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LÉVY PROCESSES THROUGH TIME SHIFT ON OSCILLATOR WEYL ALGEBRA

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Dedicated to Professor K. R. Parthasarathy on the occasion of his 75th birthday

Abstract. We extend the Lie–algebra time shift technique, introduced in [2], from the usual Weyl algebra (associated to the additive group of a Hilbert space $\mathcal{H}$) to the generalized Weyl algebra (oscillator algebra), associated to the semi–direct product of the additive group $\mathcal{H}$ with the unitary group on $\mathcal{H}$ (the *Euclidean group of* $\mathcal{H}$*, in the terminology of* [14]). While in the usual Weyl algebra the possible quantum extensions of the time shift are essentially reduced to isomorphic copies of the Wiener process, in the case of the oscillator algebra a larger class of Lévy process arises.Our main result is the proof of the fact that the generators of the quantum Markov semigroups, associated to these time shifts, share with the quantum extensions of the Laplacian the important property that the generalized Weyl operators are eigenoperators for them and the corresponding eigenvalues are explicitly computed in terms of the Lévy–Khintchin factor of the underlying classical Lévy process.

1. Introduction

The classical heat semigroup on $\mathbb{R}$ is the Markov semigroup canonically associated to the classical, real valued Brownian motion (W_t) via the basic formula of Markov processes:

$$E_{0]}(v_t^\circ(f(W_0))) = E_{0]}(f(W_t)) = P^t f = e^{t\Delta} f \qquad ; \quad t \geq 0,$$

where $E_{0]}$ denotes the W-conditional expectation onto the past σ-algebra of time 0, v_t° is the usual time shift in the Wiener space and f is any Borel measurable function. In the quantum formulation of the classical, real valued Brownian motion there is also a time shift u_t° and $E_{0]}$ denotes the (restriction of the) vacuum conditional expectation. However u_t° acts trivially on the initial algebra and therefore the generator of the corresponding semigroup is zero, in particular it cannot coincide with the time shift of the classical Brownian motion. P.A. Meyer noticed this discrepancy and, in the Oberwolfach 1987 quantum probability workshop, posed the question if there exists a quantum extension of the classical time shift in Wiener space. For any such extension the generator of the associated quantum

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Markov semigroup would provide a quantum extension of the classical laplacian. A solution of Meyer's problem was given by Accardi in the same workshop (see [2]) and it was based on the idea that the Fock space time shift u_t° is the time shift of the increment process associated to the classical Brownian motion, i.e., the (integrated) white noise, thus its action on the algebra of measurable functionals of the process is characterized by the property of being the unique (continuous) endomorphism satisfying:

$$u_t^\circ\left(W_s - W_r\right) := W_{s+t} - W_{r+t}$$

On the other hand the usual time shift v_t° in Wiener space is the unique (continuous) endomorphism of the associated algebra of measurable functions given by the property:

$$v_t^\circ(W_s) := W_{s+t}$$

Therefore, denoting $\hat{j}_t$ the restriction of the Wiener time shift on the time zero algebra, v_t° is uniquely determined by the pair $(\hat{j}_t, u_t^\circ)$ through the identity

$$v_t^\circ\left(W_s\right) := W_{s+t} = W_t + \left(W_{s+t} - W_t\right) = \hat{j}_t\left(W_0\right) + u_t^\circ\left(W_s - W_0\right).$$

But it is known that, in the quantum formulation of the classical Wiener process, the initial random variable is identified to position operator q_0 and the increment (noise) is identified to the momentum process

$$W_t - W_0 = P_{(0,t]}.$$

This leads to this identification

$$W_s = W_0 + W_s - W_0 = q_0 \otimes 1 + 1_0 \otimes P_{(0,s]}$$

So that

$$v_t^\circ(W_s) = v_t^\circ(q_0 \otimes 1 + 1_0 \otimes P_{(0,s]}) = \hat{j}_t(q_0 \otimes 1) + 1_0 \otimes u_t^\circ(P_{(0,s]}) = \hat{j}_t(q_0) + 1_0 \otimes P_{(t,t+s]}$$

