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ASYMPTOTIC SPECTRAL ANALYSIS OF A DISTANCE k-GRAPH OF N-FOLD DIRECT PRODUCT OF A GRAPH

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ABSTRACT. We construct $G_N^{(k)}$, for $1 \le k \le N$, as a generalization of N-fold direct product of a graph G. We shall determine the asymptotic spectral distributions of $G_N^{(2)}$, $G_N^{(N-1)}$, and $G_N^{(N)}$, for $G = S_d$, where S_d is a star graph with d degrees, and of $G_N^{(2)}$ for $G = K_q$, where K_q is a complete graph with q vertices. Especially, we obtain that the asymptotic spectral distributions of $G_N^{(2)}$ for $G = S_d$, and for $G = K_q$, are the normalized χ^2-distributions with 1 degree of freedom independent of d and q, using the method of quantum decomposition.

1. Introduction

Let $G = (V, E)$ be a graph with a fixed origin o. We assume that the graph is *simple*, which has no loop and no multiple-edge, and is undirected. We also assume that the graph is *locally finite*: $\kappa(x) = |\{y \in V : \partial_G(y, x) = 1\}| < \infty$ for any $x \in V$, where ∂_G is the graph distance on G. The graph is said to be *regular* if $\kappa(x) \equiv \kappa$ is constant.

Let A be the adjacency matrix of the graph G. We consider an algebraic probability space $(\mathcal{A}(G), \varphi)$, where $\mathcal{A}(G)$ is the $*$-algebra generated by A, and $\varphi(a) = \langle \delta_o, a\delta_o \rangle$ for $a \in \mathcal{A}(G)$ and $\delta_o \in l^2(V)$. The distribution μ satisfying

$$\varphi(A^m) = \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m = 1, 2, \ldots, \tag{1.1}$$

is called the *spectral distribution* of G.

There are several products of graphs, for example, direct product, free product, star product, comb product and so on. The adjacency matrix of N-fold direct product of the graph $G = (V, E)$ is

$$A_N = \sum_{i=1}^{N} I \otimes \cdots \otimes I \otimes \overset{(i-th)}{A} \otimes I \otimes \cdots \otimes I. \tag{1.2}$$

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As a generalization of N-fold direct product, we define new graphs $G_N^{(k)} = (V^N, E_N^{(k)})$, for $1 \le k \le N$, by the following adjacency matrix:

$$A_N^{(k)} = \sum_{1 \le j_1 < \cdots < j_k \le N} I \otimes \cdots \otimes I \otimes \overset{(j_1-th)}{A} \otimes I \otimes \cdots \otimes I \otimes \overset{(j_k-th)}{A} \otimes I \otimes \cdots \otimes I. \quad (1.3)$$

We are interested in the asymptotic spectral distribution of $G_N^{(k)}$ with respect to the state $\varphi_N(a) = \langle \delta_o \otimes \cdots \otimes \delta_o, a(\delta_o \otimes \cdots \otimes \delta_o)\rangle$, $a \in \mathcal{A}(G_N^{(k)})$. Namely, we would like to determine the distribution μ satisfying

$$\lim_{N\to\infty} \varphi_N \left(\left(\frac{A_N^{(k)}}{\sqrt{\binom{N}{k}\kappa(o)^k}} \right)^m \right) = \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m = 1, 2, \ldots. \quad (1.4)$$

As for the connections with independences, $G_N^{(1)}$ is N-fold direct product of the graph G, and $A_N^{(1)}$ can be regarded as a sum of commutative independent random variables. It is known that the asymptotic spectral distribution of the normalized sum of commutative independent random variables is the standard Gaussian distribution. Therefore, the asymptotic spectral distribution of $G_N^{(1)}$ is the standard Gaussian distribution, i.e.,

$$\lim_{N\to\infty} \varphi_N \left(\left(\frac{A_N^{(1)}}{\sqrt{N\kappa(o)}} \right)^m \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^m e^{-x^2/2} dx, \quad m = 1, 2, \ldots. \quad (1.5)$$

