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Pao-Liu Chow

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SOLUTIONS OF LINEAR ELLIPTIC EQUATIONS IN GAUSS-SOBOLEV SPACES

PAO-LIU CHOW

Abstract. The paper is concerned with a class of linear elliptic equations in a Gauss-Sobolev space setting. They arise from the stationary solutions of the corresponding parabolic equations. For nonhomogeneous elliptic equations, under appropriate conditions, the existence and uniqueness theorem for strong solutions is given. Then it is shown that the associated resolvent operator is compact. Based on this result, we shall prove a Fredholm Alternative theorem for the elliptic equation and a Sturm-Liouville type of theorem for the eigenvalue problem of a symmetric elliptic operator.

1. Introduction

The subject of parabolic equations in infinite dimensions has been studied by many authors, (see e.g., the papers [6, 7, 17, 3] and in the books [8, 9]). As in finite dimensions, an elliptic equation may be regarded as the equation for the stationary solution of some parabolic equation, if exists, when the time goes to infinity. For early works, in the abstract Wiener space, the infinite-dimensional Laplace equation was treated in the context of potential theory by Gross [13] and a nice exposition of the connection between the infinite-dimensional elliptic and parabolic equations was given by Daleskii [6]. More recently, in the book [9] by Da Prato and Zabczyk, the authors gave a detailed treatment of infinite-dimensional elliptic equations in the spaces of continuous functions, where the solutions are considered as the stationary solutions of the corresponding parabolic equations. Similarly, in [4], we considered a class of semilinear parabolic equations in an L^2-Gauss-Sobolev space and showed that, under suitable conditions, their stationary solutions are the mild solutions of the related elliptic equations. So far, in studying the elliptic problem, most results rely on its connection to the parabolic equation which is the Kolmogorov equation of some diffusion process in a Hilbert space. However, for partial differential equations in finite dimensions, the theory of elliptic equations is considered in its own rights, independent of related parabolic equations [12]. Therefore it is worthwhile to generalize this approach to elliptic equations in infinite dimensions. In the present paper, similar to the finite-dimensional case, we shall begin with a class of linear elliptic equations in a L^2-Sobolev space setting with respect to a suitable Gaussian measure. It will be

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shown that several basic results for linear elliptic equations in finite dimensions can be extended to the infinite-dimensional counter parts. In passing it is worth noting that the infinite-dimensional Laplacians on a Lévy - Gel'fand triple was treated by Barhoumi, Kuo and Ouerdian [1].

To be specific, the paper is organized as follows. In Section 2, we recall some basic results in Gauss-Sobolev spaces to be needed in the subsequent sections. Section 3 pertains to the strong solutions of some linear elliptic equations in a Gauss-Sobolev space, where the existence and uniqueness Theorem 3.2 is proved. Section 4 contains a key result (Theorem 4.1) showing that the resolvent of the elliptic operator is compact. Based on this result, the Fredholm Alternative Theorem 4.4 is proved. In Section 5, we first characterize the spectral properties of the elliptic operator in Theorem 5.1. Then the eigenvalue problem for the symmetric part of the elliptic operator is studied and the results are summarized in Theorem 5.2 and Theorem 5.3. They show that the eigenvalues are positive, nondecreasing with finite multiplicity, and the set of normalized eigenfunctions forms a complete orthonormal basis in the Hilbert space $\mathcal{H}$ consisting of all $L^2(\mu)-$functions, where μ is an invariant measure defined in Theorem 2.1. Moreover the principal eigenvalue is shown to be simple and can be characterized by a variational principle.

2. Preliminaries

Let H be a real separable Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. Let $V \subset H$ be a Hilbert subspace with norm $\|\cdot\|$. Denote the dual space of V by V' and their duality pairing by $\langle\cdot,\cdot\rangle$. Assume that the inclusions $V \subset H \cong H' \subset V'$ are dense and continuous [15].

Suppose that $A : V \to V'$ is a continuous closed linear operator with domain $\mathcal{D}(A)$ dense in H, and W_t is a H-valued Wiener process with the covariance operator R. Consider the linear stochastic equation in a distributional sense:

$$\begin{aligned} du_t &= Au_t\,dt + d\,W_t, \quad t > 0, \\ u_0 &= h \in H. \end{aligned} \tag{2.1}$$

Assume that the following conditions (A) hold:

(A.1) Let $A : V \to V'$ be a self-adjoint, coercive operator such that

$$\langle Av, v\rangle \le -\beta\|v\|^2,$$

for some $\beta > 0$, and $(-A)$ has positive eigenvalues $0 < \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le \cdots$, counting the finite multiplicity, with $\alpha_n \uparrow \infty$ as $n \to \infty$. The corresponding orthonormal set of eigenfunctions $\{e_n\}$ is complete.

(A.2) The resolvent operator $\mathcal{R}_\lambda(A)$ and covariance operator R commute, so that $\mathcal{R}_\lambda(A)R = R\,\mathcal{R}_\lambda(A)$, where $\mathcal{R}_\lambda(A) = (\lambda I - A)^{-1}$, $\lambda \ge 0$, with I being the identity operator in H.

(A.3) The covariance operator $R : H \to H$ is a self-adjoint operator with a finite trace such that $Tr\,R < \infty$.

It follows from (A.2) and (A.3) that $\{e_n\}$ is also the set of eigenfunctions of R with eigenvalues $\{\rho_n\}$ such that

$$R\,e_n \;=\; \rho_n\,e_n, \quad n = 1, 2, \cdots, n, \cdots, \tag{2.2}$$

where $\rho_n > 0$ and $\sum_{n=1}^{\infty} \rho_n < \infty$.

By applying Theorem 4.1 in [5] for invariant measures and a direct calculation, we have the following theorem.

Theorem 2.1. *Under conditions (A), the stochastic equation (2.1) has a unique invariant measure μ on H, which is a centered Gaussian measure with covariance operator $\Gamma = -\frac{1}{2}A^{-1}R$.*

Remark 2.2. We make the following two remarks:

(1) It is easy to check that $e_n{}'s$ are also eigenfunctions of Γ so that

$$\Gamma\,e_n \;=\; \gamma_n\,e_n, \quad n = 1, 2, \cdots, n, \cdots, \tag{2.3}$$

where $\gamma_n = \dfrac{\rho_n}{2\alpha_n}$.

(2) Let e^{tA}, $t \geq 0$, denote the semigroup of operators on H generated by A. Without condition (A.2), the covariance operator of the invariant measure μ is given by $\Gamma = \int_0^\infty e^{tA}Re^{tA}dt$, which cannot be evaluated in a closed form. Though an $L^2(\mu)-$ theory can be developed in the subsequent analysis, one needs to impose some conditions which are not easily verifiable.

