Communications on Stochastic Analysis

Volume 6 | Number 2 Article 4

6-1-2012

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Communications on Stochastic Analysis
Vol. 6, No. 2 (2012) 223-245

Serials Publications
www.serialspublications.com

CONSTRUCTION OF THE PATHS OF BROWNIAN MOTIONS ON STAR GRAPHS I

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Abstract. In the present article and its follow-up article [23] pathwise constructions of Brownian motions which satisfy all possible boundary conditions at the vertex of star graphs are given.

1. Introduction and Preliminaries

In the recent years there was a growing interest in *metric graphs* because of their wide range of important applications, see, e.g., the articles in [8] and the references given there. The simplest metric graphs are *star* or *single vertex graphs*: They can be defined as a set having a finite collection of subsets isomorphic to $\mathbb{R}_+$, called *external edges*, where the points corresponding to the origin of $\mathbb{R}_+$ under these isomorphims are identified and form the *vertex* of the graph. One may visualize a star graph as a finite number of rays in $\mathbb{R}^2$ going out from the origin.

On the other hand, in his pioneering articles [9–11] Feller investigated the problem of characterization and construction of all Brownian motions on intervals. This problem found a complete solution in the work of Itô and McKean [17,18]. In particular, in [17] Itô and McKean solved the problem of construction of the paths of all Brownian motions on the semi-line $\mathbb{R}_+$ employing the theory of the local time of Brownian motion [27], and the theory of (strong) Markov processes [2,4–6,14].

Therefore it is natural to investigate Feller's problem on metric graphs, and in particular on star graphs. In fact, the Walsh process introduced by Walsh in [33] as a generalization of the skew Brownian motion [18] is the most basic example of a Brownian motion on a star graph, and in the present article it will serve — together with its local time at the vertex — as the main building block of our constructions. The Brownian motions constructed here on star graphs are then the basic building blocks of Brownian motions on general, finite metric graphs in the article [24] of the present authors.

The main ideas for the construction of Brownian motions with boundary conditions at the vertex compatible with Feller's theorem (see below, theorem 1.5) are those which can be found in the above mentioned work by Itô and McKean (cf. also [20, Chapter 6]): The reflecting Brownian motion in the case of $\mathbb{R}_+$ is replaced by a Walsh process [33] (cf. also, e.g., [1]) on the single vertex graph, and then slowing down and the killing of this process on the scale of its local time

Received 2011-11-4; Communicated by Hui-Hsiung Kuo.
2010 *Mathematics Subject Classification.* 60J65, 60J45, 60H99, 58J65, 35K05, 05C99.
Key words and phrases. Metric graphs, Brownian motion, Feller processes, Feller's theorem.

at the vertex are used to construct processes implementing the various forms of the Wentzell boundary condition. We provide a number of arguments which — at least on a technical level — are rather different from those found in the standard literature. For example, whenever possible, we use arguments based on Dynkin's formula to derive the domain of the generator (i.e., the boundary conditions at the vertex). This approach appears to be much simpler and more intuitive than the one with standard arguments [17, 18, 20] for the semi-line $\mathbb{R}_+$, which is based on rather tricky calculation of heat kernels with the help of Lévy's theorem. Moreover, for the case of killing, instead of using the standard first passage time formula for the hitting time of the vertex we use a first passage time formula for the lifetime of the process. In the opinion of the authors this leads to much simpler computations of the transition kernels than those in [17, 18, 20] for a Brownian motion on $\mathbb{R}_+$. In all cases we also derive explicit expressions of the analogues of the *quantum mechanical scattering matrix* on star graphs.

The article is organized as follows. In several subsections of the present section we set up our notation and discuss some preparatory results. In section 2 we recall the construction of a Walsh process on a metric graph and derive a number of its properties. In section 3 we construct a Walsh process on the single vertex graph with an elastic boundary condition at the vertex. In the follow-up article [23] we construct a Walsh process with a sticky boundary condition at the vertex, and finally the most general Brownian motion on a star graph.

1.1. Brownian Motion on a Star Graph and Feller's Theorem. From now we shall consider a fixed star graph $\mathcal{G}$ with vertex v and $n \in \mathbb{N}$ external edges l_1, …, l_n. $\mathcal{G}$ is equipped with the natural metric d which is induced by the metric that each external edge inherits from being isomorphic to $\mathbb{R}_+$. Thus $(\mathcal{G}, d)$ is a locally compact, complete metric space, and we shall always consider $\mathcal{G}$ as equipped with its Borel σ–algebra. $\mathcal{G}^\circ$ denotes the set $\mathcal{G} \setminus \{v\}$ which we also call — by abuse of language — the *open interior* of $\mathcal{G}$. It is the disjoint union of n copies l_k°, $k = 1$, …, n, of the interval $(0, +\infty)$. Every $\xi \in \mathcal{G}^\circ$ is in one-to-one correspondence with its *local coordinates* (k, x), where $k \in \{1, \dots, n\}$ is the index of the edge ξ belongs to, and $x > 0$ denotes the distance of ξ to v. For simplicity we shall often write $\xi = (k, x)$.

The following definition of a Brownian motion on $\mathcal{G}$ is the generalization of the definition of a Brownian motion on the half line $\mathbb{R}_+$ as given by Knight in [20, Chapter 6].

Definition 1.1. A *Brownian motion* $X = (X_t,\, t \in \mathbb{R}_+)$ on $\mathcal{G}$ is a diffusion process on $\mathcal{G}$, such that X with absorption at v is equivalent to a Brownian motion on the half line $\mathbb{R}_+$ with absorption at the origin.

Remarks 1.2. By a diffusion process we mean a strong Markov process (e.g., in the sense of [3]), which a.s. has càdlàg paths and a.s. the paths are continuous on $[0, \zeta)$, where ζ is its lifetime. Moreover, in definition 1.1 we have — as we shall usually do without any danger of confusion — identified the external leg l_k, $k = 1$, …, n, on which the process starts with the corresponding copy of $\mathbb{R}_+$. Throughout we shall assume without loss of generality that the filtration of a Brownian motion on

$\mathcal{G}$ satisfies the "usual conditions", i.e., it is right continuous and complete relative to the underlying family $(P_\xi,\, \xi \in \mathcal{G})$ of probability measures.

Let $C_0(\mathcal{G})$ denote the Banach space of continuous functions on $\mathcal{G}$ which vanish at infinity equipped with the sup-norm.

Define $C_0^2(\mathcal{G})$ as the subspace of $C_0(\mathcal{G})$ consisting of those functions $f \in C_0(\mathcal{G})$ which are twice continuously differentiable on $\mathcal{G}^\circ$, such that f'' extends from $\mathcal{G}^\circ$ to $\mathcal{G}$ as a function in $C_0(\mathcal{G})$. The following lemma states some of the properties of functions in $C_0^2(\mathcal{G})$. It can be proved with the help of applications of the fundamental theorem of calculus and the mean value theorem, and the proof is omitted here.

Lemma 1.3. *Suppose that $f \in C_0^2(\mathcal{G})$, $k \in \{1, \dots, n\}$. Then the limit of $f'(\xi)$ as ξ converges to v along the open edge l_k° exists. The directional derivatives $f^{(i)}(v_k)$, $i = 1,\, 2$, of first and second order of f at the vertex v in direction of the edge l_k exist, and the equalities*

$$f^{(i)}(v_k) = \lim_{\xi \to v,\, \xi \in l_k^\circ} f^{(i)}(\xi), \qquad i = 1,\, 2,$$

hold true. Moreover, f' vanishes at infinity.

Remark 1.4. By definition we have that for every $f \in C_0^2(\mathcal{G})$ and all $j,\, k = 1, \dots,$ n, $f''(v_j) = f''(v_k)$, and henceforth we shall simply write $f''(v)$ for this quantity. On the other hand, in general $f'(v_j) \neq f'(v_k)$ for $j \neq k$.

It is not hard to show that every Brownian motion on a star graph is a Feller process. A convenient way to prove this is to show that its resolvent maps $C_0(\mathcal{G})$ into itself by arguments similar to those in [18, Section 3.6], and to observe that the path properties imply for all $\xi \in \mathcal{G}$, $f \in C_0(\mathcal{G})$, $P_t f(\xi)$ converges to $f(\xi)$ as t decreases to 0, where $(P_t,\, t \in \mathbb{R}_+)$ denotes the semigroup generated by the Brownian motion. Then one can use well-known arguments (for example, a complete proof can be found in Appendix B of [24]) to conclude the Feller property in its usual form, e.g., [29].

The analogue of Feller's theorem [20, Theorem 6.2] for a Brownian motion on the single vertex graph $\mathcal{G}$ reads as follows:

Theorem 1.5. *Assume that X is a Brownian motion on $\mathcal{G}$. Then there exist constants a, b_k, $c \in [0, 1]$, $k = 1, \dots, n$, with*

$$a + c + \sum_{k=1}^{n} b_k = 1, \quad a \neq 1, \tag{1.1a}$$

such that the domain $\mathcal{D}(A)$ of the generator A of X in $C_0(\mathcal{G})$ consists exactly of those $f \in C_0^2(\mathcal{G})$ for which

$$a f(v) + \frac{c}{2} f''(v) = \sum_{k=1}^{n} b_k f'(v_k) \tag{1.1b}$$

holds true. Moreover, for $f \in \mathcal{D}(A)$, $Af = 1/2 f''$.

