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QWN-CONSERVATION OPERATOR AND ASSOCIATED WICK DIFFERENTIAL EQUATION

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Abstract. In this paper we introduce the quantum white noise (QWN) conservation operator N^Q acting on nuclear algebra of white noise operators $\mathcal{L}(\mathcal{F}_\theta(\mathcal{S}'_{\mathbb{C}}(\mathbb{R})), \mathcal{F}^*_\theta(\mathcal{S}'_{\mathbb{C}}(\mathbb{R})))$ endowed with the Wick product. Similarly to the classical case, we give a useful integral representation in terms of the QWN-derivatives $\{D_t^-, D_t^+;\ t \in \mathbb{R}\}$ for the QWN-conservation operator from which it follows that the QWN-conservation operator is a Wick derivation. Via this property, a relation with the Cauchy problem associated to the QWN-conservation operator and the Wick differential equation is worked out.

1. Introduction

Piech [21] initiated the study of an infinite dimensional analogue of a finite dimensional Laplacian on infinite dimensional abstract Wiener space. This infinite dimensional Laplacian (called the number operator) has been extensively studied in [16, 17] and the references cited therein. In particular, Kuo [15] formulated the number operator as continuous linear operator acting on the space of test white noise functionals (E). By virtue of the general theory based on an infinite dimensional analogue of Schwarz's distribution theory, where the Lebesgue measure on $\mathbb{R}$ and the Gel'fand triple

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R}) \tag{1.1}$$

are replaced respectively by the Gaussian measure μ on $\mathcal{S}'(\mathbb{R})$ and the following Gel'fand triple of test function space $\mathcal{F}_\theta(\mathcal{S}'_{\mathbb{C}}(\mathbb{R}))$ and generalized function space $\mathcal{F}^*_\theta(\mathcal{S}'_{\mathbb{C}}(\mathbb{R}))$

$$\mathcal{F}_\theta(\mathcal{S}'_{\mathbb{C}}(\mathbb{R})) \subset L^2(\mathcal{S}'(\mathbb{R}), \mu) \subset \mathcal{F}^*_\theta(\mathcal{S}'_{\mathbb{C}}(\mathbb{R})), \tag{1.2}$$

see for more details [5] and if we employ a discrete coordinate, the number operator N has the following expressions:

$$N = \sum_{k=1}^{\infty} \partial^*_{e_k}\, \partial_{e_k}, \tag{1.3}$$

where $\{e_n;\ n \geq 0\}$ is an arbitrary orthonormal basis for $L^2(\mathbb{R})$, ∂_{e_k} denotes the derivative in the direction e_k acting on $\mathcal{F}_\theta(\mathcal{S}'_{\mathbb{C}}(\mathbb{R}^d))$ and $\partial^*_{e_k}$ is the adjoint

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of ∂_{e_k}. For details see [16], [17]. In [2], the conservation operator $N(K)$, for $K \in \mathcal{L}(\mathcal{S}'_{\mathbb{C}}(\mathbb{R}), \mathcal{S}'_{\mathbb{C}}(\mathbb{R}))$, is given by

$$N(K)\varphi(x) = \sum_{n=0}^{\infty} n\langle x^{\otimes n}, (K\widehat{\otimes} I^{\otimes(n-1)})\varphi_n\rangle, \tag{1.4}$$

from which it is obvious that $N(I) = N$. Using the S-transform it is well known that $N(K)$ is a wick derivation of distributions, see [4], moreover we have

$$N(K) = \int_{\mathbb{R}^2} \tau_K(s,t)x(s) \diamond a_t ds dt. \tag{1.5}$$

In the present paper, by using the new idea of QWN-derivatives pointed out by Ji-Obata in [9, 8], we extend some results contained in [2] to the QWN domains. For $B_1, B_2 \in \mathcal{L}(\mathcal{S}'_{\mathbb{C}}(\mathbb{R}), \mathcal{S}'_{\mathbb{C}}(\mathbb{R}))$, the QWN-analogous $N^Q_{B_1,B_2}$ stands for appropriate QWN counterpart of the conservation operator in (1.3). In the first main result we show that $N^Q_{B_1,B_2}$ has functional integral representations in terms of the QWN-derivatives $\{D^-_t,\ D^+_t;\ t \in \mathbb{R}\}$ and a suitable Wick product $\diamond$ on the class of white noise operators as a quantum white noise analogue of (1.5). The second remarkable feature is that $N^Q_{B_1,B_2}$ behaviors as a Wick derivation of operators. This enable us to give a relation between the Cauchy problem associated to $N^Q_{B_1,B_2}$

$$\frac{\partial}{\partial t} U_t = N^Q_{B_1,B_2} U_t\,, \quad U_0 \in \mathcal{L}(\mathcal{F}_\theta(\mathcal{S}'_{\mathbb{C}}(\mathbb{R})), \mathcal{F}^*_\theta(\mathcal{S}'_{\mathbb{C}}(\mathbb{R}))), \tag{1.6}$$

and the Wick differential equation introduced in [10] as follows

$$\mathfrak{D}Y = G \diamond Y, \quad G \in \mathcal{L}(\mathcal{F}_\theta(\mathcal{S}'_{\mathbb{C}}(\mathbb{R})), \mathcal{F}^*_\theta(\mathcal{S}'_{\mathbb{C}}(\mathbb{R}))) \tag{1.7}$$

where $\mathfrak{D}$ is a Wick derivation. It is well known that (see [10]) if there exists an operator Y in the algebra $\mathcal{L}(\mathcal{F}_\theta(\mathcal{S}'_{\mathbb{C}}(\mathbb{R})), \mathcal{F}^*_\theta(\mathcal{S}'_{\mathbb{C}}(\mathbb{R})))$ such that $\mathfrak{D}Y = G$ and $wexpY := e^{\diamond Y}$ is defined in $\mathcal{L}(\mathcal{F}_\theta(\mathcal{S}'_{\mathbb{C}}(\mathbb{R})), \mathcal{F}^*_\theta(\mathcal{S}'_{\mathbb{C}}(\mathbb{R})))$, then every solution to (1.7), is of the form:

$$\Xi = (wexpY) \diamond F,$$

where $F \in \mathcal{L}(\mathcal{F}_\theta(\mathcal{S}'_{\mathbb{C}}(\mathbb{R})), \mathcal{F}^*_\theta(\mathcal{S}'_{\mathbb{C}}(\mathbb{R})))$ satisfying $\mathfrak{D}F = 0$. More precisely, an important example of the Wick differential equation associated with the QWN-conservation operator is studied where the solution of a system of equations of type (1.7) is given explicitly in terms of the solution of the associated Cauchy problem associated to QWN-conservation operator.

