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CONSTRUCTION OF THE PATHS OF BROWNIAN MOTIONS ON STAR GRAPHS II

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Abstract. In this article and its predecessor [9], pathwise constructions of Brownian motions which satisfy all possible boundary conditions at the vertex of star graphs are given.

1. Introduction

In the present article we continue and complete our program of constructing all possible Brownian motions on a star graph which has been begun in [9], henceforth also cited as *article I*. The case of a general metric graph is treated in [10].

In order to make the present article self-contained, we quickly recall the most important notions and results from article I, partly in a somewhat informal way — for more details and for a more extensive introduction to the subject the reader is referred to article I. A *star graph* $\mathcal{G}$ is a finite collection $\{l_1, l_2, \dots, l_n\}$, $n \in \mathbb{N}$, of sets isomorphic to $\mathbb{R}_+$, called *external edges*, where the points corresponding to the origin of $\mathbb{R}_+$ under the isomorphims are identified and form the *vertex* v of the graph $\mathcal{G}$.

Definition 1.1. A *Brownian motion* $X = (X_t,\, t \in \mathbb{R}_+)$ on $\mathcal{G}$ is a diffusion process on $\mathcal{G}$, such that X with absorption at v is equivalent to a Brownian motion on the half line $\mathbb{R}_+$ with absorption at the origin.

We quote the analogue of Feller's theorem [7, Theorem 6.2] for a Brownian motion on the single vertex graph $\mathcal{G}$ from article I:

Theorem 1.2. *Assume that X is a Brownian motion on $\mathcal{G}$. Then there exist constants a, b_k, $c \in [0,1]$, $k = 1, \dots, n$, with*

$$a + c + \sum_{k=1}^{n} b_k = 1, \quad a \neq 1, \tag{1.1a}$$

such that the domain $\mathcal{D}(A)$ of the generator A of X in $C_0(\mathcal{G})$ consists exactly of those $f \in C_0^2(\mathcal{G})$ for which the Wentzell boundary condition

$$a f(v) + \frac{c}{2} f''(v) = \sum_{k=1}^{n} b_k f'(v_k) \tag{1.1b}$$

holds true. Moreover, for $f \in \mathcal{D}(A)$, $Af = 1/2 f''$.

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Indeed this theorem follows from our result [10, Theorem 1.3] in the situation of a general metric graph. Above we used the subspace $C_0^2(\mathcal{G})$ of the space $C_0(\mathcal{G})$ of continuous functions vanishing at infinity, which is defined as the space of those functions $f \in C_0(\mathcal{G})$ which are twice continuously differentiable on $\mathcal{G}^\circ = \mathcal{G} \setminus \{v\}$, such that f'' extends from $\mathcal{G}^\circ$ to $\mathcal{G}$ as a function in $C_0(\mathcal{G})$. As usual, $C_0(\mathcal{G})$ is equipped with the sup norm, so that it becomes a Banach space.

In article I we constructed the Walsh process on the single vertex graph $\mathcal{G}$ from a standard Brownian motion on the real line. The Walsh process on $\mathcal{G}$ implements a *Neumann* boundary condition: $a = c = 0$ in (1.1). Moreover, by killing this process on the scale of its local time at the vertex, we obtained in [9] a Brownian motion on $\mathcal{G}$ satisfying an *elastic* boundary condition, i.e., $c = 0$, $a \neq 0$ in (1.1). In section 2 of the present article we carry out the construction of a Brownian motion with a *sticky* boundary condition of the form (1.1) with $a = 0$, $c \neq 0$, $b_k \neq 0$ for at least one k. Finally in section 3 we construct the process on a single vertex graph in its most general form.

It will be convenient to refer to sections, results and formulae of article I with a prefix "I". Thus, e.g., "lemma I.1.3" means lemma 1.3 of article I, while "equation (I.1.16)" refers to equation (1.16) of article I.

2. The Walsh Process with a Sticky Vertex

In this section we construct Brownian motions on $\mathcal{G}$ with $a = 0$ in the boundary condition (1.1).

Consider the Walsh process W on $\mathcal{G}$ from section I.2 together with a right continuous, complete filtration $\mathcal{F}^w$, relative to which it is strongly Markovian. Furthermore, we denote its local time at the vertex v by L^w (cf. section I.3).

As in [9] we closely follow the recipe given by Itô and McKean in [4] (cf. also [7, Section 6.2]) for the case of a Brownian motion on the half line. For $\gamma \geq 0$ introduce a new time scale τ by

$$\tau^{-1} : t \mapsto t + \gamma L_t^w, \qquad t \geq 0. \tag{2.1}$$

Since L^w is non-decreasing, τ^{-1} is strictly increasing. Moreover, we have $\tau^{-1}(0) = 0$ and $\lim_{t\to+\infty} \tau^{-1}(t) = +\infty$, which implies that τ exists, and is strictly increasing from $\mathbb{R}+$ onto $\mathbb{R}_+$, too. As is shown in [7, p. 160], the additivity of L^w entails the additivity of τ on its own time scale, i.e.:

Lemma 2.1. *For all s, $t \geq 0$, a.s. the following formula holds true*

$$\tau(s + t) = \tau(s) + \tau(t) \circ \theta_{\tau(s)}. \tag{2.2}$$

It is easily checked that for every $t \geq 0$, $\tau(t)$ is an $\mathcal{F}^w$–stopping time, and since τ is increasing, we obtain the subfiltration $\mathcal{F}^s = (\mathcal{F}_t^s,\, t \geq 0)$ of $\mathcal{F}^w$ defined by $\mathcal{F}_t^s = \mathcal{F}_{\tau(t)}^w$, $t \in \mathbb{R}_+$. Moreover, we set $\mathcal{F}_\infty^w = \sigma(\mathcal{F}_t^w,\, t \in \mathbb{R}_+)$ and $\mathcal{F}_\infty^s = \sigma(\mathcal{F}_t^s,\, t \in \mathbb{R}_+)$, and find $\mathcal{F}_\infty^s \subset \mathcal{F}_\infty^w$. Standard calculations show that the completeness and the right continuity of $\mathcal{F}^w$ entail the same properties for $\mathcal{F}^s$. (For details of the argument in the case where $\mathcal{G} = \mathbb{R}_+$ we refer the interested reader to section 3 of [8].)

