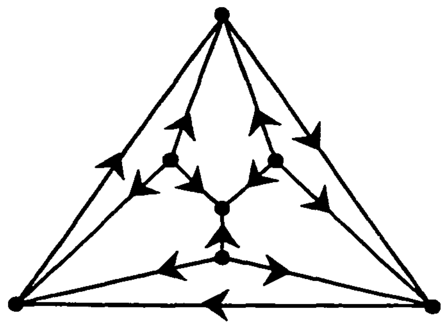

Figure 31: An orientation of H_7 without $\vec{S}_3$ as a minor

vertex of in-degree and out-degree at least three. If 1 , 6, and another vertex are identified, the resulting vertex has in-degree at most two. Therefore, $\vec{S}_3$ is not a minor of this orientation of $K_{2,2,2}$.□

Claim 4.2.1e *The orientation of H_7 given in Figure 31 does not have $\vec{S}_3$ as a minor.*

<u>Proof</u>

This orientation has three sources, and since $\vec{S}_3$ has no sources each of these must be identified with some other (non-source) vertex. Moreover, since H_7 has seven vertices, exactly three contractions can occur in attempting to create $\vec{S}_3$. Therefore, every contraction must use an arc incident with one of the three sources.

Furthermore, $\vec{S}_3$ has a vertex of in-degree and out-degree three. Any of the three sources identified with any of its neighbors has

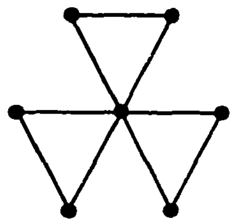

Figure 32: A subgraph of W_6

in-degree two. Identifying an additional source will not increase the in-degree. Therefore, one source must be identified with at least two non-source vertices, but then at least four contractions will be needed. Therefore, this orientation of H_7 does not have $\vec{S}_3$ as a minor.□

Thus, every 3-connected graph that does not have a W_5-minor has an orientation without an $\vec{S}_3$-minor.□

Avoiding a small wheel guarantees an orientation without an $\vec{S}_3$-minor. Every orientation of any graph with a moderately large wheel as a minor must have an $\vec{S}_3$-minor, as is shown in the following Observation.

Observation 4.2.2: *Every orientation of W_6 has a $\vec{S}_3$-minor.*

Proof

The graph shown in Figure 32 is a subgraph of W_6, which obviously has an $\vec{S}_3$-minor.□

Corollary 4.2.3: *Every orientation of a graph with a W_6-minor has an $\vec{S}_3$-minor.*

Theorem 4.2.4: *There exists k, such that any 3-connected graph with at least k vertices and an orientation without $\vec{S}_3$ as a minor is itself a minor of A_n for some n.*

Proof

Given a number n, there exists a number m such that any 3-connected graph with at least m vertices has either W_n or $K_{3,n}$ as a minor. Since every orientation of W_6 has $\vec{S}_3$ as a minor it is only necessary to consider the case of graphs with $K_{3,n}$ as a minor. Specifically it is enough to show that a 3-connected graph G, with a $K_{3,n}$-minor but not a $K_{3,n+1}$-minor, is either a minor of A_n or every orientation of G has $\vec{S}_3$ as a minor.

Let $n \geq 5$, and let G be a graph with $K_{3,n}$ but not $K_{3,n+1}$ as a minor. Suppose G is not a minor of A_n. G then has a minor H such that H is either $K_{3,n}$ plus an edge connecting two vertices of the size n vertex partition, or H is obtained from $K_{3,n}$, $A_n\backslash\{x, y\}$, $A_n\backslash\{x\}$, or from A_n by subdividing a vertex of the size three vertex partition.

If an edge joins two vertices of the size n vertex partition, then contracting any edge of the original $K_{3,n}$ results in a graph with three edge disjoint circuits meeting at a single vertex. Therefore, every orientation of $K_{3,n}$ plus any edge not in A_n has $\vec{S}_3$ as a minor.

If H is obtained from $K_{3,n}$, $A_n\backslash\{x, y\}$, $A_n\backslash\{x\}$, or from A_n by subdividing a vertex of the size three vertex partition, then H has either $K_{3,n+1}$ or one of the two graphs shown in Figure 33 as a minor. In the first case H has three circuits meeting at a single vertex (indicated by the darker edges). In the second case contraction of the dashed edge creates a minor of H with three circuits meeting at a single vertex. Therefore, for $n \geq 5$, any 3-connected graph G with a $K_{3,n}$-minor is either a minor of A_m for some m, or every orientation of G has $\vec{S}_3$ as a minor. □

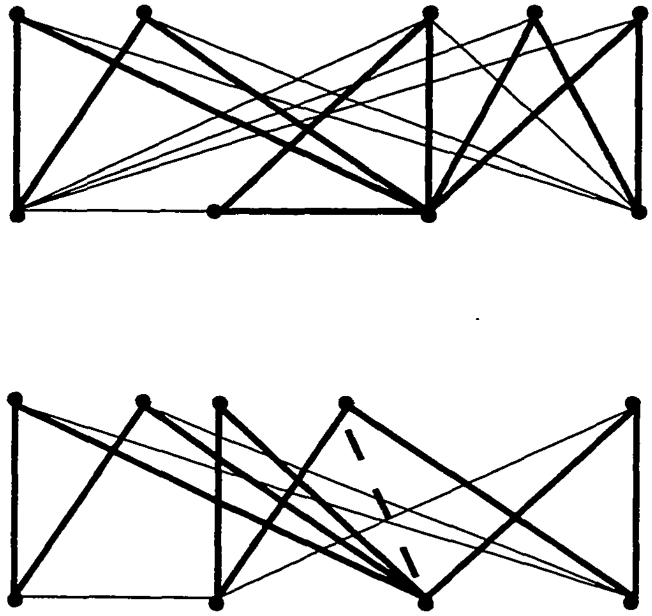

Figure 33: Two possible minors of A_n subdivide a vertex

Clearly, there can be no large 4-connected graphs having an orientation without $\vec{S}_3$ as a minor. In fact a complete list of 4-connected graphs with such an orientation can be made.

Lemma 4.2.5: *Every orientation of the graph S, shown in Figure 16, has an $\vec{S}_3$-minor.*

Proof

Let D be an orientation of S. D must have one of the partially oriented graphs shown in Figure 34 as a minor. In either case a graph that clearly must have an $\vec{S}_3$-minor is easily found to be a minor of D, as shown in Figure 35. Therefore, every orientation of S has $\vec{S}_3$ as a minor.□

Corollary 4.2.6: *Every orientation of the line graph of every cubic cyclically 4-connected graph except K_4 has $\vec{S}_3$ as a minor.*

Proof

The necessary argument to show this is given in Lemma 4.1.14 and Lemma 4.1.1.

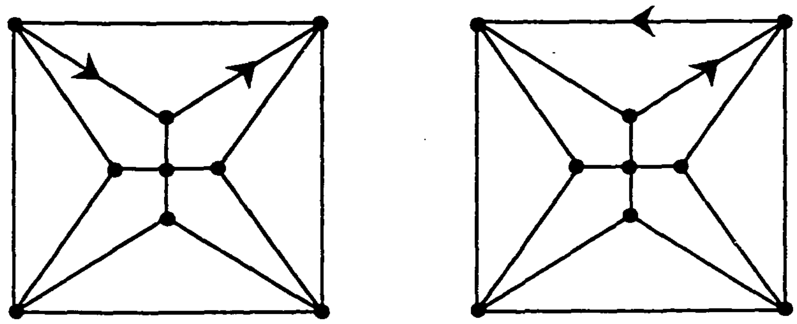

Figure 34: Two partial orientations of the graph S

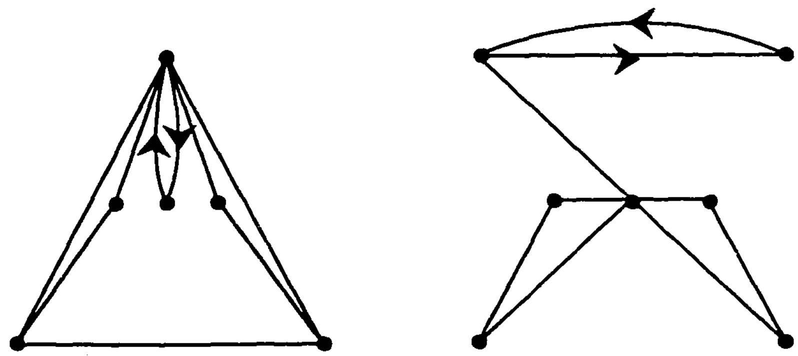

Figure 35: Minors of S which must have $\vec{S}_3$ as a minor

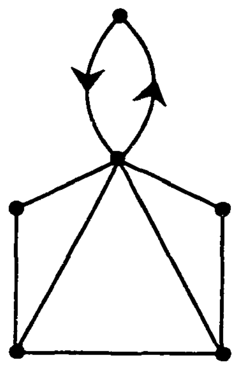

Figure 36: A partially oriented minor of C_7^2

Lemma 4.2.7: *Every orientation of* C_{2n+1}^2, $n \geq 3$ *has* $\vec{S}_3$ *as a minor.*

Proof

As shown in Lemma 4.1.5, it is enough to show that every orientation of C_7^2 has an $\vec{S}_3$-minor.