The paper is organized as follows. In Section 2, we briefly recall well-known results on nuclear algebra of entire holomorphic functions. In Section 3, we reformulate in our setting the creation derivative and annihilation derivative as well as their adjoints. Then, we introduce the QWN-conservation operator acting on $\mathcal{L}(\mathcal{F}_\theta(\mathcal{S}'_{\mathbb{C}}(\mathbb{R})), \mathcal{F}^*_\theta(\mathcal{S}'_{\mathbb{C}}(\mathbb{R})))$. As a main result, we give a useful integral representation for the QWN-conservation operator from which it follows that the QWN-conservation operator is a Wick derivation. In Section 4, we find a connection between the solution of a continuous system of QWN-differential equations and the solution of the Cauchy problem associated to the QWN-conservation operator $N^Q_{B_1,B_2}$.

2. Preliminaries

Let H be the real Hilbert space of square integrable functions on $\mathbb{R}$ with norm $|\cdot|_0$, $E \equiv \mathcal{S}(\mathbb{R})$ and $E' \equiv \mathcal{S}'(\mathbb{R})$ be the Schwarz space consisting of rapidly decreasing C^∞-functions and the space of the tempered distributions, respectively. Then, the Gel'fand triple (1.1) can be reconstructed in a standard way (see Ref. [17]) by the harmonic oscillator $A = 1 + t^2 - d^2/dt^2$ and H. The eigenvalues of A are $2n+2$, $n = 0, 1, 2, \cdots$, the corresponding eigenfunctions $\{e_n;\ n \geq 0\}$ form an orthonormal basis for $L^2(\mathbb{R})$ and each e_n is an element of E. In fact E is a nuclear space equipped with the Hilbertian norms

$$|\xi|_p = |A^p \xi|_0\,, \qquad \xi \in E, \quad p \in \mathbb{R}$$

and we have

$$E = \operatorname{proj}\lim_{p\to\infty} E_p\,, \qquad E' = \operatorname{ind}\lim_{p\to\infty} E_{-p},$$

where, for $p \geq 0$, E_p is the completion of E with respect to the norm $|\cdot|_p$ and E_{-p} is the topological dual space of E_p. We denote by $N = E + iE$ and $N_p = E_p + iE_p$, $p \in \mathbb{Z}$, the complexifications of E and E_p, respectively.

Throughout the paper, we fix a Young function θ, i.e. a continuous, convex and increasing function defined on $\mathbb{R}_+$ and satisfies the two conditions: $\theta(0) = 0$ and $\lim_{x\to\infty} \theta(x)/x = +\infty$. The polar function θ^* of θ, defined by

$$\theta^*(x) = \sup_{t\geq 0}(tx - \theta(t)), \quad x \geq 0,$$

is also a Young function. For more details , see Refs. [5], [12] and [18]. For a complex Banach space $(B, \|\cdot\|)$, let $\mathcal{H}(B)$ denotes the space of all entire functions on B, i.e. of all continuous $\mathbb{C}$-valued functions on B whose restrictions to all affine lines of B are entire on $\mathbb{C}$. For each $m > 0$ we denote by $\operatorname{Exp}(B, \theta, m)$ the space of all entire functions on B with $\theta-$exponential growth of finite type m, i.e.

$$\operatorname{Exp}(B,\theta,m) = \Big\{ f \in \mathcal{H}(B);\ \ \|f\|_{\theta,m} := \sup_{z\in B} |f(z)| e^{-\theta(m\|z\|)} < \infty \Big\}.$$

The projective system $\{\operatorname{Exp}(N_{-p},\theta,m);\ p \in \mathbb{N},\ m > 0\}$ and the inductive system $\{\operatorname{Exp}(N_p,\theta,m);\ p \in \mathbb{N},\ m > 0\}$ give the two spaces

$$\mathcal{F}_\theta(N') = \operatorname{proj}\lim_{p\to\infty; m\downarrow 0} \operatorname{Exp}(N_{-p},\theta,m)\,, \qquad \mathcal{G}_\theta(N) = \operatorname{ind}\lim_{p\to\infty; m\to 0} \operatorname{Exp}(N_p,\theta,m). \tag{2.1}$$

It is noteworthy that, for each $\xi \in N$, the exponential function

$$e_\xi(z) := e^{\langle z,\xi\rangle}, \quad z \in N',$$

belongs to $\mathcal{F}_\theta(N')$ and the set of such test functions spans a dense subspace of $\mathcal{F}_\theta(N')$.

