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KRYLOV–VERETENNIKOV EXPANSION FOR COALESCING STOCHASTIC FLOWS

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Abstract. In this article we consider multiplicative operator-valued white noise functionals related to a stochastic flow. A generalization of the Krylov–Veretennikov expansion is presented. An analog of this expansion for the Arratia flow is derived

Introduction

In this article we present the form of the kernels in the Itô–Wiener expansion for functionals from a dynamical system driven by an additive Gaussian white noise. The most known example of such expansion is the Krylov–Veretennikov representation [11]:

$$f(y(t)) = T_t f(u) + \sum_{k=1}^{\infty} \int_{\Delta_k(0;t)} T_{t-\tau_k} b\partial T_{\tau_2-\tau_1} \ldots$$
$$\ldots b\partial T_{\tau_1} f(u) dw(\tau_1) \ldots dw(\tau_k).$$

where f is a bounded measurable function, y is a solution of the SDE

$$dy(t) = a(y(t))dt + b(y(t))dw(t)$$

with smooth and nondegenerate coefficients, and $\{T_t;\ t \geq 0\}$ is the semigroup of operators related to the SDE and ∂ is the symbol of differentiation.

A family of substitution operators of the SDE's solution into a function can be treated as a multiplicative Gaussian white noise functional. In the first section of this article we consider a family $\{G_{s,t};\ 0 \leq s \leq t < +\infty\}$ of strong random operators (see Definition 1.1) in the Hilbert space which is an operator-valued multiplicative functional from the Gaussian white noise. It turns out that the precise form of the kernels in the Itô–Wiener expansion can be found for a wide class of operator-valued multiplicative functionals using some simple algebraic relations. The obtained formula covers the Krylov–Veretennikov case and gives a representation for different objects such as Brownian motion in Lie group etc.

The representation obtained in the first section may be useful in studing the properties of a dynamical system with an additive Gaussian white noise. On the

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other hand, there exist cases when a dynamical system is obtained as a limit in a certain sense of systems driven by the Gaussian white noise. A limiting system could be highly irregular [2, 5, 10]. One example of such a system is the Arratia flow [2] of coalescing Brownian particles on the real line. The trajectories of individual particles in this flow are Brownian motions, but the whole flow cannot be built from the Gaussian noise in a regular way [13]. Nevertheless, it is possible to construct the $n-$point motion of the Arratia flow from the pieces of the trajectories of n independent Wiener processes. Correspondingly a function from the n-point motion of the Arratia flow has an Itô–Wiener expansion based on the initial Wiener processes. This expansion depends on the way of construction (coalescing description). We present such expansion in terms of an infinite family of expectation operators related to all manner of coalescence of the trajectories in the Arratia flow. To do this we first obtain an analog of the Krylov–Veretennikov expansion for the Wiener process stopped at zero.

This paper is divided onto three parts. The first section is devoted to multiplicative operator-valued functionals from Gaussian white noise. The second part contains the definition and necessary facts about the Arratia flow. In the last section we present a family of Krylov–Veretennikov expansions for the n-point motion of the Arratia flow.

1. Multiplicative White Noise Functionals

In this part we present the Itô–Wiener expansion for the semigroup of strong random linear operators in Hilbert space. Such operators in the space of functions can be generated by the flow of solutions to a stochastic differential equation. In this case our expansion turns into the well-known Krylov–Veretennikov representation [11]. In the case when these operators have a different origin, we obtain a new representation for the semigroup.

Let us start with the definition and examples of strong random operators in the Hilbert space. Let H denote a separable real Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. As usual $(\Omega,\mathcal{F},P)$ denotes a complete probability space.

Definition 1.1. A *strong linear random operator* in H is a continuous linear map from H to $L_2(\Omega,P,H)$.

Remark 1.2. The notion of strong random operator was introduced by A.V.Skorokhod [14]. In his definition Skorokhod used the convergence in probability, rather than convergence in the square mean.

Consider some typical examples of strong random operators.

Example 1.3. Let H be l_2 with the usual inner product and $\{\xi_n;\ n\geq 1\}$ be an i.i.d. sequence with finite second moment. Then the map

$$l_2 \ni x=(x_n)_{n\geq 1}\mapsto Ax=(\xi_n x_n)_{n\geq 1}$$

is a strong random operator. In fact

$$E\|Ax\|^2=\sum_{n=1}^{\infty}x_n^2 E\xi_1^2$$

and the linearity is obvious. Note that pathwise the operator A can be not well-defined. For example, if $\{\xi_n : n \geq 1\}$ have the standard normal distribution, then with probability one

$$\sup_{n\geq 1} |\xi_n| = +\infty.$$

An interesting set of examples of strong random operators can be found in the theory of stochastic flows. Let us recall the definition of a stochastic flow on $\mathbb{R}$ [12].

Definition 1.4. A family $\{\phi_{s,t};\ 0 \leq s \leq t\}$ of random maps of $\mathbb{R}$ to itself is referred to as a *stochastic flow* if the following conditions hold:

(1) For any $0 \leq s_1 \leq s_2 \leq \ldots s_n < \infty :\ \phi_{s_1,s_2}, \ldots, \phi_{s_{n-1},s_n}$ are independent.
(2) For any $s, t, r \geq 0 : \phi_{s,t}$ and $\phi_{s+r,t+r}$ are equidistributed.
(3) For any $r \leq s \leq t$ and $u \in \mathbb{R} :\ \phi_{r,s}\phi_{s,t}(u) = \phi_{r,t}(u)$, $\phi_{r,r}$ is an identity map.
(4) For any $u \in \mathbb{R} : \phi_{0,t}(u) \mapsto u$ in probability when $t \mapsto 0$.