Let D be an orientation of C_7^2 and note that the orientation of any odd circuit must include consecutive arcs oriented tail to head. Therefore, in any orientation D of C_7^2 a completion of the orientation of the partially oriented graph shown in Figure 36 must be a minor of D. Every completion of the orientation of this partially oriented graph clearly must have $\vec{S}_3$ as a minor. Therefore, every orientation of C_{2n+1}^2, $n \geq 3$ has $\vec{S}_3$ as a minor. □

Lemma 4.2.8: *Every orientation of* C_{2n}^2, $n \geq 4$ *has* $\vec{S}_3$ *as a minor.*

Proof

As shown in Lemma 4.1.6, it is enough to show that every orientation of C_8^2 has an $\vec{S}_3$-minor.

Let D be an orientation of C_8^2. If D has two arcs, tail to head, then the same argument given in Lemma 4.2.7 shows that every orientation of D has an $\vec{S}_3$-minor.

If the first case is avoided then D must have a completion of the orientation of the partially oriented graph given in Figure 37 as a

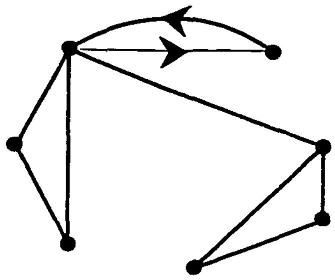

Figure 37: A partially oriented minor of C_8^2

minor. Therefore, every orientation of C_8^2 has an $\vec{S}_3$-minor, and every orientation of C_{2n}^2, $n \geq 4$ has $\vec{S}_3$ as a minor.□

Therefore, the only uncontractible 4-connected graphs that might have an orientation without $\vec{S}_3$ as a minor are C_6^2 and K_5.

Observation 4.2.9: *Every orientation of W_4^2 has $\vec{S}_3$ as a minor.*

Proof

First note that if the contraction of any spoke of an orientation of W_4^2 results in a directed graph with 2 directed 2-circuits then $\vec{S}_3$ is clearly a minor of the orientation. Similarly, if the contraction of any edge of the rim results in a digraph with two directed 2-circuits then $\vec{S}_3$ is easily seen to be a minor of the orientation.

Assume that one of the triangles consisting of two spokes and an edge on the rim is oriented as a directed circuit, and further suppose an orientation for the axle. Without loss of generality the partial orientation shown in Figure 38 may be assumed. Either choice of direction for the darker edge results in a contradiction when attempting to complete this orientation so as to avoid having an $\vec{S}_3$-minor. Therefore, it may be assumed that none of the triangles consisting of 2 spokes and a rim edge are oriented as a directed circuit. Additionally, if the contraction of the axle results in two

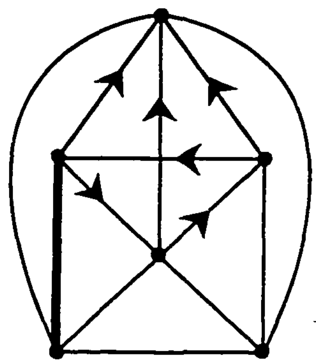

Figure 38: A possible partial orientation of W_4^2

consecutive spokes being replaced with directed 2-circuits then the existence of an $\vec{S}_3$-minor is clear.

Considering the possible orientations of the arcs incident with a hub leaves 9 possibilities, displayed in Figure 39, each of which quickly reaches an impossibility when trying to avoid the situations discussed above.

Orientations 1 and 2 deal with the case of a hub having four out (or in) spokes. In these cases the rim must be oriented as a circuit. Assume that $\langle 2,5\rangle$ is an arc. The contraction of $\langle 1,2\rangle$ then shows that $\langle 3,2\rangle$ must be an arc else two directed 2-circuits could be created by a single contraction. Similarly, the remaining edges of the rim can be oriented. In orientation 1 the arc $\langle 1,6\rangle$ is assumed. Since the contraction of any of the four spokes incident with 1 creates a directed 2-circuit, each of the spokes incident with 6 must be oriented as shown in Figure 39, orientation 1, to avoid a single contraction creating two directed 2-circuits. Contracting the axle $\langle 1,6\rangle$ now creates four directed 2-circuits, and so $\vec{S}_3$ is a minor of orientation 1. In orientation 2 the other assumption for the axle is made. In this case the contraction of (2,6) creates a

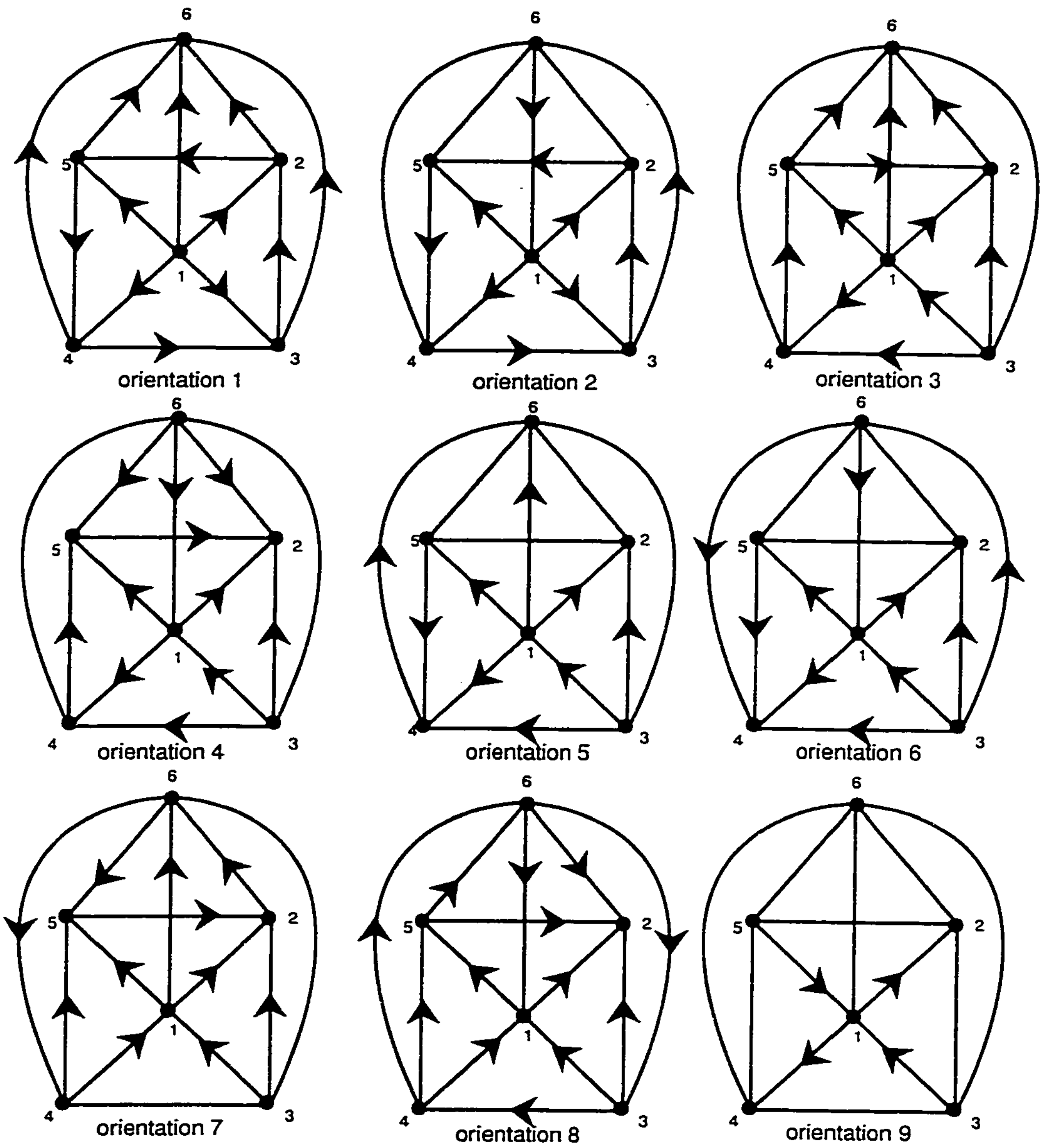

Figure 39: Nine possible partial orientations of W_4^2

2-circuit, and to avoid creating two 2-circuits the arc $\langle 3, 6\rangle$ must be the correct orientation. Now, however the contraction of (4,6) creates two directed 2-circuits, and so $\vec{S}_3$ is a minor of orientation 2, and of any orientation where all four spokes go into or out of a hub.

Orientations 3 through 6 deal with the issue of three spokes directed out from the hub and a single spoke directed into the hub. To avoid the triangle consisting of two spokes and an edge of the rim being a directed circuit, $\langle 3, 2\rangle$ and $\langle 3, 4\rangle$ must be arcs. In orientations 3 and 4 the additional assumption is made that $\langle 4, 5\rangle$ is an arc. Since $\langle 4, 5\rangle$ is an arc the contraction of $\langle 1, 5\rangle$ shows that $\langle 5, 2\rangle$ also must be an arc in orientations 3 and 4.