On the other hand

$$\begin{aligned} v_t^\circ(W_s) &= W_{s+t} = W_0 + (W_{s+t} - W_0) = q_0 \otimes 1 + 1_0 \otimes P_{(0,s+t]} \\ &= q_0 \otimes 1 + 1_0 \otimes P_{(0,t]} + 1_0 \otimes P_{(t,s+t]} \end{aligned}$$

From this it follows that

$$\hat{j}_t(q_0) = q_0 \otimes 1 + 1_0 \otimes P_{(0,t]}$$

is a possible answer to our problem. In [2] it was shown that this is indeed the case and that the generator of the associated quantum Markov semigroup is indeed a quantum generalization of the classical laplacian, in the sense that its restriction to the operators of multiplication by smooth functions coincides with the usual laplacian.

The systematic investigation of all possible solutions to this problem when the initial algebra is a general (not necessarily finite dimensional) Weyl algebra, begun in paper [3] (see also [4]), led in particular to the unexpected identification of the Lévy laplacian with a usual Volterra–Gross laplacian, corresponding to a usual Brownian motion with values in a special Hilbert space (the Cesaro Hilbert space), and opened the way to a multiplicity of new developments on the structure of the heat semigroups associated to the whole hierarchy of exotic laplacians, to which the previously developed analytical techniques did not apply (see for example [11], [6], [7]).

In the present paper, we extend the results of [2], [3], [4] from the usual Weyl algebra over an Hilbert space $\mathcal{H}$, which gives a projective representation of the additive group $\mathcal{H}$ acting on itself by translations, to the oscillator algebra, which gives a projective representation of the semi–direct product of the additive group $\mathcal{H}$ with the unitary group on $\mathcal{H}$, denoted $\mathcal{U}(\mathcal{H})$. We combine this extension of the quantum time shifts techniques with the techniques developed by Araki, Woods, Parthasarathy and Schmidt (see [15] for a systematic exposition and bibliography), to construct quantum Markov processes of *random walk type*, i.e., obtained by adding, to an initial operator process, the increments of a quantum independent increment operator process. This simple additive picture at Lie algebra level produces, after exponentiation, a projective representation of Lie groups. While in the usual Weyl algebra the possible quantum extensions of the classical time shift are essentially reduced to isomorphic copies of the Wiener process, in the case of the oscillator algebra a larger class of Lévy process arises as possible candidates for quantum extensions of the time shift. We determine the explicit form of these time shifts and construct the associated Markov cocycles hence, via the quantum Feynman–Kac technique ([1]), the associated quantum Markov semigroup. From this we deduce the main result of the present paper which can be described as follows. The quantum Markov semigroup associated to the usual quantum Brownian motion is the quantum heat semigroup and usual Weyl operators are eigenoperators of its generator (the quantum laplacian). Analogously the generalized Weyl operators, associated to the oscillator algebra, are eigenoperators of the generator of the quantum Markov semigroups canonically associated to the Lie algebra time shifts of the oscillator algebra (which are given by classical Lévy process). Furthermore the corresponding eigenvalues are explicitly computed in terms of the Lévy–Khintchin factor of the underlying classical Lévy process. The standard GKSL form of these generators can be easily computed using stochastic calculus. Generally these generators are unbounded even in the case of finite dimensional Brownian motion. The fact that they have a total, self-adjoint, set of linearly independent eigenoperators singles out an interesting new class of quantum Markov semigroups (up to now the only non trivial known example in this class was the quantum laplacian) and can be probably exploited to achieve a deeper understanding of the analytical structure of this class.

2. Notations and Preliminaries

2.1. Boson Fock space. In the following all Hilbert spaces are assumed to be complex and separable with inner product linear in the second variable denoted, $\langle\cdot,\cdot\rangle$. For any Hilbert space $\mathcal{H}$, we denote:

- $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$
- $\Gamma(\mathcal{H})$ the symmetric (boson) Fock space over $\mathcal{H}$
- ψ_u ($u \in \mathcal{H}$) the exponential vector associated with u:

$$\psi_u := \sum_{n\geq 0} \frac{u^{\otimes n}}{\sqrt{n!}} \in \Gamma(\mathcal{H}) \quad ; \quad \psi_0 = \Phi \text{ vacuum vector.}$$