Stochastic flows arise as solutions to stochastic differential equations with smooth coefficients. Namely, if $\phi_{s,t}(u)$ is a solution to the stochastic differential equation

$$dy(t) = a(y(t))dt + b(y(t))dw(t) \tag{1.1}$$

starting at the point u in time s and considered in time t, then under smoothness conditions on the coefficients a and b the family $\{\phi_{s,t}\}$ will satisfy the conditions of Definition 1.4 [12]. Another example of a stochastic flow is the Harris flow consisting of Brownian particles [5]. In this flow $\phi_{0,t}(u)$ for every $u \in \mathbb{R}$ is a Brownian martingale with respect to a common filtration and

$$d\langle \phi_{0,t}(u_1),\ \phi_{0,t}(u_2)\rangle = \Gamma(\phi_{0,t}(u_1) - \phi_{0,t}(u_2))dt$$

for some positive definite function Γ with $\Gamma(0) = 1$.

For a given stochastic flow one can try to construct a corresponding family of strong random operators as follows.

Example 1.5. Let $H = L_2(\mathbb{R})$. Define

$$G_{s,t}f(u) = f(\phi_{s,t}(u)).$$

Let us check that in the both cases mentioned above $G_{s,t}$ satisfies Definition 1.1. For the Harris flow we have

$$E\int_{\mathbb{R}} f(\phi_{s,t}(u))^2 du = \int_{\mathbb{R}}\int_{\mathbb{R}} f(v)^2 p_{t-s}(u-v)dudv = \int_{\mathbb{R}} f(v)^2 dv.$$

Here p_r denotes the Gaussian density with zero mean and variance r.

To get an estimation for the flow generated by a stochastic differential equation let us suppose that the coefficients a and b are bounded Lipschitz functions and b is separated from zero. Under such conditions $\phi_{s,t}(u)$ has a density, which can be estimated from above by a Gaussian density [1]. Consequently we will have the inequality $E\int_{\mathbb{R}} f(\phi_{s,t}(u))^2 du \leq c\int_{\mathbb{R}} f(v)^2 dv$.

As it was shown in Example 1.3, a strong random operator in general is not a family of bounded linear operators in H indexed by the points of probability space. Despite of this the composition of such operators can be properly defined (see [3] for detailed construction in case of dependent nonlinear operators via Wick product). Here we will consider only the case when strong random operators A and B are independent. In this case both A and B have measurable modifications and one can define for $u \in H,\ \omega \in \Omega$

$$AB(u,\omega) := A(B(u,\omega),\omega)$$

and prove that the value $AB(u)$ does not depend on the choice of modifications. Note that the operators from the previous example satisfy the semigroup property, and that for the flow generated by a stochastic differential equation these operators are measurable with respect to increments of the Wiener process. In this section we will consider a general situation of this kind and study the structure of the semigroup of strong random operators measurable with respect to a Gaussian white noise. The white noise framework is presented in [3, 6, 8], here we just recall necessary facts and definitions.

Let's start with a description of the noise. Let H_0 be a separable real Hilbert space. Consider a new Hilbert space $\tilde{H} = H_0 \otimes L_2([0;\ +\infty])$, where an inner product is defined by the formula

$$\tilde{H} \ni f, g \mapsto < f,\ g\rangle = \int_0^\infty (f(t),\ g(t))_0 dt.$$

Definition 1.6. *Gaussian white noise* ξ in $\tilde{H}$ is a family of jointly Gaussian random variables $\{\langle \xi, h\rangle;\ h \in \tilde{H}\}$ which is linear with respect to $h \in \tilde{H}$ and for every h, $\langle \xi, h\rangle$ has mean zero and variance $\|h\|^2$.

Let $\tilde{H}_{s,t}$ be the product $H_0 \otimes L_2([s;\ t])$, which can be naturally considered as a subspace of $\tilde{H}$. Define the σ-fields $\mathcal{F}_{s,t} = \sigma\{\langle \xi, h\rangle;\ h \in \tilde{H}_{s,t}\},\ 0 \le s \le t < +\infty$.

Definition 1.7. A family $\{G_{s,t};\ 0 \le s \le t < +\infty\}$ of strong random operators in H is a *multiplicative functional* from ξ if the following conditions hold:

1) $G_{s,t}$ is measurable with respect to $\mathcal{F}_{s,t}$,
2) $G_{s,s}$ is an identity operator for every s,
3) $G_{s_1,s_3} = G_{s_2,s_3} G_{s_1,s_2}$ for $s_1 \le s_2 \le s_3$.

Remark 1.8. Taking an orthonormal basis $\{e_n\}$ in H_0 one can replace ξ by a sequence of independent Wiener processes $\{w_n(t) = \langle e_n \otimes 1_{[0;\ t]};\ \xi\rangle;\ t \ge 0\}$. We use ξ in order to simplify notations and consider simultaneously both cases of finite and infinite number of the processes $\{w_n\}$.

Example 1.9. Let us define $x(u,s,t)$ as a solution to the Cauchy problem for (1.1) which starts from the point u at the moment s. Using the flow property one can easily verify that the family of operators $\{G_{s,t} f(u) = f(x(u,s,t))\}$ in $L_2(\mathbb{R})$ is a multiplicative functional from the Gaussian white noise $\dot{w}$ in $L_2([0;+\infty])$.