In orientation 3 the axle is assumed to be arc $\langle 1, 6\rangle$. Contracting arcs $\langle 1, 2\rangle$ and $\langle 1, 5\rangle$ shows that arcs $\langle 2, 6\rangle$ and $\langle 5, 6\rangle$ may be assumed. In this case the contraction of (3,6) creates two directed 2-circuits, demonstrating that $\vec{S}_3$ is a minor of orientation 3.

In orientation 4 the axle is assumed to be arc $\langle 6, 1\rangle$. The contractions of arcs $\langle 1, 2\rangle$ and $\langle 1, 5\rangle$ show that arcs $\langle 6, 2\rangle$ and $\langle 6, 5\rangle$ may be assumed. Now the contraction of $\langle 6, 2\rangle$ shows that $\vec{S}_3$ is a minor of orientation 4.

In orientations 5 and 6 the arc $\langle 5, 4\rangle$ is assumed. In orientation 5 the arc $\langle 1, 6\rangle$ is also assumed. The contraction of $\langle 1, 4\rangle$ creates a directed 2-circuit, and to avoid creating two directed 2-circuits, $\langle 4, 6\rangle$ may be assumed. Now the contraction of (3,6) creates two directed 2-circuits, and therefore $\vec{S}_3$ is a minor of orientation 5.

In orientation 6, $\langle 6, 1\rangle$ is assumed to be an arc. Contracting arc $\langle 1, 4\rangle$ demonstrates that $\langle 6, 4\rangle$ also must be an arc. With $\langle 6, 4\rangle$ as

an arc contracting (2,6) shows that $\langle 3,6\rangle$ also must be an arc. In this case, however, contracting $\langle 3,4\rangle$ creates two directed 2-circuits, and therefore $\vec{S}_3$ is a minor of orientation 6.

Orientations 7 through 9 deal with the case of a hub having two inward and two outward spokes. Two different cases are considered, the first in orientation 7 and 8 and the second in orientation 9. In orientation 7 and 8 the arcs $\langle 3,2\rangle$ and $\langle 4,5\rangle$ may be assumed so as to avoid orienting a triangle created from two spokes and a rim edge as a directed circuit. In addition, without loss of generality, the arc $\langle 5,2\rangle$ may be assumed. In orientation 7 the arc $\langle 1,6\rangle$ is assumed. In this case the contraction of $\langle 1,2\rangle$ shows that $\langle 2,6\rangle$ also must be an arc. Then the contraction of (3,6) results in two directed 2-circuits, showing that an $\vec{S}_3$ minor must exist in orientation 7.

In orientation 8 the arc $\langle 6,1\rangle$ is assumed. The contraction of $\langle 1,2\rangle$ shows the existence of $\langle 6,2\rangle$. Now contracting $\langle 3,2\rangle$ shows the existence of $\langle 6,3\rangle$. The contraction of $\langle 6,2\rangle$ shows the existence of $\langle 5,6\rangle$. The contraction of $\langle 5,6\rangle$ shows the existence of $\langle 4,6\rangle$. Now the contraction of $\langle 4,1\rangle$ shows the existence of $\langle 3,4\rangle$. Finally, contracting $\langle 4,6\rangle$ creates the two directed 2-circuits, showing the existence of an $\vec{S}_3$-minor in orientation 8.

In orientation 9 the contraction of any of the rim edges (2,3), (3,4), (4,5), or (2,5) creates a directed 2-circuit. Therefore, in an attempt to avoid creating two directed 2-circuits every spoke incident with vertex 6 must be oriented out of 6, or they all must be oriented into 6. This, however, creates a hub with all spokes going out or all spokes going in, a case that was handled in orientations 1 and 2. Therefore, every orientation of W_4^2 has $\vec{S}_3$ as a minor.□

Lemma 4.2.10: *If G is a 4-connected graph obtained by subdividing a vertex of C_6^2, then every orientation of G has $\vec{S}_3$ as a minor.*

Proof

Let G be a 4-connected graph created by the subdivision of a vertex of C_6^2, then G has one of the graphs shown in Figure 40 as a subgraph. Note that case 3 is isomorphic to case 2, case 4 has a C_7^2-minor, and cases 6 and 7 are not 4-connected. Making case 6 4-connected makes it have either case 2 or case 5 as a minor. Making case 7 4-connected makes it have either case 1 or case 8 as a minor. Therefore, it is enough to show that cases 1, 2, 5, and 8 must all have an $\vec{S}_3$-minor.

In the first case consider the triangle (3,4,5). If the contraction of (3,4) creates a directed 2-circuit rooted between 34 and 5, then the triangles (1,2,34) and (34,6,7) show that $\vec{S}_3$ is a minor of the orientation. If the contraction of (3,5) creates a directed 2-circuit rooted between 35 and 4, then the triangles (1,2,35) and (35,6,7) show that $\vec{S}_3$ is a minor of the orientation. If the contraction of (4,5) creates a directed 2-circuit rooted between 45 and 3, then the triangles (45,6,7) and (1,2,45) show that $\vec{S}_3$ is a minor of the orientation. Since at least one of the three edges of (3,4,5) will contract to form a directed 2-circuit in any orientation, every orientation of case 1 has $\vec{S}_3$ as a minor.

In the second case consider the triangle (2,4,6). If the contraction of (2,6) creates a directed 2-circuit rooted between 26 and 4, then the triangles (1,26,7) and (26,3,5) show that $\vec{S}_3$ is a minor of the orientation. If the contraction of (2,4) creates a directed 2-circuit rooted between 24 and 6, then the triangles (1,24,3) and (24,5,7)

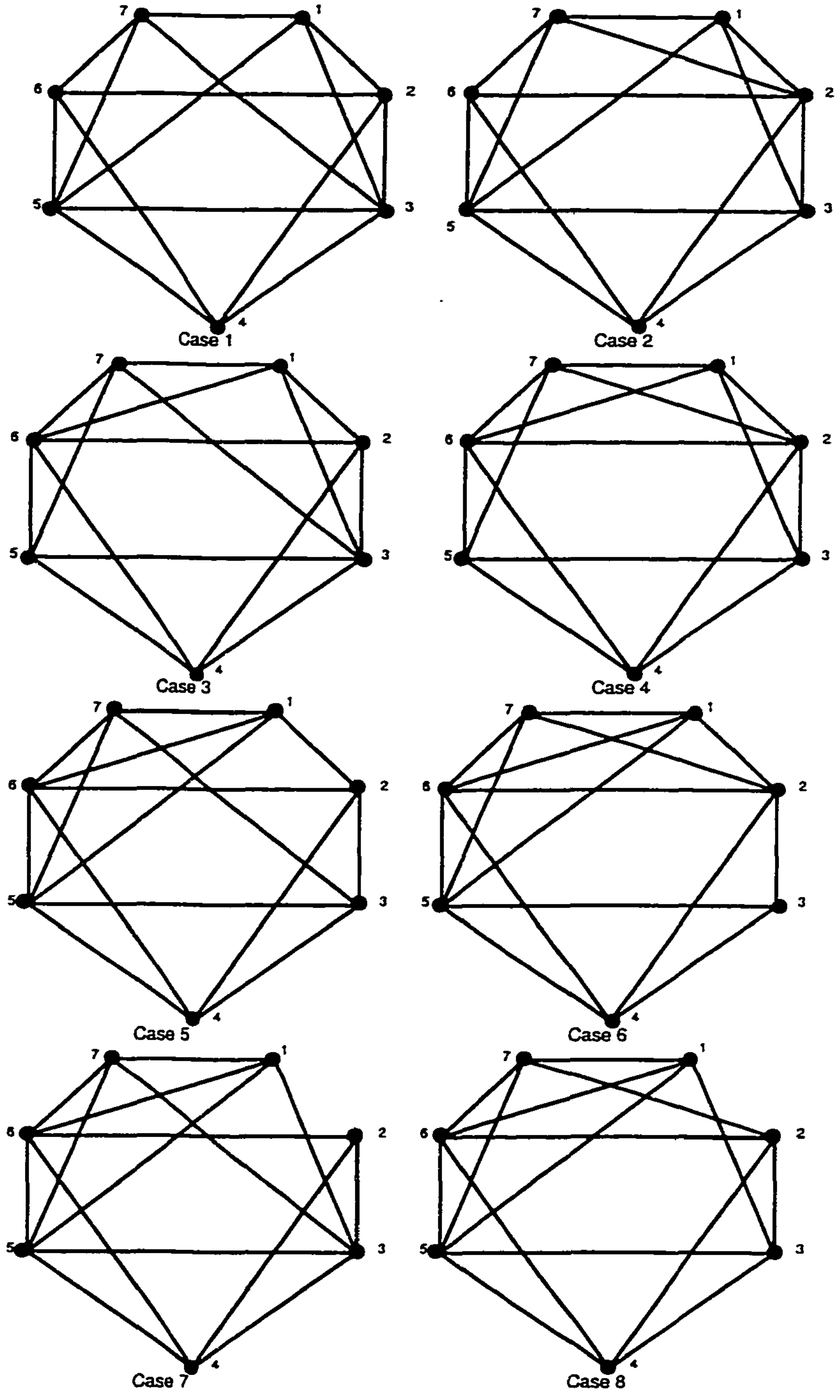

Figure 40: Eight possible subgraphs of C_6^2 with a vertex subdivided

show that $\vec{S}_3$ is a minor of the orientation. Now consider the triangle (4,5,6). If the contraction of (4,5) creates a directed 2-circuit rooted between 45 and 6, then the triangles (2,3,45) and (1,45,7) show that $\vec{S}_3$ is a minor of the orientation. If the contraction of (5,6) creates a directed 2-circuit rooted between 56 and 4, then the triangles (1,56,7) and (2,3,56) show that $\vec{S}_3$ is a minor of the orientation. These four observations imply that the contraction of (4,6) must result in two directed 2-circuits, one rooted to 2 and 46 and the other rooted to 46 and 5, but now the circuit (46,7,1,3) shows that $\vec{S}_3$ is a minor of every orientation of case 2.