We are interested in continuous linear operators from $\mathcal{F}_\theta(N')$ into its topological dual space $\mathcal{F}^*_\theta(N')$. The space of such operators is denoted by $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}^*_\theta(N'))$ and assumed to carry the bounded convergence topology. A typical examples of elements in $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}^*_\theta(N'))$, that will play a key role in our development, are Hida's white noise operators a_t. For $z \in N'$ and $\varphi(x)$ with Taylor expansions

$\sum_{n=0}^{\infty}\langle x^{\otimes n}, f_n\rangle$ in $\mathcal{F}_\theta(N')$, the holomorphic derivative of φ at $x \in N'$ in the direction z is defined by

$$(a(z)\varphi)(x) := \lim_{\lambda\to 0}\frac{\varphi(x+\lambda z)-\varphi(x)}{\lambda}. \tag{2.2}$$

We can check that the limit always exists, $a(z) \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta(N'))$ and $a^*(z) \in \mathcal{L}(\mathcal{F}_\theta^*(N'), \mathcal{F}_\theta^*(N'))$, where $a^*(z)$ is the adjoint of $a(z)$, i.e., for $\Phi \in \mathcal{F}_\theta^*(N')$ and $\phi \in \mathcal{F}_\theta(N')$, $\langle\langle a^*(z)\Phi, \phi\rangle\rangle = \langle\langle \Phi, a(z)\phi\rangle\rangle$. If $z = \delta_t \in E'$ we simply write a_t instead of $a(\delta_t)$ and the pair $\{a_t,\ a_t^*\}$ will be referred to as the QWN-process. In quantum field theory a_t and a_t^* are called the annihilation operator and creation operator at the point $t \in \mathbb{R}$. By a straightforward computation we have

$$a_t e_\xi = \xi(t)\, e_\xi\,, \quad \xi \in N. \tag{2.3}$$

Similarly as above, for $\psi \in \mathcal{G}_{\theta^*}(N)$ with Taylor expansion $\psi(\xi) = \sum_n \langle \psi_n, \xi^{\otimes n}\rangle$ where $\psi_n \in N'^{\otimes n}$, we use the common notation $a(z)\psi$ for the derivative (2.2).

The Wick symbol of $\Xi \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$ is by definition [17] a $\mathbb{C}$-valued function on $N \times N$ defined by

$$\sigma(\Xi)(\xi,\eta) = \langle\langle \Xi e_\xi, e_\eta\rangle\rangle e^{-\langle \xi,\eta\rangle}, \quad \xi,\eta \in N. \tag{2.4}$$

By a density argument, every operator in $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$ is uniquely determined by its Wick symbol. Moreover, if $\mathcal{G}_{\theta^*}(N \oplus N)$ denotes the nuclear space obtained as in (2.1) by replacing N_p by $N_p \oplus N_p$, see [12], we have the following characterization theorem for operator Wick symbols.

Theorem 2.1. *(See Refs.* [12]*) The Wick symbol map yields a topological isomorphism between $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$ and $\mathcal{G}_{\theta^*}(N \oplus N)$.*

In the remainder of this paper, for the sake of readers convenience, we simply use the name symbol for the transformation σ.

Let μ be the standard Gaussian measure on E' uniquely specified by its characteristic function

$$e^{-\frac{1}{2}|\xi|_0^2} = \int_{E'} e^{i\langle x,\xi\rangle}\mu(dx), \quad \xi \in E.$$

In all the remainder of this paper we assume that the Young function θ satisfies the following condition

$$\limsup_{x\to\infty}\frac{\theta(x)}{x^2} < +\infty. \tag{2.5}$$

It is shown in Ref. [5] that, under this condition, we have the nuclear Gel'fand triple (1.2). Moreover, we observe that $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta(N'))$, $\mathcal{L}(\mathcal{F}_\theta^*(N'), \mathcal{F}_\theta(N'))$ and $\mathcal{L}(L^2(E',\mu), L^2(E',\mu))$ can be considered as subspaces of $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$. Furthermore, identified with its restriction to $\mathcal{F}_\theta(N')$, each operator Ξ in the space $\mathcal{L}(\mathcal{F}_\theta^*(N'), \mathcal{F}_\theta^*(N'))$ will be considered as an element of $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$, so that we have the continuous inclusions

$$\mathcal{L}(\mathcal{F}_\theta^*(N'), \mathcal{F}_\theta^*(N')) \subset \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N')),$$

$$\mathcal{L}(\mathcal{F}_\theta^*(N'), \mathcal{F}_\theta(N')) \subset \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N')).$$

It is a fundamental fact in QWN theory [17] (see, also Ref. [12]) that every white noise operator $\Xi \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$ admits a unique Fock expansion

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \tag{2.6}$$

where, for each pairing $l, m \geq 0$, $\kappa_{l,m} \in (N^{\otimes(l+m)})'_{sym(l,m)}$ and $\Xi_{l,m}(\kappa_{l,m})$ is the integral kernel operator characterized via the symbol transform by

$$\sigma(\Xi_{l,m}(\kappa_{l,m}))(\xi, \eta) = \langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle, \quad \xi, \eta \in N. \tag{2.7}$$

This can be formally reexpressed as

$$\Xi_{l,m}(\kappa_{l,m}) = \int_{\mathbb{R}^{l+m}} \kappa_{l,m}(s_1, \cdots, s_l, t_1, \cdots, t_m) \\ a^*_{s_1} \cdots a^*_{s_l} a_{t_1} \cdots a_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m.$$

In this way $\Xi_{l,m}(\kappa_{l,m})$ can be considered as the operator polynomials of degree $l+m$ associated to the distribution $\kappa_{l,m} \in (N^{\otimes(l+m)})'_{sym(l,m)}$ as coefficient; and therefore every white noise operator is a "function" of the QWN. This gives a natural idea for defining the derivatives of an operator $\Xi \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$ with respect to the QWN coordinate system $\{a_t,\ a_t^*;\ t \in \mathbb{R}\}$.