The proof in [20] for the case where $\mathcal{G}$ has only one external edge, i.e., $\mathcal{G} = \mathbb{R}_+$, is readily modified for a general star graph $\mathcal{G}$. So a very short sketch of the argument should suffice here. Indeed, when the vertex is a trap or there is exponential holding, then the edges are decoupled and we can use the arguments in [20]. For the other case one computes the Dynkin generator of X as in [20], where one uses the first hitting time of the ball of radius $\epsilon > 0$ around the vertex. As a result on obtains that every f in $\mathcal{D}(A)$ satisfies a boundary condition of the form (1.1). Let $\mathcal{H}_{a,b,c}$ denote the subspace of functions $f \in C_0^2(\mathcal{G})$ satisfying a boundary condition of the form (1.1). Then we have shown that $\mathcal{D}(A) \subset \mathcal{H}_{a,b,c}$. Now suppose that this inclusion is strict. Then this entails that for every $\lambda > 0$ the mapping $f \mapsto \lambda f - 1/2\, f''$ is not injective from $\mathcal{H}_{a,b,c}$ onto $C_0(\mathcal{G})$. Let $\lambda > 0$ and assume that $f_\lambda \neq 0$ is in the kernel. Since f_λ is bounded and continuous on $\mathcal{G}$, it follows that

$$f_\lambda(a) = C_\lambda\, e^{-\sqrt{2\lambda} d(v,a)}, \qquad a \in \mathcal{G},$$

for $C_\lambda \neq 0$, where $d(v,a)$ denotes the natural distance of $a \in \mathcal{G}$ to the vertex v. The boundary condition (1.1) for f_λ reads as follows

$$a + c\lambda + \sqrt{2\lambda} \sum_{k=1}^{n} b_k = 0.$$

Since this has to hold for all $\lambda > 0$, we arrived at a contradiction.

Remarks 1.6. Observe that theorem 1.5 in particular states that every boundary condition of the form (1.1) uniquely defines the domain $\mathcal{D}(A)$ of the generator A of X. Theorem 1.5 also follows from Feller's theorem in the case of a general metric graph [24, Theorem 2.5].

1.2. Standard Brownian Motion on the Real Line. The construction of Brownian motions on a single vertex graph with infinitesimal generator whose domain consists of functions f which satisfy the boundary conditions (1.1) is quite similar to the construction carried out for the half-line in [17], [18], [20]. This in turn is based on the properties of a standard Brownian motion on the real line, cf., e.g., [12, 13, 15, 19, 29, 34], and the works cited above. For the convenience of the reader, and for later reference, we collect the pertinent notions, tools and results here.

Let $(Q_x,\, x \in \mathbb{R})$ denote a family of probability measures on a measurable space $(\Omega', \mathcal{A}')$, and let $B = (B_t,\, t \in \mathbb{R}_+)$ denote a standard Brownian motion defined on $(\Omega', \mathcal{A}')$ with $Q_x(B_0 = x) = 1$, $x \in \mathbb{R}$. It will be convenient to assume throughout that B exclusively has continuous paths. Whenever it is notationally convenient, we shall also write $B(t)$ for B_t, $t \geq 0$. Furthermore, we may suppose that there is a shift operator $\theta : \mathbb{R}_+ \times \Omega \to \Omega$, such that for all s, $t \geq 0$, $B_s \circ \theta_t = B_{s+t}$.

We shall always understand the Brownian family $(B, (Q_x,\, x \in \mathbb{R}))$ to be endowed with a filtration $\mathcal{J} = (\mathcal{J}_t,\, t \geq 0)$ which is right continuous and complete for the family $(Q_x,\, x \in \mathbb{R})$. (For example, $\mathcal{J}$ could be chosen as the usual augmentation of the natural filtration of B (e.g., [19, Sect. 2.7] or [29, Sect.'s I.4, III.2]).)

For any $A \subset \mathbb{R}$, we denote by H_A^B the hitting time of A by B,

$$H_A^B = \inf\{t > 0,\, B_t \in A\}, \tag{1.2}$$

and we note that for all A belonging to the Borel σ–algebra $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$, H_A^B is a stopping time with respect to $\mathcal{J}$ (e.g., [29, Theorem III.2.17]). In the case where $A = \{x\}$, $x \in \mathbb{R}$, we also simply write H_x^B for $H_{\{x\}}^B$. We shall also denote these stopping times by $H^B(A)$ and $H^B(x)$, respectively, whenever it is typographically more convenient. The following particular cases deserve special attention. Let $x \in \mathbb{R}$. Then we have (e.g., [32], [18, Sect. 1.7], [19, Sect. 2.6], [29, Sect.'s II.3, III.3])

$$Q_0(H_x^B \in dt) = Q_x(H_0^B \in dt) = \frac{|x|}{t}\, g(t,x)\, dt, \qquad t > 0, \tag{1.3}$$

where g is the Gauß-kernel

$$g(t,x) = \frac{1}{\sqrt{2\pi t}}\, e^{-x^2/2t}, \qquad t > 0,\, x \in \mathbb{R}, \tag{1.4}$$

and

$$E_0^Q\big(e^{-\lambda H_x^B}\big) = E_x^Q\big(e^{-\lambda H_0^B}\big) = e^{-\sqrt{2\lambda}|x|}, \qquad \lambda > 0. \tag{1.5}$$

Moreover, for $a < x < b$ the law of $H_{\{a,b\}}^B$ under Q_x is well-known (e.g., [18, Problem 6, Sect. 1.7]), and its expectation is given by

$$E_x^Q\big(H_{\{a,b\}}^B\big) = (x-a)(b-x). \tag{1.6}$$

Denote by $L^B = (L_t^B,\, t \geq 0)$ the local time of B at zero, where we choose the normalization as in, e.g., [29] (and which thus differs by a factor 2 from the one used in, e.g., [15, 19]): for $x \in \mathbb{R}$, P_x–a.s.

$$L_t^B = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon}\, \lambda\big(\big\{s \leq t,\, |B_s| \leq \epsilon\big\}\big), \qquad t \geq 0, \tag{1.7}$$

and here λ denotes the Lebesgue measure. Thus, in terms of its α–potential (cf. [3, Theorem V.3.13]) we have

$$u_{L^B}^\alpha(x) = E_x\Big(\int_0^\infty e^{-\alpha t}\, dL_t^B\Big) = \frac{1}{\sqrt{2\alpha}}\, e^{-\sqrt{2\alpha}|x|}, \qquad \alpha > 0,\, x \in \mathbb{R}, \tag{1.8}$$

which provides an efficient way to compare the various normalizations of the local time used in the literature. Slightly informally we can write

$$L_t^B = \int_0^t \delta_0(B_s)\, ds, \tag{1.9}$$

where δ_0 is the Dirac distribution concentrated at 0. L^B is adapted to $\mathcal{J}$, and non-decreasing. Moreover, for every $x \in \mathbb{R}$, P_x–a.s. the paths of L^B are continuous, and L^B is additive in the sense that

$$L_{t+s}^B = L_t^B + L_s^B \circ \theta_t, \qquad s,t \in \mathbb{R}_+. \tag{1.10}$$

Similarly as above, we shall occasionally take the notational freedom to rewrite L_t^B as $L^B(t)$.

We will need the following well-known result (e.g., [18, Section 2.2, Problem 3]):

Lemma 1.7. *The joint law of* $|B_t|$ *and* L_t, $t>0$, *under* Q_0 *is given by*

$$Q_0\bigl(|B_t| \in dx,\, L_t^B \in dy\bigr) = 2\,\frac{x+y}{\sqrt{2\pi t^3}}\, e^{-(x+y)^2/2t}\, dx\, dy, \qquad x, y \geq 0. \tag{1.11}$$

Let $K^B = (K_r^B,\, r \geq 0)$ denote the right continuous pseudo-inverse of L,

$$K_r^B = \inf\bigl\{t \geq 0,\, L_t^B > r\bigr\}, \qquad r \geq 0. \tag{1.12}$$

Note that due to the a.s. continuity of L^B we have a.s. $L_{K_r}^B = r$. In appendix B of [21] the present authors proved the following

Lemma 1.8. *For any* $r \geq 0$

$$Q_0\bigl(K_r^B \in dt\bigr) = \frac{r}{t}\, g(t,r)\, dt, \qquad t > 0, \tag{1.13}$$

and

$$E_0^Q\bigl(e^{-\lambda K_r^B}\bigr) = e^{-\sqrt{2\lambda} r}, \qquad \lambda > 0 \tag{1.14}$$

holds.

Moreover, we shall make use of the following lemma, which is similar to results in Section 6.4 of [19], and which is proved in appendix B of [21], too.

Lemma 1.9. *Under* Q_0, $L^B\bigl(H_{\{-x,+x\}}^B\bigr)$, $x > 0$, *is exponentially distributed with mean* x.

1.3. First Passage Time Formula for Single Vertex Graphs. In this subsection we set up some additional notation which will be used throughout this article. Also we record a special form of the well-known first passage time formula, e.g., [18, 28].