In case 5 consider the triangle (4,5,6). If the contraction of (4,5) creates a directed 2-circuit rooted between 45 and 6, then the triangles (2,3,45) and (1,45,7) show that $\vec{S}_3$ is a minor of the orientation. If the contraction of (4,6) creates a directed 2-circuit rooted between 46 and 5, then the triangles (1,46,7) and (2,3,46) show that $\vec{S}_3$ is a minor of the orientation. If the contraction of (5,6) creates a directed 2-circuit rooted between 56 and 4, then the triangles (1,56,7) and (2,3,56) show that $\vec{S}_3$ is a minor of the orientation. Therefore, every orientation of case 5 has $\vec{S}_3$ as a minor.

In case 8 consider the triangle (4,5,6). If the contraction of (4,5) creates a directed 2-circuit rooted between 45 and 6, then the triangles (1,45,7) and (2,3,45) show that $\vec{S}_3$ is a minor of the orientation. If the contraction of (4,6) creates a directed 2-circuit rooted between 46 and 6, then the triangles (1,46,7) and (2,3,46) show that $\vec{S}_3$ is a minor of the orientation. If the contraction of (5,6) creates a directed 2-circuit rooted between 56 and 4, then the triangles (1,56,7)

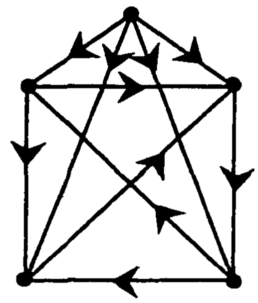

Figure 41: An orientation of K_5 without $\vec{S}_3$ as a minor

and (2,3,56) show that $\vec{S}_3$ is a minor of the orientation. Therefore, every orientation of case 8 has $\vec{S}_3$ as a minor.

Therefore, if G is a 4-connected graph obtained by subdividing a vertex of C_6^2, then every orientation of G has $\vec{S}_3$ as a minor.□

Observation 4.1.17 showed that the only 4-connected graph that can be obtained from K_5 by splitting a vertex is W_2^4, which is also the graph obtained by adding an edge to C_6^2. Therefore, this section ends with showing that every orientation of almost every 4-connected graph has an $\vec{S}_3$-minor.

Theorem 4.2.11: *The only 4-connected graphs that have an orientation without $\vec{S}_3$ as a minor are K_5 and C_6^2.*

Proof

The fact that every orientation of any 4-connected graph besides K_5 and C_6^2 has $\vec{S}_3$ as a minor is an immediate result of Observation 4.2.9, Lemma 4.2.10, and the description of uncontractible 4-connected graphs given in [7]. All that remains to be shown is that there is an orientation of each graph without $\vec{S}_3$ as a minor.

In the case of K_5 consider the orientation given in Figure 41. Only one contraction can be used in an attempt to obtain $\vec{S}_3$, and this contraction must identify the source with another vertex. The

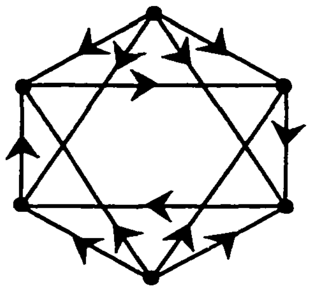

Figure 42: An orientation of C_6^2 without $\vec{S}_3$ as a minor

source identified with another vertex, however, creates a vertex of in-degree at most 2. As there is no vertex of in-degree and out-degree at least 3, this orientation does not have $\vec{S}_3$ as a minor.

In the case of C_6^2 consider the orientation given in Figure 42. Here two contractions are all that an attempt to obtain $\vec{S}_3$ can use, but there are two sources, and each must be identified with another vertex. Identifying a source with another vertex only results in a vertex of in-degree 2, so once again the vertex of in-degree and out-degree at least 3 cannot be obtained. Therefore, this orientation of C_6^2 does not have $\vec{S}_3$ as a minor.

Therefore, the only 4-connected graphs which have an orientation without $\vec{S}_3$ as a minor are K_5 and C_6^2□

4.3 Orienting Graphs to Avoid a $\vec{P}_3$-minor

Finally, consider the last of the digraphs with three 2-circuits, $\vec{P}_3$, shown in Figure 43.

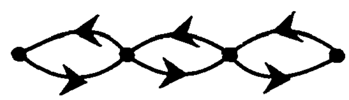

Figure 43: $\vec{P}_3$

First observe that $\vec{P}_3$ has two vertex disjoint circuits; therefore, every orientation of K_5, a wheel, or A_n does not have $\vec{P}_3$ as a minor.

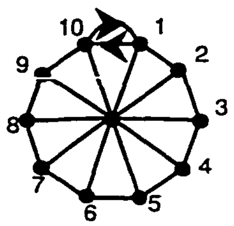

Figure 44: W_{10} with a directed 2-circuit in place of a rim edge

In the case of the family of wheels, however, something interesting can be observed.

Theorem 4.3.1: *There exists n, such that any 3-connected graph which has W_n as a minor, and can be oriented to not have $\vec{P}_3$ as a minor, must be a wheel.*

Proof

Suppose that G is a 3-connected graph with a W_{40}-minor, but that G is not a wheel. G must have either W_{20} plus an edge, or W_{40} with the hub split as a minor. If G has W_{20} plus an edge as a minor, then any orientation of G must have a completion of the orientation of either the partially oriented graphs W_{10} with one edge of the rim replaced with a directed 2-circuit or the one-sum of a directed 2-circuit and W_{10} as a minor. If G is oriented so that it has W_{10} with a directed 2-circuit in place of a rim edge as a minor, then label the vertices of W_{10} 1 through 10 and c, with 1 and 10 being the vertices in the directed 2-circuit, the rest of the vertices on the rim being numbered in the obvious way, and c being the hub of the wheel.

Consider the three contractions $(5, c)$, $(6, c)$, and (5,6). For any orientation one of these must create a directed 2-circuit out of the triangle $(5, 6, c)$. Checking all three cases (see Figure 45) quickly shows that any orientation must have a $\vec{P}_3$-minor.

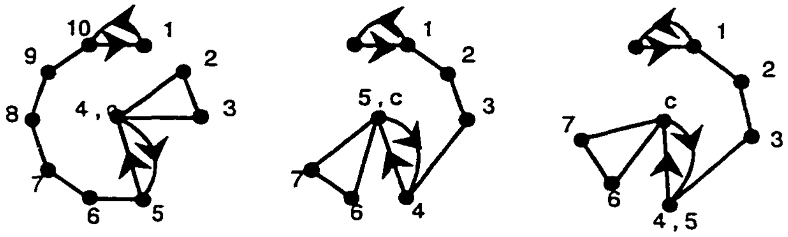

Figure 45: Three minors of W_{10} plus directed 2-circuit

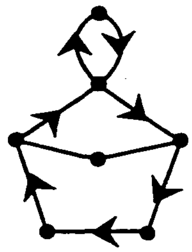

Figure 46: A partially oriented minor of W_{10} plus a 2-circuit.

If G is oriented to have the one-sum of W_{10} and a directed 2-circuit as a minor, then there must be at least 5 edges along the rim of W_{10} oriented as a directed circuit. Therefore, this orientation of G must have a completion of the orientation of the partially oriented graph shown in Figure 46 as a minor, and $\vec{P}_3$ must be a minor of G. Therefore, if G has W_{20} plus an edge as a minor, then every orientation of G has $\vec{P}_3$ as a minor.

If G has W_{40} with the hub split as a minor then G has W_{20} plus an edge as a minor, and so has $\vec{P}_3$ as a minor. Therefore, any 3-connected graph with a large wheel as a minor, but orientable to not have a $\vec{P}_3$-minor must be a wheel.□

With Theorem 4.3.1 it is easy to show the following.