Define a stochastic process W^s on $\mathcal{G}$, called *Walsh process with sticky vertex*, by

$$W_t^s = W_{\tau(t)}, \qquad t \in \mathbb{R}_+. \tag{2.3}$$

Observe that when W is away from the vertex, L^w is constant, and therefore in this case τ^{-1} grows with rate 1. On the other hand, when W is at the vertex, τ^{-1} grows faster than with rate 1, and therefore τ increases slower than the deterministic time scale $t \mapsto t$. Thus W^s "experiences a slow down in time" until W has left the vertex. In this heuristic sense the vertex is "sticky" for W^s, because it spends more time there than W.

Note that because L^w has continuous paths, τ^{-1} and therefore also τ are pathwise continuous. Consequently, W^s has continuous sample paths. Since W has continuous paths, it is a measurable process, and hence for every $t \geq 0$, $W_{\tau(t)}$ is $\mathcal{F}^w_{\tau(t)}$–measurable, that is, W^s is $\mathcal{F}^s$–adapted. Set $\theta_t^s = \theta_{\tau(t)}$. With the additivity (2.2) of τ we immediately find

$$W_s^s \circ \theta_t^s = W_{s+t}^s, \qquad s,\, t \in \mathbb{R}_+. \tag{2.4}$$

Thus $\theta^s = (\theta_t^s,\, t \in \mathbb{R}_+)$ is a family of shift operators for W^s.

Next we show the strong Markov property of W^s relative to $\mathcal{F}^s$ following the argument briefly sketched in section 6.2 of [7] for the case $\mathcal{G} = \mathbb{R}_+$. First we prove the simple Markov property of W^s with respect to $\mathcal{F}^s$. To this end, let $s,\, t \geq 0$, $\xi \in \mathcal{G}$, and $C \in \mathcal{B}(\mathcal{G})$. Then we get with (2.4)

$$\begin{aligned} P_\xi\big(W_{t+s}^s \in C \,\big|\, \mathcal{F}_t^s\big) &= P_\xi\big(W_s^s \circ \theta_t^s \in C \,\big|\, \mathcal{F}_t^s\big) \\ &= P_\xi\big(W_{\tau(s)} \circ \theta_{\tau(t)} \in C \,\big|\, \mathcal{F}^w_{\tau(t)}\big) \\ &= P_{W_{\tau(t)}}\big(W_{\tau(s)} \in C\big) \\ &= P_{W_t^s}\big(W_s^s \in C\big), \end{aligned}$$

where we used the strong Markov property of W with respect to $\mathcal{F}^w$. As a next step we prove that W^s has the strong Markov property for its hitting time H_v^s of the vertex. By construction, W^s and W have the same paths up to the hitting time of the vertex, and in particular H_v^s is also the hitting time of the vertex by W, that is, $H_v^s = H_v$. Moreover, since $L^w(H_v) = 0$, we get that $\tau^{-1}(H_v) = H_v = \tau(H_v)$, as well as $\theta^s(H_v) = \theta(H_v)$. Assume now that $t \geq 0$, $\xi \in \mathcal{G}$, and $C \in \mathcal{B}(\mathcal{G})$. Then on $\{H_v < +\infty\}$ we can compute with the strong Markov property of W as follows

$$\begin{aligned} P_\xi\big(W_{t+H_v}^s \in C \,\big|\, \mathcal{F}_{H_v}^w\big) &= P_\xi\big(W_t^s \circ \theta_{H_v}^s \in C \,\big|\, \mathcal{F}_{H_v}^w\big) \\ &= P_\xi\big(W_{\tau(t)} \circ \theta_{H_v} \in C \,\big|\, \mathcal{F}_{H_v}^w\big) \\ &= P_v\big(W_{\tau(t)} \in C\big) \\ &= P_v\big(W_t^s \in C\big). \end{aligned}$$

It is readily checked that $\mathcal{F}_{H_v}^s \subset \mathcal{F}_{H_v}^w$, and therefore we get in particular the strong Markov property of W^s with respect to $H_v^s = H_v$ in the form

$$P_\xi\big(W_{t+H_v^s}^s \in C \,\big|\, \mathcal{F}_{H_v}^s\big) = P_v\big(W_t^s \in C\big). \tag{2.5}$$

Finally, with the strong Markov property of the standard one-dimensional Brownian motions on every edge and the strong Markov property (2.5) just proved we can apply the arguments similar to those in [5, Section 3.6] to conclude that W^s is a Feller process. Hence it is strongly Markovian relative to the filtration $\mathcal{F}^s$.

By construction, W^s is up to time H_v^s equivalent to a standard one-dimensional Brownian motion, and it has continuous sample paths. Hence, altogether we have shown that W^s is a Brownian motion on $\mathcal{G}$ in the sense of definition 1.1.

Now we want to compute the generator of W^s, and first we argue that v is not a trap for W^s. To this end, we may consider W as constructed from a standard Brownian motion B as described in section I.2. Let Z denote the zero set of B. Given $s \geq 0$ we can choose $t_0 \geq s$ in the complement Z^c of Z. Consider $t = \tau^{-1}(t_0)$, i.e., $t = t_0 + \gamma L_{t_0}^w$. Obviously $t \geq s$, and $\tau(t) \in Z^c$. Therefore $B_{\tau(t)} \neq 0$, and consequently $W_t^s = W_{\tau(t)} \neq v$.

Theorem 2.2. *Consider the boundary condition* (1.1) *with* $a = 0$, $c \in (0,1)$, *and* $b \in [0,1]^n$. *Set*

$$w_k = \frac{b_k}{1-c},\ k = 1, \ldots, n, \qquad \gamma = \frac{c}{1-c}, \tag{2.6}$$

and let W^s *be the sticky Walsh process as constructed above with these parameters. Then the generator* A^s *of* W^s *is* $1/2$ *times the second derivative on* $\mathcal{G}$ *with domain consisting of those* $f \in C_0^2(\mathcal{G})$ *which satisfy condition* (1.1b).

Before we prove theorem 2.2 we first prepare two preliminary results. Let $\epsilon > 0$, and let $H_{v,\epsilon}^s$ denote the hitting time of the complement of the open ball $B_\epsilon(v)$ with radius ϵ and center v by W^s. Recall that $H_{v,\epsilon}^w$ denotes the corresponding first hitting time for the Walsh process W.

Lemma 2.3. P_v*–a.s., the formula*

$$H_{v,\epsilon}^s = H_{v,\epsilon}^w + \gamma\, L_{H_{v,\epsilon}^w}^w \tag{2.7}$$

holds true.