Let X be a Brownian motion on $\mathcal{G}$ in the sense of definition 1.1 defined on a family $\bigl(\Omega, \mathcal{A}, \mathcal{F} = (\mathcal{F}_t,\, t \geq 0), (P_\xi,\, \xi \in \mathcal{G})\bigr)$ of filtered probability spaces. Let H_v be the hitting time of the vertex v. It follows from definition 1.1 that for all $\xi \in \mathcal{G}$, $P_\xi(H_v < +\infty) = 1$. For $\lambda > 0$, set

$$e_\lambda(\xi) = E_\xi\bigl(\exp(-\lambda H_v)\bigr) = e^{-\sqrt{2\lambda} d(\xi,v)}, \qquad \xi \in \mathcal{G}, \tag{1.15}$$

where $E_\xi(\,\cdot\,)$ denotes expectation with respect to P_ξ. The last equality follows from formula (1.5).

Recall that we denote the natural metric on $\mathcal{G}$ by d. We introduce another symmetric map d_v from $\mathcal{G} \times \mathcal{G}$ to $\mathbb{R}_+$ defined by

$$d_v(\xi,\eta) = d(\xi,v) + d(v,\eta), \qquad \xi,\, \eta \in \mathcal{G}, \tag{1.16}$$

which is the "distance from ξ to η via the vertex v". Observe that if ξ, $\eta \in \mathcal{G}$ do not belong to the same edge, then $d_v(\xi,\eta) = d(\xi,\eta)$ holds.

Next we define two heat kernels on $\mathcal{G}$ by

$$p(t,\xi,\eta) = \sum_{k=1}^n 1_{l_k}(\xi)\, g\bigl(t, d(\xi,\eta)\bigr)\, 1_{l_k}(\eta), \tag{1.17}$$

$$p_v(t,\xi,\eta) = \sum_{k=1}^n 1_{l_k}(\xi)\, g\bigl(t, d_v(\xi,\eta)\bigr)\, 1_{l_k}(\eta), \tag{1.18}$$

with $t > 0$, $\xi, \eta \in \mathcal{G}$. g is the Gauß-kernel defined in equation (1.4). Hence, in local coordinates $\xi = (k, x)$, $\eta = (m, y)$, x, $y \geq 0$, k, $m \in \{1, 2, \dots, n\}$, these kernels read

$$p\big(t, (k, x), (m, y)\big) = \frac{1}{\sqrt{2\pi t}}\, e^{-(x-y)^2/2t}\, \delta_{km} \tag{1.19}$$

$$p_v\big(t, (k, x), (m, y)\big) = \frac{1}{\sqrt{2\pi t}}\, e^{-(x+y)^2/2t}\, \delta_{km}. \tag{1.20}$$

The *Dirichlet heat kernel* p^D on $\mathcal{G}$ is then given by

$$p^D(t, \xi, \eta) = p(t, \xi, \eta) - p_v(t, \xi, \eta), \qquad t > 0,\ \xi,\ \eta \in \mathcal{G}. \tag{1.21}$$

It is the transition density of a strong Markov process with state space $\mathcal{G}^\circ \cup \{\Delta\}$ which on every edge of $\mathcal{G}^\circ$ is equivalent to a Brownian motion until the moment of reaching the vertex when it is killed, and Δ denotes a universal cemetery state for all stochastic processes considered adjoined to $\mathcal{G}$ as an isolated point. (Observe that this process is *not* a Brownian motion on $\mathcal{G}$ in the sense of definition 1.1.)

The *Dirichlet resolvent* $R^D = (R^D_\lambda,\ \lambda > 0)$ on $\mathcal{G}$ is defined by

$$R^D_\lambda f(\xi) = E_\xi\Big(\int_0^{H_v} e^{-\lambda t} f(X_t)\, dt\Big), \qquad \lambda > 0,\ \xi \in \mathcal{G},\ f \in B(\mathcal{G}). \tag{1.22}$$

It is easy to see that R^D_λ has the following integral kernel on $\mathcal{G}$

$$r^D_\lambda(\xi, \eta) = r_\lambda(\xi, \eta) - r_{v,\lambda}(\xi, \eta), \qquad \xi,\ \eta \in \mathcal{G}, \tag{1.23}$$

where for ξ, $\eta \in \mathcal{G}$,

$$r_\lambda(\xi, \eta) = \sum_{k=1}^n 1_{l_k}(\xi)\, \frac{e^{-\sqrt{2\lambda}\, d(\xi,\eta)}}{\sqrt{2\lambda}}\, 1_{l_k}(\eta), \tag{1.24}$$

and

$$\begin{aligned} r_{v,\lambda}(\xi, \eta) &= \sum_{k=1}^n 1_{l_k}(\xi)\, \frac{e^{-\sqrt{2\lambda}\, d_v(\xi,\eta)}}{\sqrt{2\lambda}}\, 1_{l_k}(\eta) \\ &= \sum_{k=1}^n 1_{l_k}(\xi)\, \frac{1}{\sqrt{2\lambda}}\, e_\lambda(\xi)\, e_\lambda(\eta)\, 1_{l_k}(\eta). \end{aligned} \tag{1.25}$$

In particular, r^D_λ is the Laplace transform of the Dirichlet heat kernel (1.21) at $\lambda > 0$.

In the present context the well-known first passage time formula, e.g., [18, 28], reads as follows

$$R_\lambda f(\xi) = E_\xi\Big(\int_0^S e^{-\lambda t} f(X_t)\, dt\Big) + E_\xi\big(e^{-\lambda S}\, R_\lambda f(X_S)\big),$$

where S is any P_ξ–a.s. finite stopping time relative to $\mathcal{J}$. The choice $S = H_v$ gives the following result.

Lemma 1.10. *Let X be a Brownian motion on $\mathcal{G}$ with resolvent $R = (R_\lambda,\ \lambda > 0)$. Then for all $\lambda > 0$, $\xi \in \mathcal{G}$, $f \in B(\mathcal{G})$, we have*

$$R_\lambda f(\xi) = R^D_\lambda f(\xi) + e_\lambda(\xi)\, R_\lambda f(v). \tag{1.26}$$

The following notation will be convenient. For real valued measurable functions f, g on $\mathcal{G}$, with restrictions f_k, g_k, $k \in \{1, 2, \dots, n\}$, to the edges l_k we set

$$(f, g) = \int_{\mathcal{G}} f(\xi)\, g(\xi)\, d\xi = \sum_{k=1}^{n} (f_k, g_k),$$

where the integration is with respect to the Lebesgue measure on $\mathcal{G}$, and

$$(f_k, g_k) = \int_0^\infty f_k(x)\, g_k(x)\, dx,$$

whenever the integrals exist.

Assume that $f \in C_0(\mathcal{G})$. Then for $\lambda > 0$, $R_\lambda f$ belongs to the domain of the generator of X, and therefore to $C_0^2(\mathcal{G})$ (cf. subsection 1.1). It is straightforward to compute the derivative of the right hand side of formula (1.26), and we obtain the

Corollary 1.11. *For every Brownian motion X on $\mathcal{G}$ with resolvent $R = (R_\lambda,\, \lambda > 0)$, and all $f \in C_0(\mathcal{G})$,*

$$(R_\lambda f)'(v_k) = 2(e_{\lambda,k}, f_k) - \sqrt{2\lambda} R_\lambda f(v), \qquad k \in \{1, 2, \dots, n\}, \tag{1.27}$$

holds true.

1.4. The Case $\boldsymbol{b = 0}$. The case, where all parameters b_k, $k = 1, \dots, n$, in equation (1.1) vanish, is trivial in the sense that the associated Brownian motion can be constructed by a stochastic process living only on the edge where it started, and therefore it is just a classical Brownian motion on $\mathbb{R}_+$ in the sense of [20, Section 6.1]. This case is also discussed briefly in [20], but for the sake of completeness we include it here in somewhat more detail than in [20].

Consider a standard Brownian motion on $\mathbb{R}$ as before, and without loss of generality assume in addition that the underlying sample space is large enough such that all constant paths in $\mathbb{R}$ can be realized as paths of the Brownian motion. Construct from the Brownian motion a new process by stopping it when it reaches the origin of $\mathbb{R}$, and then kill it after an exponential holding time (independent of the Brownian motion) with rate $\beta \geq 0$. We shall only consider starting points $x \in \mathbb{R}_+$. If $\beta = 0$, then the process is simply a Brownian motion with absorption at the origin. For example, it follows from Theorem 10.1 and Theorem 10.2 in [4] that for every $\beta \geq 0$ this process is a strong Markov process, and obviously it has the path properties which make it a Brownian motion on $\mathbb{R}_+$ in the sense of [20, Section 6.1]. Thus, if $\xi \in \mathcal{G}$, $\xi \in l_k$, $k = 1, \dots, n$, then we just have to map this process with the isomorphisms between the edges l_k, $k = 1, \dots, n$, and the interval $[0, +\infty)$ into $\mathcal{G}$ to obtain a Brownian motion on $\mathcal{G}$ with start in ξ, such that it is stopped when reaching the vertex, and then is killed there after an exponential holding time with rate $\beta \geq 0$.