Corollary 4.3.2: *For all $k \geq 40$ there is an integer N such that every 3-connected graph with at least N vertices, that can be oriented to not have a $\vec{P}_3$-minor, is either a wheel or has a subgraph*

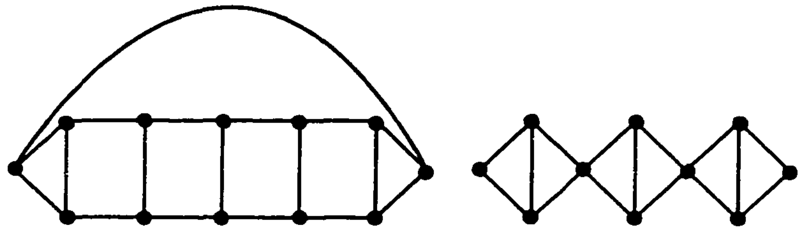

Figure 47: V_7 and a minor of V_7 which must have a $\vec{P}_3$-minor

isomorphic to a subdivision of $K_{3,k}$

Proof

Let G be a 3-connected graph with an orientation without $\vec{P}_3$ as a minor. Oporowski, Oxley, and Thomas show in [8] that for every k there is an integer N such that every 3-connected graph with at least N vertices has a subgraph isomorphic to a subdivision of either W_k, V_k, or $K_{3,k}$. Theorem 4.3.1 shows that if $k \geq 40$ and G has a subgraph isomorphic to a subdivision of W_k, then G is isomorphic to W_k. If $k \geq 7$, then V_7 is a minor of V_k, and using Lemma 2.2 it is trivial to see that the minor of V_7 shown in Figure 47 has $\vec{P}_3$ as a minor. Therefore, for all $k \geq 40$, there is an integer N such that every 3-connected graph with at least N vertices, that can be oriented to avoid $\vec{P}_3$, is either a wheel or has a subgraph isomorphic to a subdivision of $K_{3,k}$□

But while there are infinite families of 3-connected graphs that can be oriented to not have this small directed minor, almost all 4-connected graphs must have it. Consider the 4-connected situation, and once again use the result that the only uncontractible 4-connected graphs are C_n^2 for $n \geq 5$ and the line graphs of the cubic cyclically 4-connected graphs, begin with the line graph of $K_{3,3}$.

Lemma 4.3.3: *Every orientation of* $L(K_{3,3})$ *has* $\vec{P}_3$ *as a minor.*

Proof

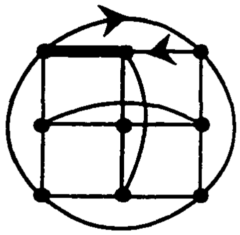

Figure 48: A partial orientation of $L(K_{3,3})$

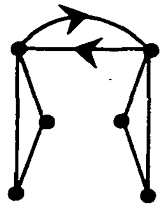

Figure 49: A minor of $L(K_{3,3})$ with a $\vec{P}_3$-minor.

Note first that all the edges of $L(K_{3,3})$ are symmetric. Therefore, the partial orientation of $L(K_{3,3})$ given in Figure 48 may be supposed. Contracting the indicated edge results in a partially oriented graph with a partially oriented graph as a minor that obviously has $\vec{P}_3$ as a minor, no matter how the orientation is completed, as shown in Figure 49. Therefore, every orientation of $L(K_{3,3})$ has $\vec{P}_3$ as a minor.□

Now consider the case of the line graph of $(K_5\backslash e)^*$, shown in Figure 15.

Lemma 4.3.4: *Every orientation of* $L((K_5\backslash e)^*)$ *has* $\vec{P}_3$ *as a minor.*

Proof

Every orientation of $L((K_5\backslash e)^*)$ must have a completion of one of the partially oriented graphs shown in Figure 50 as a minor. Each of these graphs has $\vec{P}_3$ as a minor. Therefore, every orientation of $L(K_{3,3})$ has $\vec{P}_3$ as a minor.□

These two results combine to show that the only uncontractible 4-connected graphs with orientations without $\vec{P}_3$ as a minor are C_n^2

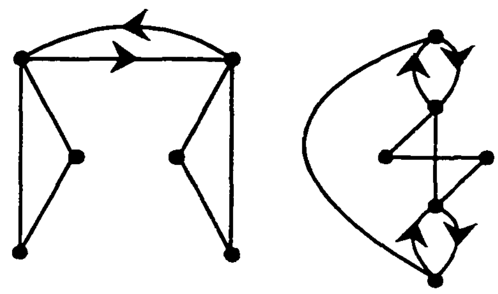

Figure 50: Two partially oriented minors of $L((K_5\backslash e)^*)$

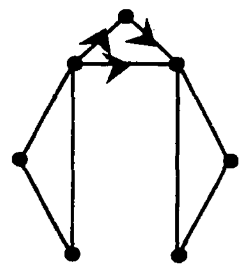

Figure 51: A partially oriented subgraph of C_n^2

for $n \geq 5$.

Lemma 4.3.5: *Every orientation of C_n^2 for $n \geq 7$ has $\vec{P}_3$ as a minor.*

<u>Proof</u>

Consider the subgraph of C_n^2 for $n \geq 7$, shown in Figure 51. It is clear that to not have $\vec{P}_3$ as a minor it is necessary that the partial orientation given be used. Therefore, there is a unique orientation of C_n^2 that might not have $\vec{P}_3$ as a minor, but $\vec{P}_3$ may be found as shown in Figure 52. Therefore, every orientation of C_n^2 for $n \geq 7$ has $\vec{P}_3$ as a minor.□

Lemma 4.3.6: *Let G be a 4-connected graph with edge e such that G/e is isomorphic to C_6^2. Every orientation of G has $\vec{P}_3$ as a minor.*

<u>Proof</u>

G can be labeled so that it must have the graph shown in Figure 53 as a subgraph. Additionally, there must be at least two more edges adjacent with vertex 1 and three more edges adjacent with

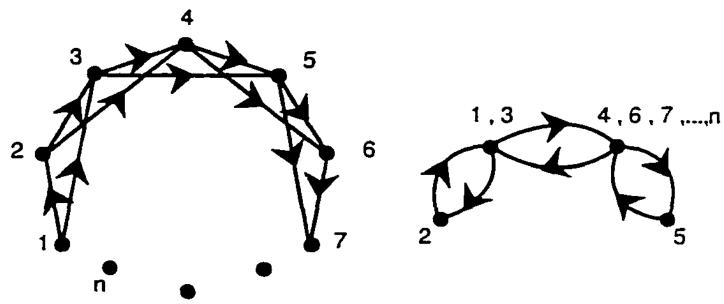

Figure 52: An orientation of C_n^2 with a $\vec{P}_3$-minor.

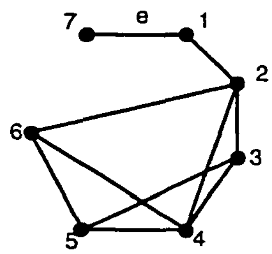

Figure 53: A subgraph of G

vertex 7, and vertices 3, 5, and 6 need one more edge each. As a first case, suppose that the edges (5,7) and (1,3) are in G, as shown in Figure 54.

Note that if (6,7)$\in E(G)$, then C_7^2 is a minor of G, so 7 is adjacent to three of the vertices 2,3,4, and 5. This also shows that $(1,6) \in E(G)$. Now consider the three possible ways to produce a directed 2-circuit from the triangle (2,3,4). If contracting either the edge (2,3) or the edge (3,4) creates a directed 2-circuit from the triangle (2,3,4), then $\vec{P}_3$ is a minor of the orientation of G. Therefore, the contraction

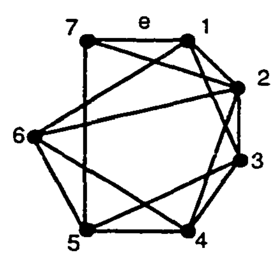

Figure 54: A possible subgraph of G

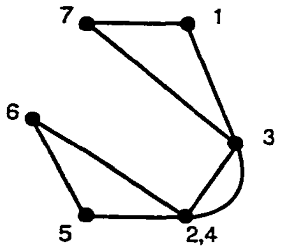

Figure 55: A possible minor of $G/(2,4)$ with a $\vec{P}_3$-minor

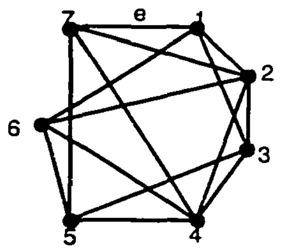

Figure 56: Another possible subgraph of G

of (2,4) must produce a directed 2-circuit from the triangle (2,3,4). If $(3,7) \in E(G)$, then contracting (2,4) produces a graph that clearly has $\vec{P}_3$ as a minor, as shown in Figure 55.

Thus, it may be assumed that $(4,7),(2,7) \in E(G)$, as shown in Figure 56. In this graph consider the triangle (2,4,7). Whichever edge of this triangle can be contracted to create a directed 2-circuit allows $\vec{P}_3$ to be obtained easily, as shown in Figure 57. Therefore, if (5,7) and (1,3) are edges of G every orientation of G has $\vec{P}_3$ as a minor.

Now consider the case, shown in Figure 58, where (3,7) and (1,5) are edges of G. The vertex 7 must be adjacent to two of 2,4,5,and 6,

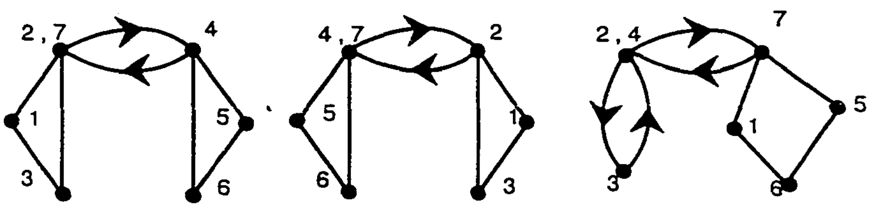

Figure 57: Subgraphs of G contract each edge of triangle (2,4,7)

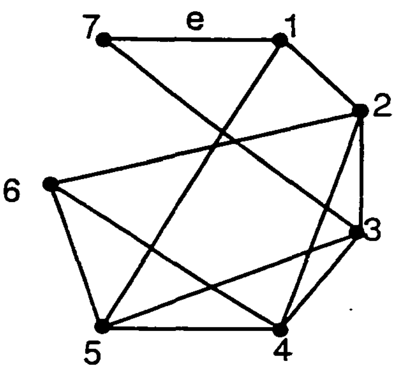

Figure 58: G with (3,7) and (1,5) as edges

which gives six possibilities. Additionally, if (6,7) is not an edge of G, then (1,6) must be an edge of G. Also, at least one of (1,3), (1,4), and (1,6) must be an edge of G. Note first that if $(1,3) \in E(G)$, then considering the contraction of each of the edges of triangle (3,4,5) shows that $\vec{P}_3$ is a minor of every orientation of G.