For a given graph $G = (V, E)$, its distance k-graph $G^{[k]} = (V, E^{[k]})$ is defined by

$$E^{[k]} = \{\{x, y\} : x, y \in V, \partial_G(x, y) = k\}.$$

Though $G_N^{(k)}$ is not necessarily the distance k-graph of $G_N^{(1)}$, it is so in cases of $G = S_d$, where S_d is a star graph with d degrees, and of $G = K_q$, where K_q is a complete graph with q vertices. The graph $G_N^{(1)}$ for $G = K_q$ is called Hamming graph. Especially, $G_N^{(1)}$ for $G = K_2$ is the N-dimensional hypercube. In Obata [4], the asymptotic spectral distributions of distance k-graphs of N-dimensional hypercubes are obtained for arbitrary k, however, his argument is not the quantum probabilistic approach. In section 4, using the method of quantum decomposition, we obtain the asymptotic spectral distribution of distance 2-graph of Hamming graph, as a generalization of [3].

In section 2, we review the method of quantum decomposition. In section 3, we consider $G_N^{(2)}$, $G_N^{(N-1)}$, and $G_N^{(N)}$ for $G = S_d$. In such cases, the associated Hilbert spaces $\Gamma(G_N^{(2)})$, $\Gamma(G_N^{(N-1)})$, $\Gamma(G_N^{(N)})$, are invariant, so the method is applicable. On the contrary, in section 4, we consider $G_N^{(2)}$ for $G = K_q$. In such a case, the associated Hilbert space $\Gamma(G_N^{(2)})$ is not invariant. Thus we deal with it as a growing regular graph, following [2]. As a result, in both cases of $G = S_d$, and of $G = K_q$, we obtain that the asymptotic spectral distributions of $G_N^{(2)}$ are the normalized χ^2-distributions with 1 degree of freedom independent of d and q. Thus we expect that the asymptotic spectral distribution of $G_N^{(2)}$ is identical for arbitrary G. This conjecture will be proven in our next work.

2. Quantum Decomposition

In order to obtain the spectral distribution of A, we use the method of quantum decomposition. We review it, following [1].

For a given graph $G = (V, E)$, we define the n-th stratum by

$$V_n = \{x \in V : \partial_G(x, o) = n\}.$$

We introduce the stratification $V = \bigcup_{n=0}^{\infty} V_n$ into the graph G.

We define $\Phi_n \in l^2(V)$, for $n = 0, 1, 2, \ldots$, by

$$\Phi_n = |V_n|^{-1/2} \sum_{x \in V_n} \delta_x, \quad n = 0, 1, 2, \ldots, \tag{2.1}$$

and $\Gamma(G) \subset l^2(V)$, where $\Gamma(G)$ is a Hilbert space associated with the graph G, by

$$\Gamma(G) = \operatorname{span}\left\{\Phi_n = |V_n|^{-1/2} \sum_{x \in V_n} \delta_x : n = 0, 1, 2, \ldots\right\}. \tag{2.2}$$

According to the stratification, the adjacency matrix A has a *quantum decomposition* $A = A^+ + A^- + A^\circ$, which is defined by

$$\begin{aligned} A \sum_{x \in V_n} \delta_x &= (A^+ + A^- + A^\circ) \sum_{x \in V_n} \delta_x \\ &= \sum_{y \in V_{n+1}} \omega_-(y)\delta_y + \sum_{z \in V_{n-1}} \omega_+(z)\delta_z + \sum_{w \in V_n} \omega_\circ(w)\delta_w, \end{aligned}$$

where $\omega_\epsilon(x) = |\{y \in V_{n+\epsilon} : \partial_G(y, x) = 1\}|$ for $x \in V_n$ and $\epsilon \in \{+, -, \circ\}$.