Let $\mathcal{H} = L^2(H,\mu)$ be a Hilbert space consisting of real-valued functions Φ on H with norm defined by

$$\|\!|\Phi|\!\| \;=\; \{\int_H |\Phi(v)|^2\mu(dv)\}^{1/2},$$

and the inner product $[\cdot,\cdot]$ given by

$$[\Theta,\Phi] = \int_H \Theta(v)\Phi(v)\mu(dv), \quad \text{for}\,\Theta,\Phi \in \mathcal{H}.$$

Let $\mathbf{n} = (n_1, n_2, \cdots, n_k, \cdots)$, where $n_k \in \mathbb{Z}^+$, be a sequence of nonnegative integers, and let $\mathbf{Z} = \{\mathbf{n} : n = |\mathbf{n}| = \sum_{k=1}^{\infty} n_k < \infty\}$, so that $n_k = 0$ except for a finite number of $n_k's$. Let $h_m(r)$ be the normalized

one-dimensional Hermite polynomial of degree m. For $v \in H$, define a Hermite (polynomial) functional of degree n by

$$H_{\mathbf{n}}(v) = \prod_{k=1}^{\infty} h_{n_k}[\ell_k(v)],$$

where we set $\ell_k(v) = (v, \Gamma^{-1/2}e_k)$ and $\Gamma^{-1/2}$ denotes a pseudo-inverse of $\Gamma^{1/2}$. For a smooth functional Φ on H, let $D\Phi$ and $D^2\Phi$ denote the Fréchet derivatives of the first and second orders, respectively. The differential operator

$$\mathcal{A}\Phi(v) = \frac{1}{2}Tr[RD^2\Phi(v)] + \langle Av, D\Phi(v)\rangle \tag{2.4}$$

is well defined for a polynomial functional Φ with $D\Phi(v)$ lies in the domain $\mathcal{D}(A)$ of A. However this condition is rather restrictive on Φ. For ease of calculations, in place of Hermite functionals, introduce an exponential family $\mathcal{E}_A(H)$ of functionals as follows [8]:

$$\mathcal{E}_A(H) \doteq Span\{Re\,\Phi_h,\, Im\,\Phi_h : h \in D(A)\}, \tag{2.5}$$

where $\Phi_h(v) \doteq \exp\{i(h,v)\}$. It is known that $\mathcal{E}_A(H) \subset D(\mathcal{A})$ is dense in $\mathcal{H}$. For $v \in \mathcal{E}_A(H)$, the equation (2.4) is well defined.

Returning to the Hermite functionals, it is known that the following holds [2]:

Proposition 2.3. *The set of all Hermite functionals $\{H_{\mathbf{n}} : \mathbf{n} \in \mathbf{Z}\}$ forms a complete orthonormal system in $\mathcal{H}$. Moreover we have*

$$\mathcal{A}H_{\mathbf{n}}(v) = -\lambda_{\mathbf{n}} H_{\mathbf{n}}(v), \quad \forall\, \mathbf{n} \in \mathbf{Z},$$

where $\lambda_{\mathbf{n}} = \sum_{k=1}^{\infty} n_k \alpha_{n_k}$.

We now introduce the L^2-Gauss-Sobolev spaces. For $\Phi \in \mathcal{H}$, by Proposition 2.2, it can be expressed as

$$\Phi = \sum_{\mathbf{n}\in\mathbf{Z}} \Phi_{\mathbf{n}} H_{\mathbf{n}},$$

where $\Phi_{\mathbf{n}} = [\Phi, H_{\mathbf{n}}]$ and $\|\Phi\|^2 = \sum_{\mathbf{n}} |\Phi_{\mathbf{n}}|^2 < \infty$.

Let $\mathcal{H}_m$ denote the Gauss-Sobolev space of order m defined by

$$\mathcal{H}_m = \{\Phi \in \mathcal{H} : |||\Phi|||_m < \infty\}$$

for any integer m, where the norm

$$\|\Phi\|_m = \|(I-\mathcal{A})^{m/2}\Phi\| = \{\sum_{\mathbf{n}} (1+\lambda_{\mathbf{n}})^m |\Phi_{\mathbf{n}}|^2\}^{1/2}, \tag{2.6}$$

with I being the identity operator in $\mathcal{H} = \mathcal{H}_0$. For $m \ge 1$, the dual space $\mathcal{H}'_m$ of $\mathcal{H}_m$ is denoted by $\mathcal{H}_{-m}$, and the duality pairing between them will be denoted by $\langle\langle\cdot,\cdot\rangle\rangle_m$ with $\langle\langle\cdot,\cdot\rangle\rangle_1 = \langle\langle\cdot,\cdot\rangle\rangle$. Clearly, the sequence of norms $\{\|\Phi\|_m\}$ is increasing, that is,

$$\|\Phi\|_m < \|\Phi\|_{m+1},$$

for any integer m, and, by identify $\mathcal{H}$ with its dual $\mathcal{H}'$, we have

$$\mathcal{H}_m \subset \mathcal{H}_{m-1} \subset \cdots \subset \mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_{-1} \subset \cdots \subset \mathcal{H}_{-m+1} \subset \mathcal{H}_{-m}, \quad \text{for}\, m \ge 1,$$

and the inclusions are dense and continuous. Of course the spaces $\mathcal{H}_m$ can be defined for any real number m, but they are not needed in this paper.

Owing to the use of the invariant measure μ, it is possible to develop a L^2-theory of infinite-dimensional parabolic and elliptic equations connected to stochastic PDEs. To do so, similar to the finite-dimensional case, the integration by parts is an indispensable technique. In the abstract Wiener Space, the integration by parts with respect the Wiener measure was obtained by Kuo [14]. As a generalization to the Gaussian invariant measure μ, the following integration by parts formula holds (see Lemma 9.2.3 [9]). In this case, instead of the usual derivative $D\Phi$, it is more natural to use the $R-$derivative

$$D_R\Phi = R^{1/2} D\Phi,$$

which can be regarded as a Gross derivative or the derivative of Φ in the direction of $\mathcal{H}_R \doteq \overline{R^{1/2}\mathcal{H}}$.