From Refs. [7] and [8], (see also Refs. [9] and [1]), we summarize the novel formalism of QWN-derivatives. For $\zeta \in N$, then $a(\zeta)$ extends to a continuous linear operator from $\mathcal{F}_\theta^*(N')$ into itself (denoted by the same symbol) and $a^*(\zeta)$ (restricted to $\mathcal{F}_\theta(N')$) is a continuous linear operator from $\mathcal{F}_\theta(N')$ into itself. Thus, for any white noise operator $\Xi \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$, the commutators

$$[a(\zeta), \Xi] = a(\zeta)\Xi - \Xi\, a(\zeta), \quad [a^*(\zeta), \Xi] = a^*(\zeta)\Xi - \Xi\, a^*(\zeta),$$

are well defined white noise operators in $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$. The *QWN-derivatives* are defined by

$$D_\zeta^+ \Xi = [a(\zeta), \Xi], \quad D_\zeta^- \Xi = -[a^*(\zeta), \Xi]. \tag{2.8}$$

These are called the *creation derivative* and *annihilation derivative* of Ξ, respectively.

3. QWN-Conservation Operator

In the following technical lemma, by using the symbol transform σ, we reformulate the QWN-derivatives $D_z^\pm$ as natural QWN counterparts of the partial derivatives $\partial_{1,x_1} \equiv \frac{\partial}{\partial x_1}$ and $\partial_{2,x_2} \equiv \frac{\partial}{\partial x_2}$ on the space of entire functions with two variables $g(x_1, x_2)$ in $\mathcal{G}_{\theta^*}(N \oplus N)$. More precisely, for $x_1, x_2, z \in N$,

$$(\partial_{1,z} g)(x_1, x_2) := \lim_{\lambda \to 0} \frac{g(x_1 + \lambda z, x_2) - g(x_1, x_2)}{\lambda}, \tag{3.1}$$

$$(\partial_{2,z} g)(x_1, x_2) := \lim_{\lambda \to 0} \frac{g(x_1, x_2 + \lambda z) - g(x_1, x_2)}{\lambda}. \tag{3.2}$$

Then, in view of Theorem 2.1 and using the same technic of calculus used in [3], we have the following

Lemma 3.1. *Let be given $z \in N$. The creation derivative and annihilation derivative of $\Xi \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$ are given by*

$$D_z^- \Xi = \sigma^{-1} \partial_{1,z} \sigma(\Xi) \quad \text{and} \quad D_z^+ \Xi = \sigma^{-1} \partial_{2,z} \sigma(\Xi).$$

Moreover, their dual adjoints are given by

$$(D_z^-)^* \Xi = \sigma^{-1} \partial_{1,z}^* \sigma(\Xi) \quad \text{and} \quad (D_z^+)^* \Xi = \sigma^{-1} \partial_{2,z}^* \sigma(\Xi).$$

In the remainder of this paper we need to use the action of $D_z^\pm$ on the operator $\Xi_{l,m}(\kappa_{l,m})$ for a given $l, m \geq 0$ and $\kappa_{l,m}$ in $(N^{\otimes(l+m)})'_{sym(l,m)}$. Therefore, for $z \in N$, by direct computation, the partial derivatives of the identity (2.7) in the direction z are given by

$$\begin{aligned} \partial_{1,z} \sigma(\Xi_{l,m}(\kappa_{l,m}))(\xi, \eta) &= m \langle \kappa_{l,m}, \eta^{\otimes l} \otimes (\xi^{\otimes(m-1)} \widehat{\otimes} z) \rangle \\ &= \sigma(m \Xi_{l,m-1}(\kappa_{l,m} \otimes_1 z))(\xi, \eta) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \partial_{2,z} \sigma(\Xi_{l,m}(\kappa_{l,m}))(\xi, \eta) &= l \langle \kappa_{l,m}, (z \widehat{\otimes} \eta^{\otimes(l-1)}) \otimes \xi^{\otimes m} \rangle \\ &= \sigma(l \Xi_{l-1,m}(z \otimes^1 \kappa_{l,m}))(\xi, \eta), \end{aligned} \tag{3.4}$$

where, for $z_p \in (N^{\otimes p})'$, and $\xi_{l+m-p} \in N^{\otimes(l+m-p)}$, $p \leq l+m$, the contractions $z_p \otimes_p \kappa_{l,m}$ and $\kappa_{l,m} \otimes^p z_p$ are defined by

$$\langle z_p \otimes^p \kappa_{l,m}, \xi_{l-p+m} \rangle = \langle \kappa_{l,m}, z_p \otimes \xi_{l-p+m} \rangle$$

$$\langle \kappa_{l,m} \otimes_p z_p, \xi_{l+m-p} \rangle = \langle \kappa_{l,m}, \xi_{l+m-p} \otimes z_p \rangle.$$

Similarly, if we denote $\partial_{1,z}^*$ and $\partial_{2,z}^*$ the adjoint operators of $\partial_{1,z}$ and $\partial_{2,z}$ respectively, we get

$$\partial_{1,z}^* \sigma(\Xi_{l,m}(\kappa_{l,m}))(\xi, \eta) = \sigma(\Xi_{l,m+1}(\kappa_{l,m} \otimes z))(\xi, \eta) \tag{3.5}$$

$$\partial_{2,z}^* \sigma(\Xi_{l,m}(\kappa_{l,m}))(\xi, \eta) = \sigma(\Xi_{l+1,m}(z \otimes \kappa_{l,m}))(\xi, \eta). \tag{3.6}$$

Note that, from [3] and the above discussion, for $z \in N$, the `QWN`-derivatives $D_z^\pm$ and $(D_z^\pm)^*$ are continuous linear operators from $\mathcal{L}(\mathcal{F}_\theta^*(N'), \mathcal{F}_\theta(N'))$ into itself and from $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$ into itself, i.e., $D_z^\pm, (D_z^\pm)^* \in \mathcal{L}(\mathcal{L}(\mathcal{F}_\theta^*(N'), \mathcal{F}_\theta(N'))) \cap \mathcal{L}(\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N')))$.