For any dense linear subspace S of $\mathcal{H}$ the family $\{\psi_u, u \in S\}$ is total and linearly independent in $\Gamma(\mathcal{H})$. We denote by $\mathcal{E}(S)$ the vector space algebraically generated by it. If S is as above a linear operator may be defined densely on $\Gamma(\mathcal{H})$ by giving arbitrarily its action on the family $\{\psi_u, u \in S\}$. We simply use the notation $\mathcal{E}$ when $S = \mathcal{H}$. The annihilation, creation and Weyl operators are defined respectively by:

$$A^-(v)\psi_u := \langle v, u\rangle \psi_u \,; \qquad A^+(v)\psi_u := \frac{d}{ds}\,|_{s=0}\; \psi_{u+sv}\,,$$

$$W(v)\psi_u = e^{-\frac{1}{2}\|v\|^2 - \langle v,u\rangle}\psi_{u+v}\,; \qquad \forall v \in \mathcal{H}.$$

The boson creation and annihilation operators satisfy the *canonical commutations relations* (CCR):

$$\begin{aligned} [A^-(u), A^+(v)] &= \; < u, v > 1_0 && (2.1)\\ [A^-(u), A^-(v)] &= \; [A^+(u), A^+(v)] = 0\,, && (2.2)\end{aligned}$$

for any $u, v \in \mathcal{H}$, where $[x, y] := xy - yx$ is the commutator and 1_0 is the identity operator on $\Gamma(\mathcal{H})$.

The second quantized $\Gamma(T)$ of a self-adjoint bounded operator T on $\mathcal{H}$ is given by the relation:

$$\Gamma(T)\psi_u := \psi_{Tu}.$$

The differential second quantization operator $\Lambda(T)$ (or the number operator) of T is defined via the Stone theorem by:

$$\Gamma(e^{itT}) =: e^{it\Lambda(T)}\,, \qquad t \in \mathbb{R}.$$

Its action on $\mathcal{E}(S)$ is given by:

$$\Lambda(T)\psi_u = \frac{1}{i}\frac{d}{ds}|_{s=0}\psi_{e^{isT}u}.$$

If T is a bounded but not necessarily self-adjoint operator on $\mathcal{H}$, then by writing T as "sum" of two self-adjoint operators

$$T = T_1 + iT_2 = \frac{T + T^*}{2} + i\frac{T - T^*}{2i},$$

then we can define $\Lambda(T)$ by:

$$\Lambda(T) := \Lambda(T_1) + i\Lambda(T_2).$$

Hence $\Lambda(T)$ is linear in T and the following canonical commutations relations hold weakly on the set of the exponential vectors

$$[\Lambda(T), A^+(u)] = A^+(Tu), \;\; [\Lambda(T), A(u)] = -A(T^*u) \tag{2.3}$$

and

$$[\Lambda(T_1), \Lambda(T_2)] = \Lambda([T_1, T_2]). \tag{2.4}$$

2.2. Markov flows on white noise spaces. In the notations of section (2.1), if the space $\mathcal{H}$ has the form

$$L^2(\mathbb{R}_+, \mathcal{H}_0) \equiv L^2(\mathbb{R}_+) \otimes \mathcal{H}_0$$

where $\mathcal{H}_0$ is an Hilbert space, the associated Fock space is called a *white noise space* and the space $\mathcal{H}_0$ a *multiplicity (or polarization) space*. Usually an *initial (or system)* space is added to the white noise space and in the following we will fix the choice

$$\mathcal{H}_w := \Gamma(\mathcal{H}_0) \otimes \Gamma\big(L^2(\mathbb{R}_+, \mathcal{H}_0)\big).$$