Proposition 2.4. *Let $g \in \mathcal{H}_R$ and $\Phi, \Psi \in \mathcal{H}_1$. Then we have*

$$\int_H (D_R\Phi, g)\,\Psi\, d\mu = -\int_H \Phi(D_R\Psi, g)\, d\mu + \int_H (v, \Gamma^{-1/2} g)\Phi\,\Psi\, d\mu. \tag{2.7}$$

The following properties of $\mathcal{A}$ are crucial in the subsequent analysis. For now let the differential operator $\mathcal{A}$ given by (2.4) be defined in the set of Hermite polynomial functionals. In fact it can be extended to a self-adjoint linear operator in $\mathcal{H}$. To this end, let $\mathcal{P}_N$ be a projection operator in $\mathcal{H}$ onto its subspace spanned by the Hermite polynomial functionals of degree N and define $\mathcal{A}_N = \mathcal{P}_N\mathcal{A}$. Then the following theorem holds (Theorem 3.1, [2]).

Theorem 2.5. *The sequence $\{\mathcal{A}_N\}$ converges strongly to a linear symmetric operator $\mathcal{A} : \mathcal{H}_2 \to \mathcal{H}$, so that, for $\Phi, \Psi \in \mathcal{H}_2$, the second integration by parts formula holds:*

$$\int_H (\mathcal{A}\Phi, \Psi)\, d\mu = \int_H (\mathcal{A}\Psi)\Phi\, d\mu = -\frac{1}{2}\int_H (D_R\Phi, D_R\Psi)\, d\mu, \tag{2.8}$$

Moreover $\mathcal{A}$ has a self-adjoint extension, still denoted by $\mathcal{A}$ with domain dense in $\mathcal{H}$.

In particular, for $m = 2$, it follows from (2.6) and (2.8) that

Corollary 2.6. *The $\mathcal{H}_1$-norm can be defined as*

$$\|\Phi\|_1 = \{\|\Phi\|^2 + \frac{1}{2}\|D_R\Phi\|^2\}^{1/2}, \tag{2.9}$$

for all $\Phi \in \mathcal{H}_1$, where $D_R\,\Phi \doteq R^{1/2}\, D\Phi$.

Remark 2.7. In (2.9) the factor $\frac{1}{2}$ was not deleted for convenience. Also it becomes clear that the space $\mathcal{H}_1$ consists of all $L^2(\mu)$-functions whose R-derivatives are $\mu-$square-integrable.

Let the functions $F : H \to H$ and $\mathcal{G} : H \to \mathbb{R}$ be bounded and continuous. For $Q \in L^2((0,T); \mathcal{H})$ and $\Theta \in \mathcal{H}$, consider the initial-value problem for the parabolic equation:

$$\begin{aligned} \frac{\partial}{\partial t}\Psi_t(v) &= \mathcal{A}\,\Psi_t(v) - (F(v), D_R\Psi_t(v)) - \mathcal{G}(v)\Psi_t(v) + Q_t(v), \\ \Psi_0(v) &= \Theta(v), \end{aligned} \tag{2.10}$$

for $0 < t < T$, $v \in H$, where $\mathcal{A}$ is given by (2.4). Suppose that the conditions for Theorem 4.2 in [3] are met. Then the following proposition holds.

Proposition 2.8. *The initial-value problem for the parabolic equation (2.10) has a unique solution* $\Psi \in \mathbf{C}([0,T];\mathcal{H}) \cap L^2((0,T);\mathcal{H}_1)$ *such that*

$$\sup_{0\le t\le T} \|\Psi_t\|^2 + \int_0^T \|\Psi_s\|_1^2\, ds \le K(T)\{1 + \|\Theta\|^2 + \int_0^T \|Q_s\|^2\, ds\}, \tag{2.11}$$

where $K(T)$ *is a positive constant depending on* T*.*

Moreover, when $Q_t = Q$ independent of t, it was shown that, as $t \to \infty$, the solution Φ_t of (2.10) approaches the mild solution Φ of the linear elliptic equation

$$-\mathcal{A}\,\Phi(v) + (F(v), D_R\Phi(v)) + \mathcal{G}(v)\Phi(v) = Q(v), \tag{2.12}$$

or, for $\alpha > 0$ and $\mathcal{A}_\alpha = \mathcal{A} + \alpha$, Φ satisfies the equation

$$\Phi(v) = \mathcal{A}_\alpha^{-1}\,\{(F(v), D_R\Phi(v)) + \mathcal{G}(v)\Phi(v) - Q(v)\}, \quad v \in H.$$

In what follows, we shall study the strong solutions (to be defined) of equation (2.12) in an L^2-Gauss-Sobolev space setting and the related eigenvalue problems.

3. Solutions of Linear Elliptic Equations

Let $\mathcal{L}$ denote an linear elliptic operator defined by

$$\mathcal{L}\,\Phi = -\mathcal{A}\,\Phi + \mathcal{F}\,\Phi + \mathcal{G}\,\Phi, \qquad \Phi \in \mathcal{E}_A(H), \tag{3.1}$$

where $\mathcal{A}$ is given by (2.4) and

$$\mathcal{F}\,\Phi = (F\,(\cdot),\, D_R\,\Phi(\cdot)). \tag{3.2}$$

Then, for $\Phi \in \mathcal{H}_1$ and $Q \in \mathcal{H}$, the elliptic equation (2.12) can be written as

$$\mathcal{L}\,\Phi \,=\, Q, \tag{3.3}$$

in a generalized sense. Multiplying the equation (3.1) by $\Psi \in \mathcal{H}_1$ and integrating the resulting equation with respect to μ, we obtain

$$\int_H (\mathcal{L}\Phi)\,\Psi\, d\mu \,=\, \int_H \{\,\frac{1}{2}(D_R\Phi, D_R\Psi) \,+\, (\mathcal{F}\Phi)\,\Psi \,+\, (\mathcal{G}\,\Phi)\,\Psi\,\}d\mu, \tag{3.4}$$

where the second integration by parts formula (2.8) was used.

Associated with $\mathcal{L}$, we define a bilinear form $\mathbf{B}(\cdot,\cdot) : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathbb{R}$ as follows

$$\begin{aligned} \mathbf{B}(\Phi,\Psi) \;&=\; \int_H \{\,\frac{1}{2}(D_R\Phi, D_R\Psi) \,+\, (\mathcal{F}\Phi)\,\Psi \,+\, (\mathcal{G}\,\Phi)\,\Psi\,\}\, d\mu \\ &=\; \frac{1}{2}[D_R\Phi, D_R\Psi] \,+\, [\mathcal{F}\Phi,\Psi] + [\mathcal{G}\Phi,\Psi], \end{aligned} \tag{3.5}$$

for $\Phi, \Psi \in \mathcal{H}_1$.