Now we are going to introduce the notion of a homogeneous multiplicative functional. Let us recall, that every square integrable random variable α measurable with respect to ξ can be uniquely expressed as a series of multiple Wiener integrals [8]

$$\alpha = E\alpha + \sum_{k=1}^{\infty} \int_{\Delta_k(0;+\infty)} a_k(\tau_1,\ldots,\tau_k)\xi(d\tau_1)\ldots\xi(d\tau_k),$$

where

$$\Delta_k(s;t) = \{(\tau_1, \ldots, \tau_k) : \ s \le \tau_1 \le \ldots \le \tau_k \le t\},$$
$$a_k \in L_2(\Delta_k(0;+\infty),\ H_0^{\otimes k}),\ k \ge 1.$$

Here in the multiple integrals we consider the white noise ξ as Gaussian H_0-valued random measure on $[0;\ +\infty)$. In the terms of the mentioned above orthonormal basis $\{e_n\}$ in H_0 and the sequence of the independent Wiener processes $\{w_n\}$ one can rewrite the above multiple integrals as

$$\int_{\Delta_k(0;+\infty)} a_k(\tau_1, \ldots, \tau_k)\xi(d\tau_1)\ldots\xi(d\tau_k)$$
$$= \sum_{n_1,\ldots,n_k} \int_{\Delta_k(0;+\infty)} a_k(\tau_1, \ldots, \tau_k)(e_{n_1}, ..., e_{n_k})dw_{n_1}(\tau_1)...dw_{n_k}(\tau_k).$$

Define the shift of α for $r \ge 0$ as follows

$$\theta_r\alpha = E\alpha + \sum_{k=1}^{\infty} \int_{\Delta_k(r;+\infty)} a_k(\tau_1 - r, \ldots, \tau_k - r)\xi(d\tau_1)\ldots\xi(d\tau_k).$$

Definition 1.10. A multiplicative functional $\{G_{s,t}\}$ is *homogeneous* if for every $s \le t$ and $r \ge 0$

$$\theta_r G_{s,t} = G_{s+r,t+r}.$$

Note that the family $\{G_{s,t}\}$ from Example 1.9 is a homogeneous functional. From now on, we will consider only homogeneous multiplicative functionals from ξ. For a homogeneous functional $\{G_{s,t}\}$ one can define the expectation operators

$$T_t u = EG_{0,t}u,\ u \in H,\ t \ge 0.$$

Since the family $\{G_{s,t}\}$ is homogeneous, then $\{T_t\}$ is the semigroup of bounded operators in H. Under the well-known conditions the semigroup $\{T_t\}$ can be described by its generator. However the family $\{G_{s,t}\}$ cannot be recovered from this semigroup. The following simple example shows this.

Example 1.11. Define $\{G^1_{s,t}\}$ and $\{G^2_{s,t}\}$ in the space $L_2(\mathbb{R})$ as follows

$$G^1_{s,t}f(u) = T_{t-s}f(u),$$

where $\{T_t\}$ is the heat semigroup, and

$$G^2_{s,t}f(u) = f(u + w(t) - w(s)),$$

where w is a standard Wiener processes. It is evident, that

$$EG^2_{s,t}f(u) = T_{t-s}f(u) = EG^1_{s,t}f(u).$$

To recover multiplicative functional uniquely we have to add some information to $\{T_t\}$. It can be done in the following way. For $f \in H$ define an operator which acts from H_0 to H by the rule

$$A(f)(h) \doteq \lim_{t\to 0+} \frac{1}{t} EG_{0,t}f(\xi,\ h \otimes 1_{[0;\ t]}). \tag{1.2}$$

Example 1.12. Let the family $\{G_{s,t}\}$ be defined as in Example 1.9. Now $H = L_2(\mathbb{R})$ and the noise ξ is defined on $L_2([0;\, +\infty)$ as $\dot{w}$. Then for $f \in L_2(\mathbb{R})$ (now $H_0 = \mathbb{R}$ and it makes sense only to take $h = 1$)

$$A(f)(u) = \lim_{t \to 0+} \frac{1}{t} Ef(x(u,t))w(t).$$

Suppose that f has two bounded continuous derivatives. Then using Itô's formula one can get

$$Ef(x(u,t))w(t) = \int_0^t Ef'(x(u,s))\varphi(x(u,s))ds,$$

and

$$\frac{1}{t} Ef(x(\bullet,t))w(t) \overset{L_2(\mathbb{R})}{\to} f'(\bullet)b(\bullet),\ t \to 0+\,.$$

Consequently, for "good" functions

$$Af = bf'.$$

Definition 1.13. An element u of H belongs to the domain of definition $D(A)$ of A if the limit (1.2) exists for every $h \in H_0$ and defines a Hilbert–Schmidt operator $A(u) : H_0 \to H$. The operator A is refereed to as the *random generator* of $\{G_{s,t}\}$.

Now we can formulate the main statement of this section, which describes the structure of homogeneous multiplicative functionals from ξ.

Theorem 1.14. *Suppose, that for every $t > 0$, $T_t(H) \subset D(A)$ and the kernels of the Itô-Wiener expansion for $G_{0,t}$ are continuous with respect to time variables. Then $G_{0,t}$ has the following representation*

$$G_{0,t}(u) = T_t u + \sum_{k=1}^{\infty} \int_{\Delta_k(0;t)} T_{t-\tau_k} A T_{\tau_k - \tau_{k-1}} \dots A T_{\tau_1} u d\xi(\tau_1) \dots d\xi(\tau_k). \tag{1.3}$$

Proof. Let us denote the kernels of the Itô–Wiener expansion for $G_{0,t}(u)$ as $\{a_k^t(u, \tau_1, \dots, \tau_k);\ k \geq 0\}$. Since

$$a_0^t(u) = EG_{0,t}(u),$$

then

$$a_0^t(u) = T_t u.$$

Since

$$G_{0,t+s}(u) = G_{t,t+s}(G_{0,t}(u)),$$

and $G_{t,t+s} = \theta_t G_{0,s}$, then

$$a_1^{t+s}(u, \tau_1) = T_s a_1^t(u, \tau_1) 1_{\tau_1 < t} + a_1^s(T_t u, \tau_1 - t) 1_{t \leq \tau_1 \leq t+s}. \tag{1.4}$$