Notice that we choose the initial space to be the Fock space over the multiplicity space. This is a common feature in the theory of quantum time shifts (see [3]) whose motivation will be clear from the following development. This space has two natural Hilbert space filtrations (past and future) defined (asymmetrically) by

$$\mathcal{H}_{t]} := \Gamma(\mathcal{H}_0) \otimes \Gamma\big(L^2([0,t], \mathcal{H}_0)\big) \quad ; \quad \mathcal{H}_{[t} = \Gamma\big(L^2([t, +\infty[, \mathcal{H}_0)\big) \quad ; \quad t \geq 0$$

where, here and in the following, $\mathcal{H}_{t]}$ (resp. $\mathcal{H}_{[t}$) will be identified to the subspace $\Phi_0 \otimes \mathcal{H}_{t]} \otimes \Phi_{[t}$ (resp. $\Phi_{t]} \otimes \mathcal{H}_{[t}$), $\Phi_{[t}$ (resp. $\Phi_0, \Phi_{t]}$) being the vacuum vector in $\mathcal{H}_{[t}$ (resp. $\mathcal{H}_0, \mathcal{H}_{t]}$). With these notations the following factorization property holds

$$\mathcal{H}_w = \Gamma(\mathcal{H}_0) \otimes \Gamma\big(L^2(\mathbb{R}_+, \mathcal{H}_0)\big) = \mathcal{H}_{t]} \otimes \mathcal{H}_{[t}.$$

Similarly we define the Von Neumann algebra pure noise filtration:

$$\mathcal{B}_{t]} = \mathcal{B}\big(\Gamma\big(L^2([0,t], \mathcal{H}_0)\big)\big) \equiv \mathcal{B}\big(\Gamma\big(L^2([0,t], \mathcal{H}_0)\big)\big) \otimes 1_{[t},$$

$$\mathcal{B}_{[t} = \mathcal{B}\big(\Gamma\big(L^2([t, +\infty[, \mathcal{H}_0)\big)\big) \equiv 1_{t]} \otimes \mathcal{B}(\mathcal{H}_{[t})$$

and

$$\mathcal{B} = \mathcal{B}\big(\Gamma(L^2(\mathbb{R}_+), \mathcal{H}_0)\big) \equiv \mathcal{B}_{t]} \otimes \mathcal{B}_{[t},$$

where $1_{t]}$ and $1_{[t}$ are respectively the identities on the spaces $\Gamma\big(L^2([0,t], \mathcal{H}_0)\big)$ and $\Gamma\big(L^2([t, +\infty[, \mathcal{H}_0)\big)$. If $\mathcal{A}_0$ is a C^*–subalgebra of $\mathcal{B}(\Gamma(\mathcal{H}_0))$, we define the filtrations:

$$\mathcal{A}_{t]} := \mathcal{A}_0 \otimes \mathcal{B}_{t]},\ \mathcal{A}_{[t} := \mathcal{B}_{[t},\ \mathcal{A} = \mathcal{A}_{t]} \otimes \mathcal{A}_{[t} = \mathcal{A}_0 \otimes \mathcal{B}$$

and denote by 1_0 (resp. 1) the identity on $\Gamma(\mathcal{H}_0)$ (resp. $\Gamma\big(L^2(\mathbb{R}_+, \mathcal{H}_0)\big)$). The noise creation, annihilation and conservation increment processes, acting on the space $\Gamma\big(L^2(\mathbb{R}_+, \mathcal{H}_0)\big)$, will be denoted respectively

$$A^+_{s,t}(\xi) := A^+(\chi_{[s,t]} \otimes \xi), \quad A^-_{s,t}(\xi) := A^-(\chi_{[s,t]} \otimes \xi), \quad \Lambda_{s,t}(\xi) := \Lambda(M_{\chi_{[s,t]}} \otimes T). \tag{2.5}$$

If $s = 0$, we simply write

$$A^+_t(\xi) := A^+_{0,t}(\xi), \quad A^-_t(\xi) := A^-_{0,t}(\xi), \quad \Lambda_t(T) := \Lambda_{0,t}(T) \tag{2.6}$$

and, when no ambiguity is possible, the same notation will be used for their natural action on $\Gamma(\mathcal{H}_0) \otimes \Gamma\big(L^2(\mathbb{R}_+, \mathcal{H}_0)\big)$. Denote by θ_t, the right shift on $L^2(\mathbb{R}_+, \mathcal{H}_0)$, so that $\forall t \geq 0$

$$\theta_t f(s) = \begin{cases} f(s-t), & \text{if } s \geq t \geq 0; \\ 0, & \text{if } 0 \leq s \leq t. \end{cases} \tag{2.7}$$