Now consider a generalized solution of the elliptic equation (3.3). There are several versions of generalized solutions, such as mild solution, strict solution and so on (see [9]). Here, for $Q \in \mathcal{H}_{-1}$, a generalized solution Φ is said to be a strong (or variational) solution of problem (3.3) if $\Phi \in \mathcal{H}_1$ and it satisfies the following equation

$$\mathbf{B}(\Phi,\Psi) \,=\, \langle\!\langle Q, \Psi\rangle\!\rangle, \quad \text{for all } \Psi \in \mathcal{H}_1. \tag{3.6}$$

Lemma 3.1. *(Energy inequalities) Suppose that $F : H \to H$ and $\mathcal{G} : H \to \mathbb{R}$ are bounded and continuous. Then the following inequalities hold.*

There exists a constant $b > 0$ such that

$$|\mathbf{B}(\Phi, \Psi)| \leq b \,\|\Phi\|_1 \,\|\Psi\|_1, \quad \textit{for } \Phi, \Psi \in \mathcal{H}_1, \tag{3.7}$$

and, for any $\varepsilon \in (0, 1/2)$, $\mathbf{B}$ satisfies the coercivity condition:

$$\mathbf{B}(\Phi, \Phi) \geq (\frac{1}{2} - \varepsilon)\|D_R\Phi\|^2 + (\delta - \frac{\beta^2}{4\varepsilon})\|\Phi\|^2, \quad \textit{for } \Phi \in \mathcal{H}_1, \tag{3.8}$$

where $\beta = \sup_{v\in H} |F(v)|$ and $\delta = \inf_{v\in H} \mathcal{G}(v)$.

Proof. From the equations (3.2) and (3.5), we have

$$\begin{aligned} |\mathbf{B}(\Phi, \Psi)| &= |\int_H \{ \frac{1}{2}(D_R\Phi, D_R\Psi) + (F, D_R\Phi)\,\Psi + (\mathcal{G}\,\Phi)\,\Psi \} d\mu| \\ &\leq \frac{1}{2}\|D_R\Phi\| \,\|D_R\Psi\| + \|(F, D_R\Phi)\| \,\|\Psi\| + \|\mathcal{G}\Phi\| \,\|\Psi\| \\ &\leq \frac{1}{2}\|D_R\Phi\| \,\|D_R\Psi\| + \beta \,\|D_R\Phi\| \,\|\Psi\| + \gamma \,\|\Phi\| \,\|\Psi\|, \end{aligned} \tag{3.9}$$

where $\beta = \sup_{v\in H} |F(v)|$ and $\gamma = \sup_{v\in H} |\mathcal{G}(v)|$. It follows from (3.9) that

$$|\mathbf{B}(\Phi, \Psi)| \leq b \,\|\Phi\|_1 \,\|\Psi\|_1,$$

for some suitable constant $b > 0$.

By setting $\Psi = \Phi$ in (3.2) and (3.5), we obtain

$$\begin{aligned} \mathbf{B}(\Phi, \Phi) &= \int_H \{ \frac{1}{2}(D_R\Phi, D_R\Phi) + (F, D_R\Phi)\,\Phi + (\mathcal{G}\,\Phi)\,\Phi \} d\mu \\ &= \frac{1}{2}\|D_R\Phi\|^2 + [(F, D_R\Phi), \Phi] + [\mathcal{G}\Phi, \Phi] \\ &\geq \frac{1}{2}\|D_R\Phi\|^2 - \beta \,\|D_R\Phi\| \,\|\Phi\| + \delta \,\|\Phi\|^2, \end{aligned} \tag{3.10}$$

where $\delta = \inf_{v\in H} \mathcal{G}(v)$.

For any $\varepsilon > 0$, we have

$$\beta \,\|D_R\Phi\| \,\|\Phi\| \leq \varepsilon \,\|D_R\Phi\|^2 + \frac{\beta^2}{4\varepsilon}\|\Phi\|^2. \tag{3.11}$$

By making use of (3.11) in (3.10), we can get the desired inequality (3.8):

$$\mathbf{B}(\Phi, \Phi) \geq (\frac{1}{2} - \varepsilon)\|D_R\Phi\|^2 + (\delta - \frac{\beta^2}{4\varepsilon})\|\Phi\|^2,$$

which completes the proof. □

With the aid of the energy estimates, under suitable conditions on F and $\mathcal{G}$, the following existence theorem can be established.

Theorem 3.2. *(Existence of strong solutions) Suppose the functions $F : H \to H$ and $\mathcal{G} : H \to \mathbb{R}$ are bounded and continuous. Then there is a constant $\alpha_0 \geq 0$ such that for each $\alpha > \alpha_0$ and for any $Q \in \mathcal{H}_{-1}$, the following elliptic problem*

$$\mathcal{L}_\alpha \Phi \doteq \mathcal{L}\,\Phi + \alpha\,\Phi = Q \tag{3.12}$$

has a unique strong solution $\Phi \in \mathcal{H}_1$.

Proof. By definition of a strong solution, we have to show the that there exists a unique solution $\Phi \in \mathcal{H}_1$ satisfying the variational equation

$$\mathbf{B}_\alpha(\Phi, \Psi) \doteq \mathbf{B}(\Phi, \Psi) + \alpha\,[\Phi, \Psi] = \langle\!\langle Q, \Psi \rangle\!\rangle, \tag{3.13}$$

for all $\Psi \in \mathcal{H}_1$.

To this end we will apply the Lax-Milgram Theorem [18] in the real separable Hilbert space $\mathcal{H}_1$. By Lemma 3.1, the inequality (3.7) holds similarly for $\mathbf{B}_\alpha$ with a different constant $b_1 > 0$,

$$|\mathbf{B}_\alpha(\Phi, \Psi)| \leq b_1 \,\|\Phi\|_1 \,\|\Psi\|_1, \quad \text{for } \Phi, \Psi \in \mathcal{H}_1. \tag{3.14}$$

In particular we take $\varepsilon = \dfrac{1}{4}$ and $\alpha_0 = |\delta - \beta^2|$ in the inequality (3.11), which is then used in (3.13) to give

$$\mathbf{B}_\alpha(\Phi, \Phi) \geq \frac{1}{4}\,\|D_R\Phi\|^2 + \eta\,\|\Phi\|^2, \tag{3.15}$$

where $\eta = \alpha - \alpha_0 > 0$ by assumption. It follows that

$$\mathbf{B}_\alpha(\Phi, \Phi) \geq \kappa\,\|\Phi\|_1^2, \tag{3.16}$$

for $\kappa = \min\{\dfrac{1}{4}, \eta\}$. In view of (3.14) and (3.15), the bilinear form $\mathbf{B}_\alpha(\cdot, \cdot)$ satisfies the hypotheses for the Lax-Milgram Theorem.