Theorem 3.2. *For $z \in N$ and $\Xi \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$, we have*

$$(D_z^+)^* \Xi = a^*(z) \diamond \Xi, \quad (D_z^-)^* \Xi = a(z) \diamond \Xi.$$

Proof. Let $\Xi = \sum_{l,m=0} \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$. Using (3.5), (3.6) and Lemma 3.1, we get

$$(D_z^+)^* \Xi = \sum_{l,m=0}^\infty \Xi_{l+1,m}(z \otimes \kappa_{l,m}) \tag{3.7}$$

and

$$(D_z^-)^* \Xi = \sum_{l,m=0}^\infty \Xi_{l,m+1}(\kappa_{l,m} \otimes z). \tag{3.8}$$

On the other hand,

$$\begin{aligned}\sigma(a^*(z)\diamond\Xi)(\xi,\eta) &= \sigma(a^*(z))(\xi,\eta).\sigma(\Xi)(\xi,\eta)\\ &= \langle z,\eta\rangle\sum_{l,m=0}^{\infty}\langle\kappa_{l,m},\eta^{\otimes l}\otimes\xi^{\otimes m}\rangle\\ &= \sum_{l,m=0}^{\infty}\langle z\otimes\kappa_{l,m},\eta^{\otimes l+1}\otimes\xi^{\otimes m}\rangle\\ &= \sigma\left(\sum_{l,m=0}^{\infty}\Xi_{l+1,m}(z\otimes\kappa_{l,m})\right)(\xi,\eta).\end{aligned}$$

Then, for $\Xi=\sum_{l,m=0}^{\infty}\Xi_{l,m}(\kappa_{l,m})\in\mathcal{L}(\mathcal{F}_\theta(N'),\mathcal{F}_\theta^*(N'))$, we get

$$a^*(z)\diamond\Xi=\sum_{l,m=0}^{\infty}\Xi_{l+1,m}(z\otimes\kappa_{l,m}). \tag{3.9}$$

Similarly, we obtain

$$\begin{aligned}\sigma(a(z)\diamond\Xi)(\xi,\eta) &= \sigma(a(z))(\xi,\eta)\sigma(\Xi)(\xi,\eta)\\ &= \langle z,\xi\rangle\sum_{l,m=0}^{\infty}\langle\kappa_{l,m},\eta^{\otimes l}\otimes\xi^{\otimes m}\rangle\\ &= \sum_{l,m=0}^{\infty}\langle\kappa_{l,m}\otimes z,\eta^{\otimes l}\otimes\xi^{\otimes m+1}\rangle\\ &= \sigma\left(\sum_{l,m=0}^{\infty}\Xi_{l,m+1}(\kappa_{l,m}\otimes z)\right)(\xi,\eta).\end{aligned}$$

From which, we get

$$a(z)\diamond\Xi=\sum_{l,m=0}^{\infty}\Xi_{l,m+1}(\kappa_{l,m}\otimes z). \tag{3.10}$$

Hence, by (3.7), (3.8), (3.9) and (3.10) we get the desired statement. □

3.1. Representation of the QWN-Conservation operator. For Locally convex spaces $\mathfrak{X}$ and $\mathfrak{Y}$ we denote by $\mathcal{L}(\mathfrak{X},\mathfrak{Y})$ the set of all continuous linear operators from $\mathfrak{X}$ into $\mathfrak{Y}$. Let B_1 and B_2 in $\mathcal{L}(N',N')$. For $\Xi\in\mathcal{L}(\mathcal{F}_\theta(N'),\mathcal{F}_\theta^*(N'))$, define $\widetilde{N}^Q_{B_1,B_2}(\sigma(\Xi))$ to be

$$\widetilde{N}^Q_{B_1,B_2}(\sigma(\Xi))(\xi,\eta)=\sum_{j=0}^{\infty}\partial^*_{1,e_j}\partial_{1,B_2^*e_j}\sigma(\Xi)(\xi,\eta)+\sum_{j=0}^{\infty}\partial^*_{2,e_j}\partial_{2,B_1^*e_j}\sigma(\Xi)(\xi,\eta). \tag{3.11}$$

Using a technic of calculus used in [2, 3, 12, 13] one can show that $\widetilde{N}^Q_{B_1,B_2}(\sigma(\Xi))$ belongs to $\mathcal{G}_{\theta^*}(N\oplus N)$ which gives us an essence to the following

Definition 3.3. For $\Xi \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}^*_\theta(N'))$, we define the *QWN-conservation operator* at Ξ by

$$N^Q_{B_1,B_2}\Xi = \sigma^{-1}(\widetilde{N}^Q_{B_1,B_2}(\sigma(\Xi))). \tag{3.12}$$

As a straightforward fact, the QWN-conservation operator is a continuous linear operator from $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}^*_\theta(N'))$ into itself.

For a later use, define the operator $\Xi^{a,b}$ for $a,\, b \in N'$ by

$$\Xi^{a,b} \equiv \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}(a,b)) \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}^*_\theta(N')),$$

where $\kappa_{l,m}(a,b) = \dfrac{1}{l!m!} a^{\otimes l} \otimes b^{\otimes m}$. It is noteworthy that $\{\Xi^{a,b};\ a,\, b \in N'\}$ spans a dense subspace of $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}^*_\theta(N'))$.