The operator θ_t is isometric with $\theta_t^* f(s) = f(s+t)$. The white noise time shift is the 1–parameter endomorphism semigroup of $\mathcal{B}\big(\Gamma(\mathcal{H}_0)\big) \otimes \mathcal{B}\Big(\Gamma\big(L^2(\mathbb{R}_+, \mathcal{H}_0)\big)\Big)$ characterized by the property that, for all $b_0 \in \mathcal{B}(\Gamma(\mathcal{H}_0))$, $b \in \mathcal{B}\Big(\Gamma\big(L^2(\mathbb{R}_+, \mathcal{H}_0)\big)\Big)$ and $t \geq 0$ one has:

$$u_t^\circ(b_0 \otimes b) = b_0 \otimes \Gamma(\theta_t) b \Gamma(\theta_t^*). \tag{2.8}$$

The shift u_t° is a normal, injective $*$–endomorphism and satisfies the following:

(1) For $s, t \geq 0$, $u_s^\circ u_t^\circ = u_{s+t}^\circ$.
(2) For $s \geq 0$, $u_s^\circ(\mathcal{A}) = \mathcal{A}_0 \otimes 1_{s]} \otimes \mathcal{B}_{[s}$.

The vacuum conditional expectations, defined on the algebra $\mathcal{A}$ and with values in the past filtration algebra are defined by:

$$E_{t]}(b_{t]} \otimes b_{[t}) := \langle \Phi_{[0}, b_{[t} \Phi_{[0} \rangle b_{t]}, \quad b_{t]} \in \mathcal{A}_{t]}, \quad b_{[t} \in \mathcal{A}_{[t}. \tag{2.9}$$

Definition 2.1. A stochastic process on $\mathcal{A}_0$ is a family $\{j_t : \mathcal{A}_0 \longrightarrow \mathcal{A}_{t]}, t \geq 0\}$ of $*$–homomorphisms with $j_0(x) = x \otimes 1$. A stochastic process is called normal if for each $t \geq 0$, j_t is σ-weakly continuous.

Let $(j_t)_{t \geq 0}$ be a normal stochastic process on $\mathcal{A}_0$. We define $\tilde{j}_t$ as the unique normal $*$–homomorphism $\tilde{j}_t : \mathcal{A}_0 \otimes 1_{[0,t]} \otimes \mathcal{B}_{[t} \longrightarrow \mathcal{A}$ characterized by

$$\tilde{j}_t(x \otimes 1) = j_t(x), \quad \tilde{j}_t(1 \otimes b_{[t}) = j_t(1) \otimes b_{[t}. \tag{2.10}$$

Each $\tilde{j}_t$ can be extended in an obvious way to the algebraic linear span of the elements of the form $x \otimes 1_{[0,t]} \otimes Y_{[t}$, where $x \in \mathcal{A}_0$, $Y_{[t}$ is an operator on $\mathcal{H}_{[t}$.

Definition 2.2. A stochastic process on $\mathcal{A}_0$ is said to be a Markov cocycle if, in the notations of (2.10), it satisfies the cocycle equation: for all $s, t \geq 0$ and $x \in \mathcal{A}_0$

$$j_0(x) = x \otimes 1 \ ; \ j_{s+t}(x) = \tilde{j}_s \circ u_s^\circ \circ j_t(x). \tag{2.11}$$

It is said to be σ-weakly continuous if the map $(t, x) \longmapsto j_t(x)$ is continuous w.r.t the σ-weak topology of the Von Neumann algebras involved.

According to the Feynman-Kac formula to every Markov cocycle (j_t) on $\mathcal{A}_0$, one can associate Markov semigroup (P^t) on $\mathcal{A}_0$, characterized by the identity:

$$P^t(x) = E_{0]} j_t(x) \tag{2.12}$$

for all $t \geq 0$, $x \in \mathcal{A}_0$. Moreover the cocycle identity (2.11) and condition (2.12) imply that for each $s, t \geq 0$

$$E_{s]} \circ j_{s+t} = j_s \circ P^t. \tag{2.13}$$

In the following we will take the initial space $\Gamma(\mathcal{H}_0)$ to be $\Gamma(\mathcal{H})$.

3. The Generalized Weyl Operator

In this section we introduce the *generalized Weyl operator* associated to the Fock representation of the oscillator Weyl algebra. We use the notations of section (2.1).