For $Q \in \mathcal{H}_{-1}$, $\langle\!\langle Q, \cdot \rangle\!\rangle$ defines a bounded linear functional on $\mathcal{H}_1$. Hence there exists a function $\Phi \in \mathcal{H}_1$ which is the unique solution of the equation

$$\mathbf{B}_\alpha(\Phi, \Psi) = \langle\!\langle Q, \Psi \rangle\!\rangle$$

for all $\Psi \in \mathcal{H}_1$. □

Remark 3.3. By writing

$$\mathbf{B}_\alpha(\Phi, \Psi) = \langle\!\langle \mathcal{L}_\alpha \Phi\,,\, \Psi \rangle\!\rangle,$$

it follows from Theorem 3.2 that the mapping $\mathcal{L}_\alpha : \mathcal{H}_1 \to \mathcal{H}_{-1}$ is an isomorphism.

Corollary 3.4. *Suppose that $F \in \mathbf{C}_b(H; H)$ and $\mathcal{G} \in \mathbf{C}_b(H)$ such that*

$$\inf_{v \in H} \mathcal{G}(v) > \sup_{v \in H} |F(v)|^2. \tag{3.17}$$

Then there exists a unique strong solution $\Phi \in \mathcal{H}_1$ of the equation

$$\mathcal{L}\,\Phi = Q.$$

Proof. This follows from the fact that, under condition (3.17), the inequality (3.16) holds with $\alpha = 0$. □

4. Compact Resolvent and Fredholm Alternative

For $U \in \mathcal{H}$, consider the elliptic problem:

$$\mathcal{L}_\lambda \Phi \doteq \mathcal{L}\Phi + \lambda\Phi = U, \tag{4.1}$$

where $\lambda > \alpha_0$ is a real parameter. By Theorem 3.2, the problem (4.1) has a unique strong solution $\Phi \in \mathcal{H}_1$ satisfying

$$\mathbf{B}_\lambda(\Phi, \Psi) = \langle\!\langle \mathcal{L}_\lambda \Phi \,,\, \Psi \rangle\!\rangle = \langle\!\langle U, \Psi \rangle\!\rangle, \tag{4.2}$$

for all $\Psi \in \mathcal{H}_1$. For each $U \in \mathcal{H}$, let us express the solution of (4.1) as

$$\Phi = \mathcal{L}_\lambda^{-1} U. \tag{4.3}$$

Denote the resolvent operator $\mathcal{K}_\lambda$ of $\mathcal{L}$ on $\mathcal{H}$ by

$$\mathcal{K}_\lambda U \doteq \mathcal{L}_\lambda^{-1} U \tag{4.4}$$

for all $U \in \mathcal{H}$.

In the following theorem, we will show that the resolvent operator $\mathcal{K}_\lambda : \mathcal{H} \to \mathcal{H}$ is compact.

Theorem 4.1. *(Compact Resolvent) Under the conditions of Theorem 3.2 with $\lambda = \alpha$, the resolvent operator $\mathcal{K}_\alpha : \mathcal{H} \to \mathcal{H}$ is bounded, linear and compact.*

Proof. By the estimate (3.16) and equation (3.13), we have

$$\begin{aligned} \kappa \,|\!|\!| \Phi |\!|\!|_1^2 \quad &\le \quad \mathbf{B}_\alpha(\Phi, \Phi) = [U \,,\, \Phi] \\ &\le \quad |\!|\!| U |\!|\!| \, |\!|\!| \Phi |\!|\!|_1, \end{aligned}$$

which, in view of (4.3) and (4.4), implies

$$|\!|\!| \Phi |\!|\!|_1 = |\!|\!| \mathcal{K}_\alpha U |\!|\!|_1 \le C \,|\!|\!| U |\!|\!|, \tag{4.5}$$

for $C = \dfrac{1}{\kappa}$. Hence the linear operator $\mathcal{K}_\alpha : \mathcal{H} \to \mathcal{H}$ is bounded.

To show compactness, let $\{U_n\}$ be a bounded sequence in $\mathcal{H}$ with $|\!|\!| U_n |\!|\!| \le C_0$ for some $C_0 > 0$ and for each $n \ge 1$. Define $\Phi_n = \mathcal{K}_\alpha U_n$. Then, by (4.5), we obtain

$$|\!|\!| \Phi_n |\!|\!|_1 \le \frac{1}{\kappa} \,|\!|\!| U_n |\!|\!| \le C_1, \tag{4.6}$$

where $C_1 = \dfrac{C_0}{\kappa}$. It follows that $\{\Phi_n\}$ is a bounded sequence in the separable Hilbert space $\mathcal{H}_1$ and, hence, there exists a subsequence, to be denoted by $\{\Phi_k\}$ for simplicity, which converges weakly to Φ or $\Phi_k \rightharpoonup \Phi$ in $\mathcal{H}_1$. To show that the subsequence will converge strongly in $\mathcal{H}$, by Proposition 2.2, we can express

$$\Phi = \sum_{\mathbf{n} \in \mathbf{Z}} \phi_{\mathbf{n}} H_{\mathbf{n}} \quad \text{and} \quad \Phi_k = \sum_{\mathbf{n} \in \mathbf{Z}} \phi_{\mathbf{n},k} H_{\mathbf{n}}, \tag{4.7}$$

where $\phi_{\mathbf{n}} = [\Phi, H_{\mathbf{n}}]$ and $\phi_{\mathbf{n},k} = [\Phi_k, H_{\mathbf{n}}]$. For any integer $N > 0$, let $\mathbf{Z}_N = \{\mathbf{n} \in \mathbf{Z} : 1 \le |\mathbf{n}| \le N\}$ and $\mathbf{Z}_N^+ = \{\mathbf{n} \in \mathbf{Z} : |\mathbf{n}| > N\}$. By the orthogonality of Hermite functionals $H'_{\mathbf{n}}s$, we can get

$$\begin{aligned} \|\Phi - \Phi_k\|^2 &= \sum_{\mathbf{n}\in\mathbf{Z}_N} (\phi_{\mathbf{n}} - \phi_{\mathbf{n},k})^2 + \sum_{\mathbf{n}\in\mathbf{Z}_N^+} (\phi_{\mathbf{n}} - \phi_{\mathbf{n},k})^2 \\ &\le \sum_{\mathbf{n}\in\mathbf{Z}_N} (\phi_{\mathbf{n}} - \phi_{\mathbf{n},k})^2 + \frac{1}{N} \sum_{\mathbf{n}\in\mathbf{Z}_N^+} (1+\lambda_{\mathbf{n}})(\phi_{\mathbf{n}} - \phi_{\mathbf{n},k})^2 \qquad (4.8) \\ &\le \sum_{\mathbf{n}\in\mathbf{Z}_N} [\Phi_{\mathbf{n}} - \Phi_{\mathbf{n},k}\,,\, H_{\mathbf{n}}]^2 + \frac{\alpha_1}{N} \|\Phi - \Phi_k\|_1^2. \end{aligned}$$