If $\Gamma(G)$ is invariant under A^+, A^-, and A°, namely $A^\epsilon\Gamma(G) \subset \Gamma(G)$ for $\epsilon \in \{+, -, \circ\}$ (these conditions are equivalent that $\omega_\epsilon(x)$, $\epsilon \in \{+, -, \circ\}$, are constants on each stratum), we have

$$\begin{aligned} A^+\Phi_n &= \sqrt{\omega_{n+1}}\Phi_{n+1}, \\ A^-\Phi_n &= \sqrt{\omega_n}\Phi_{n-1}, \\ A^\circ\Phi_n &= \alpha_{n+1}\Phi_n, \end{aligned}$$

where

$$\omega_n = \frac{|V_n|}{|V_{n-1}|}\omega_-(x)^2, \quad x \in V_n, \tag{2.3}$$

$$\alpha_n = \omega_\circ(x), \quad x \in V_{n-1}. \tag{2.4}$$

This pair of two sequences $(\{\omega_n\}, \{\alpha_n\})$ is called the *Jacobi coefficient*.

The Stieltjes transform $G_\mu(z)$ of spectral distribution of G is expanded into the continued fraction by using the Jacobi coefficient:

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} \tag{2.5}$$

$$= \cfrac{1}{z-\alpha_1-\cfrac{\omega_1}{z-\alpha_2-\cfrac{\omega_2}{z-\alpha_3-\cdots}}}. \tag{2.6}$$

3. Cases of $G = S_d$ for $d \geq 1$

First, we consider $G_N^{(2)}$, $G_N^{(N-1)}$, and $G_N^{(N)}$, for $G = S_d$. The star graph $S_d = (V, E)$ is defined by $V = \{0, 1, \ldots, d\}$ and $E = \{\{0, x\} : x \in \{1, \ldots, d\}\}$. The adjacency matrix A of S_d is defined by $A\delta_0 = \sum_{k=1}^d \delta_k$ and $A\delta_j = \delta_0$ for $1 \leq j \leq d$.

Proposition 3.1. *The asymptotic spectral distribution of $G_N^{(N-1)}$ for $G = S_d$ is the standard Gaussian distribution, namely,*

$$\lim_{N\to\infty} \varphi_N \left(\left(\frac{A_N^{(N-1)}}{\sqrt{Nd^{N-1}}} \right)^m \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^m e^{-x^2/2} dx \tag{3.1}$$

holds for $m = 1, 2, \ldots$.

Proof. Let the stratification of $G_N^{(N-1)} = (V^N, E_N^{(N-1)})$ be $V^N = \bigcup_{n=0}^\infty V_n^{(N,N-1)}$. We shall calculate $\omega_\epsilon(x)$ for $x \in V_n^{(N,N-1)}$ and $\epsilon \in \{+, -, \circ\}$.

The origin $\mathbb{O} = \delta_0 \otimes \cdots \otimes \delta_0$ is adjacent to the elememts which contain one δ_0. Thus we obtain $\kappa(\mathbb{O}) = |V_1^{(N,N-1)}| = Nd^{N-1}$, and $\omega_+(\mathbb{O}) = Nd^{N-1}$, $\omega_-(\mathbb{O}) = 0$, $\omega_\circ(\mathbb{O}) = 0$.

For a large enough number N, an element of $V_1^{(N,N-1)}$ is adjacent to the origin, $\binom{N}{2}d^2$ elements which contain $(N-2)$ δ_0's. Thus we obtain $|V_2^{(N,N-1)}| = \binom{N}{2}d^2$, and $\omega_+(x) = (N-1)d$, $\omega_-(x) = 1$, $\omega_\circ(x) = 0$, for $x \in V_1^{(N,N-1)}$.

Similarly, an element of $V_2^{(N,N-1)}$ is adjacent to $(N-2)d^{N-3}$ elemets which contain three δ_0's, $2d^{N-2}$ elements which contain one δ_0. Thus we obtain $|V_3^{(N,N-1)}| = \binom{N}{3}d^{N-3}$, and $\omega_+(x) = (N-2)d^{N-3}$, $\omega_-(x) = 2d^{N-2}$, $\omega_\circ(x) = 0$, for $x \in V_2^{(N,N-1)}$.