Proposition 3.4. *The QWN-conservation operator admits on $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}^*_\theta(N'))$ the following representation*

$$N^Q_{B_1,B_2} \quad = \quad \sum_{j=1}^{\infty} (D^-_{e_j})^* D^-_{B_2^* e_j} + \sum_{j=1}^{\infty} (D^+_{e_j})^* D^+_{B_1^* e_j}. \tag{3.13}$$

Proof. From the fact

$$\sigma(\Xi^{a,b})(\xi,\eta) = \exp\{\langle a,\eta\rangle + \langle b,\xi\rangle\}\,, \quad \xi,\, \eta \in N,\ a,\, b \in N'$$

by using (3.11), we compute

$$\begin{aligned}
\widetilde{N}^Q_{B_1,B_2}\left(\sigma\left(\Xi^{a,b}\right)\right)(\xi,\eta) \quad &= \quad e^{\langle a,\eta\rangle + \langle b,\xi\rangle} \sum_{j=0}^{\infty} \Big(\langle e_j, B_2 b\rangle \langle e_j,\xi\rangle + \langle e_j, B_1 a\rangle\langle e_j,\eta\rangle \Big) \\
&= \quad \left(\langle B_2 b,\xi\rangle + \langle B_1 a,\eta\rangle\right) e^{\langle a,\eta\rangle + \langle b,\xi\rangle} \\
&= \quad \left(\langle B_2 b,\xi\rangle + \langle B_1 a,\eta\rangle\right) \sigma\left(\Xi^{a,b}\right)(\xi,\eta).
\end{aligned} \tag{3.14}$$

On the other hand, we get

$$\sigma\left(\sum_{j=1}^{\infty} (D^-_{e_j})^* D^-_{B_2^* e_j} \Xi^{a,b}\right)(\xi,\eta) = \sum_{j=1}^{\infty} \langle e_j,\xi\rangle \langle B_2^* e_j, b\rangle e^{\langle a,\eta\rangle + \langle b,\xi\rangle}$$

$$\sigma\left(\sum_{j=1}^{\infty} (D^+_{e_j})^* D^+_{B_1^* e_j} \Xi^{a,b}\right)(\xi,\eta) = \sum_{j=1}^{\infty} \langle e_j,\eta\rangle \langle B_1^* e_j, a\rangle e^{\langle a,\eta\rangle + \langle b,\xi\rangle}$$

which gives that

$$\begin{aligned}
&\sigma\left(\textstyle\sum_{j=1}^{\infty} (D^-_{e_j})^* D^-_{B_2^* e_j} \Xi^{a,b} + \sum_{j=1}^{\infty} (D^+_{e_j})^* D^+_{B_1^* e_j} \Xi^{a,b}\right)(\xi,\eta) \\
&\qquad = (\langle B_1 a,\eta\rangle + \langle B_2 b,\xi\rangle)\sigma(\Xi^{a,b})(\xi,\eta).
\end{aligned}$$

Hence, the representation (3.13) follows by (3.17) and density argument. □

Remark 3.5. Note that, by a straightforward calculus using the symbol map we see that the QWN-conservation operator defined in this paper on $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$ coincides on $\mathcal{L}(\mathcal{F}_\theta^*(N'), \mathcal{F}_\theta(N'))$ with the QWN-conservation operator defined in [3] and coincides with its adjoint on $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$, which shows that the QWN-conservation operator is symmetric.

3.2. QWN-Conservation operator as a Wick derivation. It is shown (see Refs. [12]) that $\mathcal{G}_{\theta^*}(N \oplus N)$ is closed under pointwise multiplication. Then, for any $\Xi_1,\ \Xi_2 \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$, there exists a unique $\Xi \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$ such that $\sigma(\Xi) = \sigma(\Xi_1)\sigma(\Xi_2)$. The operator Ξ will be denoted $\Xi_1 \diamond \Xi_2$ and it will be referred to as *the Wick product* of Ξ_1 and Ξ_2. It is noteworthy that, endowed with the Wick product $\diamond$, $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$ becomes a commutative algebra.

Since $(\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N')), \diamond)$ is a topological algebra, each white noise operator Ξ_0 in $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$ gives rise to an operator-valued Wick operator

$$\Xi \mapsto \Xi_0 \diamond \Xi \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N')), \quad \Xi \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N')).$$

In fact this is a continuous operator. We then adopt the following slightly general definition: a linear operator $\mathfrak{D}$ from $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$ into itself is called a Wick derivation (see [11]) if

$$\mathfrak{D}(\Xi_1 \diamond \Xi_2) = \mathfrak{D}(\Xi_1) \diamond \Xi_2 + \Xi_1 \diamond \mathfrak{D}(\Xi_2), \quad \Xi_1,\ \Xi_2 \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N')).$$

As a non trivial example, we study, in this paper the QWN-conservation operator $N^Q_{B_1,B_2}$ on an appropriate subset of $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$; more precisely, from $\mathcal{L}(\mathcal{F}_\theta^*(N'), \mathcal{F}_\theta(N'))$ into itself. As in [11], we can prove that $\mathfrak{D}$ is a continuous Wick derivation from $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$ into itself if and only if there exist a white noise operator coefficients $\mathfrak{F}, \mathfrak{G} \in N \otimes \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$ such that

$$\mathfrak{D} = \int_{\mathbb{R}} \mathfrak{F}(t) \diamond D_t^+ dt + \int_{\mathbb{R}} \mathfrak{G}(t) \diamond D_t^- dt, \tag{3.15}$$

where $\mathfrak{F}(t), \mathfrak{G}(t) \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$ are identified with QWN Wick operators with parameter t. In fact $t \mapsto \mathfrak{F}(t)$ and $t \mapsto \mathfrak{G}(t)$ are $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$ $-$valued processes on $\mathbb{R}$, namely, $\mathfrak{F}(t,x)$ and $\mathfrak{G}(t,x)$ are elements in $N \otimes \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N')) \cong N \otimes \mathcal{F}_\theta^*(N') \otimes \mathcal{F}_\theta^*(N')$.