Let $U^0 = (U_t^0,\, t \geq 0)$ denote the semigroup associated with this process. It is obvious that for $f \in C_0(\mathcal{G})$ we get $U_t^0 f(v) = \exp(-\beta t) f(v)$, $t \geq 0$. Thus for the corresponding resolvent $R^0 = (R_\lambda^0,\, \lambda > 0)$, and $f \in C_0(\mathcal{G})$ one finds

$$\lambda R_\lambda^0 f(v) - f(v) + \beta R_\lambda^0 f(v) = 0, \qquad \lambda > 0. \tag{1.28}$$

Let A^0 be the generator of this process, and recall from theorem 1.5, that for all $f \in \mathcal{D}(A^0)$, $A^0 f(\xi) = 1/2\, f''(\xi)$, $\xi \in \mathcal{G}$. But then the identity $\lambda R^0_\lambda = A^0 R^0_\lambda + \mathrm{id}$ implies the following formula

$$\frac{1}{2}\left(R^0_\lambda f\right)''(v) + \beta\, R^0_\lambda f(v) = 0.$$

For every $\lambda > 0$ R^0_λ maps $C_0(\mathcal{G})$ onto $\mathcal{D}(A^0)$. With the choice $a = (1+\beta)^{-1}\,\beta$, $c = (1+\beta)^{-1}$ this shows that the process realizes the boundary conditions of equations (1.1) with $b_k = 0$, $k = 1, \ldots, n$.

Moreover, we can now use equation (1.26) combined with formula (1.28), to obtain the following explicit expression for the resolvent with $f \in C_0(\mathcal{G})$, $\lambda > 0$:

$$R^0_\lambda f(\xi) = R^D_\lambda f(\xi) + \frac{1}{\beta + \lambda}\, e^{-\sqrt{2\lambda} d(\xi,v)}\, f(v), \qquad \xi \in \mathcal{G}, \tag{1.29}$$

where, as before, R^D_λ is the Dirichlet resolvent.

In order to compute the heat kernel associated with this process on $\mathcal{G}$, we invert the Laplace transforms in equation (1.29). For the first term on the right hand side this is trivial, and gives the Dirichlet heat kernel p^D, cf. equation (1.21). The second term could be handled by a formula which can be found in the tables (e.g., [7, eq. (5.6.10)]). But this formula involves the complementary error function erfc at complex arguments, and does not yield a very intuitive expression. Instead, we can simply use the observation that $t \mapsto \exp(-\beta t)$ is the inverse Laplace transform of $\lambda \mapsto (\beta + \lambda)^{-1}$. Moreover, the well-known formula for the density of the hitting time of the origin by a Brownian motion on the real line (e.g., [18, p. 25], [19, p. 96], [29, p. 102]) provides us with the following expression for the density of the first hitting time of the vertex

$$P_\xi(H_v \in ds) = \frac{d(\xi,v)}{\sqrt{2\pi s^3}}\, e^{-d(\xi,v)^2/2s}\, ds, \qquad s \geq 0. \tag{1.30}$$

Using the well-known Laplace transform (e.g., [7, eq. (4.5.28)])

$$\int_0^\infty e^{-\lambda s}\, \frac{a}{\sqrt{2\pi s^3}}\, e^{-a^2/2s}\, ds = e^{-\sqrt{2\lambda} a}, \qquad a > 0,\ \lambda > 0, \tag{1.31}$$

we infer that the inverse Laplace transform of the exponential in (1.29) is given by $P_\xi(H_v \in ds)$. Thus we obtain the following heat kernel

$$\begin{aligned} p^0(t,\xi,d\eta) &= p^D(t,\xi,\eta)\, d\eta - \Bigl(\int_0^t e^{-\beta(t-s)}\, P_\xi(H_v \in ds)\Bigr)\, \epsilon_v(d\eta), \\ &= p^D(t,\xi,\eta)\, d\eta - \Bigl(\int_0^t e^{-\beta(t-s)}\, \frac{d(\xi,v)}{\sqrt{2\pi s^3}}\, e^{-d(\xi,v)^2/2s}\, ds\Bigr)\, \epsilon_v(d\eta), \end{aligned} \tag{1.32}$$

with $\xi, \eta \in \mathcal{G}$, $t > 0$, and ϵ_v is the Dirac measure at the vertex v.

1.5. Killing via the Local Time at the Vertex. We recall from remark 1.2, that we may and will consider every Brownian motion X on $\mathcal{G}$ with respect to a filtration $\mathcal{F} = (\mathcal{F}_t,\, t \geq 0)$ which is right continuous and complete relative to $(P_\xi,\, \xi \in \mathcal{G})$, and such that X is strongly Markovian with respect to $\mathcal{F}$.

In this subsection we suppose that X is a Brownian motion on the single vertex graph $\mathcal{G}$ with infinite lifetime, and such that the vertex is not absorbing. This entails (e.g., [29, Proposition II.2.19]) that X leaves the vertex immediately and begins a standard Brownian excursion into one of the edges. Therefore we get in this case for the hitting time H_v of the vertex $P_v(H_v = 0) = 1$, i.e., v is regular for $\{v\}$ in the sense of [3]. Consequently X has a local time $L = (L_t,\, t \geq 0)$ at the vertex (e.g., [3, Theorem V.3.13]). Without loss of generality, we suppose throughout this subsection that L is a *perfect continuous homogeneous additive functional (PCHAF)* of X in the sense of [34, Section III.32]. That is, L is a non-decreasing process, which is adapted to $\mathcal{F}$, and such that it is a.s. continuous, additive, i.e., $L_{t+s} = L_t + L_s \circ \theta_t$, and for all $\xi \in \mathcal{G}$, $P_\xi\bigl(L_0 = 0\bigr) = 1$ holds true. Moreover we may and will assume from now on that X and L are *pathwise* continuous.

Killing X exponentially on the scale of L, we can construct a new Brownian motion $\hat{X}$ on $\mathcal{G}$. We shall do this using the method of [19, 20].

Let $K = (K_s,\, s \in \mathbb{R}_+)$ denote the right continuous pseudo-inverse of L:

$$K_s = \inf\{t \geq 0,\, L_t > s\}, \qquad s \in \mathbb{R}_+, \tag{1.33}$$

where — as usual — we make the convention that $\inf \emptyset = +\infty$. The continuity of L entails that for every $s \in \mathbb{R}_+$, $L_{K_s} = s$. Clearly, K is increasing, and due to its right continuity it is a measurable stochastic process. Fix $s \in \mathbb{R}_+$. It is straightforward to check that for every $t \in \mathbb{R}_+$,

$$\{K_s < t\} = \{L_t > s\}. \tag{1.34}$$

Because L is adapted, the set on the right hand side belongs to $\mathcal{F}_t$, and since $\mathcal{F}$ is right continuous, equality (1.34) shows that for every $s \in \mathbb{R}_+$, K_s is a stopping time relative to $\mathcal{F}$. We remark that since L only increases when X is at the vertex v, the continuity of X implies that for every $s \in \mathbb{R}_+$ we get $X(K_s) = v$ on $\{K_s < +\infty\}$. On the other hand, we shall argue below that L a.s. increases to $+\infty$, so that we get $X(K_s) = v$ a.s. for all $s \in \mathbb{R}_+$.

Let $\beta > 0$. Bring in the additional probability space $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), P_\beta)$, where P_β is the exponential law with rate β. Let S denote the associated coordinate random variable $S(s) = s$, $s \in \mathbb{R}_+$. Define

$$\hat{\Omega} = \Omega \times \mathbb{R}_+, \quad \hat{\mathcal{A}} = \mathcal{A} \otimes \mathcal{B}(\mathbb{R}_+), \quad \hat{P}_\xi = P_\xi \otimes P_\beta,\ \xi \in \mathcal{G}.$$

We extend X, L, K, and S in the canonical way to these enlarged probability spaces, but for simplicity keep the same notation for these quantities.

Set

$$\zeta_\beta = \inf\bigl\{t \geq 0,\, L_t > S\bigr\}, \tag{1.35}$$

and observe that since K is measurable we may write $\zeta_\beta = K_S$. Thus as above we get $X(\zeta_\beta) = v$. Define the *killed process*

$$\hat{X}_t = \begin{cases} X_t, & t < \zeta_\beta, \\ \Delta, & t \geq \zeta_\beta. \end{cases} \tag{1.36}$$

Since this prescription for killing the process X via the (PCHAF) L is slightly different from the method used in [3, 34], we cannot directly use the results proved there to conclude that the subprocess $\hat{X}$ of X is still a strong Markov process. However, it has been proved in [21, Appendix A] that the strong Markov property is preserved under this method of killing, i.e., $\hat{X}$ is a strong Markov process relative to its natural filtration (actually relative to a larger filtration, but we will not use this here). Now we may employ the arguments in section 2.7 of [19], or in section III.2 of [29] to conclude that $\hat{X}$ is a strong Markov process with respect to the universal right continuous and complete augmentation of its natural filtration.