Generally, for $1 \leq n < N/2$, we obtain

$$|V_{2n}^{(N,N-1)}| = \binom{N}{2n} d^{2n}, \tag{3.2}$$

$$\omega_+ = (N-2n)d^{N-2n-1}, \tag{3.3}$$

$$\omega_- = 2nd^{N-2n}, \tag{3.4}$$

$$\omega_\circ = 0, \text{ for } x \in V_{2n}^{(N,N-1)}, \tag{3.5}$$

and

$$\begin{aligned} |V_{2n+1}^{(N,N-1)}| &= \binom{N}{2n+1} d^{N-(2n+1)}, &(3.6)\\ \omega_+ &= (N-2n-1)d^{2n+1}, &(3.7)\\ \omega_- &= (2n+1)d^{2n}, &(3.8)\\ \omega_\circ &= 0,\ \text{for } x \in V_{2n+1}^{(N,N-1)}. &(3.9)\end{aligned}$$

Since $\omega_\epsilon(x)$, $\epsilon \in \{+,-,\circ\}$, are constans for any $x \in V_n^{(N,N-1)}$, $\Gamma(G_N^{(N-1)})$ is invariant. By means of (2.3), (2.4), the Jacobi coefficient is

$$\begin{aligned} \omega_n^{(N)} &= n(N-n+1)d^{N-1},\ \text{for } x \in V_n^{(N,N-1)}, &(3.10)\\ \alpha_n^{(N)} &= 0,\ \text{for } x \in V_{n-1}^{(N,N-1)}. &(3.11)\end{aligned}$$

Normalizing $A_N^{(N-1)}$ by $\sqrt{\kappa(\mathbb{O})} = \sqrt{Nd^{N-1}}$, we can reduce the Jacobi coefficient $(\{\bar{\omega}_n^{(N)}\}, \{\bar{\alpha}_n^{(N)}\})$ of $A_N^{(N-1)}/\sqrt{\kappa(\mathbb{O})}$ to

$$\begin{aligned} \bar{\omega}_n^{(N)} &= \frac{n(N-n+1)}{N}, &(3.12)\\ \bar{\alpha}_n^{(N)} &= 0. &(3.13)\end{aligned}$$

Letting $N \to \infty$, we have, for all n,

$$\begin{aligned} \omega_n &= \lim_{N\to\infty} \bar{\omega}_n^{(N)} = n, &(3.14)\\ \alpha_n &= \lim_{N\to\infty} \bar{\alpha}_n^{(N)} = 0. &(3.15)\end{aligned}$$

We know that the distribution with the Jacobi coefficient (3.14),(3.15) is the standard Gaussian distribution. Therefore, the proof is completed. □

Remark 3.2. In the above proposition, though the asymptotic spectral distribution of $G_N^{(N-1)}$ is the standard Gaussian distribution, $A_N^{(N-1)}$ is not a sum of commutative independent random variables. Because they do not satisfy the singleton condition in spite that the random variables themselves are commutative.

Proposition 3.3. *The asymptotic spectral distribution of $G_N^{(N)}$ for $G = S_d$ is the symmetric Bernoulli distribution, namely,*

$$\lim_{N\to\infty} \varphi\left(\left(\frac{A_N^{(N)}}{\sqrt{d^N}}\right)^m\right) = \frac{1}{2}\int_{-\infty}^{+\infty} x^m(\delta_{+1}(x) + \delta_{-1}(x))dx \qquad (3.16)$$

holds for $m = 1, 2, \ldots$.

Proof. For any N, the connected component containing the origin $\mathbb{O} = \delta_0 \otimes \cdots \otimes \delta_0$ of $G_N^{(N)}$ for $G = S_d$ is isomorphic to a star graph S_{d^N}.

It is easy to show that the Jacobi codfficient of S_{d^N} is $(\{\omega_1^{(N)} = d^N, \omega_2^{(N)} = \omega_3^{(N)} = \cdots = 0\}, \{\alpha_n^{(N)} = 0)\}$.