Theorem 3.6. *For $B_1,\ B_2 \in \mathcal{L}(N', N')$, the QWN-conservation operator is a Wick derivation with coefficients*

$$\mathfrak{F} = \int_{\mathbb{R}} \tau_{B_1}(s,.) a_s^*\, ds \quad \textit{and} \quad \mathfrak{G} = \int_{\mathbb{R}} \tau_{B_2}(s,.) a_s\, ds,$$

i.e., the QWN-conservation operator admits, on $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$, the following integral representation

$$N^Q_{B_1,B_2} = \int_{\mathbb{R}^2} \tau_{B_1}(s,t) a_s^* \diamond D_t^+\, dsdt + \int_{\mathbb{R}^2} \tau_{B_2}(s,t) a_s \diamond D_t^-\, dsdt. \tag{3.16}$$

Proof. By straightforward computation, by using (3.11), we obtain

$$\begin{aligned}\widetilde{N}^{Q}_{B_1,B_2}\left(\sigma\left(\Xi^{a,b}\right)\right)(\xi,\eta) &= e^{\langle a,\eta\rangle+\langle b,\xi\rangle}\sum_{j=0}^{\infty}\Big(\langle e_j,B_2b\rangle\langle e_j,\xi\rangle+\langle e_j,B_1a\rangle\langle e_j,\eta\rangle\Big)\\ &= \left(\langle B_2b,\xi\rangle+\langle B_1a,\eta\rangle\right)e^{\langle a,\eta\rangle+\langle b,\xi\rangle}\\ &= \left(\langle B_2b,\xi\rangle+\langle B_1a,\eta\rangle\right)\sigma\left(\Xi^{a,b}\right)(\xi,\eta). \qquad (3.17)\end{aligned}$$

On the other hand, denote

$$N^{Q-}(B_2)\equiv\int_{\mathbb{R}^2}\tau_{B_2}(s,t)a_s\diamond D_t^{-}dsdt$$
$$N^{Q+}(B_1)\equiv\int_{\mathbb{R}^2}\tau_{B_1}(s,t)a_s^{*}\diamond D_t^{+}dsdt.$$

Then, from the fact

$$\sigma(\Xi^{a,b})(\xi,\eta)=\exp\{\langle a,\eta\rangle+\langle b,\xi\rangle\}\,,\quad \xi,\ \eta\in N,\ a,\ b\in N'$$

and using (3.3) and (3.5) we compute

$$\begin{aligned}&\sigma\left(N^{Q-}(B_2)\Xi^{a,b}\right)(\xi,\eta)\\ &= \sum_{l,m}\int_{\mathbb{R}^2}\tau_{B_2}(s,t)\sigma\left(a_s)(\xi,\eta)\sigma(D_t^{-}\Xi_{l,m}(\kappa_{l,m}(a,b))\right)(\xi,\eta)dsdt\\ &= e^{\langle a,\eta\rangle+\langle b,\xi\rangle}\int_{\mathbb{R}^2}\tau_{B_2}(s,t)b(t)\xi(s)dsdt\\ &= \langle B_2b,\xi\rangle\sigma(\Xi^{a,b})(\xi,\eta). \qquad (3.18)\end{aligned}$$

Similarly, using (3.4) and (3.6) we obtain

$$\sigma\left((N^{Q+}(B_1)\Xi^{a,b}\right)(\xi,\eta)=\langle B_1a,\eta\rangle\sigma(\Xi^{a,b})(\xi,\eta). \qquad (3.19)$$

Hence, by Theorem 2.1 and a density argument, we complete the proof. □

4. Application to Wick Differential Equation

In this section we give an important example of the differential equation associated with the QWN-conservation operator where the solution of (1.7) is given explicitly in terms of the solution of the associated Cauchy problem (1.6). Let us start by studying the Cauchy problem. Let $B_1,B_2\in\mathcal{L}(N',N')$ such that $\{B_1^n;\ n=1,2,\cdots\}$ and $\{B_2^n;\ n=1,2,\cdots\}$ are equi-continuous. For $\Xi=\sum_{l,m=0}^{\infty}\Xi_{l,m}(\kappa_{l,m})$ in $\mathcal{L}(\mathcal{F}_\theta(N'),\mathcal{F}_\theta^*(N'))$ and $\kappa_{l,m}\in(N^{\otimes(l+m)})'_{sym(l,m)}$, the transformation $\widetilde{G}_t^Q$ is defined by

$$\widetilde{G}_t^Q\sigma(\Xi)(\xi,\eta):=\sum_{l,m=0}^{\infty}\left\langle\left((e^{tB_1})^{\otimes l}\otimes(e^{tB_2})^{\otimes m}\right)\kappa_{l,m},\eta^{\otimes l}\otimes\xi^{\otimes m}\right\rangle. \qquad (4.1)$$

Note that it is easy to show that $\widetilde{G}_t^Q\sigma(\Xi)$ belongs to $\mathcal{G}_{\theta^*}(N\oplus N)$, see [2]. Then using Theorem 2.1, there exists a continuous linear operator G_t^Q acting on the nuclear algebra $\mathcal{L}(\mathcal{F}_\theta(N'),\mathcal{F}_\theta^*(N'))$ such that

$$G_t^Q\Xi=\sigma^{-1}\widetilde{G}_t^Q\sigma(\Xi).$$

Similarly to the classical case studied in [2] and the scalar case studied in [6], we have the following

Lemma 4.1. *The solution of the Cauchy problem associated with the QWN- conservation operator (1.6) is given by* $U_t = G_t^Q U_0$.

Let β be a Young function satisfying the condition (2.5) and put $\theta = (e^{\beta^*} - 1)^*$. For $\Upsilon \in \mathcal{L}(\mathcal{F}_\beta(N'), \mathcal{F}_\beta^*(N'))$ the exponential Wick $wexp(\Upsilon)$ defined by

$$wexp(\Upsilon) = \sum_{n=0}^{\infty} \frac{1}{n!} \Upsilon^{\diamond n},$$

belongs to $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N'))$, see [12]. In the following we study the system of Wick differential equations for white noise operators of the form

$$D_t \Xi_t = \Pi_t \diamond \Xi_t, \quad \Xi_0 \in \mathcal{L}(\mathcal{F}_\beta(N'), \mathcal{F}_\beta^*(N')) \tag{4.2}$$

where $\Pi_t \in \mathcal{L}(\mathcal{F}_\beta(N'), \mathcal{F}_\beta^*(N'))$ and

$$D_t := \frac{\partial}{\partial t} - N_{B_1,B_2}^Q.$$

Eq. (4.2) is referred to as a *Wick differential equation* associated to the QWN-conservation operator.