Now consider the six possible sets of neighbors for vertex 7. If $(2,7), (5,7) \in E(G)$, then the contraction of each of the edges of triangle (2,3,4) shows that $\vec{P}_3$ is a minor of every orientation of G. If $(2,7), (6,7) \in E(G)$, then the contraction of the edges of triangle (2,4,6) shows that $\vec{P}_3$ is a minor of every orientation of G. If $(2,7), (4,7) \in E(G)$, then if G contains the edge (6,7) $\vec{P}_3$ would be a minor of G, so (1,6) must be an edge of G. In this case considering triangle (3,4,5) shows that every orientation of G has $\vec{P}_3$ as a minor. If $(4,7), (5,7), (1,6) \in E(G)$, then considering triangle (2,3,4) shows that $\vec{P}_3$ is a minor of every orientation of G. If $(4,7), (5,7) \in E(G)$, but (1,6) is not an edge of G, then $(1,4), (6,7) \in E(G)$. In this case, considering triangle (4,5,6) proves the existence of a $\vec{P}_3$ minor. Finally consider the two cases $(5,7), (6,7), (1,6) \in E(G)$ and $(5,7), (6,7), (1,4) \in E(G)$. In the first case consider triangle (5,6,7),

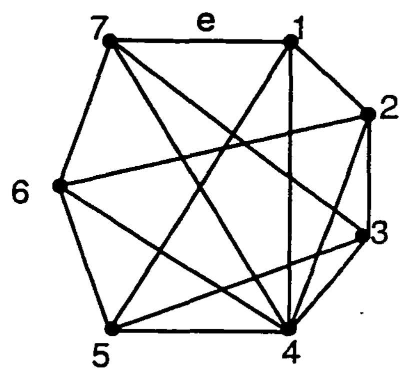

Figure 59: The remaining case for G with (3,7) and (1,5) as edges

in the second case consider triangle (1,4,5) and the existence of a $\vec{P}_3$ minor for every orientation can be easily shown.

A single possibility is left that assumes (1,5) and (3,7) are edges of G. This possibility is shown in Figure 59. If the contraction of (2,3) in triangle (2,3,4), (3,5) in triangle (3,4,5), (5,6) in triangle (4,5,6), (1,7) in triangle (1,4,7), (1,5) in triangle (1,4,5), (2,6) in triangle (2,4,6), or (6,7) in triangle (4,6,7) results in a directed 2-circuit in the appropriate triangle, then a completion of the orientation of the partially oriented graph shown in Figure 48 is a minor of G, and therefore every orientation of G would have $\vec{P}_3$ as a minor. In addition, if the contraction of any of the edges adjacent with vertex 4 results in two directed 2-circuits then $\vec{P}_3$ must be a minor of that orientation of G. Combining these two observations results in the partial orientation of G given in Figure 60 with (3,7) and (1,5) as edges that might not have $\vec{P}_3$ as a minor. In this case, however, no matter how the edge (2,6) is oriented there is a contraction (either (2,4) or (4,6)) which results in two directed 2-circuits. Therefore, every orientation of G with (3,7) and (1,5) as edges has $\vec{P}_3$ as a minor.

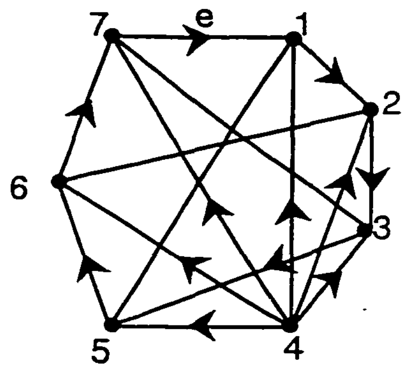

Figure 60: A partial orientation of G with (3,7) and (1,5) as edges

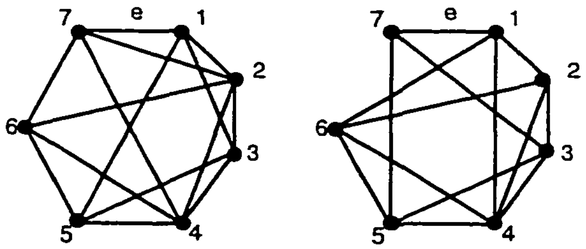

Figure 61: G with 7 and 1 adjacent to neither 3 nor 5

Avoiding these two cases requires either that 7 be adjacent to neither 3 nor 5, or that 1 be adjacent to neither 3 nor 5, as shown in Figure 61. In the case that neither 3 nor 5 are adjacent to 7, considering the three edges of the triangle (2,3,4) shows that $\vec{P}_3$ must be a minor of G. In the case where vertex 1 is adjacent to neither 3 nor 5, the three possible contractions of the triangle (4,5,6) show that $\vec{P}_3$ is a minor of G.

Therefore, if G is a 4-connected graph with edge e such that G/e is isomorphic to C_6^2, then every orientation of G has $\vec{P}_3$ as a minor.

□

Observation 4.3.7: *There exists an orientation of* $C_6^2 + e$ *without a* $\vec{P}_3$*-minor.*

Proof

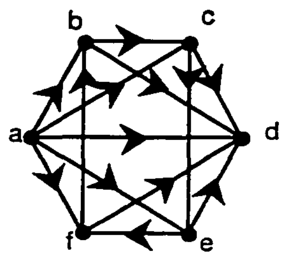

Figure 62: An orientation of $C_6^2 + e$ without a $\vec{P}_3$-minor

Consider the orientation of $C_6^2 + e$ shown in Figure 62. Since this orientation has a source, and there is no source in $\vec{P}_3$, one of the edges incident with the source must be contracted. This leads to five possibilities, shown in Figure 63. Each quickly can be seen to not have $\vec{P}_3$ as a minor.□

Lemma 4.3.8: *Every orientation of C_6^2 plus two edges has $\vec{P}_3$ as a minor.*

Proof

Draw and label C_6^2 plus two edges as shown in Figure 64. Note that if any contraction of an orientation of this graph results in two directed 2-circuits then that orientation has $\vec{P}_3$ as a minor. For a first case suppose that $\langle 1,5\rangle, \langle 3,1\rangle$ are arcs of an orientation. If $\langle 4,5\rangle$ and $\langle 3,4\rangle$, or $\langle 5,4\rangle$ and $\langle 4,3\rangle$ are arcs of the orientation then the contraction of (3,5) results in a partially oriented graph that clearly must have $\vec{P}_3$ as a minor. Therefore, the orientations of the edges (4,5) and (3,4) are either both into vertex 4 or both out of vertex 4. In either case, contracting the edge (1,4) results in a graph with a 2-circuit, and together with the need to avoid the contraction of (2,6) creating two 2-circuits, this shows that one of the four partial orientations given in Figure 65 is necessary. The second partial orientation must have a $\vec{P}_3$-minor since the contraction of