Normalizing $A_N^{(N)}$ by $\sqrt{\kappa(\mathbb{O})} = \sqrt{d^N}$, we can reduce the Jacobi coefficient $(\{\bar{\omega}_n^{(N)}\}, \{\bar{\alpha}_n^{(N)}\})$ of $A_N^{(N)}/\sqrt{\kappa(\mathbb{O})}$ to $(\{\bar{\omega}_1^{(N)} = 1, \bar{\omega}_2^{(N)} = \bar{\omega}_3^{(N)} = \cdots = 0\}, \{\bar{\alpha}_n^{(N)} = 0)\}$. Letting $N \to \infty$, we have $(\{\omega_1 = 1, \omega_2 = \omega_3 = \cdots = 0\}, \{\alpha_n = 0)\}$.

It is known that the Jacobi coefficient of symmetric Bernoulli distribution is $(\{\omega_1 = 1, \omega_2 = \omega_3 = \cdots = 0\}, \{\alpha_n = 0)\}$. Thus the proof is completed. □

Theorem 3.4. *The asymptotic spectral distribution of $G_N^{(2)}$ for $G = S_d$ is the normalized χ^2-distribution with 1 degree of freedom, namely,*

$$\lim_{N\to\infty} \varphi\left(\left(\frac{A_N^{(2)}}{\sqrt{\binom{N}{2}d^2}}\right)^m\right) = \frac{1}{\sqrt{\pi(\sqrt{2}x+1)}}\int_{-1/\sqrt{2}}^{+\infty} x^m e^{-(\sqrt{2}x+1)/2}dx \qquad (3.17)$$

holds for $m = 1, 2, \ldots$.

Proof. Let the stratification of $G_N^{(2)} = (V^N, E_N^{(2)})$ be $V^N = \bigcup_{n=0}^{\infty} V_n^{(N,2)}$. We shall calculate $\omega_\epsilon(x)$ for $x \in V_n^{(N,2)}$ and $\epsilon \in \{+, -, \circ\}$.

The origin $\mathbb{O} = \delta_0 \otimes \cdots \otimes \delta_0$ is adjacent to the elememts which contain $(N-2)$ δ_0's. Thus we obtain $\kappa(\mathbb{O}) = |V_1^{(N,2)}| = \binom{N}{2}d^2$, and $\omega_+(\mathbb{O}) = \binom{N}{2}d^2$, $\omega_-(\mathbb{O}) = 0$, $\omega_\circ(\mathbb{O}) = 0$.

For a large enough number N, an element of $V_1^{(N,2)}$ is adjacent to the origin, $2(N-2)d$ elements which contain $(N-2)$ δ_0's, and $\binom{N-2}{2}d^2$ elements which contain $(N-4)$ δ_0's. Thus we obtain $|V_2^{(N,2)}| = \binom{N}{4}d^4$, and $\omega_+(x) = \binom{N-2}{2}d^2$, $\omega_-(x) = 1$, $\omega_\circ(x) = 2(N-2)d$, for $x \in V_1^{(N,2)}$.

Similarly, an element of $V_2^{(N,2)}$ is adjacent to $\binom{4}{2}$ elemets which contain $(N-2)$ δ_0's, $4(N-4)d$ elements which contain $(N-4)$ δ_0's, and $\binom{N-4}{2}d^6$ elements which contain $(N-6)$ δ_0's. Thus we obtain $|V_3^{(N,2)}| = \binom{N}{6}d^6$, and $\omega_+(x) = \binom{N-4}{2}d^2$, $\omega_-(x) = \binom{4}{2}$, $\omega_\circ(x) = 4(N-4)d$, for $x \in V_2^{(N,2)}$.

Generally, for $1 \le n < N/4$, we obtain $|V_n^{(N,2)}| = \binom{N}{2n}d^{2n}$, and $\omega_+(x) = \binom{N-2n}{2}d^2$, $\omega_-(x) = \binom{2n}{2}$, $\omega_\circ(x) = 2n(N-2n)d$, for $x \in V_n^{(N,2)}$.