Proof. Let W, and therefore also W^s, start in the vertex v. Since W^s and W have continuous paths with infinite lifetime we have for all $\gamma \geq 0$

$$H_{v,\epsilon}^s = \inf\bigl\{t > 0,\, d(v, W_{\tau(t)}) = \epsilon\bigr\},$$

and in particular for $\gamma = 0$,

$$H_{v,\epsilon}^w = \inf\bigl\{t > 0,\, d(v, W_t) = \epsilon\bigr\}.$$

Moreover, as argued above, both infima are a.s. finite. Set

$$\sigma = H_{v,\epsilon}^w + \gamma\, L_{H_{v,\epsilon}^w}^w.$$

Then $\tau(\sigma) = H_{v,\epsilon}^w$, and therefore

$$\begin{aligned} d\bigl(v, W_\sigma^s\bigr) &= d\bigl(v, W_{\tau(\sigma)}\bigr) \\ &= d\bigl(v, W_{H_{v,\epsilon}^w}\bigr) \\ &= \epsilon. \end{aligned}$$

Consequently we get $H_{v,\epsilon}^s \leq \sigma$. To derive the converse inequality we remark that

$$\begin{aligned} \epsilon &= d\bigl(v, W_{H_{v,\epsilon}^s}^s\bigr) \\ &= d\bigl(v, W_{\tau(H_{v,\epsilon}^s)}\bigr), \end{aligned}$$

which implies

$$\tau\big(H^s_{v,\epsilon}\big) \geq H^w_{v,\epsilon}.$$

Since τ is strictly increasing this entails

$$H^s_{v,\epsilon} \geq \tau^{-1}\big(H^w_{v,\epsilon}\big) = \sigma,$$

and the proof is finished. □

Corollary 2.4. *For every $\gamma \geq 0$,*

$$E_v\big(H^s_{v,\epsilon}\big) = \epsilon^2 + \gamma\epsilon \tag{2.8}$$

holds.

Proof. By construction, the paths of W starting in v hit the complement of $B_\epsilon(v)$ exactly when the underlying standard Brownian motion B (cf. section I.2) starting at the origin hits one of the points $\pm\epsilon$ on the real line. Thus under P_v, $L^w(H^w_{v,\epsilon})$ has the same law as $L^B(H^B_{\{-\epsilon,\epsilon\}})$ under P_0. Lemma I.1.9 states that under P_0 this random variable is exponentially distributed with mean ϵ. Then equation (2.8) follows directly from lemma 2.3, and lemma I.2.1. □

Given these results, we come to the

Proof of theorem 2.2. Let w_k, $k = 1, \dots, n$, and γ be defined as in (2.6), and note that due to the condition (1.1a) on b_k, $k = 1, \dots, n$, and c, we have $w_k \in [0, 1]$, $k = 1, \dots, n$, $\sum_k w_k = 1$, as well as $\gamma > 0$. Hence we can construct the associated sticky Walsh process W^s as above.

Let A^s denote the generator of W^s with domain $\mathcal{D}(A^s)$. Then we have for $f \in \mathcal{D}(A^s)$, $A^s f(v) = 1/2 f''(v)$ (cf. theorem 1.2). On the other hand, we can compute $A^s f(v)$ via Dynkin's formula as follows

$$\begin{aligned} A^s f(v) &= \lim_{\epsilon\downarrow 0} \frac{E_v\Big(f\big(W^s(H^s_{v,\epsilon})\big)\Big) - f(v)}{E_v\big(H^s_{v,\epsilon}\big)} \\ &= \lim_{\epsilon\downarrow 0} \frac{\sum_k w_k f_k(\epsilon) - f(v)}{\epsilon^2 + \gamma\epsilon}, \end{aligned}$$

where we used corollary 2.4. Since the directional derivatives of f at v

$$f'(v_k) = \lim_{\xi\to v,\, \xi\in l_k} \frac{f(\xi) - f(v)}{d(\xi, v)}, \qquad k \in \{1, 2, \dots, n\},$$

exist (cf. lemma I.1.3), we obviously get the boundary condition

$$\frac{1}{2}\, f''(v) = \frac{1}{\gamma} \sum_{k=1}^{n} w_k f'(v_k) \tag{2.9}$$

as a necessary condition. Finally, inserting of the values (2.6) of the parameters w_k, $k = 1, \dots, n$, and γ into equation (2.9) we obtain the boundary conditions as stated in theorem 2.2. The proof of theorem 2.2 is completed by the remark that boundary conditions of the form (1.1) uniquely characterize the domain of the generator A^s, cf. remark I.1.6. □

Next we shall compute the resolvent R^s of the Walsh process with sticky vertex. Similarly to the alternative proof of theorem I.3.2, for the elastic Walsh process, as a byproduct we obtain an alternative proof of theorem 2.2. We begin with the following

Lemma 2.5. *Let $\lambda > 0$, $f \in C_0(\mathcal{G})$. Then*

$$\frac{1}{2}\big(R^s_\lambda f\big)''(v) = \frac{1}{\sqrt{2\lambda}+\gamma\lambda}\Big(2\lambda\,(e^w_\lambda, f) - \sqrt{2\lambda} f(v)\Big) \tag{2.10}$$

holds, where

$$e^w_\lambda(\xi) = w_k\, e^{-\sqrt{2\lambda}\, d(\xi,v)}, \qquad \xi \in l_k,\ k = 1,\dots,n. \tag{2.11}$$

Proof. Let A^s be the generator of W^s on $C_0(\mathcal{G})$. From the identity $A^s R^s_\lambda = \lambda\, R^s_\lambda - \mathrm{id}$, and the definition of τ we get

$$\begin{aligned}
\frac{1}{2}\,\big(R^s_\lambda f\big)''(v) &= \lambda E_v\Big(\int_0^\infty e^{-\lambda t}\big(f(W^s_t) - f(v)\big)\,dt\Big)\\
&= \lambda E_v\Big(\int_0^\infty e^{-\lambda(s+\gamma L^w_s)}\big(f(W_s) - f(v)\big)\,(ds + \gamma dL^w_s)\Big)\\
&= \lambda E_v\Big(\int_0^\infty e^{-\lambda(s+\gamma L^w_s)}\big(f(W_s) - f(v)\big)\,ds\Big).
\end{aligned}$$