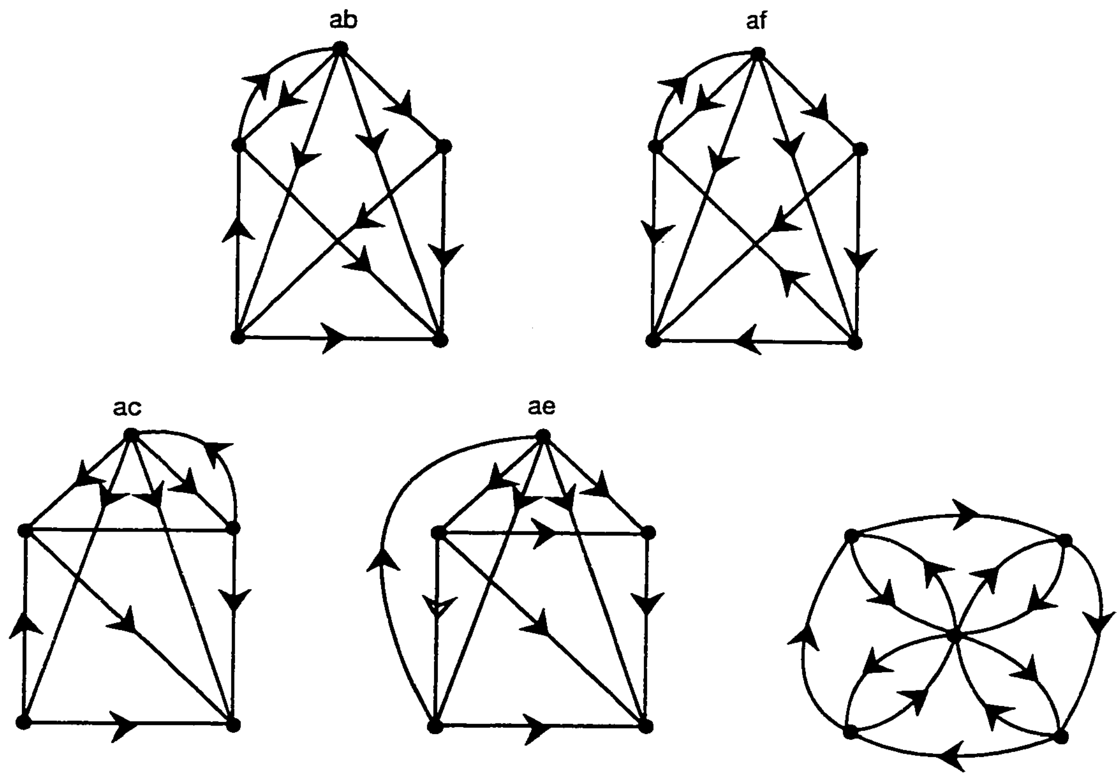

Figure 63: 5 minors of an orientation of $C_6^2 + e$

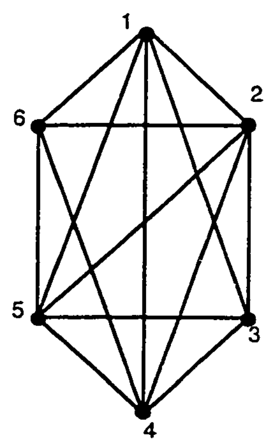

Figure 64: C_6^2 plus two edges

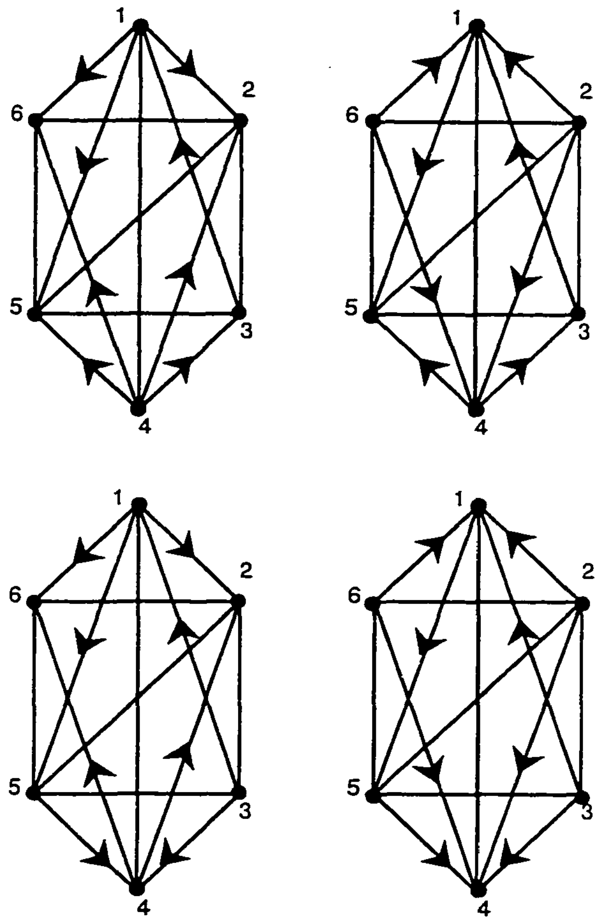

Figure 65: Four possible partial orientations of C_6^2 plus two edges

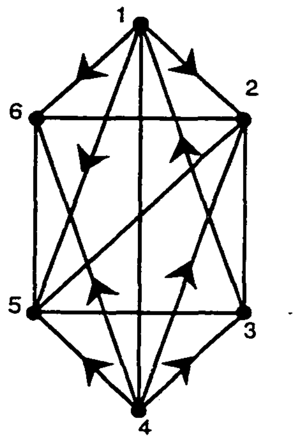

Figure 66: A partial orientation of C_6^2 plus two edges

(5,6) results in two 2-circuits. Similarly, contracting edge (2,3) in the third orientation creates two 2-circuits. Moreover, the fourth case can be made from the first by reversing the directions of every arc and relabeling 3 as 5, 5 as 3, 2 as 6, and 6 as 2. Therefore, the partial orientation shown in Figure 66 may be assumed.

From here consider four possibilities. If $\langle 2,6\rangle$ and $\langle 5,6\rangle$ are arcs, then contracting $\langle 4,6\rangle$ creates two 2-circuits. If $\langle 6,2\rangle, \langle 3,2\rangle$ are arcs, then contracting $\langle 4,2\rangle$ creates two 2-circuits. If $\langle 2,6\rangle, \langle 6,5\rangle$ are arcs, then the contraction of (3,6) shows that $\langle 2,3\rangle, \langle 3,5\rangle$ must be arcs, and then the contraction of $\langle 4,5\rangle$ creates two 2-circuits. Finally, if $\langle 6,2\rangle, \langle 2,3\rangle$ are arcs, then contracting (3,6) creates two 2-circuits. Therefore, if $\langle 1,5\rangle, \langle 3,1\rangle$ are arcs, then the orientation must have a $\vec{P}_3$-minor. In the graph C_6^2 plus two edges, as labeled, the triangles (1,2,6), (1,3,5), (1,2,3), (1,5,6), (2,3,4), (2,4,6), (3,4,5), and (4,5,6) are all interchangeable. Therefore, by case 1, if an orientation is to not have a $\vec{P}_3$-minor it is necessary that in all these triangles the only edge that can be contracted to create a directed 2-circuit is the one joining two vertices of degree 5 (1 to 2, 1 to 5,

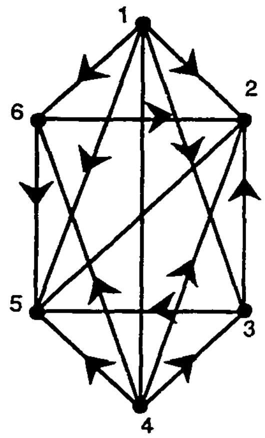

Figure 67: A partial orientation of C_6^2 plus two edges with $\langle 1,5\rangle$

2 to 4, or 4 to 5). The only possible orientation with the arc $\langle 1,5\rangle$ is shown in Figure 67. Contracting $\langle 1,2\rangle$ results in two 2-circuits; therefore, this orientation has a $\vec{P}_3$-minor, and every orientation of C_6^2 plus two edges has $\vec{P}_3$ as a minor.□

Theorem 4.3.9: *The only 4-connected graphs having an orientation without $\vec{P}_3$ as a minor are K_5, C_6^2, and $C_6^2 + e$.*

Proof

Lemmas 4.3.3, 4.3.4, and 4.3.5 show that the only uncontractible 4-connected graphs with an orientation without $\vec{P}_3$ as a minor are K_5 and C_6^2. In Observation 4.1.17 it was shown that the only 4-connected graph obtained by splitting a vertex of K_5 is $W_4^2 = C_6^2 + e$. This along with Lemma 4.3.8 shows that the three listed graphs are the only 4-connected graphs with an orientation without a $\vec{P}_3$-minor.

5 Orientations of Graphs with More Than One of $\vec{K}_3$, $\vec{S}_3$, and $\vec{P}_3$ as Minors

This final chapter considers graphs that may be oriented to avoid combinations of the three digraphs discussed in chapter 4. In considering possible pairings of the directed graphs $\vec{K}_3$, $\vec{S}_3$, and $\vec{P}_3$ the primary results are given in Corollary 5.2, and Theorems 5.3, 5.4, and 5.5. Corollary 5.2 shows that only small 3-connected graphs can be oriented to avoid both $\vec{S}_3$ and $\vec{K}_3$. Theorems 5.3, 5.4, and 5.5 show infinite families of 3-connected graphs that avoid the pairs of $\vec{P}_3$ and $\vec{K}_3$ or $\vec{S}_3$ and $\vec{P}_3$. First consider the 3-connected graphs that do not contain W_5 as a minor.

Observation 5.1: *Every orientation of $K_{3,4}$ has either $\vec{S}_3$ or $\vec{K}_3$ as a minor.*

Proof

Label $K_{3,4}$ as shown in Figure 68 and first consider the case that none of b_1, b_2, or b_3 is either a source or a sink. In this case the contraction of the edges $(a_1, b_4), (a_2, b_4)$, and(a_3, b_4) creates $\vec{S}_3$.

Now consider the case that both b_1 and b_2 are sinks. Contracting the arcs $\langle a_1, b_1\rangle$ and $\langle a_2, b_2\rangle$ and the edges (a_3, b_3) and (a_3, b_4) leads to the partially oriented graph shown in Figure 69. So as not to have a $\vec{K}_3$-minor, either $\langle b_3, a_1\rangle$ and $\langle b_4, a_1\rangle$, or $\langle b_3, a_2\rangle$ and $\langle b_4, a_2\rangle$ are arcs. In these two cases, however, $\vec{K}_3$ can be obtained as shown in Figure 70.