Theorem 4.2. *The unique solution of the Wick differential equation (4.2) is given by*

$$\Xi_t = G_t^Q \left(\Xi_0 \diamond wexp \left(\int_0^t G_{-s}^Q(\Pi_s) ds \right) \right) = G_t^Q(\Xi_0) \diamond wexp \left(\int_0^t G_{t-s}^Q(\Pi_s) ds \right). \tag{4.3}$$

Proof. Applying the operator $\frac{\partial}{\partial t}$ to the right hand side of Eq. (4.3), we get

$$\begin{aligned} \frac{\partial}{\partial t}\Xi_t &= \frac{\partial}{\partial t} G_t^Q(\Xi_0) \diamond wexp \left(\int_0^t G_{t-s}^Q(\Pi_s) ds \right) \\ &\quad + G_t^Q(\Xi_0) \diamond N_{B_1,B_2}^Q \left(\int_0^t G_{t-s}^Q(\Pi_s) ds \right) \diamond wexp \left(\int_0^t G_{t-s}^Q(\Pi_s) \right) ds \\ &\quad + G_t^Q(\Xi_0) \diamond \Pi_t \diamond wexp \left(\int_0^t G_{t-s}^Q(\Pi_s) \right) ds. \end{aligned}$$

Then, using Lemma 4.1 and Theorem 3.6 , we get

$$\frac{\partial}{\partial t}\Xi_t = N_{B_1,B_2}^Q(\Xi_t) + \Pi_t \diamond \Xi_t.$$

Which shows that Ξ_t is solution of (4.2). Let Ξ_t be an arbitrary solution of (4.2) and put

$$F_t = \Xi_t \diamond wexp \left(- \int_0^t G_{t-s}^Q(\Pi_s) ds \right).$$

Then, we have

$$\begin{aligned}\frac{\partial}{\partial t}F_t &= \frac{\partial}{\partial t}\Xi_t \diamond wexp\left(-\int_0^t G^Q_{t-s}(\Pi_s)ds\right)\\ &\quad + N^Q_{B_1,B_2}\left(-\int_0^t G^Q_{t-s}(\Pi_s)ds\right)\diamond F_t - \Pi_t \diamond F_t\\ &= \left(\frac{\partial}{\partial t}\Xi_t - \Pi_t\diamond \Xi_t\right)\diamond wexp\left(-\int_0^t G^Q_{t-s}(\Pi_s)ds\right)\\ &\quad + N^Q_{B_1,B_2} wexp\left(-\int_0^t G^Q_{t-s}(\Pi_s)ds\right)\diamond \Xi_t.\end{aligned}$$

Using Eq. (4.2) and the fact that $N^Q_{B_1,B_2}$ is a Wick derivation, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}F_t &= N^Q_{B_1,B_2}(\Xi_t)\diamond wexp\left(-\int_0^t G^Q_{t-s}(\Pi_s)ds\right)\\ &\quad + \Xi_t \diamond N^Q_{B_1,B_2} wexp\left(-\int_0^t G^Q_{t-s}(\Pi_s)ds\right)\\ &= N^Q_{B_1,B_2}(F_t).\end{aligned}$$

From which we deduce that $D_t F_t = 0$. Then, by Lemma 4.1, we get

$$F_t = G^Q_t(F_0) = G^Q_t(\Xi_0).$$

Therefore, we deduce that

$$\Xi_t = G^Q_t(\Xi_0)\diamond wexp\left(\int_0^t G^Q_{t-s}(\Pi_s)ds\right).$$

Now, using (4.1), we obtain

$$\sigma\left(G^Q_t(\Xi)\right)(\xi,\eta) = \sigma\big(\Xi\big)\left((e^{tB_2})^*\xi,(e^{tB_1})^*\eta\right).$$

Then, using the definition of the Wick product of two operators, for every Ξ_1, $\Xi_2 \in \mathcal{L}(\mathcal{F}_\beta(N'),\mathcal{F}^*_\beta(N'))$, we get

$$\begin{aligned}&\sigma\left(G^Q_t(\Xi_1\diamond\Xi_2)\right)(\xi,\eta)\\ &= \sigma\big(\Xi_1\diamond\Xi_2\big)\left((e^{tB_2})^*\xi,(e^{tB_1})^*\eta\right)\\ &= \sigma\big(\Xi_1\big)\left((e^{tB_2})^*\xi,(e^{tB_1})^*\eta\right).\sigma\big(\Xi_2\big)\left((e^{tB_2})^*\xi,(e^{tB_1})^*\eta\right)\\ &= \sigma\left(G^Q_t(\Xi_1)\right)(\xi,\eta).\sigma\left(G^Q_t(\Xi_2)\right)(\xi,\eta).\end{aligned}$$

From which we deduce that

$$G^Q_t(\Xi_1\diamond\Xi_2) = G^Q_t(\Xi_1)\diamond G^Q_t(\Xi_2).$$

Hence, we get

$$\begin{aligned} & G_t^Q\left(\Xi_0 \diamond wexp\left(\int_0^t G_{-s}^Q(\Pi_s)ds\right)\right) \\ = \quad & G_t^Q(\Xi_0) \diamond wexp\left(\int_0^t G_t^Q G_{-s}^Q(\Pi_s)ds\right) \\ = \quad & G_t^Q(\Xi_0) \diamond wexp\left(\int_0^t G_{t-s}^Q(\Pi_s)ds\right). \end{aligned}$$

Which completes the proof. □

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