Using this relation one can get

$$\begin{aligned} a_1^t(u, 0) &= T_{t-\tau_1} a_1^{\tau_1}(u, 0), \\ a_1^t(u, \tau_1) &= a_1^{t-\tau_1}(T_{\tau_1} u, 0). \end{aligned} \tag{1.5}$$

The condition of the theorem implies that for $v = T_{\tau_1} u$ and every $h \in H_0$ there exists the limit

$$A(v)h = \lim_{t \to 0+} \frac{1}{t} EG_{0,t}(v)(\xi, h \otimes 1_{[0;\, t]})$$

$$= \lim_{t\to 0+} \frac{1}{t} \int_0^t a_1^t(v,\tau_1) h d\tau_1.$$

Now, by continuity of a_1,

$$a_1^0(T_{\tau_1}u, 0) = A(T_{\tau_1}u).$$

Finally,

$$a_1^t(u,\tau_1) = T_{t-\tau_1} A T_{\tau_1} u.$$

The case $k \geq 2$ can be proved by induction. Suppose, that we have the representation (1.3) for a_j^t, $j \leq k$. Consider a_{k+1}^{t+s}. Using the multiplicative and homogeneity properties one can get

$$\begin{aligned}
&a_{k+1}^{t+s}(u,\tau_1,\ldots,\tau_{k+1}) 1_{\{0\leq\tau_1\leq\ldots\leq\tau_k\leq t\leq\tau_{k+1}\leq t+s\}} \\
&= a_1^s(a_k^t(u,\tau_1,\ldots,\tau_k),\tau_{k+1}-t) \\
&= T_{s+t-\tau_{k+1}} A T_{\tau_{k+1}-t} a_k^t(u,\tau_1,\ldots,\tau_k) \\
&= T_{s+t-\tau_{k+1}} A T_{\tau_{k+1}-t} T_{t-\tau_k} A \ldots A T_{\tau_1} u \\
&= T_{s+t-\tau_{k+1}} A T_{\tau_{k+1}-\tau_k} A \ldots A T_{\tau_1} u.
\end{aligned}$$

The theorem is proved. □

Consider some examples of application of the representation (1.3).

Example 1.15. Consider the multiplicative functional from Example 1.9. Suppose that the coefficients a, b have infinitely many bounded derivatives. Now it can be proved, that $x(u,t)$ has infinitely many stochastic derivatives [15]. Consequently for a smooth function f the first kernel in the Itô–Wiener expansion of $f(x(u,t))$ can be expressed as follows

$$a_1^t(u,\tau) = EDf(x(u,t))(\tau). \tag{1.6}$$

Indeed, for an arbitrary $h \in L_2([0;+\infty))$

$$\begin{aligned}
\int_0^t a_1^t(u,\tau)h(\tau)d\tau &= Ef(x(u,t)) \int_0^t h(\tau)dw(\tau) \\
&= E\int_0^t Df(x(u,t))(\tau)h(\tau)d\tau,
\end{aligned}$$

which gives us the expression (1.6). The required continuity of a_1 follows from a well-known expression for the stochastic derivative of x [8]. As it was mentioned in Example 1.12, the operator A coincides with $b\frac{d}{du}$ on smooth functions. Finally, the expression (1.3) turns into the well-known Krylov–Veretennikov expansion [11] for $f(x(u,t))$

$$\begin{aligned}
f(x(u,t)) = T_t f(u) + \sum_{k=1}^{\infty} \int_{\Delta_k(0;t)} T_{t-\tau_k} b\partial T_{\tau_2-\tau_1} \ldots \\
\ldots b\partial T_{\tau_1} f(u) dw(\tau_1) \ldots dw(\tau_k).
\end{aligned}$$

Remark 1.16. The expression (1.3) can be applied to multiplicative functionals, which are not generated by a stochastic flow.

Example 1.17. Let $\mathcal{L}$ be a matrix Lie group with the corresponding Lie algebra $\mathcal{A}$ with $\dim \mathcal{A} = n$. Consider an $\mathcal{L}$-valued homogeneous multiplicative functional $\{G_{s,t}\}$ from ξ. Suppose that $\{G_{0,t}\}$ is a semimartingale with respect to the filtration generated by ξ. Let $\{G_{s,t}\}$ be continuous with respect to s, t with probability one. It means, in particular, that $\{G_{0,t}\}$ is a multiplicative Brownian motion in $\mathcal{L}$ [7]. Then $G_{0,t}$ is a solution to the following SDE

$$dG_{0,t} = G_{0,t} dM_t,$$
$$G_{0,0} = I.$$

Here $\{M_t; t \geq 1\}$ is an $\mathcal{A}$-valued Brownian motion obtained from G by the rule [7]

$$M_t = P\text{-}\lim_{\Delta \to 0+} \sum_{k=0}^{[\frac{t}{\Delta}]} (G_{k\Delta,(k+1)\Delta} - I). \tag{1.7}$$

Since $G_{0,t}$ is a semimartingale with respect to the filtration of ξ, then M_t also has the same property. The representation (1.7) shows that $M_t - M_s$ is measurable with respect to the σ-field $\mathcal{F}_{s,t}$ and for arbitrary $r \geq 0$

$$\theta_r(M_t - M_s) = M_{t+r} - M_{s+r}.$$

Considering the Itô–Wiener expansion of $M_t - M_s$ one can easily check, that

$$M_t = \int_0^t Z d\xi(\tau) \tag{1.8}$$

with a deterministic matrix Z. We will prove (1.8) for the one-dimensional case. Suppose that M_t has the following Itô-Wiener expansion with respect to ξ

$$M_t = \sum_{k=1}^{\infty} \int_{\Delta_k(t)} a_k(t, \tau_1, \ldots, \tau_k) d\xi(\tau_1) \ldots d\xi(\tau_k).$$

Then for $k \geq 2$ the corresponding kernel a_k satisfies relation

$$a_k(t+s, \tau_1, \ldots, \tau_k) = a_k(t, \tau_1, \ldots, \tau_k) 1_{\{\tau_1, \ldots, \tau_k \leq t\}} + a_k(s, \tau_1 - t, \ldots, \tau_k - t) 1_{\{\tau_1, \ldots, \tau_k \geq t\}}.$$

Iterating this relation for $t = \sum_{j=1}^{n} \frac{t}{n}$ one can verify that $a_k \equiv 0$. For $k = 1$ the same arguments give $a_k \equiv const$.