The only remaining case to be considered is that in which b_1 is a sink and b_2 is a source. Consider the three partially oriented graphs

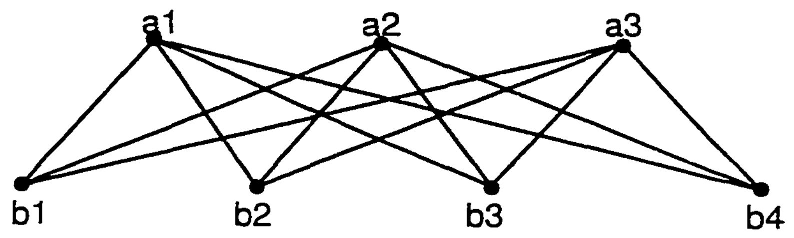

Figure 68: A labeling of $K_{3,4}$

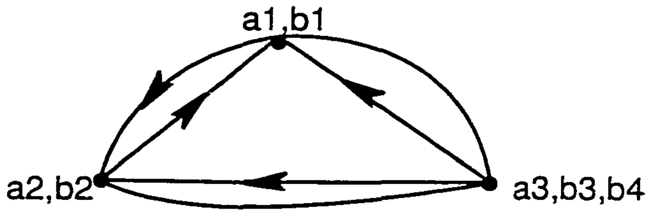

Figure 69: $K_{3,4}$ contract $\langle a_1, b_1\rangle$, $\langle a_2, b_2\rangle$, (a_3, b_3) and (a_3, b_4)

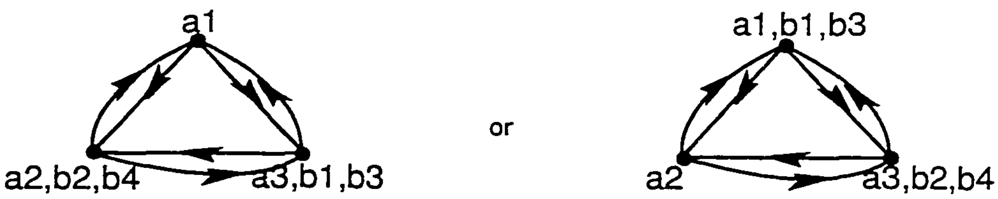

Figure 70: How to obtain $\vec{K}_3$ from $K_{3,4}$ with two sinks

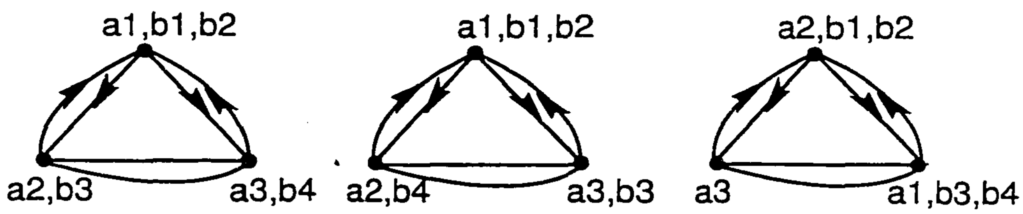

Figure 71: Three partial orientations of $K_{3,4}$ with a sink and a source

Figure 72: How to obtain $\vec{K}_3$ from $K_{3,4}$ with a sink and a source

in Figure 71. $\{\langle b_3, a_3\rangle, \langle b_4, a_3\rangle, \langle a_2, b_3\rangle, \langle a_2, b_4\rangle\}$ or the reverse of all the above arcs must be a subset of the arcs, or there would be a $\vec{K}_3$-minor. In either of these cases, however, $\vec{K}_3$ can be obtained as shown in Figure 72. □

Corollary 5.2: *There exists a number n such that every orientation of any 3-connected graph with at least n vertices must have $\vec{S}_3$ or $\vec{K}_3$ as a minor.*

Proof

By the previously mentioned result of Oporowski, Oxley, and Thomason there exists n such that any 3-connected graph on at least n vertices has either W_6 or $K_{3,4}$ as a minor. Therefore, by Observation 4.2.2 and Observation 5.1 every orientation of a 3-connected graph on at least n vertices has $\vec{S}_3$ or $\vec{K}_3$ as a minor.□

The following two results describe infinite families of 3-connected graphs without either $\vec{K}_3$ or $\vec{P}_3$ as a minor.

Theorem 5.3: *For all n there exists an orientation of W_n without either $\vec{P}_3$ or $\vec{K}_3$ as a minor. Moreover, there exists m such that if a 3-connected graph G has a W_m-minor and there is an orientation of G without either $\vec{P}_3$ or $\vec{K}_3$ as a minor, then G is a wheel.*

Proof

In Theorem 4.1.7 an orientation is given of the double wheel without $\vec{K}_3$ as a minor; therefore, an orientation of the wheel without

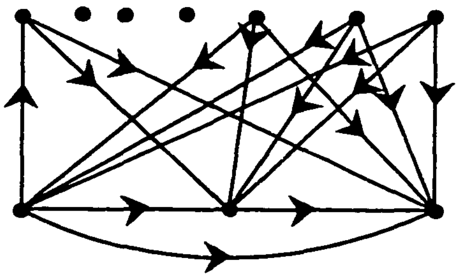

Figure 73: An orientation of A_n with no $\vec{P}_3$-minor or $\vec{K}_3$-minor

a $\vec{K}_3$-minor exists. Additionally, since any two circuits of a wheel share a vertex no orientation of the wheel can have a $\vec{P}_3$-minor. Therefore, there exists an orientation of W_n with neither $\vec{K}_3$ nor $\vec{P}_3$ as a minor. Theorem 4.3.1 also showed that for sufficiently large m the only 3-connected graphs with a W_m-minor which have an orientation without a $\vec{P}_3$-minor are the wheels. Therefore, there exists m such that if a 3-connected graph G has a W_m-minor and there is an orientation of G without either $\vec{P}_3$ or $\vec{K}_3$ as a minor, then G is a wheel.□

Theorem 5.4: *For all n there exists an orientation of A_n without either $\vec{P}_3$ or $\vec{K}_3$ as a minor.*

Proof

Orient A_n as indicated in Figure 73, with all of the vertices of the size n vertex partition (the vertices of degree 3) oriented identically.

Since there is a vertex which is a source, if this orientation has either of the forbidden minors then one of the arcs incident with the source can be contracted and the resulting graph will also have one of the forbidden minors. The three different choices of arc to contract result in the three digraphs shown in Figure 74.

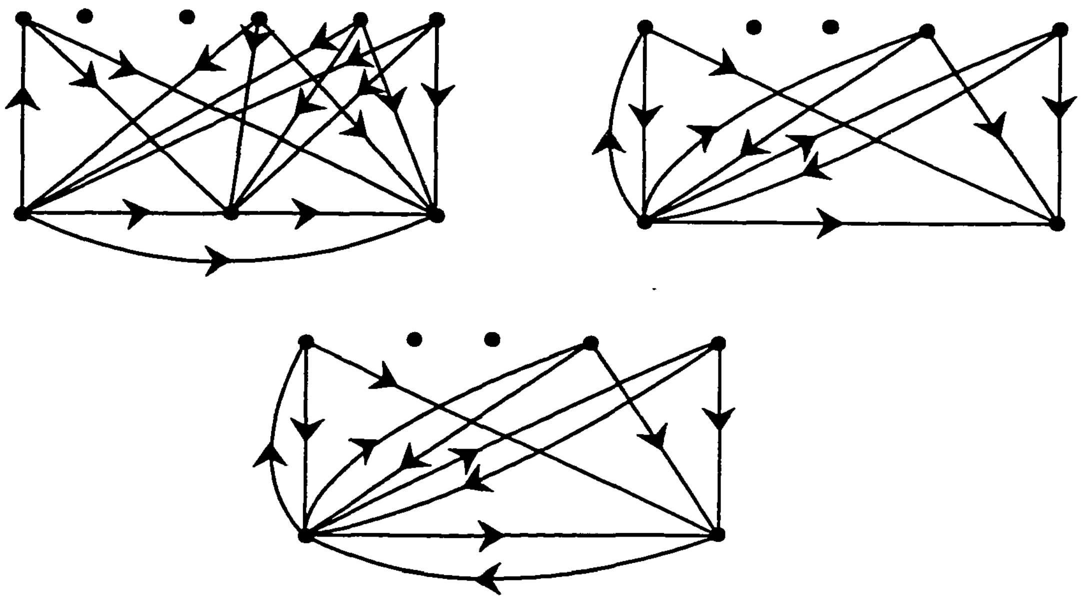

Figure 74: Three possible minors of the orientation of A_n

In the first case the graph is A_{n-1} with the same orientation, and A_1 oriented as given has neither minor; therefore, this digraph is going to have $\vec{P}_3$ or $\vec{K}_3$ as a minor only if one of the other two digraphs does. The other two digraphs, however, clearly do not have either of these directed graphs as a minor. Therefore, for all n there exists an orientation of A_n without either $\vec{P}_3$ or $\vec{K}_3$.□

Theorem 5.5: *For all n there exists an orientation of A_n without either $\vec{P}_3$ or $\vec{S}_3$ as a minor.*

Proof

Orient A_n so that each of the n vertices is a source. Since there are no sources in either $\vec{P}_3$ or $\vec{S}_3$ every one of these vertices must be identified by contraction to some other vertex of A_n, which leaves only three vertices and therefore neither $\vec{P}_3$ nor $\vec{S}_3$ is a minor of this orientation of A_n.□

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Vita

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