Since $\Phi_k \rightharpoonup \Phi$, by a theorem on the weak convergence in $\mathcal{H}_1$ (p.120, [18]), the subsequence $\{\Phi_k\}$ is bounded such that

$$\sup_{k\ge 1} \|\Phi_k\| \le C, \text{ and } \|\Phi\| \le C, \text{ some constant } C > 0.$$

For the last term in the inequality (4.8), given any $\varepsilon > 0$, there is an integer $N_0 > 0$ such that

$$\frac{\alpha_1}{N} \|\Phi - \Phi_k\|_1^2 \le \frac{2\alpha_1 C^2}{N} < \frac{\varepsilon}{2}, \quad \text{for} \quad N > N_0. \qquad (4.9)$$

Again, by the weak convergence of $\{\Phi_k\}$, for the given N_0, we have

$$\lim_{k\to\infty} \sum_{\mathbf{n}\in\mathbf{Z}_{N_0}} (\phi_{\mathbf{n}} - \phi_{\mathbf{n},k})^2 = \lim_{k\to\infty} \sum_{\mathbf{n}\in\mathbf{Z}_{N_0}} [\Phi_{\mathbf{n}} - \Phi_{\mathbf{n},k}, H_{\mathbf{n}}]^2 = 0.$$

Therefore there is an integer $m > 0$

$$\sum_{\mathbf{n}\in\mathbf{Z}_N} [\Phi_{\mathbf{n}} - \Phi_{\mathbf{n},k}, H_{\mathbf{n}}]^2 < \frac{\varepsilon}{2} \qquad (4.10)$$

for $k > m$. Now the estimates (4.8)–(4.10) implies

$$\lim_{k\to\infty} \|\Phi - \Phi_k\| = 0,$$

which proves the compactness of the resolvent $\mathcal{K}\gamma$. □

Due to the coercivity condition (3.16), the compactness of the resolvent operator implies the following fact.

Theorem 4.2. *The embedding of $\mathcal{H}_1$ into $\mathcal{H}$ is compact.*

In the Wiener space, a direct proof of this important result was given in [10] and [16].

To define the adjoint operator of $\mathcal{L}$, we first introduce the divergence operator $D^\star$. Let $F \in \mathbf{C}_b^1(H; H)$ be expanded in terms of the eigenfunctions $\{e_k\}$ of A:

$$F = \sum_{k=1}^{\infty} f_k\, e_k,$$

where $f_k = (F, e_k)$. Then the divergence $D^\star F$ of F is defined as

$$D^\star F \doteq Tr\,(DF) = \sum_{k=1}^{\infty} (D\, f_k, e_k).$$

Recall the covariant operator $\Gamma = (-1)A^{-1}R$ for μ. We shall need the following integration by parts formula.

Lemma 4.3. *Suppose that the function $F : H \to \Gamma(H) \subset H$ is bounded continuous and differentiable such that*

$$\sup_{v\in H} |TrDF(v)| < \infty. \tag{4.11}$$

and

$$\sup_{v\in H} |(\Gamma^{-1}F(v), v)| < \infty. \tag{4.12}$$

Then, for $\Phi, \Psi \in \mathcal{H}_1$, the following equation holds

$$\int_H (D\Phi, F)\Psi\, d\mu = -\int_H \Phi\, D^\star(\Psi\, F)\, d\mu\, + \int_H \langle\, \Gamma^{-1}F(v), v\,\rangle \Phi\, \Psi\, d\mu. \tag{4.13}$$

Proof. The proof is similar to that of Lemma 9.2.3 in [8], it will only be sketched. Let

$$F_n \doteq \sum_{k=1}^{n} f_k\, e_k, \tag{4.14}$$

with $f_k = (F, e_k)$. Then the sequence $\{F_n\}$ converges strongly to F in $\mathcal{H}$. In view of (4.14), we have

$$\int_H (D\Phi, F_n)\Psi\, d\mu = \sum_{k=1}^{n} \int_H (D\Phi, e_k) f_k \Psi\, d\mu. \tag{4.15}$$

By invoking the first integral-by-parts formula (2.7),

$$\int_H (D\Phi, e_k) f_k \Psi\; = -\int_H \Phi(D(\Psi f_k), e_k)\, d\mu + \int_H (v, \Gamma^{-1}e_k)) f_k \Phi\Psi\, d\mu,$$

so that (4.15) yields

$$\begin{aligned}\int_H (D\Phi, F_n)\Psi\, d\mu = &-\sum_{k=1}^{n} \int_H \Phi D^\star(\Psi f_k e_k)\, d\mu \\ &+\sum_{k=1}^{n} \int_H (v, \Gamma^{-1}e_k) f_k \Phi\Psi\, d\mu \qquad (4.16)\\ = -\int_H \Phi D^\star(\Psi F_n)\, d\mu\, &+ \int_H (v, \Gamma^{-1}F_n(v))\, \Phi\Psi d\mu(v).\end{aligned}$$

Now the formula (4.13) follows from (4.15) by taking the limit termwise as $n \to \infty$. □

Let $\Phi, \Psi \in \mathcal{E}_A(H)$ and let $F, \mathcal{G}$ be given as in by Lemma 4.3. we can write

$$[\mathcal{L}\Phi, \Psi] = [\Phi, \mathcal{L}^\star\Psi],$$

where $\mathcal{L}^\star$ is the formal adjoint of $\mathcal{L}$ defined by

$$\mathcal{L}^\star\Psi \doteq -\mathcal{A}\Psi + D^\star(\Psi\, F)\, - \langle\Gamma^{-1}F(v), v\rangle\, \Psi - \mathcal{G}\, \Psi. \tag{4.17}$$

The associated bilinear form $\mathbf{B}^\star : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathbb{R}$ is given by

$$\mathbf{B}^\star(\Phi, \Psi)\, =\, \mathbf{B}(\Psi, \Phi),$$

for all $\Phi, \Psi \in \mathcal{H}_1$. For $Q \in \mathcal{H}$, consider the adjoint problem

$$\mathcal{L}^\star \Psi \; = Q. \tag{4.18}$$

A function $\Psi \in \mathcal{H}_1$ is said to be a strong solution of (4.16) provided that

$$\mathbf{B}^\star(\Psi, \Phi) \;\; = [Q, \Phi]$$

for all $\Phi \in \mathcal{H}_1$.