Consequently, the equation for G can be rewritten using ξ as

$$dG_{0,t} = G_{0,t} Z d\xi(t). \tag{1.9}$$

Now the elements of the Itô–Wiener expansion from Theorem 1.14 can be determined as follows

$$T_t = EG_{0,t}, \ A = Z.$$

Consequently,

$$G_{0,t} = T_t + \sum_{k=1}^{\infty} \int_{\Delta_k(0;t)} T_{t-\tau_k} Z T_{\tau_k - \tau_{k-1}} \ldots Z T_{\tau_1} d\xi(\tau_1) \ldots d\xi(\tau_k).$$

2. The Arratia Flow

When trying to obtain an analog of the representation (1.3) for a stochastic flow which is not generated by a stochastic differential equation with smooth coefficients, we are faced with the difficulty that there is no such a Gaussian random vector field, which would generate the flow. This circumstance arise from the possibility of coalescence of particles in the flow. We will consider one of the best known examples of such stochastic flows, the Arratia flow. Let us start with the precise definition.

Definition 2.1. The *Arratia flow* is a random field $\{x(u,t);\ u \in \mathbb{R},\ t \geq 0\}$, which has the properties

1) all $x(u,\cdot),\ u \in \mathbb{R}$ are Wiener martingales with respect to the join filtration,
2) $x(u,0) = u,\ u \in \mathbb{R}$,
3) for all $u_1 \leq u_2,\ t \geq 0$

$$x(u_1,t) \leq x(u_2,t),$$

4) the joint characteristics of $x(u_1,t)$ and $x(u_2,t)$ equals

$$\langle x(u_1,\cdot),\ x(u_2,\cdot)\rangle_t = \int_0^t 1_{\{\tau(u_1,u_2)\leq s\}} ds,$$

where

$$\tau(u_1,u_2) = \inf\{t:\ x(u_1,t) = x(u_2,t)\}.$$

It follows from the properties 1)–3), that individual particles in the Arratia flow move as Brownian particles and coalesce after meeting. Property 4) reflects the independence of the particles before meeting. It was proved in [4], that the Arratia flow has a modification, which is a cdlg process on $\mathbb{R}$ with the values in $C([0;+\infty)])$. From now on, we assume that we are dealing with such a modification. We will construct the Arratia flow using a sequence of independent Wiener processes $\{w_k :\ k \geq 1\}$. Suppose that $\{r_k;\ k \geq 1\}$ are rational numbers on $\mathbb{R}$. To construct the Arratia flow put $w_k(0) = r_k,\ k \geq 1$ and define

$$x(r_1,t) = w_1(t),\ t \geq 0.$$

If $x(r_1,\cdot),\ldots,\ x(r_n,\cdot)$ have already been constructed, then define

$$\sigma_{n+1} = \inf\{t:\ \prod_{k=1}^{n}(x(r_k,t) - w_{n+1}(t)) = 0\},$$

$$x(r_{n+1},t) = \begin{cases} w_{n+1}(t),\ t \leq \sigma_{n+1} \\ x(r_{k^*},t),\ t \geq \sigma_{n+1}, \end{cases}$$

where

$$w_{n+1}(\sigma_{n+1}) = x(r_{k^*},\sigma_{n+1}),$$

$$k = \min\{l:\ w_{n+1}(\sigma_{n+1}) = x(r_l,\sigma_{n+1})\}.$$

In this way we construct a family of the processes $x(r,\cdot),\ r \in \mathbb{Q}$ which satisfies conditions 1)–4) from Definition 2.1.

Lemma 2.2. *For every $u \in \mathbb{R}$ the random functions $x(r,\cdot)$ uniformly converge on compacts with probability one as $r \to u$. For rational u the limit coincides with $x(u,\cdot)$ defined above. The resulting random field $\{x(u,t);\ u \in \mathbb{R},\ t \geq 0\}$ satisfies the conditions of Definition 2.1.*

Proof. Consider a sequence of rational numbers $\{r_{n_k};\ k \geq 1\}$ which converges to some $u \in \mathbb{R} \setminus \mathbb{Q}$. Without loss of generality one can suppose that this sequence decreases. For every $t \geq 0$, $\{x(r_{n_k}, t);\ k \geq 1\}$ converges with probability one as a bounded monotone sequence. Denote

$$x(u,t) = \lim_{k\to\infty} x(r_{n_k}, t).$$

Note that for arbitrary $r',\ r'' \in \mathbb{Q}$ and $t \geq 0$

$$E \sup_{[0;\ t]} (x(r',s) - x(r'',s))^2 \leq C \cdot (|r'-r''| + (r'-r'')^2). \tag{2.1}$$

Here the constant C does not depend on r' and r''. Inequality (2.1) follows from the fact, that the difference $x(r',\cdot) - x(r'',\cdot)$ is a Wiener process with variance 2, started at $r'-r''$ and stopped at 0. Monotonicity and (2.1) imply that the first assertion of the lemma holds. Note that for every $t \geq 0$

$$\begin{aligned}\mathcal{F}_t &= \sigma(x(r,s);\ r \in \mathbb{Q},\ s \in [0;t]) \\ &= \sigma(x(r,s);\ r \in \mathbb{R},\ s \in [0;t]).\end{aligned}$$