Definition 3.1. For all unitary operator U, on $\mathcal{H}$ and $u, v \in \mathcal{H}, z \in \mathbb{C}$, define the *exponential operator* on the set of the exponential vectors by:

$$\Gamma(u, U, v, z) := e^{A^+(u)} \Gamma(U) e^{A^-(v)} e^z. \tag{3.1}$$

In the following we need some known results which we sum up in Lemmas (3.2) and (3.3).

Lemma 3.2. *For all unitary operator U on $\mathcal{H}$ and $u, v \in \mathcal{H}$, one has*

$$\Gamma(u,U,v,z)\psi_x = e^{z+\langle v,x\rangle}\psi_{u+Ux} \quad ; \ x \in \mathcal{H}. \tag{3.2}$$

Proof.

$$\begin{aligned}\Gamma(u,U,v,z)\psi_x &= e^z e^{A^+(u)}\Gamma(U)e^{A^-(v)}\psi_x = e^z e^{A^+(u)}\Gamma(U)\big(e^{\langle v,x\rangle}\psi_x\big)\\ &= e^{z+\langle v,x\rangle}e^{A^+(u)}\big(\psi_{Ux}\big) = e^{z+\langle v,x\rangle}\psi_{u+Ux}.\end{aligned}$$

□

Lemma 3.3. *For all unitary operators U_j, on $\mathcal{H}$ and $u_j, v_j \in \mathcal{H}, z_j \in \mathbb{C}$; $j = 1, 2$, we have the following relation:*

$$\Gamma(u_1,U_1,v_1,z_1)\Gamma(u_2,U_2,v_2,z_2) = \Gamma(u,U,v,z), \tag{3.3}$$

where

$$u = u_1 + U_1u_2 \quad ; \ U = U_1U_2 \quad ; \ v = v_2 + U_2^*v_1 \quad ; \ z = z_1 + z_2 + \langle v_1, u_2\rangle. \tag{3.4}$$

Proof. Let $x \in \mathcal{H}$. Using Lemma (3.2) one finds

$$\begin{aligned}\Gamma_1\Gamma_2\psi_x &:= \Gamma(u_1,U_1,v_1,z_1)\Gamma(u_2,U_2,v_2,z_2)\psi_x\\ &= \Gamma(u_1,U_1,v_1,z_1)\big(e^{z_2+\langle v_2,x\rangle}\psi_{u_2+U_2x}\big)\\ &= e^{z_2+\langle v_2,x\rangle}\Gamma(u_1,U_1,v_1,z_1)\psi_{u_2+U_2x}\\ &= e^{z_2+\langle v_2,x\rangle}e^{z_1+\langle v_1,u_2+U_2x\rangle}\psi_{u_1+U_1(u_2+U_2x)}\\ &= e^{z_1+z_2+\langle v_1,u_2\rangle+\langle v_2+U_2^*v_1,x\rangle}\psi_{u_1+U_1u_2+U_1U_2x}\\ &= \Gamma(u_1+U_1u_2, U_1U_2, v_2+U_2^*v_1, z_1+z_2+\langle v_1,u_2\rangle)\psi_x.\end{aligned}$$

□

Lemma 3.4. *Let $f(z) = \sum_{n\geq 0} a_nz^n$, be an entire function. Denoting*

$$\bar{f}(z) := \sum_{n\geq 0}\overline{a}_nz^n\,, \quad |\, f\,|\,(z) := \sum_{n\geq 0}|\,a_n\,|\,z^n$$

-(i) Let T be a bounded operator on $\mathcal{H}$, then the operator

$$f(T) := \sum_{n\geq 0} a_nT^n$$

is well defined and bounded on $\mathcal{H}$ with norm less than $|\, f\,|\,(\|T\|)$. Moreover, the adjoint of $f(T)$ is $\bar{f}(T^*)$.
-(ii) If f and g are two analytic functions then the operator $(fg)(T) := f(T)g(T)$ is well defined and it is bounded on $\mathcal{H}$.
-(iii) If e_1 and e_2 are the analytic functions defined respectively by:

$$e_1(z) := \sum_{n=1}^{+\infty}\frac{z^{n-1}}{n!} = \frac{e^z-1}{z}, \tag{3.5}$$

$$e_2(z) := \sum_{n=2}^{+\infty} \frac{z^{n-2}}{n!} = \frac{e^z - z - 1}{z^2}, \tag{3.6}$$

then the operators $e_1(T)$ and $e_2(T)$ enjoy the following properties:

$$(e_1(T))^* = e_1(T^*), \tag{3.7}$$

$$(e_2(T))^* = e_2(T^*), \tag{3.8}$$

$$e^{-T} e_1(T) = e_1(-T), \tag{3.9}$$

$$e_1(-T)e_1(T) = e_2(T) + e_2(-T), \tag{3.10}$$

$$\|e_1(T)\| \leq e_1(\|T\|) \qquad ; \qquad \|e_2(T)\| \leq e_2(\|T\|). \tag{3.11}$$

Proof. The statements (i) and (ii) are clear. (3.7), (3.8) and (3.11) follow from (i).(3.9) and (3.10) follow from the identities

$$e^{-x} e_1(x) = e_1(-x) \quad ; \quad e_1(-x)e_1(x) = e_2(-x) + e_2(x), \ x \in \mathbb{C}.$$

□

In the following, we denote by $\mathcal{B}_s(\mathcal{H})$, the set of all bounded self-adjoint operators on $\mathcal{H}$.

Definition 3.5. For $\xi \in \mathcal{H}$ and $T \in \mathcal{B}_s(\mathcal{H})$, denote

$$u_{\xi,T} := ie_1(iT)\xi \quad ; \quad v_{\xi,T} = -ie_1(-iT)\xi \quad ; \quad z_{\xi,T} := -\langle \xi, e_2(iT)\xi\rangle. \tag{3.12}$$

The operator $W(\xi, T)$ defined by

$$W(\xi, T) := \Gamma\big(u_{\xi,T}, e^{iT}, v_{\xi,T}, z_{\xi,T}\big) \tag{3.13}$$

is called *generalized Weyl operator* over $\mathcal{H}$.

Proposition 3.6. *For all pair $(\xi, T) \in \mathcal{H} \times \mathcal{B}_s(\mathcal{H}), W(\xi, T)$ is a unitary operator on $\Gamma(\mathcal{H})$ whose action on the exponential vectors is given, in the notation (3.12), by*

$$W(\xi, T)\psi_x = e^{z_{\xi,T} + \langle v_{\xi,T}, x\rangle} \psi_{u_{\xi,T} + e^{iT}x}. \tag{3.14}$$

Proof. (3.14) follows from (3.2). Since the domain $\mathcal{E}$ is dense in $\Gamma(\mathcal{H})$, unitarity is equivalent to

$$\langle W(\xi, T)\varphi, W(\xi, T)\psi\rangle = \langle \varphi, \psi\rangle \quad \forall\, \varphi, \psi \in \mathcal{E}. \tag{3.15}$$

Let $x, y \in \mathcal{H}$, then (3.14) gives

$$\begin{aligned}
\langle W(\xi, T)\psi_x, W(\xi, T)\psi_y\rangle &= e^{\overline{z_{\xi,T} + \langle v_{\xi,T}, x\rangle} + z_{\xi,T} + \langle v_{\xi,T}, y\rangle} \\
&\times \ \langle \psi_{u_{\xi,T} + e^{iT}x}, \psi_{u_{\xi,T} + e^{iT}y}\rangle \\
&= e^{z_{\xi,T} + \overline{z_{\xi,T}} + \langle x, v_{\xi,T}\rangle + \langle v_{\xi,T}, y\rangle + \langle u_{\xi,T} + e^{iT}x, u_{\xi,T} + e^{iT}y\rangle} \\
&= e^{h_{\xi,T}(x,y)},
\end{aligned} \tag{3.16}$$

where

$$h_{\xi,T}(x, y) = z_{\xi,T} + \overline{z_{\xi,T}} + \langle x, v_{\xi,T}\rangle + \langle v_{\xi,T}, y\rangle + \langle u_{\xi,T} + e^{iT}x, u_{\xi,T} + e^{iT}y\rangle.$$