It is clear that $\hat{X}$ has a.s. right continuous paths which admit left limits, and that its paths on $[0, \zeta_\beta)$ are equal to those of X, and thus are continuous on this random time interval. Moreover, it is obvious that for every $\xi \in \mathcal{G}^\circ$, we have $P_\xi(\zeta_\beta \geq H_v) = 1$. Therefore, up to its hitting time of the vertex, $\hat{X}$ is equivalent to a Brownian motion on the edge to which ξ belongs, because so is X. Altogether we have proved that — under the hypothesis that L is a PCHAF, which will be argued below in all cases that we consider — $\hat{X}$ is a Brownian motion on $\mathcal{G}$ in the sense of definition 1.1.

There is a simple, useful relationship between the resolvents R and $\hat{R}$ of the processes X and $\hat{X}$, respectively. Recall our convention that all functions f on $\mathcal{G}$ are extended to $\mathcal{G} \cup \{\Delta\}$ by $f(\Delta) = 0$.

Lemma 1.12. *For all $\lambda > 0$, $f \in B(\mathcal{G})$, $\xi \in \mathcal{G}$,*

$$\hat{R}_\lambda f(\xi) = R_\lambda f(\xi) - e_\lambda(\xi)\, \hat{E}_v\big(e^{-\lambda\zeta_\beta}\big)\, R_\lambda f(v) \tag{1.37}$$

holds true, where e_λ is defined in equation (1.15).

Proof. For $\lambda > 0$, $f \in B(\mathcal{G})$, $\xi \in \mathcal{G}$

$$\begin{aligned}\hat{R}_\lambda f(\xi) &= \hat{E}_\xi\Big(\int_0^{\zeta_\beta} e^{-\lambda t} f(X_t)\, dt\Big)\\ &= R_\lambda f(\xi) - \hat{E}_\xi\Big(e^{-\lambda\zeta_\beta} \int_0^\infty e^{-\lambda t}\, f\big(X_{t+\zeta_\beta}\big)\, dt\Big).\end{aligned}$$

By construction, the last expectation value is equal to

$$\beta \int_0^\infty e^{-\beta s} \int_0^\infty e^{-\lambda t} E_\xi\Big(e^{-\lambda K_s}\, f\big(X_{t+K_s}\big)\Big)\, dt\, ds,$$

where we used Fubini's theorem. Consider the expectation value under the integrals, and recall that for fixed $s \in \mathbb{R}_+$, K_s is an $\mathcal{F}$–stopping time, while X is strongly Markovian relative to $\mathcal{F}$. Hence we can compute as follows

$$\begin{aligned}E_\xi\Big(e^{-\lambda K_s}\, f\big(X_{t+K_s}\big)\Big) &= E_\xi\Big(e^{-\lambda K_s}\, E_\xi\Big(f\big(X_{t+K_s}\big) \,\big|\, \mathcal{F}_{K_s}\Big)\Big)\\ &= E_\xi\Big(e^{-\lambda K_s}\, E_{X(K_s)}\big(f(X_t)\big)\Big)\\ &= E_\xi\big(e^{-\lambda K_s}\big)\, E_v\big(f(X_t)\big),\end{aligned}$$

where we used the fact that a.s. $X(K_s) = v$. So far we have established

$$\hat{R}_\lambda f(\xi) = R_\lambda f(\xi) - \hat{E}_\xi\big(e^{-\lambda\zeta_\beta}\big)\, R_\lambda f(v).$$

In order to compute the expectation value on the right hand side, we first remark that because L is zero until X hits the vertex for the first time, we find that for given $s \in \mathbb{R}_+$, $K_s \geq H_v$, and therefore $K_s = H_v + K_s \circ \theta_{H_v}$. Hence, and again by the strong Markov property,

$$\begin{aligned} E_\xi\bigl(e^{-\lambda K_s}\bigr) &= E_\xi\bigl(e^{-\lambda H_v}\, e^{-\lambda K_s \circ H_v}\bigr) \\ &= E_\xi\bigl(e^{-\lambda H_v}\, E_\xi\bigl(e^{-\lambda K_s \circ H_v} \bigm| \mathcal{F}_{H_v}\bigr)\bigr) \\ &= E_\xi\bigl(e^{-\lambda H_v}\bigr) E_v\bigl(e^{-\lambda K_s}\bigr), \end{aligned}$$

Integrating the last identity against the exponential law in the variable s, we find with formula (1.15)

$$\hat{E}_\xi\bigl(e^{-\lambda \zeta_\beta}\bigr) = e_\lambda(\xi)\, \hat{E}_v\bigl(e^{-\lambda \zeta_\beta}\bigr),$$

and the proof is finished. □

Remark 1.13. Formula (1.37) is quite useful, because if the resolvent of X is known, then — in view of equation (1.15) — it reduces the calculation of $\hat{R}_\lambda$ to the computation of the Laplace transform of the density of ζ_β under $\hat{P}_v$.

2. The Walsh Process

The most basic process — which on a single vertex graph plays the same role as a reflected Brownian motion on the half line — is the well-known *Walsh process*, which we denote by $W = (W_t,\, t \geq 0)$. It corresponds to the case where the parameters a and c in the boundary condition (1.1) both vanish. This process has been introduced by Walsh in [33] as a generalization of the skew Brownian motion discussed in [18, Chapter 4.2] to a process in $\mathbb{R}^2$ which only moves on rays connected to the origin.

A pathwise construction of the Walsh process in the present context is as follows. Consider the paths of the standard Brownian motion $B = (B_t,\, t \geq 0)$ on $\mathbb{R}$, and its associated reflected Brownian motion $|B| = (|B_t|,\, t \geq 0)$, where $|\cdot|$ denotes absolute value. Let $Z = \{t \geq 0,\, B_t = 0\}$. Then its complement Z^c is open, and hence it is the pairwise disjoint union of a countable family of *excursion intervals* $I_j = (t_j, t_{j+1})$, $j \in \mathbb{N}$. Let $R = (R_j,\, j \in \mathbb{N})$ be an independent sequence of identically distributed random variables, independent of B, with values in $\{1, 2, \ldots, n\}$ such that R_j, $j \in \mathbb{N}$, takes the value $k \in \{1, 2, \ldots, n\}$ with probability $w_k \in [0, 1]$, $\sum_k w_k = 1$. Now define $W_t = v$ if $t \in Z$, and if $t \in I_j$, and $R_j = k$ set $W_t = (k, |B_t|)$. In other words, when starting at $\xi \in \mathcal{G}^\circ$, the process moves as a Brownian motion on the edge containing ξ until it hits the vertex at time H_v, and then W performs Brownian excursions from the vertex v into the edges l_k, $k \in \{1, 2, \ldots, n\}$, whereby the edge l_k is selected with probability w_k.

As for the standard Brownian motion on $\mathbb{R}$ (cf. subsection 1.2), we may and will assume without loss of generality that W has exclusively continuous paths.

Walsh has remarked in the epilogue of [33], cf. also [1], that it is not completely straightforward to prove that this stochastic process is strongly Markovian. A proof of the strong Markov property based on Itô's excursion theory [16] has been given in [30, 31]. A construction of this process via its Feller semigroup can be found in [1] (cf. also the references quoted there for other approaches).

Next we check that the Walsh process has a generator with boundary condition at the vertex given by (1.1) with $a = c = 0$. Let $f \in \mathcal{D}(A^w)$. At the vertex v

Dynkin's form for the generator reads

$$A^w f(v) = \lim_{\epsilon \downarrow 0} \frac{E_v\Bigl(f\bigl((X(H^w_{v,\epsilon})\bigr)\Bigr) - f(v)}{E_v(H^w_{v,\epsilon})}, \tag{2.1}$$

where $H^w_{v,\epsilon}$ is the hitting time of the complement of the open ball $B_\epsilon(v)$ of radius $\epsilon > 0$ around v.

Lemma 2.1. *For the Walsh process* $E_v(H^w_{v,\epsilon}) = \epsilon^2$.

Proof. Since by construction W has infinite lifetime, $H^w_{v,\epsilon}$ is the hitting time of the set of the n points with local coordinates (k,ϵ), $k = 1, \dots, n$. Therefore, by the independence of the choice of the edge for the values of the excursion, it follows that under P_v the stopping time $H^w_{v,\epsilon}$ has the same law as the hitting time of the point $\epsilon > 0$ of a reflected Brownian motion on $\mathbb{R}_+$, starting at 0. Thus the statement of the lemma follows from equation (1.6). □

From the construction of W we immediately get

$$E_v\Bigl(f\bigl(W(H^w_{v,\epsilon})\bigr)\Bigr) = \sum_{k=1}^n w_k f_k(\epsilon),$$

with the notation $f_k(x) = f(k,x)$, $x \in \mathbb{R}_+$. Inserting this into equation (2.1) we obtain

$$A^w f(v) = \lim_{\epsilon \downarrow 0} \epsilon^{-2} \sum_{k=1}^n w_k \bigl(f_k(\epsilon) - f(v)\bigr),$$

and since $f'(v_k)$ exists (cf. lemma 1.3) it is obvious that this entails the condition

$$\sum_{k=1}^n w_k f'(v_k) = 0. \tag{2.2}$$

For later use we record this result as

Theorem 2.2. *Consider the boundary condition* (1.1) *with* $a = c = 0$*, and* $b \in [0,1]^n$*. Let* W *be a Walsh process as constructed above with the choice* $w_k = b_k$*,* $k \in \{1, 2, \dots, n\}$*. Then the generator* A^w *of* W *is* $1/2$ *times the second derivative on* $\mathcal{G}$ *with domain consisting of those* $f \in C^2_0(\mathcal{G})$ *which satisfy condition* (1.1b).