Now, for $U \in \mathcal{H}$, consider the nonhomogeneous problem

$$\mathcal{L}\,\Phi \; = \; U, \tag{4.19}$$

and the related homogeneous problems

$$\mathcal{L}\,\Phi \; = \; 0, \tag{4.20}$$

and

$$\mathcal{L}^\star\,\Psi \; = 0. \tag{4.21}$$

Let $\mathcal{N}$ and $\mathcal{N}^\star$ denote, respectively, the subspaces of solutions of (4.20) and (4.21) in $\mathcal{H}_1$. Then, by applying the Fredholm theory of compact operators [18], we can prove the following theorem.

Theorem 4.4. *(Fredholm Alternative) Let $\mathcal{L}$ and $\mathcal{L}^\star$ be defined by (3.1) and (4.17) respectively, in which F satisfies the conditions (4.11) and (4.12) in Lemma 4.3.*

(1) *Exactly one of the following statements is true:*
(a) For each $Q \in \mathcal{H}$, the nonhomogeneous problem (4.19) has a unique strong solution.
(b) The homogeneous problem (4.20) has a nontrivial solution.
(2) *If case (b) holds, the dimension of null space $\mathcal{N}$ is finite and equals to the dimension of $\mathcal{N}^\star$.*
(3) *The nonhomogeneous problem (4.19) has a solution if and only if*

$$[U, \Psi] \; = \; 0, \quad \textit{for all } \Psi \in \mathcal{N}^\star.$$

Proof. To prove the theorem, we shall convert the differential equations into equivalent Fredholm type of equations involving a compact operator. To proceed let α be given as in Theorem 4.1 and rewrite equation (4.19) as

$$\mathcal{L}_\alpha \Phi \; \doteq \; \mathcal{L}\Phi + \alpha\Phi = \alpha\Phi + U.$$

By theorem 4.1, the equation (4.19) is equivalent to the equation

$$\Phi \; = \; \mathcal{K}_\alpha\,(\alpha\Phi + U),$$

which can be rewritten as the Fredholm equation

$$(I - \mathcal{T})\,\Phi \; = \; Q, \tag{4.22}$$

where I is the identity operator on $\mathcal{H}$,

$$\mathcal{T} = \alpha\,\mathcal{K}_\alpha$$

and

$$Q \; = \; \mathcal{K}_\alpha U.$$

Since $\mathcal{K}_\alpha : \mathcal{H} \to \mathcal{H}$ is compact, $\mathcal{T}$ is also compact and Q belongs to $\mathcal{H}$. By applying the Fredholm Alternative Theorem [11] to equation (4.22), the equivalent statements (1)–(3) hold for the Fredholm operator $(I-\mathcal{T})$. Due to the equivalence of the problems (4.19) and (4.22), the theorem is thus proved. □

5. Spectrum and Eigenvalue Problem

For $\lambda \in \mathbb{R}$, consider strong solutions of the eigenvalue problem:

$$\mathcal{L}\,\Phi \,=\, \lambda\Phi. \tag{5.1}$$

Here, for simplicity, we only treat the case of real solutions. As usual a nontrivial solution Φ of (5.1) is called an eigenfunction and the corresponding real number λ, an eigenvalue of $\mathcal{L}$. The (real) spectrum Σ of $\mathcal{L}$ consists of all of its eigenvalues.

Theorem 5.1. *(Spectral Property) The spectrum Σ of $\mathcal{L}$ is at most countable. If the set Σ is infinite, then $\Sigma = \{\lambda_k \in \mathbb{R} : k \geq 1\}$ with $\lambda_k \leq \lambda_{k+1}$, each with a finite multiplicity, for $k \geq 1$, and $\lambda_k \to \infty$ as $k \to \infty$.*

Proof. By taking a real number α, rewrite the equation (5.1) as

$$\mathcal{L}_\alpha \doteq \mathcal{L}\,\Phi + \alpha\Phi \,=\, (\lambda+\alpha)\,\Phi. \tag{5.2}$$

By taking $\alpha > \alpha_0$ as in Theorem 4.2, the equation (5.2) can be converted into the eigenvalue problem for the resolvent operator

$$\mathcal{K}_\alpha\,\Phi \,=\, \rho\,\Phi, \tag{5.3}$$

where

$$\rho = \frac{1}{\lambda+\alpha}. \tag{5.4}$$

By Theorem 4.1, the resolvent operator $\mathcal{K}_\alpha$ on $\mathcal{H}$ is compact. Therefore its spectrum Σ_α is discrete. If the spectrum is infinite, then $\Sigma_\alpha = \{\rho_k \in \mathbb{R} : k \geq 1\}$ with $\rho_k \geq \rho_{k+1}$, each of a finite multiplicity, and $\lim\limits_{k\to\infty} \rho_k = 0$. Now it follows from equation (5.4) that spectrum Σ of $\mathcal{L}$ has the asserted property with $\lambda_k = \dfrac{1}{\rho_k} - \alpha$. □

Remark 5.2. As in finite dimensions, the eigenvalue problem (5.1) may be generalized to the case of complex-valued solutions in a complex Hilbert space. In this case the eigenvalues λ_k may be complex.

As a special case, set $F \equiv 0$ in $\mathcal{L}$ and the reduced operator $\mathcal{L}_0$ is given by

$$\mathcal{L}_0\,\Phi \,=\, (-\mathcal{A})\,\Phi \,+\, \mathcal{G}\,\Phi.$$

Clearly $\mathcal{L}_0$ is a formal self-adjoint operator, or $\mathcal{L}_0 = \mathcal{L}_0^\star$. The corresponding bilinear form

$$\mathbf{B}_0(\Phi,\Psi) \,=\, \langle\!\langle \mathcal{L}_0\,\Phi\,,\,\Psi \rangle\!\rangle \,=\, \frac{1}{2}\,[RD\Phi, D\Psi] \,+\, [G\Phi,\Psi], \tag{5.5}$$

for $\Phi,\Psi \in \mathcal{H}_1$

Consider the eigenvalue problem:

$$\mathcal{L}_0\,\Phi \,=\, \lambda\,\Phi. \tag{5.6}$$

For the special case when $\mathcal{G} = 0$, $\mathcal{L}_0 = -\mathcal{A}$. The eigenvalues and the eigenfunctions of $(-\mathcal{A})$ were given explicitly in Proposition 2.2. The results show that the eigenvalues can be ordered as a non-increasing sequence $\{\lambda_k\}$ and the corresponding eigenfunctions are orthonormal Hermite polynomial functionals. With a smooth perturbation by $\mathcal{G}$, similar results will hold for the eigenvalue problem (5.6) as stated in the following theorem.