In the last equality we used the fact that L^w only grows when W is at the vertex v. By construction of the Walsh process W we have

$$\begin{aligned}
&E_v\Big(e^{-\lambda\gamma L^w_s}\big(f(W_s) - f(v)\big)\Big)\\
&\quad = \sum_{k=1}^n w_k\, E_0\Big(e^{-\lambda\gamma L^B_s}\big(f_k(|B_s|) - f_k(0)\big)\Big)\\
&\quad = 2\sum_{k=1}^n w_k \int_0^\infty\int_0^\infty e^{-\lambda\gamma y}\big(f_k(x) - f_k(0)\big)\,\frac{x+y}{\sqrt{2\pi s^3}}\,e^{-(x+y)^2/2s}\,dx\,dy,
\end{aligned}$$

where we used lemma I.1.7. We insert the last expression above, and use formula (I.1.31). This gives

$$\begin{aligned}
\frac{1}{2}\,\big(R^s_\lambda f\big)''(v) &= 2\lambda\sum_{k=1}^n w_k\,\frac{1}{\sqrt{2\lambda}+\gamma\lambda}\int_0^\infty e^{-\sqrt{2\lambda}x}\big(f_k(x) - f_k(0)\big)\,dx\\
&= \frac{1}{\sqrt{2\lambda}+\gamma\lambda}\big(2\lambda\,(e^w_\lambda, f) - \sqrt{2\lambda} f(v)\big).
\end{aligned}$$

□

From the identity $A^s R^s_\lambda = \lambda\, R^s_\lambda - \mathrm{id}$ and some simple algebra we get the

Corollary 2.6. *Let $\lambda > 0$, and $f \in C_0(\mathcal{G})$. Then*

$$R^s_\lambda f(v) = \frac{1}{\sqrt{2\lambda}+\gamma\lambda}\big(2\,(e^w_\lambda, f) + \gamma f(v)\big) \tag{2.12}$$

holds.

Since formula (I.1.27) in corollary I.1.10 is valid for the resolvent of every Brownian motion on $\mathcal{G}$, we may use that formula for $R^s_\lambda f$, sum it against the weights w_k, $k = 1, \dots, n$, and insert the right hand side of equation (2.12). This results in

$$\sum_{k=1}^{n} w_k \bigl(R^s_\lambda f\bigr)'(v_k) = \gamma\, \frac{1}{\sqrt{2\lambda} + \gamma\lambda}\, \bigl(2\lambda\, (e^w_\lambda, f) - \sqrt{2\lambda} f(v)\bigr),$$

and a comparison with formula (2.10) shows that equation (2.9) holds true for f replaced by $R^s_\lambda f$ for arbitrary $f \in C_0(\mathcal{G})$. As promised we thus have another proof of theorem 2.2.

With the help of the first passage time formula we can now provide explicit expressions for the resolvent R^s, its kernel r^s and the transition kernel p^s of W^s. Inserting the right hand side of equation (2.12) into the first passage time formula (I.1.26), we immediately obtain for $f \in C_0(\mathcal{G})$, $\lambda > 0$,

$$R^s_\lambda f(\xi) = R^D_\lambda f(\xi) + \frac{1}{\sqrt{2\lambda} + \gamma\lambda}\, e_\lambda(\xi)\bigl(2\, (e^w_\lambda, f) + \gamma f(v)\bigr), \quad \xi \in \mathcal{G}, \tag{2.13}$$

where R^D is the Dirichlet resolvent (I.1.22). Using formula (I.1.23) for the kernel of R^D together with (I.1.26), and (I.1.27) , we get the following result.

Corollary 2.7. *For ξ, $\eta \in \mathcal{G}$, $\lambda > 0$, the resolvent kernel r^s_λ, of the Walsh process with sticky vertex is given by*

$$\begin{aligned} r^s_\lambda(\xi, d\eta) = r^D_\lambda(\xi, \eta)\, d\eta + \sum_{k,m=1}^{n} e_{\lambda,k}(\xi)\; 2w_m\, \frac{1}{\sqrt{2\lambda} + \gamma\lambda}\, e_{\lambda,m}(\eta)\, d\eta \\ + \frac{\gamma}{\sqrt{2\lambda} + \gamma\lambda}\, e_\lambda(\xi)\, \epsilon_v(d\eta), \end{aligned} \tag{2.14}$$

with r^D_λ defined in (I.1.23), and ϵ_v denotes the Dirac measure in v. Alternatively, r^s_λ is given by

$$\begin{aligned} r^s_\lambda(\xi, d\eta) = r_\lambda(\xi, \eta)\, d\eta + \sum_{k,m=1}^{n} e_{\lambda,k}(\xi)\, S^s_{km}(\lambda)\, \frac{1}{\sqrt{2\lambda}}\, e_{\lambda,m}(\eta)\, d\eta \\ + \frac{\gamma}{\sqrt{2\lambda} + \gamma\lambda}\, e_\lambda(\xi)\, \epsilon_v(d\eta), \end{aligned} \tag{2.15a}$$

where r_λ is defined in equation (I.1.24), and

$$S^s_{km}(\lambda) = 2\, \frac{\sqrt{2\lambda}}{\sqrt{2\lambda} + \gamma\lambda}\, w_m - \delta_{km}. \tag{2.15b}$$

Remark 2.8. When all w_m, $m = 1, \dots, n$, are equal to $1/n$, the matrix $S^s(\lambda)$ takes the form

$$S^s(\lambda) = -1 + \frac{2\sqrt{2\lambda}}{\sqrt{2\lambda} + \gamma\lambda}\, P_n$$

which reduces to (I.2.9) when $\gamma = 0$. $S^s(\lambda)$ is unitary for all $\lambda < 0$. Also the $S^s(\lambda)$ for different λ all commute. As a consequence $S^s(\lambda)$ has the interpretation of a quantum scattering matrix in the sense of [11]. More precisely, $S^s(\lambda)$ stems from the Schrödinger operator $-\Delta^s$, where Δ^s is a self-adjoint Laplace operator on $L^2(\mathcal{G})$ with boundary conditions of the form (I.2.5) with the choice

$$\begin{aligned} A &= -\frac{1}{2}\bigl(S^s(\lambda_0)+1\bigr),\\ B &= -\frac{1}{2\sqrt{2\lambda_0}}\bigl(S^s(\lambda_0)-1\bigr), \end{aligned} \tag{2.16}$$

for any λ_0 for which $\sqrt{2\lambda_0}+\gamma\lambda_0 \neq 0$. We emphasize that the Schrödinger operator $-\Delta^s$ and the generator A^s of the Walsh process are quite different: Not only do they act on different Banach spaces, but also the functions in the intersection of their domains satisfy different boundary conditions at the vertex v. As matter of fact, the integral kernel of the resolvent $(-\Delta^s + 2\lambda)^{-1}$ of the Schrödinger operator $-\Delta^s$ is given by, see Lemma 4.2 in [12],

$$\frac{1}{2}\Bigl(r_\lambda(\xi,\eta) + \sum_{k,m=1}^{n} e_{\lambda,k}(\xi)\, S^s_{km}(\lambda)\, \frac{1}{\sqrt{2\lambda}}\, e_{\lambda,m}(\eta)\Bigr),$$

that is — up to a factor 2 — by the right hand side of (2.15a) *without* the last term.