For the remainder of this section we make the choice $a = c = 0$, $w_k = b_k$, $k \in \{1, 2, \dots, n\}$ in (1.1).

Next we compute the resolvent of W. Let $\lambda > 0$, $f \in C_0(\mathcal{G})$, and consider first $\xi = v$. Without loss of generality, we may assume that W has been constructed pathwise from a standard Brownian motion B as described above, and that B is as in subsection 1.2. Then we get

$$E_v\bigl(f(W_t)\bigr) = \sum_{m=1}^n b_m E^Q_0\bigl(f_m(|B_t|)\bigr).$$

Hence we find for the resolvent R^w of the Walsh process

$$R^w_\lambda f(v) = \int_{\mathcal{G}} r^w_\lambda(v,\eta)\, f(\eta)\, d\eta \tag{2.3a}$$

with resolvent kernel $r_\lambda^w(v,\eta)$, $\eta \in \mathcal{G}$, given by

$$r_\lambda^w(v,\eta) = \sum_{m=1}^n 2b_m \frac{e^{-\sqrt{2\lambda}\, d(v,\eta)}}{\sqrt{2\lambda}} 1_{l_m}(\eta), \tag{2.3b}$$

and where the integration in (2.3a) is with respect to the Lebesgue measure on $\mathcal{G}$.

Now let $\xi \in \mathcal{G}$. We use the first passage time formula (1.26) together with formulae (1.15) and (2.3), and obtain

Lemma 2.3. *The resolvent of the Walsh process on $\mathcal{G}$ is given by*

$$R_\lambda^w f(\xi) = \int_{\mathcal{G}} r_\lambda^w(\xi,\eta) f(\eta)\, d\eta, \qquad \lambda > 0,\ \xi \in \mathcal{G},\ f \in B(\mathcal{G}), \tag{2.4a}$$

with

$$r_\lambda^w(\xi,\eta) = r_\lambda(\xi,\eta) + \sum_{k,m=1}^n e_{\lambda,k}(\xi)\, S_{km}^w \frac{1}{\sqrt{2\lambda}} e_{\lambda,m}(\eta), \tag{2.4b}$$

$$S_{km}^w = 2w_m - \delta_{km}, \tag{2.4c}$$

where r_λ is defined in equation (1.24)*, and where $e_{\lambda,k}$, $e_{\lambda,m}$ denote the restrictions of e_λ (cf.* (1.15)*) to the edges l_k, l_m respectively.*

Remark 2.4. The matrix $S^w = \bigl(S_{km}^w,\ k,m = 1,\dots,n\bigr)$ is the *scattering matrix* as defined in quantum mechanics. We briefly recall its construction in the present context, for more details the interested reader is referred to [25]. S^w is obtained from the boundary conditions at the vertex v in the following way. Consider a function f on $\mathcal{G}$ which is continuously differentiable in $\mathcal{G}^\circ = \mathcal{G} \setminus \{v\}$, and such that for all $k = 1,\dots,n$ the limits

$$F_k = f(v_k) = \lim_{\xi \to v,\, \xi \in l_k^\circ} f(\xi)$$
$$F_k' = f'(v_k) = \lim_{\xi \to v,\, \xi \in l_k^\circ} f'(\xi)$$

exist. Define two column vectors F, $F' \in \mathbb{C}^n$, having the components F_k and F_k', $k = 1,\dots,n$, respectively. Furthermore, consider boundary conditions of the following form

$$AF + BF' = 0, \tag{2.5}$$

where A and B are complex $n \times n$ matrices. The *on-shell scattering matrix at energy $E > 0$* is defined as

$$S_{A,B}(E) = -(A + i\sqrt{E}B)^{-1}(A - i\sqrt{E}B), \tag{2.6}$$

which exists and is unitary, provided the $n \times 2n$ matrix (A,B) has maximal rank (i.e., rank n) and $AB^\dagger$ is hermitian. These requirements for A and B guarantee that the corresponding Laplace operator is self-adjoint on $L^2(\mathcal{G})$ (with Lebesgue measure). Observe that under these conditions the boundary conditions (2.5) are equivalent to any boundary conditions of the form $CAF + CBF' = 0$ where C

is invertible. Also $S_{CA,CB}(E) = S_{A,B}(E)$ holds true. For the Walsh process at hand, concrete choices for A and B are given by

$$A^w = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \end{pmatrix}, \; B^w = \begin{pmatrix} b_1 & b_2 & b_3 & \dots & b_{n-1} & b_n \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Then (2.5) is the condition that f is actually continuous at the vertex v, i.e., $f(v_k) = f(v_m)$, k, $m = 1, \dots, n$, and that (2.2) is valid (with $w_k = b_k$, $k \in \{1, 2, \dots, n\}$). Obviously, (A^w, B^w) has maximal rank. However, $A^w(B^w)^\dagger$ is hermitian if and only if all b_k are equal (i.e., $b_k = 1/n$, $k = 1, \dots, n$). Nevertheless, (2.6) exists also in the non-hermitian case, and $S_{A^w,B^w}(E) = S^w$ holds for all $E > 0$ due to the relations $A^w S^w = -A^w$, and $B^w S^w = B^w$. In addition, the following relations are valid:

$$S^w = \left(S^w\right)^{-1}, \tag{2.7}$$

$$\det S^w = (-1)^{n+1}. \tag{2.8}$$

Furthermore, S^w is a contraction, and the associated Laplace operator is m-dissipative on $L^2(\mathcal{G})$ since trivially $\operatorname{Im}(AB^\dagger) = 0$, cf. Theorem 2.5 in [22]. When all b_k are equal, such that $A^w(B^w)^\dagger = 0$, then S^w is an involutive, orthogonal matrix of the form

$$S^w = -1 + 2P_n. \tag{2.9}$$

P_n is the matrix whose entries are equal to $1/n$. P_n is a real orthogonal projection, that is $P_n = P_n^\dagger = P_n^t = P_n^2$. It is also of rank 1, that is $\dim \operatorname{Ran} P_n = 1$. The relation (2.4b) giving the resolvent in terms of the scattering matrix is actually valid in the more general context of arbitrary metric graphs and boundary conditions of the form (2.5), see [22, 26].

It is straightforward to compute the inverse Laplace transform of the right hand side of formula (2.4b), and this yields the following result.

Lemma 2.5. *For $t > 0$, ξ, $\eta \in \mathcal{G}$ the transition density of the Walsh process on $\mathcal{G}$ is given by*

$$p^w(t,\xi,\eta) = p^D(t,\xi,\eta) + \sum_{k,m=1}^{n} 1_{l_k}(\xi)\, 2w_m\, g\big(t, d_v(\xi,\eta)\big)\, 1_{l_m}(\eta), \tag{2.10}$$

$$= p(t,\xi,\eta) + \sum_{k,m=1}^{n} 1_{l_k}(\xi)\, S^w_{km} g\big(t, d_v(\xi,\eta)\big)\, 1_{l_m}(\eta). \tag{2.11}$$

$p(t,\xi,\eta)$ is defined in equation (1.17), $p^D(t,\xi,\eta)$ in equation (1.21), g is the Gauß-kernel (1.4), and d_v is defined in equation (1.16).

Remark 2.6. Alternatively $p^w(t,\xi,\eta)$ can also be written as

$$p^w(t,\xi,\eta) = p(t,\xi,\eta) + \sum_{k,m=1}^{n} 1_{l_k}(\xi) \int_0^t P_\xi(H_v \in ds)\, S^w_{km}\, g\bigl(t-s, d(v,\eta)\bigr)\, 1_{l_m}(\eta). \tag{2.12}$$

Even though this formula appears somewhat more complicated than (2.11), it exhibits the role of the scattering matrix S^w, that is, it describes more clearly what happens when the process hits the vertex.

3. The Elastic Walsh Process

In this section we consider the boundary conditions (1.1) with $0 < a < 1$ and $c = 0$. The corresponding stochastic process, which we will denote by W^e, is constructed from the Walsh process W of the previous section in a similar way as the elastic Brownian motion on $\mathbb{R}_+$ is constructed from a reflected Brownian motion (cf., e.g., [17], [18, Chapter 2.3], [20, Chapter 6.2], [19, Chapter 6.4]).

In more detail, the construction is as follows. Consider the Walsh process W as discussed in the previous section. We may continue to suppose that W has been constructed pathwise from a standard Brownian motion B, as it has been described there. But then the local time of W at the vertex, denoted by L^w, is pathwise equal to the local time of the Brownian motion at the origin (and we continue to use the normalization determined by (1.8)). It is well-known (e.g., [18–20, 29]) that L^w has all properties of a PCHAF as formulated in subsection 1.5 for the construction of a subprocess by killing W at the vertex. We continue to denote the rate of the exponential random variable S used there by $\beta > 0$. Let W^e be the subprocess so obtained. In particular (cf. 1.5), W^e is a Brownian motion on $\mathcal{G}$, and in analogy with the case of a Brownian motion on the real line we call this stochastic process the *elastic Walsh process*. We write $\zeta_{\beta,0}$ for the lifetime of W^e (i.e., for the random time corresponding to ζ_β in subsection 1.5).