Theorem 5.3. *(Symmetric Eigenvalue Problem) Suppose that $\mathcal{G} : H \to \mathbb{R}^+$ be a bounded, continuous and positive function, and there is a constant $\delta > 0$ such that $\mathcal{G}(v) \geq \delta$, $\forall\, v \in H$. Then the following statements hold:*

(1) *Each eigenvalue of $\mathcal{L}_0$ is positive with finite multiplicity. The set Σ_0 of eigenvalues forms a nondecreasing sequence (counting multiplicity)*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \lambda_k \leq \cdots$$

such that $\lambda_k \to \infty$ as $k \to \infty$.

(2) *There exists an orthonormal basis $\Phi_k, k = 1, 2, \cdots$ of $\mathcal{H}$, where Φ_k is an eigenfunction as a strong solution of*

$$\mathcal{L}_0\, \Phi_k \;=\; \lambda_k\, \Phi_k,$$

for $k = 1, 2, \cdots$.

Proof. By the assumption on $\mathcal{G}$, it is easy to check that the bilinear form $\mathbf{B}_0 : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathbb{R}$ satisfies the conditions of Theorem 4.1. Hence the inverse $\mathcal{K}_0 \doteq \mathcal{L}_0^{-1}$ is a self-adjoint compact operator in $\mathcal{H}$. Similar to the proof of Theorem 5.1, by converting the problem (5.6) into an equivalent eigenvalue problem for $\mathcal{K}_0$, the statements (1) and (2) are well-known spectral properties of a self-adjoint, compact operator in a separable Hilbert space. □

As in finite dimensions the smallest or the principal eigenvalue λ_1 can be characterized by a variational principle.

Theorem 5.4. *(The Principal Eigenvalue) The principal eigenvalue can be obtained by the variational formula*

$$\lambda_1 \;=\; \inf_{\Phi \in \mathcal{H}_1, |\!|\!|\Phi|\!|\!| \neq 0} \frac{\mathbf{B}_0(\Phi, \Phi)}{|\!|\!|\Phi|\!|\!|^2}. \tag{5.7}$$

Proof. To verify the variation principle, a strong solution Φ of equation (5.6) must satisfy

$$\mathbf{B}_0(\Phi, \Phi) \;=\; \lambda\, [\Phi, \Phi]$$

or

$$\lambda \;=\; \mathbf{J}(\Phi) \;\doteq\; \frac{\mathbf{B}_0(\Phi, \Phi)}{|\!|\!|\Phi|\!|\!|^2}, \tag{5.8}$$

provided that $|\!|\!|\Phi|\!|\!| \neq 0$. In view of the coercivity condition

$$\mathbf{B}_0(\Phi, \Phi) \;\geq\; \kappa\, |\!|\!|\Phi|\!|\!|_1^2 \tag{5.9}$$

for some $\kappa > 0$, J is bounded from below so that the minimal value of $\mathbf{J}$ is given by

$$\lambda^\star \;=\; \inf_{\Psi \in \mathcal{H}_1, |\!|\!|\Psi|\!|\!| \neq 0} \mathbf{J}(\Psi), \tag{5.10}$$

which can also be written as

$$\lambda^\star = \inf_{\Psi\in\mathcal{H}_1, \|\Psi\|=1} \mathbf{Q}(\Psi), \tag{5.11}$$

where we set $\mathbf{Q}(\Phi) = \mathbf{B}_0(\Phi, \Phi)$.

To show that $\lambda^\star = \lambda_1$ and the minimizer of $\mathbf{Q}$ gives rise to the principal eigenfunction Φ_1, choose a minimizing sequence $\{\Psi_n\}$ in $\mathcal{H}_1$ with $\|\Psi_n\|_1 = 1$ such that $\mathbf{Q}(\Psi_n) \to \lambda_1$ as $n \to \infty$. By (5.9) and the boundedness of the sequence $\{\Psi_n\}$ in $\mathcal{H}_1$, the compact embedding Theorem 4-2 implies the existence of a subsequence, to be denoted by $\{\Psi_k\}$, which converges to a function $\Psi \in \mathcal{H}_1$ with $\|\Psi\|_1 = 1$. Since $\mathbf{Q}$ is a quadratic functional, by the Parallelogram Law and equation (5.11), we have

$$\begin{aligned}\mathbf{Q}(\frac{\Psi_j - \Psi_k}{2}) &= \frac{1}{2}(\mathbf{Q}(\Psi_j) + \mathbf{Q}(\Psi_k)) - \mathbf{Q}(\frac{\Psi_j + \Psi_k}{2}) \\ &\le \frac{1}{2}(\mathbf{Q}(\Psi_j) + \mathbf{Q}(\Psi_k) - \lambda^\star \|(\frac{\Psi_j + \Psi_k}{2})\| \to 0\end{aligned}$$

as $j, k \to \infty$. Again, by (5.9), we deduce that $\{\Psi_k\}$ is a Cauchy sequence in $\mathcal{H}_1$ which converges to $\Psi \in \mathcal{H}_1$ and $\mathbf{Q}(\Psi) = \lambda^\star$. Now, for $\Theta \in \mathcal{H}_1$ and $t \in \mathbb{R}$, let

$$f(t) = \mathbf{J}(\Psi + t\,\Theta).$$

As well-known in the calculus of variations, for $\mathbf{J}$ to attain its minimum at Ψ, it is necessary that

$$f'(0) = 2\{\mathbf{B}_0(\Psi, \Theta) - \lambda^\star [\Phi, \Theta]\} = 0,$$

which shows Ψ is the strong solution of

$$\mathcal{L}_0(\Psi) = \lambda^\star \Psi.$$

Hence, in view of (5.8), we conclude that $\lambda^\star = \lambda_1$ and Ψ is the eigenfunction associated with the principal eigenvalue. □

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P.L. Chow: Department of Mathematics, Wayne State University, Detroit, Michigan 48202, USA

E-mail address: plchow@math.wayne.edu