Using standard arguments one can easily verify, that for every $u \in \mathbb{R}$, $x(u,\cdot)$ is a Wiener martingale with respect to the flow $(\mathcal{F}_t)_{t\geq 0}$, and that the inequality

$$x(u_1,t) \leq x(u_2,t)$$

remains to be true for all $u_1 \leq u_2$. Consequently, for all $u_1, u_2 \in \mathbb{R}$, $x(u_1,\cdot)$ and $x(u_2,\cdot)$ coincide after meeting. It follows from (2.1) and property 4) for $x(r,\cdot)$ with rational r, that

$$\langle x(u_1,\cdot), x(u_2,\cdot)\rangle_t = 0$$

for

$$t < \inf\{s:\ x(u_1,s) = x(u_2,s)\}.$$

Hence, the family $\{x(u,t);\ u \in \mathbb{R},\ t \geq 0\}$ satisfies Definition 2.1. □

This lemma shows that the Arratia flow is generated by the initial countable system of independent Wiener processes $\{w_k; k \geq 1\}$.

From this lemma one can easily obtain the following statement.

Corollary 2.3. *The σ-field*

$$\mathcal{F}^x_{0+} := \bigcap_{t>0} \sigma(x(u,s);\ u \in \mathbb{R},\ 0 \leq s \leq t)$$

is trivial modulo P.

The proof of this statement follows directly from the fact that the Wiener process has the same property [9].

3. The Krylov–Veretennikov Expansion for the n-point Motion of the Arratia Flow

We begin this section with an analog of the Krylov–Veretennikov expansion for the Wiener process stopped at zero. For the Wiener process w define the moment of the first hitting zero

$$\tau = \inf\{t: \ w(t) = 0\}$$

and put $\widetilde{w}(t) = w(\tau \wedge t)$. For a measurable bounded $f : \mathbb{R} \to \mathbb{R}$ define

$$\widetilde{T}_t(f)(u) = E_u f(\widetilde{w}(t)).$$

The following statement holds.

Lemma 3.1. *For a measurable bounded function $f : \mathbb{R} \to \mathbb{R}$ and $u \geq 0$*

$$\begin{aligned} f(\widetilde{w}(t)) &= \widetilde{T}_t f(u) \\ &+ \sum_{k=1}^{\infty} \int_{\Delta_k(t)} \widetilde{T}_{t-r_k} \frac{\partial}{\partial v_k} \widetilde{T}_{r_k - r_{k-1}} \ldots \frac{\partial}{\partial v_1} \widetilde{T}_{r_1} f(v_1) \\ &\qquad dw(r_1) \ldots dw(r_k). \end{aligned} \tag{3.1}$$

Proof. Let us use the Fourier–Wiener transform. Define for $\varphi \in C([0; +\infty), \ \mathbb{R}) \bigcap L_2([0; +\infty), \ \mathbb{R})$ the stochastic exponential

$$\mathcal{E}(\varphi) = \exp\left\{ \int_0^{+\infty} \varphi(s) dw(s) - \frac{1}{2} \int_0^{+\infty} \varphi(s)^2 ds \right\}.$$

Suppose that a random variable α has the Itô–Wiener expansion

$$\alpha = a_0 + \sum_{k=1}^{\infty} \int_{\Delta_k(t)} a_j(r_1, \ldots, r_k) dw(r_1) \ldots dw(r_k).$$

Then

$$\begin{aligned} E\alpha\mathcal{E}(\varphi) &= a_0 \\ &+ \sum_{k=1}^{\infty} \int_{\Delta_k(t)} a_k(r_1, \ldots, r_k) \varphi(r_1) \ldots \varphi(r_k) dr_1 \ldots dr_k. \end{aligned} \tag{3.2}$$

Consequently, to find the Itô–Wiener expansion of α it is enough to find $E\alpha\mathcal{E}(\varphi)$ as an analytic functional from φ. Note that

$$E_u f(\widetilde{w}(t)) \mathcal{E}(\varphi) = E_u f(\widetilde{y}(t)),$$

where the process $\widetilde{y}$ is obtained from the process

$$y(t) = w(t) + \int_0^t \varphi(r) dr$$

in the same way as $\widetilde{w}$ from w. To find $E_u f(\widetilde{y}(t))$ consider the case when f is continuous bounded function with $f(0) = 0$. Let F be the solution to the following boundary problem on $[0; \ +\infty) \times [0; \ T]$

$$\begin{gathered} \frac{\partial}{\partial t} F(u,t) = -\frac{1}{2} \frac{\partial^2}{\partial u^2} F(u,t) - \varphi(t) \frac{\partial}{\partial u} F(u,t), \\ F(u,T) = f(u), \ F(0,s) = 0, \ s \in [0; \ T], \end{gathered} \tag{3.3}$$

$$F \in C^2((0;\ +\infty) \times (0;\ T)) \cap C([0;\ +\infty) \times [0;\ T]).$$

Then $F(u,0) = E_u f(\widetilde{y}(T))$. To check this relation note, that F satisfies the relation

$$\frac{\partial}{\partial u}F(0,s) = \frac{\partial^2}{\partial u^2}F(0,s) = 0,\ s \in [0;\ T].$$