We proceed to show that the elastic Walsh process W^e has a generator A^e with domain $\mathcal{D}(A^e)$ which satisfies the boundary conditions as claimed. In other words, we claim that there exist $a \in (0,1)$ and $b_k \in (0,1)$, $k \in \{1,2,\dots,n\}$, with $a + \sum_k b_k = 1$, so that for all $f \in \mathcal{D}(G)$,

$$af(v) = \sum_{k=1}^{n} b_k f'(v_k) \tag{3.1}$$

holds. To this end, we calculate $A^e f(v)$ in Dynkin's form. We shall use a notation similar to the one used in subsection 1.5. Namely, let $\hat{P}_v$ and $\hat{E}_v$ denote the probability and expectation, respectively, on the probability space extended by $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), P_\beta)$, while the corresponding symbols without $\hat{\ }$ are those for the Walsh process without killing.

For $\epsilon > 0$ and under W^e let $H^e_{v,\epsilon}$ denote the hitting time of the complement $B_\epsilon(v)^c$ of the open ball $B_\epsilon(v)$ of radius $\epsilon > 0$ with center v. Then $H^e_{v,\epsilon} = H^w_{v,\epsilon} \wedge \zeta_{\beta,0}$, where as before $H^w_{v,\epsilon}$ is the hitting time of $B_\epsilon(v)^c$ by the Walsh process W. (Note that $B_\epsilon(v)^c$ contains the cemetery Δ.) We find

$$\hat{E}_v\Big(f\big(W^e(H^e_{v,\epsilon})\big)\Big) = \Big(\sum_{k=1}^n w_k f_k(\epsilon)\Big)\,\hat{P}_v\big(H^w_{v,\epsilon} < \zeta_{\beta,0}\big). \tag{3.2}$$

The probability in the last expression is taken care of by the following lemma.

Lemma 3.1. *For all ϵ, $\beta > 0$,*

$$\hat{P}_v\big(H^w_{v,\epsilon} < \zeta_{\beta,0}\big) = \frac{1}{1+\epsilon\beta}. \tag{3.3}$$

Proof. We may consider the Walsh process W as being pathwise constructed from a standard Brownian motion B on the real line as in the previous section, and we shall use the notations and conventions from there. Then it is clear that under P_v and under $\hat{P}_v$, $H^w_{v,\epsilon}$ has the same law as the hitting time of the point ϵ in $\mathbb{R}_+$ by the reflecting Brownian motion $|B|$ under Q_0, that is, as $H^B_{\{-\epsilon,\epsilon\}}$ of the Brownian motion B itself under Q_0. Let K^w denote the right continuous pseudo-inverse of L^w. For fixed $s \in \mathbb{R}_+$ we get

$$\{K^w_s < H^w_{v,\epsilon}\} = \{L^w(H^w_{v,\epsilon}) > s\}.$$

Hence

$$\begin{aligned} P_v\big(K^w_s < H^w_{v,\epsilon}\big) &= P_v\big(L^w(H^w_{v,\epsilon}) > s\big) \\ &= Q_0\big(L^B(H^B_{\{-\epsilon,\epsilon\}}) > s\big). \end{aligned}$$

In appendix B of [21] is shown with the method in [19, Section 6.4] that under Q_0 the random variable $L^B(H^B_{\{-\epsilon,\epsilon\}})$ is exponentially distributed with mean ϵ. So we find

$$P_v\big(K^w_s < H^w_{v,\epsilon}\big) = e^{-s/\epsilon}.$$

We integrate this relation against the exponential law with rate β in the variable s, and obtain

$$\begin{aligned} \hat{P}_v\big(\zeta_{\beta,0} > H^w_{v,\epsilon}\big) &= 1 - \beta\int_0^\infty e^{-\beta s}\, P_v\big(K^W_s < H^w_{v,\epsilon}\big)\, ds \\ &= \frac{1}{1+\epsilon\beta}. \end{aligned}$$

We used the fact that due to the continuity of the paths of W we have $\zeta_{\beta,0} \neq H^w_{v,\epsilon}$. $\square$

We insert formula (3.3) into equation (3.2), and obtain

$$\begin{aligned} A^e f(v) &= \lim_{\epsilon\downarrow 0} \frac{1}{\hat{E}_v(H^e_{v,\epsilon})}\Big(\hat{E}_v\Big(f\big(W^e(H^e_{v,\epsilon})\big)\Big) - f(v)\Big) \\ &= \lim_{\epsilon\downarrow 0} \frac{1}{\hat{E}_v(H^e_{v,\epsilon})}\Big(\frac{1}{1+\epsilon\beta}\sum_{k=1}^n w_k f_k(\epsilon) - f(v)\Big) \\ &= \lim_{\epsilon\downarrow 0} \frac{\epsilon}{\hat{E}_v(H^e_{v,\epsilon})}\,\frac{1}{1+\epsilon\beta}\Big(\sum_{k=1}^n w_k\,\frac{f_k(\epsilon)-f(v)}{\epsilon} - \beta f(v)\Big). \end{aligned}$$

Obviously $\hat{E}_v(H^e_{v,\epsilon}) \leq E_v(H^w_{v,\epsilon}) = \epsilon^2$ (cf. lemma 2.1). Since the last limit and $f'(v_k)$, $k \in \{1,2,\ldots,n\}$, exist and are finite, we get as a necessary condition

$$\sum_{k=1}^{n} w_k f'(v_k) - \beta f(v) = 0. \tag{3.4}$$

Thus we have proved the following theorem.

Theorem 3.2. *Consider the boundary condition* (1.1) *with* $a \in (0,1)$*,* $b \in [0,1]^n$*, and* $c = 0$*. Set*

$$w_k = \frac{b_k}{1-a},\ k = 1,\ldots,n, \quad \beta = \frac{a}{1-a}, \tag{3.5}$$

and let W^e *be the elastic Walsh process as constructed above with these parameters. Then the generator* A^e *of* W^e *is* $1/2$ *times the second derivative on* $\mathcal{G}$ *with domain consisting of those* $f \in C_0^2(\mathcal{G})$ *which satisfy condition* (1.1b)*.*

Remark 3.3. Note that condition (1.1a) entails that if w_k and β are defined by (3.5) then $w_k \in [0,1]$, $k = 1,\ldots,n$, $\sum_k w_k = 1$, and $\beta > 0$. Therefore the choice (3.5) is consistent with the conditions on these parameters required by the construction of the elastic Walsh process W^e.

Next we compute the resolvent R^e of the elastic Walsh process. As a byproduct this will give another proof of theorem 3.2. Moreover, it will provide us with an explicit formula for the scattering matrix in this case. In contrast to the calculations in [18, Chapter 2.3], [20, Chapter 6.2] for the classical case with $\mathcal{G} = \mathbb{R}_+$, we do not use the first passage time formula (1.26), but instead we use formula (1.37). This simplifies the computation considerably.

Let $f \in C_0(\mathcal{G})$, $\lambda > 0$, and $\xi \in \mathcal{G}$. In the present context formula (1.37) reads

$$R^e_\lambda f(\xi) = R^w_\lambda f(\xi) - e_\lambda(\xi)\, \hat{E}_v\big(e^{-\lambda\zeta_{\beta,0}}\big)\, R^w_\lambda f(v),$$

where R^w is the resolvent of the Walsh process without killing, and e_λ is defined in (1.15). The Laplace transform of the density of $\zeta_{\beta,0}$ under $\hat{P}_v$ is readily computed:

Lemma 3.4. *For all* λ*,* $\beta > 0$*,*

$$\hat{E}_v\big(e^{-\lambda\zeta_{\beta,0}}\big) = \frac{\beta}{\beta + \sqrt{2\lambda}}.$$

Proof. As remarked before, we may consider L^w to be equal to the local time at the origin of the Brownian motion B underlying the construction of W, and therefore the analogous statement is true for the right continuous pseudo-inverse K^w of L^w. As above let K^B denote the right continuous pseudo-inverse of L^B (cf. 1.2). Then for $s \in \mathbb{R}_+$,

$$\begin{aligned} E_v\big(e^{-\lambda K^w_s}\big) &= E^Q_0\big(e^{-\lambda K^B_s}\big) \\ &= e^{-\sqrt{2\lambda} s}, \end{aligned}$$

where we used lemma 1.8. Hence

$$\hat{E}_v\big(e^{-\lambda\zeta_{\beta,0}}\big) = \beta \int_0^\infty e^{-(\beta+\sqrt{2\lambda})t}\, dt,$$

which proves the lemma. □

With lemma 3.4 we obtain the following formula

$$R^e_\lambda f(\xi) = R^w_\lambda f(\xi) - \frac{\beta}{\beta + \sqrt{2\lambda}}\, e_\lambda(\xi)\, R^w_\lambda f(v). \tag{3.6}$$

Note that $R^w_\lambda f$ is in the domain of the generator of the Walsh process, and therefore satisfies the boundary condition (2.2):

$$\sum_{k=1}^n w_k \bigl(R^w_\lambda f\bigr)'(v_k) = 0.$$

On the other hand, we obviously have $e'_\lambda(v_k) = -\sqrt{2\lambda}$ for all $k \in \{1, 2, \dots, n\}$. Thus with $\sum_{k=1}^n w_k = 1$ we find,

$$\sum_{k=1}^n w_k \bigl(R^e_\lambda f\bigr)'(v_k) = \beta\, \frac{\sqrt{2\lambda}}{\beta + \sqrt{2\lambda}}\, R^w_\lambda f(v),$$

while equation (3.6) yields for $\xi = v$

$$R^e_\lambda f(v) = \frac{\sqrt{2\lambda}}{\beta + \sqrt{2\lambda}}\, R^w_\lambda f(v).$$

The last two equations show that for all $f \in C_0(\mathcal{G})$, $\lambda > 0$, we have

$$\sum_{k=1}^n w_k \bigl(R^e_\lambda f\bigr)'(v_k) = \beta\, R^e_\lambda f(v).$$

Since for every $\lambda > 0$, R^e_λ maps $C_0(\mathcal{G})$ onto the domain of the generator of W^e, we have another proof of theorem 3.2.