In more detail and with the definition (I.2.6)

$$S_{A,B}\bigl(E=-2\lambda\bigr) = S^s(\lambda)$$

holds for all $\lambda > 0$. As a function of k ($k^2 = E$), S^s is meromorphic in the complex k–plane with a pole on the positive imaginary axis at $k^b = 2i/\gamma$. This corresponds to a negative eigenvalue $E^b = -4/\gamma^2$ of $-\Delta^s$. The corresponding (normalized) eigenfunction ψ^b — physically speaking a *bound state* — is given as

$$\psi^b(\xi) = \frac{1}{2}\sqrt{\frac{\gamma}{n}}\, e^{-2d(v,\xi)/\gamma}, \qquad \xi \in \mathcal{G}.$$

So quantum mechanically the vertex v acts like an attractive potential. We view this as a quantum analogue of the stickiness of the vertex v.

This analogy can be elaborated a bit further by inspecting the associated quantum mechanical time delay matrix (see, e.g., [1, 2, 6, 13–16])

$$T(k) = \frac{1}{2ik}\, S(k)^{-1} \frac{\partial}{\partial k} S(k)$$

which in the present context gives

$$T(k) = \frac{-2\gamma}{k(4+k^2\gamma^2)}\, P_n.$$

So $T(k)$ has zero as an $(n-1)$–fold eigenvalue plus the non-degenerate eigenvalue

$$\frac{-2\gamma}{k(4+k^2\gamma^2)},$$

which for $\gamma > 0$ is the signal for a strict quantum delay. Observe that for $k \to +\infty$, that is for large energies, the time delay experienced by the quantum particle tends to zero, while for $k \to 0$, i.e., for low energies, the delay becomes arbitrarily large. From the physical point of view, both effects are clearly to be expected. For comparison and in contrast to the present stochastic context, in quantum mechanics $\gamma < 0$ is also allowed for a meaningful Schrödinger operator and an associated scattering matrix.

Define for $x \geq 0$, γ, $t > 0$,

$$g_{0,\gamma}(t,x) = \frac{1}{\gamma}\,\exp\Bigl(\frac{2x}{\gamma} + \frac{2t}{\gamma^2}\Bigr)\,\mathrm{erfc}\Bigl(\frac{x}{\sqrt{2t}} + \frac{\sqrt{2t}}{\gamma}\Bigr). \tag{2.17}$$

It is not hard to check that

$$\lim_{\gamma\downarrow 0} g_{0,\gamma}(t,x) = g(t,x) = \frac{1}{\sqrt{2\pi t}}\,e^{-x^2/2t}. \tag{2.18}$$

Moreover, from [3, eq. (5.6.16)] (cf. also appendix C in [8]) the Laplace transform is

$$\mathcal{L}g_{0,\gamma}(\,\cdot\,,x)(\lambda) = \frac{1}{\sqrt{2\lambda} + \gamma\lambda}\,e^{-\sqrt{2\lambda}x}, \qquad x \geq 0. \tag{2.19}$$

Observe that in agreement with (2.18)

$$\mathcal{L}g(\,\cdot\,,x)(\lambda) = \frac{1}{\sqrt{2\lambda}}\,e^{-\sqrt{2\lambda}x}$$

holds. Now we can readily compute the inverse Laplace transform of formulae (2.14), (2.15), and obtain the following result.

Corollary 2.9. *For $t > 0$, ξ, $\eta \in \mathcal{G}$ the transition kernel of the Walsh process with sticky vertex is given by*

$$\begin{aligned} p^s(t,\xi,\eta) &= p^D(t,\xi,\eta)\,d\eta \\ &\quad + \sum_{k,m=1}^{n} 1_{l_k}(\xi)\,2w_m\,g_{0,\gamma}\bigl(t,d_v(\xi,\eta)\bigr)1_{l_m}(\eta)\,d\eta \\ &\quad + \gamma\,g_{0,\gamma}\bigl(t,d(\xi,v)\bigr)\,\epsilon_v(d\eta) \end{aligned} \tag{2.20}$$

where p^D is the heat kernel with Dirichlet boundary conditions, see (I.1.21), or alternatively by

$$\begin{aligned} p^s(t,\xi,d\eta) &= p(t,\xi,\eta)\,d\eta \\ &\quad + \sum_{k,m=1}^{n} 1_{l_k}(\xi)\,\Bigl(2w_m\,g_{0,\gamma}\bigl(t,d_v(\xi,\eta)\bigr)\,d\eta \\ &\qquad\qquad - \delta_{km}\,g\bigl(t,d_v(\xi,\eta)\bigr)\,d\eta\Bigr)\,1_{l_m}(\eta) \\ &\quad + \gamma\,g_{0,\gamma}\bigl(t,d(\xi,v)\bigr)\,\epsilon_v(d\eta), \end{aligned} \tag{2.21}$$

and $p(t,a,b)$ is given in formula (I.1.17).

We close this section with some remarks concerning the local time of W^s at the vertex v, which also serve to prepare the construction of the most general Brownian motion on the single vertex graph $\mathcal{G}$ in the next section.

Let us define

$$L^s_t = L^w_{\tau(t)}, \qquad t \geq 0, \tag{2.22}$$

where — as before — L^w denotes the local time of the Walsh process at the vertex, having (cf. section I.3) the same normalization as the local time of a standard one-dimensional Brownian motion (cf. (I.1.8)). By construction, L^s is pathwise continuous and non-decreasing. It is adapted to $\mathcal{F}^s$, and a straightforward calculation based on the additivity of L^w and formula (2.2) shows the (pathwise) additivity property

$$L^s_{s+t} = L^s_t + L^s_s \circ \theta^s_t, \qquad s,\, t \geq 0. \tag{2.23}$$

Therefore L^s is a perfect continuous homogeneous additive functional (PCHAF) of $(W^s, \mathcal{F}^s)$ in the sense of [17, Section III.32]. Furthermore, $t \geq 0$ is a point of increase for L^s if only if $\tau(t)$ is a point of increase for L^w, which only is the case if $W_{\tau(t)}$ is at the vertex, i.e., if W^s_t is at the vertex. Thus, it follows that L^s is a local time at the vertex for W^s. In order to completely identify it, it remains to compute its normalization, and it is not very hard to compute its α–potential (the interested reader can find the details for the case $\mathcal{G} = \mathbb{R}_+$ in [8]):

$$E_\xi\Bigl(\int_0^\infty e^{-\alpha t}\, dL^s_t\Bigr) = \frac{1}{\sqrt{2\alpha} + \gamma\alpha}\, e^{-\sqrt{2\alpha}\, d(\xi, v)}, \qquad \alpha > 0,\, \xi \in \mathcal{G}. \tag{2.24}$$

3. The General Brownian Motion on a Single Vertex Graph

Finally, in this subsection we construct a Brownian motion W^g by killing the Walsh process with sticky vertex of section 2 in a similar way as in the construction of the elastic Walsh process (cf. section I.3). W^g realizes the boundary condition (1.1) in its most general form.