Since $\omega_\epsilon(x)$, $\epsilon \in \{+, -, \circ\}$, are constans for any $x \in V_n^{(N,2)}$, $\Gamma(G_N^{(2)})$ is invariant. By means of (2.3), (2.4), the Jacobi coefficient is

$$\omega_n^{(N)} = \frac{1}{2}n(2n-1)(N-2n+2)(N-2n+1)d^2, \text{ for } x \in V_n^{(N,2)}, \qquad (3.18)$$

$$\alpha_n^{(N)} = 2(n-2)(N-2n+2)d, \text{ for } x \in V_{n-1}^{(N,2)}. \qquad (3.19)$$

Normalizing $A_N^{(2)}$ by $\sqrt{\kappa(\mathbb{O})} = \sqrt{\binom{N}{2}d^2}$, we can reduce the Jacobi coefficient $(\{\bar{\omega}_n^{(N)}\}, \{\bar{\alpha}_n^{(N)}\})$ of $A_N^{(2)}/\sqrt{\kappa(\mathbb{O})}$ to

$$\bar{\omega}_n^{(N)} = \frac{n(2n-1)(N-2n+2)(N-2n+1)}{N(N-1)}, \qquad (3.20)$$

$$\bar{\alpha}_n^{(N)} = \frac{2\sqrt{2}(n-2)(N-2n+2)}{\sqrt{N(N-1)}}. \qquad (3.21)$$

Letting $N \to \infty$, we have, for all n,

$$\omega_n = \lim_{N\to\infty} \bar{\omega}_n^{(N)} = n(2n-1), \qquad (3.22)$$

$$\alpha_n = \lim_{N\to\infty} \bar{\alpha}_n^{(N)} = 2\sqrt{2}(n-1). \qquad (3.23)$$

It is known that the Jacobi coefficient of normalized χ^2-distribution with 1 degree of freedom is $(\{\omega_n = n(2n-1)\}, \{\alpha_n = 2\sqrt{2}(n-1)\})$ (see e.g. [3]). Therefore, the proof is completed. □

4. Case of $G = K_q$ for $q \geq 3$

Next, we consider $G_N^{(2)}$ for $G = K_q$. The complete graph $K_q = (V, E)$ is defined by $V = \{0, 1, 2, \ldots, q-1\}$ and $E = \{\{x, y\} : x, y \in V, x \neq y\}$. The adjacency matrix A of K_q is defined by $A\delta_j = \sum_{k=0,\, k\neq j}^{q-1} \delta_k$ for $0 \leq j \leq q-1$.

On the contrary to the case of S_d, the associated Hilbert space $\Gamma(G_N^{(2)})$ is not invariant. We deal with $G_N^{(2)}$ for $G = K_q$ as a growing regular graph, where N is a growing parameter. Thanks to [2], in case of a growing regular graph $G^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$, where ν is a growing parameter, there is another method to obtain the asymptotic spectral distribution of $G^{(\nu)}$ if $\Gamma(G^{(\nu)})$ is not invariant but asymptotically invariant.

Theorem 4.1. *The asymptotic spectral distribution of $G_N^{(2)}$ for $G = K_q$ is the normalized χ^2-distribution with 1 degree of freedom, namely,*

$$\lim_{N\to\infty} \varphi\left(\left(\frac{A_N^{(2)}}{\sqrt{\binom{N}{2}(q-1)^2}}\right)^m\right) = \frac{1}{\sqrt{\pi(\sqrt{2}x+1)}} \int_{-1/\sqrt{2}}^{+\infty} x^m e^{-(\sqrt{2}x+1)/2} dx \quad (4.1)$$

holds for $m = 1, 2, \ldots$.