Consider the process $F(\widetilde{y}(s),s)$ on the interval $[0;\ T]$. Using Itô's formula one can get

$$\begin{aligned} F(\widetilde{y}(T),T) = F(u,0) + \int_0^{T\wedge\tau} &\left(\frac{1}{2}\frac{\partial^2}{\partial u^2}F(\widetilde{y}(s),s\right) \\ + \varphi(s)\frac{\partial}{\partial u}F(\widetilde{y}(s),s)) - &\left(\frac{1}{2}\frac{\partial^2}{\partial u^2}F(\widetilde{y}(s),s\right) + \varphi(s)\frac{\partial}{\partial u}F(\widetilde{y}(s),s))ds \\ &+ \int_0^{T\wedge\tau}\frac{\partial}{\partial u}F(\widetilde{y}(s),s)dw(s). \end{aligned}$$

Consequently

$$F(u,0) = E_u f(\widetilde{y}(T)).$$

The problem (3.3) can be solved using the semigroup $\{\widetilde{T}_t;\ t \geq 0\}$. It can be obtained from (3.3) that

$$F(u,s) = \widetilde{T}_{T-s}f(u) + \int_s^T \varphi(r)\widetilde{T}_{r-s}\frac{\partial}{\partial u}F(u,r)dr. \tag{3.4}$$

Solving (3.4) by the iteration method one can get the series

$$\begin{aligned} F(u,s) = &\widetilde{T}_{T-s}f(u) \\ &+ \sum_{k=1}^{\infty}\int_{\Delta_k(s;\ T)}\widetilde{T}_{r_1-s}\frac{\partial}{\partial v_1}\widetilde{T}_{r_2-r_1}\ldots \\ &\qquad \frac{\partial}{\partial v_k}\widetilde{T}_{T-r_k}f(v_k)\varphi(r_1)\ldots\varphi(r_k)dr_1\ldots dr_k. \end{aligned}$$

The last formula means that the Itô–Wiener expansion of $f(\widetilde{w}(t))$ has the form

$$\begin{aligned} f(\widetilde{w}(t)) = \widetilde{T}_t f(u) + \sum_{k=1}^{\infty}\int_{\Delta_k(t)}&\widetilde{T}_{r_1}\frac{\partial}{\partial v_1}\widetilde{T}_{r_2-r_1}\ldots \\ &\frac{\partial}{\partial v_k}\widetilde{T}_{t-r_k}f(v_k)dw(r_1)\ldots dw(r_k). \end{aligned} \tag{3.5}$$

To consider the general case note that for $t > 0$ and $c \in \mathbb{R}$

$$\frac{\partial}{\partial v}\widetilde{T}_t c \equiv 0.$$

Consequently (3.5) remains to be true for an arbitrary bounded continuous f. Now the statement of the lemma can be obtained using the approximation arguments. The lemma is proved. □

The same idea can be used to obtain the Itô–Wiener expansion for a function from the Arratia flow. The n-point motion of the Arratia flow was constructed in Section 2 from independent Wiener processes. Consequently, a function from this $n-$point motion must have the Itô–Wiener expansion in terms of these processes. We will treat such expansion as the Krylov–Veretennikov expansion for the Arratia flow.

Here there is a new circumstance compared to the case when the flow is generated by SDE with smooth coefficients. Namely, there are many different ways to construct the trajectories of the Arratia flow from the initial Wiener processes, and the form of the Itô–Wiener expansion will depend on the way of constructing the trajectories. In [2] Arratia described different ways of constructing the colliding Brownian motions from independent Wiener processes. We present here a more general approach by considering a broad class of constructions, and find the Itô–Wiener expansion for it. To describe our method we will need some preliminary notations and definitions.

Definition 3.2. An arbitrary set of the kind $\{i, i+1, \ldots, j\}$, where $i, j \in \mathbb{N}$,$i \leq j$ is called a *block*.

Definition 3.3. A representation of the block $\{1, 2, \ldots, n\}$ as a union of disjoint blocks is called a *partition* of the block $\{1, 2, \ldots, n\}$.

Definition 3.4. We say that a partition π_2 *follows* from a partition π_1 if it coincides with π_1 or if it is obtained by the union of two subsequent blocks from π_1.

We will consider a sequences of partitions $\{\pi_0, \ldots, \pi_l\}$ where π_0 is a trivial partition, $\pi_0 = \{\{1\}, \{2\}, \ldots, \{n\}\}$ and every π_{i+1} follows from π_i. The set of all such sequences will be denoted by R. Denote by R_k the set of all sequences from R that have exactly k matching pairs: $\pi_i = \pi_{i+1}$. The set R_0 of strongly decreasing sequences we denote by $\breve{R}$. For every sequence $\{\pi_0, \ldots, \pi_k\}$ from $\breve{R}$ each π_{i+1} is obtained from π_i by the union of two subsequent blocks. It is evident, that the length of every sequence from $\breve{R}$ is less or equal to n. Let us associate with every partition π a vector $\vec{\lambda_\pi} \in \mathbb{R}^n$ with the next property. For each block $\{s, \ldots, t\}$ from π the following relation holds

$$\sum_{q=s}^{t} \lambda_{\pi q}^2 = 1.$$

We will use the mapping $\vec{\lambda}$ as a rule of constructing the $n-$point motion of the Arratia flow. Suppose now, that $\{w_k;\ k = 1, \ldots, n\}$ are independent Wiener processes starting at the points $u_1 < \ldots < u_n$. We are going to construct the trajectories $\{x_1, \ldots, x_n\}$ of the Arratia flow starting at $u_1 < \ldots < u_n$ from the pieces of the trajectories of $\{w_k; k = 1, \ldots, n\}$. Assume that we have already built the trajectories of $\{x_1, \ldots, x_n\}$ up to a certain moment of coalescence τ. At this moment a partition π of $\{1, 2, \ldots, n\}$ naturally arise. Two numbers i and j belong to the same block in π if and only if $x_i(\tau) = x_j(\tau)$. Consider one block $\{s, \ldots, t\}$ in π. Define the processes $x_s, \ldots, x_t$ after the moment τ and up to the next moment of coalescence in the whole system $\{x_1, \ldots, x_n\}$ by the rule

$$x_i(t) = x_i(\tau) + \sum_{q=s}^{t} \lambda_{\pi q}(w_q(t) - w_q(\tau)).$$

Proceeding in the same way, we obtain the family $\{x_k,\ k = 1, \ldots, n\}$ of continuous square integrable martingales with respect to the initial filtration, generated by $\{w_k;\ k = 1, \ldots, n\}$ with the following properties:

1) for every $k = 1, \ldots, n, \quad x_k(0) = u_k,$
2) for every $k = 1, \ldots, n-1, \quad x_k(t) \le x_{k+1}(t),$
3) the joint characteristic of x_i and x_j satisfies relation

$$d\langle x_i,\ x_j\rangle(t) = \mathbf{1}_{t \ge \tau_{ij}},$$

where $\tau_{ij} = \inf\{s:\ x_i(s) = x_j(s)\}$.

It can be proved [10] that the processes $\{x_k,\ k = 1, \ldots, n\}$ are the $n-$point motion of the Arratia flow starting from the points $u_1 < \ldots < u_n$. We constructed it from the independent Wiener processes $\{w_k;\ k = 1, \ldots, n\}$ and the way of construction depends on the mapping $\vec{\lambda}$. To describe the Itô–Wiener expansion for functions from $\{x_k(t),\ k = 1, \ldots, n\}$ it is necessary to introduce operators related to a sequence of partitions $\tilde{\pi} \in \breve{R}$. Denote by $\tau_0 = 0 < \tau_1 < \ldots < \tau_{n-1}$ the moments of coalescence for $\{x_k(t),\ k = 1, \ldots, n\}$ and by $\tilde{\nu} = \{\pi_0, \nu_1, \ldots, \nu_{n-1}\}$ related random sequence of partitions. Namely, the numbers i and j belong to the same block in the partition ν_k if and only if $x_i(t) = x_j(t)$ for $\tau_k \le t$. Define for a bounded measurable function $f : \mathbb{R}^n \to \mathbb{R}$

$$T_t^{\tilde{\pi}} f(u_1, \ldots, u_n) = Ef(x_1(t), \ldots, x_n(t))\mathbf{1}_{\{\nu_1=\pi_1, \ldots, \nu_k=\pi_k,\ \tau_k \le t < \tau_{k+1}\}}.$$

Now let κ be an arbitrary partition and let $u_1 \le u_2 \le \ldots \le u_n$ be such, that $u_i = u_j$ if and only if i and j belong to the same block in κ. One can define formally the $n-$point motion of the Arratia flow starting at $u_1 \le u_2 \le \ldots \le u_n$, assuming that the trajectories that start at coinciding points, also coincide. Then for the strongly decreasing sequence of partitions $\tilde{\pi} = \{\kappa, \pi_1, \ldots, \pi_k\}$ the operator $T_t^{\tilde{\pi}}$ is defined by the same formula as above.

The next theorem is the Krylov–Veretennikov expansion for the $n-$point motion of the Arratia flow.

Theorem 3.5. *For a bounded measurable function $f : \mathbb{R}^n \to \mathbb{R}$ the following representation takes place*

$$f(x_1(t), \ldots, x_n(t)) = \sum_{\tilde{\pi} \in \breve{R}} T_t^{\tilde{\pi}} f(u_1, \ldots, u_n)$$

$$+ \sum_{i=1}^{n} \sum_{\tilde{\pi} \in R_1} \lambda_{\pi_1 i} \int_0^t T_{s_1}^{\tilde{\pi}_1} \partial_i T_{t-s_1}^{\tilde{\pi}_2} f(u_1, \ldots, u_n) dw_i(s_1)$$

$$+ \sum_{i_1, i_2=1}^{n} \sum_{\tilde{\pi} \in R_2} \lambda_{\pi_1 i_1} \lambda_{\pi_2 i_2} \int_{\triangle_2(t)} T_{s_1}^{\tilde{\pi}_1} \partial_{i_1} T_{s_2-s_1}^{\tilde{\pi}_2} \partial_{i_2} T_{t-s_2}^{\tilde{\pi}_3}$$

$$f(u_1, \ldots, u_n) dw_{i_1}(s_1) dw_{i_2}(s_2)$$

$$+\sum_{i_1,\dots,i_k=1}^{n}\sum_{\tilde{\pi}\in R_k}\prod_{j=1}^{k}\lambda_{\pi_j i_j}\int_{\triangle_k(t)}T_{s_1}^{\tilde{\pi}_1}\partial_{i_1}T_{s_2-s_1}^{\tilde{\pi}_2}\dots\partial_{i_k}T_{t-s_k}^{\tilde{\pi}_{k+1}}$$
$$f(u_1,\dots,u_n)dw_{i_1}(s_1)\dots dw_{i_k}(s_k)+\cdots\cdots$$

In this formula we use the following notations. For a sequence $\tilde{\pi}\in R_k$ *partitions* $\pi_1,\dots,\pi_k$ *are the left elements of equalities from* $\tilde{\pi}=\{\dots\pi_1=\dots\pi_2=\dots\pi_k=\dots\}$ *and* $\tilde{\pi}_1,\dots,\tilde{\pi}_{k+1}$ *are strictly decreasing pieces of* $\tilde{\pi}$. *The symbol* ∂_i *denotes differentiation with respect to a variable corresponding to the block of partition, which contains* i. *For example, if* $i\in\{s,\dots,t\}$ *then* $\partial_i f=\sum_{q=s}^{q=t}f'_q$.

The proof of the theorem can be obtained by induction, adopting ideas of Lemma 3.1. One has to consider subsequent boundary value problems and then use the probabilistic interpretation of the Green's functions for these problems. The corresponding routine calculations are omitted.

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