Consider the sticky Walsh process W^s with stickiness parameter $\gamma > 0$, right continuous and complete filtration $\mathcal{F}^s$, and local time L^s at the vertex. We argued in section 2 that L^s is a PCHAF for $(W^s, \mathcal{F}^s)$, and therefore we can apply the method of killing described in subsection I.1.5: We bring in the additional probability space $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), P_\beta)$ where P_β is the exponential law of rate $\beta > 0$, and the canonical coordinate random variable S. Then we take the family of product spaces $\bigl(\hat\Omega, \hat{\mathcal{A}}, (\hat P_\xi,\, \xi \in \mathcal{G})\bigr)$ of $\bigl(\Omega, \mathcal{A}, (P_\xi,\, \xi \in \mathcal{G})\bigr)$ and $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), P_\beta)$. Define the random time

$$\zeta_{\beta,\gamma} = \inf\bigl\{t \geq 0,\, L^s_t > S\bigr\}. \tag{3.1}$$

Then by the arguments given in subsection I.1.5, the stochastic process W^g defined by $W^g_t = W^s_t$ for $t \in [0, \zeta_{\beta,\gamma})$, and $W^g_t = \Delta$ for $t \geq \zeta_{\beta,\gamma}$, is again a Brownian motion on $\mathcal{G}$ in the sense of definition 1.1.

Denote by K^s the right continuous pseudo-inverse of L^s. Since L^s is continuous (cf. equation (2.22)), we get $L^s_{K^s_r} = r$ for all $r \in \mathbb{R}_+$. Recall that the right continuous pseudo-inverse of the local time L^w of the Walsh process was denoted by K^w. Then we have the following

Lemma 3.1. *For all $\gamma \geq 0$, the following relation holds true:*

$$K^s_r = K^w_r + \gamma r, \qquad r \in \mathbb{R}_+. \tag{3.2}$$

Proof. For γ, $r \in \mathbb{R}_+$ define the random subset

$$J_\gamma(r) = \{t \geq 0,\, L^s_t > r\}$$

of $\mathbb{R}_+$. Since L^s is pathwise increasing, this set is a random interval with endpoints K^s_r and $+\infty$. The relation $L^s_{K^s_r} = r$ implies that

$$J_\gamma(r) = (K^s_r, +\infty).$$

In particular, we have $J_0 = (K^w_r, +\infty)$. Now

$$t \in J_\gamma(r) \Leftrightarrow L^s_t = L^w_{\tau(t)} > r \Leftrightarrow \tau(t) \in J_0(r).$$

In other words, $J_\gamma(r) = \tau^{-1}\bigl(J_0(r)\bigr)$, and therefore $K^s_r = \tau^{-1}(K^w_r)$ holds. From the definition of τ^{-1} (see equation (2.1)), and the relation $L^s_{K^s_r} = r$ we obtain formula (3.2) □

In the proof of lemma I.3.4 the Laplace transform of the density of K^w_r, $r \ge 0$, under P_v has been determined as $\lambda \mapsto \exp(-\sqrt{2\lambda}r)$. Hence we have

$$P_v(K^w_r \in dl) = \frac{r}{\sqrt{2\pi l^3}}\, e^{-r^2/2l}\, dl, \qquad l \ge 0.$$

As a consequence we find the

Corollary 3.2. *For $r \ge 0$, K^s_r has the density*

$$P_v(K^s_r \in dl) = \frac{r}{\sqrt{2\pi(l-\gamma r)^3}}\, e^{-r^2/2(l-\gamma r)}\, dl, \qquad l \ge \gamma r. \tag{3.3}$$

Furthermore, the Laplace transform of the density of K^s_r under P_v is given by

$$E_v\bigl(e^{-\lambda K^s_r}\bigr) = e^{-(\sqrt{2\lambda}+\lambda\gamma)r}, \qquad \lambda > 0. \tag{3.4}$$

Remark 3.3. One can use lemma C.1 in [8] to check that the right hand side of equation (3.3) is indeed the inverse Laplace transform of the right hand side of formula (3.4).

Observe that $\zeta_{\beta,\gamma} = K^s_S$ and $\zeta_{\beta,0} = K^w_S$. Thus we obtain the

Corollary 3.4. *For all $\beta > 0$, $\gamma \ge 0$, the following equation holds true*

$$\zeta_{\beta,\gamma} = \zeta_{\beta,0} + \gamma S. \tag{3.5}$$

As before, $\hat{E}_\xi$ denotes the expectation with respect to $\hat{P}_\xi$, $\xi \in \mathcal{G}$.

Corollary 3.5. *For all $\beta > 0$, $\gamma \ge 0$, $\lambda > 0$, the following formula holds true*

$$\hat{E}_v\bigl(e^{-\lambda\zeta_{\beta,\gamma}}\bigr) = \beta\rho(\lambda), \tag{3.6a}$$

with

$$\rho(\lambda) = \frac{1}{\beta + \sqrt{2\lambda} + \gamma\lambda}. \tag{3.6b}$$

Proof. With corollary 3.2 and $\zeta_{\beta,\gamma} = K^s_S$ we obtain

$$\begin{aligned}\hat{E}_v\bigl(e^{-\lambda\zeta_{\beta,\gamma}}\bigr) &= \beta\int_0^\infty E_v\bigl(e^{-\lambda K^s_r}\bigr)\, e^{-\beta r}\, dr \\ &= \frac{\beta}{\beta + \sqrt{2\lambda} + \gamma\lambda}.\end{aligned}$$

□

Denote by R^g the resolvent of W^g. With lemma I.1.12 we immediately find the

Corollary 3.6. *For all $f \in C_0(\mathcal{G})$, $\lambda > 0$, $\xi \in \mathcal{G}$ the following formula holds true:*

$$R^g_\lambda f(\xi) = R^s_\lambda f(\xi) - \beta\rho(\lambda)\, e_\lambda(\xi)\, R^s_\lambda f(v). \tag{3.7}$$

Now it is easy to verify that for appropriately chosen parameters β, γ, w_k, $k = 1, \ldots, n$, the Brownian motion W_g realizes the boundary condition (1.1b).