Proof. Put $\bar{a}_i = 1$ if $a_i \in \{1, 2, \ldots, q-1\}$ and $\bar{a}_i = 0$ if $a_i = 0$. We have the stratification $V^N = \bigcup_{n=0}^{\infty} V_n^{(N,2)}$ as follows:

$$\begin{aligned}
V_0^{(N,2)} &= \{\delta_0 \otimes \cdots \otimes \delta_0\}, \\
V_1^{(N,2)} &= \{\delta_{a_1} \otimes \cdots \otimes \delta_{a_N} : \sum_{i=1}^{N} \bar{a}_i = 2, a_i \in \{0, 1, 2, \ldots, q-1\}\}, \\
V_2^{(N,2)} &= \{\delta_{a_1} \otimes \cdots \otimes \delta_{a_N} : \sum_{i=1}^{N} \bar{a}_i \in \{1, 3, 4\}, a_i \in \{0, 1, 2, \ldots, q-1\}\},
\end{aligned}$$

and for $n \geq 3$,

$$V_n^{(N,2)} = \{\delta_{a_1} \otimes \cdots \otimes \delta_{a_N} : \sum_{i=1}^{N} \bar{a}_i \in \{2n-1, 2n\}, a_i \in \{0, 1, 2, \ldots, q-1\}\}. \quad (4.2)$$

For $n \geq 3$, using (4.2), for the element $x \in V_n^{(N,2)}$ which contains $(N-2n)$ δ_0's, we have

$$\omega_-(x) = \binom{2n}{2}, \quad (4.3)$$

$$\omega_\circ(x) = \binom{2n}{2}\{(q-1)^2 - 1\} + 2n(N-2n)(q-1), \quad (4.4)$$

and for the element $y \in V_n^{(N,2)}$ which contains $(N-2n+1)$ δ_0's, we have

$$\omega_-(y) = \binom{2n-1}{2}, \tag{4.5}$$

$$\omega_\circ(y) = \binom{2n-1}{2}(q-2)^2 + (2n-1)(N-2n+1)(q-1). \tag{4.6}$$

We can easily check the conditions (A1), (A2), and (A3), in [2, Theorem 6.2]. We obtain that the Jacobi coefficient of asymptotic spectral distrbution of $G_N^{(2)}$ is

$$\begin{aligned}
\omega_n &= \lim_{N\to\infty} M(\omega_-|V_n^{(N,2)}) \\
&= \lim_{N\to\infty} \frac{\binom{2n}{2}\cdot\binom{N}{2n}(q-1)^{2n} + \binom{2n-1}{2}\cdot\binom{N}{2n-1}(q-1)^{2n-1}}{\binom{N}{2n}(q-1)^{2n} + \binom{N}{2n-1}(q-1)^{2n-1}} \\
&= n(2n-1), \\
\alpha_{n+1} &= \lim_{N\to\infty} \frac{M(\omega_\circ|V_n^{(N,2)})}{\sqrt{\binom{N}{2}(q-1)^2}} \\
&= \lim_{N\to\infty} \frac{\omega_\circ(x)\cdot\binom{N}{2n}(q-1)^{2n} + \omega_\circ(y)\cdot\binom{N}{2n-1}(q-1)^{2n-1}}{\left\{\binom{N}{2n}(q-1)^{2n} + \binom{N}{2n-1}(q-1)^{2n-1}\right\}\sqrt{\binom{N}{2}(q-1)^2}} \\
&= 2\sqrt{2}n,
\end{aligned}$$

where

$$M(\omega_\epsilon|V_n^{(N,2)}) = \frac{1}{|V_n^{(N,2)}|}\sum_{x\in V_n^{(N,2)}} \omega_\epsilon(x),$$

for $\epsilon \in \{+,-,\circ\}$.

We shall also note that $(\{\omega_n = n(2n-1)\}, \{\alpha_n = 2\sqrt{2}(n-1)\})$ holds for $n = 1, 2, 3$.

It is known that the Jacobi coefficient of normalized χ^2-distribution with 1 degree of freedom is $(\{\omega_n = n(2n-1)\}, \{\alpha_n = 2\sqrt{2}(n-1)\})$ (see e.g. [3]). Therefore, the proof is completed. □

Remark 4.2. The graph $G_N^{(2)}$ for $G = K_q$ is the distance 2-graph of Hamming graph.

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