Upon insertion of the expressions for the resolvent kernels of the Walsh process, equations (2.3), and (2.4), with the same notation as in lemma 2.3 we immediately obtain the following result:

Lemma 3.5. *For $\lambda > 0$, ξ, $\eta \in \mathcal{G}$ the resolvent kernel of the elastic Walsh process W^e is given by*

$$r^e_\lambda(\xi, \eta) = r^D_\lambda(\xi, \eta) + \sum_{k,m=1}^n e_{\lambda,k}(\xi)\, 2w_m\, \frac{1}{\beta + \sqrt{2\lambda}}\, e_{\lambda,m}(\eta) \tag{3.7a}$$

$$= r_\lambda(\xi, \eta) + \sum_{k,m=1}^n e_{\lambda,k}(\xi)\, S^e_{km}(\lambda)\, \frac{1}{\sqrt{2\lambda}}\, e_{\lambda,m}(\eta), \tag{3.7b}$$

with the scattering matrix S^e

$$S^e_{km}(\lambda) = 2\, \frac{\sqrt{2\lambda}}{\beta + \sqrt{2\lambda}}\, w_m - \delta_{km}, \qquad \lambda > 0,\ k, m \in \{1, 2, \dots, n\}. \tag{3.7c}$$

Remark 3.6. Note that in contrast to the case of the Walsh process, this time the scattering matrix is not constant with respect to $\lambda > 0$. Also, when $\beta = 0$, formula (2.4c) is recovered, as it should be. In analogy with the discussion in remark 2.4, the boundary conditions for the elastic Walsh process is given by the matrices

$$A^e = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \beta \\ 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \end{pmatrix}, \quad B^e = \begin{pmatrix} w_1 & w_2 & w_3 & \dots & w_{n-1} & w_n \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

such that

$$\begin{aligned} S^e(\lambda) &= S_{A^e,B^e}(E = -2\lambda) \\ &= -(A^e + \sqrt{2\lambda} B^e)^{-1} \, (A^e - \sqrt{2\lambda} B^e). \end{aligned}$$

Observe that for k, $m \in \{1, \dots, n\}$ the matrix element $S^e_{km}(\lambda)$ of the scattering matrix is obtained from the resolvent kernel as

$$S^e_{km}(\lambda) = \sqrt{2\lambda} \lim_{\xi,\,\eta \to v} \bigl(r^e_\lambda(\xi, \eta) - r_\lambda(\xi, \eta) \bigr),$$

where the limit on the right hand side is taken in such a way that ξ, η converge to v along the edges l_k, l_m respectively. $S^e_{km}(\lambda)$ in turn fixes the data w_m and β, e.g., via the behavior for large λ, that is from the behavior at "large energies"

$$w_m = \frac{1}{2} \bigl(\delta_{km} + \lim_{\lambda' \uparrow \infty} S^e_{km}(\lambda') \bigr), \quad \text{for all } k \in \{1, 2, \dots, n\},$$

and

$$\beta = \sqrt{2\lambda} \Bigl(\frac{\delta_{km} + \lim_{\lambda' \uparrow \infty} S^e_{km}(\lambda')}{\delta_{km} + S^e_{mm}(\lambda)} - 1 \Bigr), \quad \text{for all } \lambda, \text{ and all } k,\, m \in \{1, 2, \dots, n\}.$$

Alternatively, the data can be obtained from the small λ behavior, that is the threshold behavior, of the scattering matrix, since from

$$\frac{w_m}{\beta} = \lim_{\lambda \downarrow 0} \frac{1}{2\sqrt{2\lambda}} \bigl(S^e_{km}(\lambda) + \delta_{km} \bigr) \qquad \text{for all } k \in \{1, 2, \dots, n\},$$

we obtain

$$\beta^{-1} = \frac{1}{2\sqrt{2\lambda}} \Bigl(\sum_{m=1}^{n} S^e_{km}(\lambda) + 1 \Bigr) \qquad \text{for all } k \in \{1, 2, \dots, n\},$$

and therefore

$$w_m = \frac{\lim_{\lambda \downarrow 0} \lambda^{-1/2} \bigl(S^e_{km}(\lambda) + \delta_{km} \bigr)}{\lim_{\lambda \downarrow 0} \lambda^{-1/2} \Bigl(\sum_m S^e_{k'm}(\lambda) + 1 \Bigr)} \qquad \text{for all } k,\, k' \in \{1, 2, \dots, n\}.$$

Furthermore we remark that in the context of quantum mechanics in the self-adjoint case $w_k = 1/n$, $k = 1, \dots, n$, the boundary conditions of the elastic Walsh process are interpreted as the presence of a δ–potential of strength β at the vertex.

In order to compute expressions for the transition kernel of the elastic Walsh process, we use the following two inverse Laplace transforms which follow from formulae (5.3.4) and (5.6.12) in [7] (cf. also appendix C in [21]) ($\lambda > 0$, $t \geq 0$, $x \geq 0$):

$$\frac{\sqrt{2\lambda}}{\beta + \sqrt{2\lambda}} \quad \xrightarrow{\mathcal{L}^{-1}} \quad \epsilon_0(dt) - \beta\Big(\frac{1}{\sqrt{2\pi t}} - \frac{\beta}{2} e^{\beta^2 t/2} \operatorname{erfc}\Big(\beta\sqrt{\frac{t}{2}}\Big)\Big)\, dt, \tag{3.8}$$

$$\frac{1}{\beta + \sqrt{2\lambda}}\, e^{-\sqrt{2\lambda}x} \quad \xrightarrow{\mathcal{L}^{-1}} \quad g(t,x) - \frac{\beta}{2}\, e^{\beta x + \beta^2 t/2} \operatorname{erfc}\Big(\frac{x}{\sqrt{2t}} + \beta\sqrt{\frac{t}{2}}\Big). \tag{3.9}$$

Then the inverse Laplace transform of the scattering matrix S^e is given by the following measures on $\mathbb{R}_+$:

$$\begin{aligned} s^e_{km}(dt) = (2w_m - \delta_{km})\, \epsilon_0(dt) - 2w_m\beta\, \frac{1}{\sqrt{2\pi t}}\, dt \\ + w_m \beta^2 e^{\beta^2 t/2} \operatorname{erfc}\Big(\beta\sqrt{\frac{t}{2}}\Big)\, dt, \end{aligned} \tag{3.10}$$

with k, $m \in \{1, 2, \ldots, n\}$. Moreover, for $t > 0$, $x \geq 0$, let us introduce

$$g_{\beta,0}(t,x) = g(t,x) - \frac{\beta}{2}\, e^{\beta x + \beta^2 t/2} \operatorname{erfc}\Big(\frac{x}{\sqrt{2t}} + \beta\sqrt{\frac{t}{2}}\Big). \tag{3.11}$$

Lemma 3.7. *For $t > 0$, ξ, $\eta \in \mathcal{G}$, the transition density p^e of the elastic Walsh process is given by*

$$p^e(t,\xi,\eta) = p^D(t,\xi,\eta) + \sum_{k,m=1}^{n} 1_{l_k}(\xi)\, 2w_m\, g_{\beta,0}\big(t, d_v(\xi,\eta)\big)\, 1_{l_m}(\eta), \tag{3.12}$$

and alternatively by

$$\begin{aligned} p^e(t,\xi,\eta) = p(t,\xi,\eta) + \sum_{k,m=1}^{n} 1_{l_k}(\xi) \Big(\int_0^t P_\xi(H^w_v \in ds) \\ \times \Big(s^e_{km} * g\big(\cdot, d(v,\eta)\big)\Big)(t-s)\Big)\, 1_{l_m}(\eta), \end{aligned} \tag{3.13}$$

where $*$ denotes convolution.

Acknowledgment. It is pleasure to thank Oliver Falkenburg for pointing out reference [32] to us.

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