Theorem 3.7. *Consider the boundary condition* (1.1)*, and assume that b is not the null vector. Set $r = a + c \in (0, 1)$, and*

$$w_k = \frac{b_k}{1-r},\ k = 1, \ldots, n, \quad \beta = \frac{a}{1-r}, \quad \gamma = \frac{c}{1-r}. \tag{3.8}$$

Let W^g be the Brownian motion as constructed above with these parameters. Then the generator A^g of W^g is $1/2$ times the Laplace operator on $\mathcal{G}$ with domain $\mathcal{D}(A^g)$ consisting of those $f \in C^2_0(\mathcal{G})$ which satisfy condition (1.1b).

Proof. As in the previous cases it is readily seen that the definition (3.8) of the parameters γ, β, w_k, $k = 1, \ldots, n$, is consistent with the conditions used in the above construction of W^g.

Let A^g be the generator of W^g with domain $\mathcal{D}(A^g)$. Since W^g is a Brownian motion on $\mathcal{G}$ in the sense of definition 1.1, it follows from theorem 1.2 that $\mathcal{D}(A^g) \subset C^2_0(\mathcal{G})$, and that for all $f \in \mathcal{D}(A^g)$, $A^g f(\xi) = 1/2\, f''(\xi)$, $\xi \in \mathcal{G}$. Let $h \in C_0(\mathcal{G})$, $\lambda > 0$. Then $R^g_\lambda h \in \mathcal{D}(A^g)$, and therefore we may compute with equation (3.7) as follows

$$\begin{aligned}\frac{\gamma}{2}\left(R^g_\lambda h\right)''(v) &= \frac{\gamma}{2}\left(R^s_\lambda h\right)''(v) - \beta\,\rho(\lambda)\,2\lambda\big(R^s_\lambda h\big)(v)\\ &= \sum_{k=1}^n w_k \left(R^s_\lambda h\right)'(v_k) - \beta\,\rho(\lambda)\,\gamma\lambda\big(R^s_\lambda h\big)(v),\end{aligned}$$

where we used the fact that, since $R^s_\lambda h$ is in the domain of the generator A^s of W^s, it satisfies the boundary condition (2.9). We rewrite this equation in the following way:

$$\begin{aligned}\frac{\gamma}{2}\left(R^g_\lambda h\right)''(v) = \sum_{k=1}^n w_k \left(R^s_\lambda h\right)'(v_k) &+ \beta\sqrt{2\lambda}\,\rho(\lambda)\big(R^s_\lambda h\big)(v)\\ &- \beta\,\rho(\lambda)\big(\sqrt{2\lambda} + \gamma\lambda\big)\,\Big(R^s_\lambda h\Big)(v).\end{aligned} \tag{3.9}$$

Now we differentiate equation (3.7) at $\xi \in l_k$, $k = 1, \ldots, n$, let ξ tend to v along any edge l_k, and sum the resulting equation against the weights w_k, $k = 1, \ldots, n$. Then we get the following formula

$$\sum_{k=1}^n w_k \left(R^g_\lambda h\right)'(v_k) = \sum_{k=1}^n w_k \left(R^s_\lambda h\right)'(v_k) + \beta\sqrt{2\lambda}\,\rho(\lambda)\big(R^s_\lambda h\big)(v), \tag{3.10}$$

where we used $\sum_k w_k = 1$. On the other hand, for $\xi = v$, equation (3.7) gives

$$\big(R^g_\lambda h\big)(v) = \rho(\lambda)\big(\sqrt{2\lambda} + \gamma\lambda\big)\big(R^s_\lambda h\big)(v). \tag{3.11}$$

A comparison of equations (3.10), (3.11) with (3.9) shows that we have proved the following formula

$$\frac{\gamma}{2}\,\bigl(R_\lambda^g h\bigr)''(v) = \sum_{k=1}^{n} w_k\,\bigl(R_\lambda^g h\bigr)'(v_k) - \beta\bigl(R_\lambda^g h\bigr)(v). \tag{3.12}$$

With the values (3.8) for β, γ, and w_k, $k = 1, \dots, n$, it is obvious that $f = R_\lambda^g h$ satisfies equation (1.1b). Since R_λ^g is surjective from $C_0(\mathcal{G})$ onto the domain of the generator A^g of W^g, the proof of the boundary conditions as stated in the theorem is finished. As before, the proof of theorem 3.7 is completed by the remark that boundary conditions of the form (1.1) uniquely determine the domain of the generator A^g, see also remark I.1.6. □

Let $\lambda > 0$, $f \in C_0(\mathcal{G})$. Insertion of the right hand side of formula (2.13) for R_λ^s into equation (3.7) gives us after some simple algebra the following expression for $R_\lambda^g f$:

$$R_\lambda^g f(\xi) = R_\lambda^D f(\xi) + \rho(\lambda)\, e_\lambda(\xi)\, \bigl(2(e_\lambda^w, f) + \gamma f(v)\bigr), \qquad \xi \in \mathcal{G}, \tag{3.13}$$

where R^D is the Dirichlet resolvent, e_λ is defined in equation (I.1.15), e_λ^w in equation (2.11), and $\rho(\lambda)$ is as in formula (3.6b). From equation (3.13) we can read off the following result:

Corollary 3.8. *For ξ, $\eta \in \mathcal{G}$, $\lambda > 0$, the resolvent kernel r_λ^g of the general Brownian motion W^g on $\mathcal{G}$ is given by*

$$\begin{aligned} r_\lambda^g(\xi, d\eta) = r_\lambda^D(\xi,\eta)\, d\eta &+ \sum_{k,m=1}^{n} e_{\lambda,k}(\xi)\, 2w_m\, \rho(\lambda)\, e_{\lambda,m}(\eta)\, d\eta \\ &+ \gamma\, \rho(\lambda)\, e_\lambda(\xi)\, \epsilon_v(d\eta), \end{aligned} \tag{3.14}$$

with r_λ^D as in formula (I.1.23), and ρ is defined in equation (3.6b)*. Alternatively, r_λ^g can be written in the following form*

$$\begin{aligned} r_\lambda^g(\xi, d\eta) = r_\lambda(\xi,\eta)\, d\eta &+ \sum_{k,m=1}^{n} e_{\lambda,k}(\xi)\, S_{km}^g(\lambda)\, \frac{1}{\sqrt{2\lambda}}\, e_{\lambda,m}(\eta)\, d\eta \\ &+ \gamma\, \rho(\lambda)\, e_\lambda(\xi)\, \epsilon_v(d\eta), \end{aligned} \tag{3.15a}$$

where r_λ is defined in equation (I.1.24), and

$$S_{km}^g(\lambda) = 2\sqrt{2\lambda}\, \rho(\lambda)\, w_m - \delta_{km}. \tag{3.15b}$$

In order to invert the Laplace transforms in equations (3.14), (3.15), we define for β, $\gamma > 0$, the following function $g_{\beta,\gamma}$ on $(0, +\infty) \times \mathbb{R}_+$:

$$g_{\beta,\gamma}(t,x) = \frac{1}{\gamma^2}\, \frac{1}{\sqrt{2\pi}} \int_0^t \frac{s + \gamma x}{(t-s)^{3/2}}\, \exp\Bigl(-\frac{(s+\gamma x)^2}{2\gamma^2 (t-s)}\Bigr)\, e^{-\beta s/\gamma}\, ds, \tag{3.16}$$

with $(t,x) \in (0,+\infty) \times \mathbb{R}_+$. The heat kernel $g_{\beta,\gamma}$ is discussed in more detail in appendix C of [8]. In particular, it is outlined there that the limits of $g_{\beta,\gamma}$ as $\beta \downarrow 0$, and $\gamma \downarrow 0$, yield the kernels $g_{\beta,0}$ (equation (I.3.11)) and $g_{0,\gamma}$ (equation (2.17)), respectively. Moreover, it is proved there that the Laplace transform of $g_{\beta,\gamma}(\cdot, x)$, $x \geq 0$, is given by

$$\rho(\lambda)\, e^{-\sqrt{2\lambda}x}, \qquad \lambda > 0, \tag{3.17}$$

where ρ is defined in (3.6b). Hence we get the

Corollary 3.9. *For ξ, $\eta \in \mathcal{G}$, $t > 0$, the transition kernel of the general Brownian motion W^g on $\mathcal{G}$ is given by*

$$\begin{aligned} p^g(t,\xi,d\eta) &= p^D(t,\xi,\eta)\,d\eta \\ &\quad + \sum_{k,m=1}^{n} 1_{l_k}(\xi)\, 2w_m\, g_{\beta,\gamma}\big(t, d_v(\xi,\eta)\big)\, 1_{l_m}(\eta)\, d\eta \\ &\quad + \gamma\, g_{\beta,\gamma}\big(t, d(\xi,v)\big)\, \epsilon_v(d\eta), \end{aligned} \tag{3.18}$$

which alternatively can be written as

$$\begin{aligned} p^g(t,\xi,d\eta) &= p(t,\xi,\eta)\,d\eta \\ &\quad + \sum_{k,m=1}^{n} 1_{l_k}(\xi) \Big(2w_m\, g_{\beta,\gamma}\big(t, d_v(\xi,\eta)\big) \\ &\qquad\qquad - \delta_{km}\, g\big(t, d_v(\xi,\eta)\big)\Big)\, 1_{l_m}(\eta)\, d\eta \\ &\quad + \gamma\, g_{\beta,\gamma}\big(t, d(\xi,v)\big)\, \epsilon_v(d\eta). \end{aligned} \tag{3.19}$$

References

1. Amrein, W., Jauch, J., and Sinha, K.: *Scattering Theory in Quantum Mechanics*, Benjamin, Reading Mass., 1977.
2. Brunetti, R. and Fredenhagen, K.: Time of occurence observables in quantum mechanics, *Phys. Rev. A* **66** (2002) 044101–1–044101–3.
3. Erdélyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G.: *Tables of Integral Transforms*, vol. I, McGraw-Hill, New York, Toronto, London, 1954.
4. Itô, K. and McKean, H. P., Jr.: Brownian motions on a half line, *Illinois J. Math.* **7** (1963) 181–231.
5. Itô, K. and McKean, H. P., Jr.: *Diffusion Processes and their Sample Paths*, Springer–Verlag, Berlin, Heidelberg, New York, second edn., 1974.
6. Jauch, J., Misra, B., and Sinha, K.: Time delay in scattering processes, *Helv. Phys. Acta* **45** (1972) 398–426.
7. Knight, F.: *Essentials of Brownian Motion and Diffusion*, *Mathematical Surveys and Monographs*, vol. 18, American Mathematical Society, Providence, Rhode Island, 1981.
8. Kostrykin, V., Potthoff, J., and Schrader, R.: Brownian motions on metric graphs: Feller Brownian motions on intervals revisited, arXiv: 1008.3761, August 2010.
9. Kostrykin, V., Potthoff, J., and Schrader, R.: Construction of the Paths of Brownian Motions on Star Graphs I, in this volume.
10. Kostrykin, V., Potthoff, J., and Schrader, R.: Brownian Motions on Metric Graphs, arXiv: 1102.4937, 2012, to appear in *J. Math. Physics*, special issue celebrating the 80th birthday of Elliott Lieb.
11. Kostrykin, V. and Schrader, R.: Kirchhoff's rule for quantum wires, *J. Phys. A: Math. Gen.* **32** (1999) 595–630.
12. Kostrykin, V. and Schrader, R.: Laplacians on metric graphs: Eigenvalues, resolvents and semigroups, in: Berkolaiko, G., Carlson, R., Fulling, S., and Kuchment, P. (eds.), *Quantum Graphs and Their Applications*, *Contemp. Math.*, vol. 415, Amer. Math. Soc., Providence, RI, 2006.

13. Martin, P. A.: On the time delay of simple scattering systems, *Commun. Math. Phys.* **47** (1976) 221–227.
14. Martin, P. A.: Time delay in quantum scattering processes, in: *New developments in mathematical physics, Schladming, 1981*, vol. XXIII, 1981, pages 157–208.
15. Narnhofer, H.: Another definition of time delay, *Phys.Rev.D* **22** (1980) 2387–2390.
16. Wigner, E. P.: Lower limit for the energy derivative of the scattering phase shift, *Phys. Rev.* **98** (1955) 145–147.
17. Williams, D.: *Diffusions, Markov Processes, and Martingales*, John Wiley & Sons, Chichester, New York, Brisbane, Toronto, 1979.

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