Using the fact that $v_{\xi,T} = -e^{-iT}u_{\xi,T}$ and the properties (3.7), (3.8) and (3.10), one has

$$\begin{aligned} h_{\xi,T}(x,y) &= -\langle \xi, e_2(iT)\xi\rangle - \langle \xi, e_2(-iT)\xi\rangle - \langle e^{iT}x, u_{\xi,T}\rangle - \langle u_{\xi,T}, e^{iT}y\rangle \\ &+ \langle u_{\xi,T}, e^{iT}y\rangle + \langle e^{iT}x, u_{\xi,T}\rangle + \langle u_{\xi,T}, u_{\xi,T}\rangle + \langle e^{iT}x, e^{iT}y\rangle \\ &= -\langle \xi, (e_2(iT) + e_2(-iT))\xi\rangle + \langle \xi, e_1(-iT)e_1(iT)\xi\rangle + \langle x, y\rangle \\ &= \langle x, y\rangle. \end{aligned}$$

From the expression of $h_{\xi,T}$ and from equation (3.16), one obtains

$$\langle W(\xi,T)\psi_x, W(\xi,T)\psi_y\rangle = e^{\langle x,y\rangle} = \langle \psi_x, \psi_y\rangle$$

which can be extended by linearity to whole of $\mathcal{E}$. □

Remark 3.7. We use Lie algebra notations. The standard notation for the generalized Weyl operator, here denoted $W(\xi,T)$, is $W(\xi, e^{iT})$ (see [14]). Moreover, as shown in Theorem (3.8) below, our definition of generalized Weyl operator differs by a phase from the standard one. Our choice has the notational advantage that it is better suited for the transition between the Lie algebra and the Lie group language. For example, while in the standard notation the inverse (adjoint) of $W(u,U)$ is given by

$$\Big(W(u,U)\Big)^* = W(-U^*u, U^*),$$

in our notation it becomes

$$\Big(W(u,U)\Big)^* = W(-u, U^*)$$

or, in Lie algebra notations, identity (3.18) below.

In the following, we will use the notations $\Re(z)$ and $\Im(z)$, respectively, for real and imaginary parts of a such complex number z.

Theorem 3.8. *In the notations of Definition (3.5), and denoting W_E the standard generalized Weyl operator, one has*

$$W(\xi,T) = e^{i\Im(z_{\xi,T})}W_E(u_{\xi,T}, e^{iT}). \tag{3.17}$$

Moreover,

$$\Big(W(\xi,T)\Big)^* = W(-\xi,-T). \tag{3.18}$$

Proof. We have

$$W_E(u_{\xi,T}, e^{iT}) = \Gamma\Big(u_{\xi,T}, e^{iT}, -(e^{iT})^*u_{\xi,T}, -\frac{1}{2} \parallel u_{\xi,T} \parallel^2 \Big).$$

But

$$-(e^{iT})^*u_{\xi,T} = -e^{-iT}u_{\xi,T} = v_{\xi,T}$$

and

$$\begin{aligned}
-\frac{1}{2}\parallel u_{\xi,T}\parallel^2 &= -\frac{1}{2}\langle ie_1(iT)\xi, ie_1(iT)\xi\rangle \\
&= -\frac{1}{2}\langle \xi, e_1(-iT)e_1(iT)\xi\rangle \\
&= -\frac{1}{2}\langle \xi, \big(e_2(iT)+e_2(-iT)\big)\xi\rangle \\
&= -\frac{1}{2}\Big(\langle \xi, \big(e_2(iT)\xi\rangle + \langle \xi, e_2(-iT)\xi\rangle\Big) \\
&= \frac{1}{2}\Big(z_{\xi,T}+\overline{z_{\xi,T}}\Big) \\
&= \Re(z_{\xi,T}). \qquad (3.19)
\end{aligned}$$

Therefore

$$\begin{aligned}
W_E(u_{\xi,T}, e^{iT}) &= \Gamma\Big(u_{\xi,T}, e^{iT}, v_{\xi,T}, \Re(z_{\xi,T})\Big) \\
&= \Gamma\Big(u_{\xi,T}, e^{iT}, v_{\xi,T}, z_{\xi,T}\Big)e^{-i\Im(z_{\xi,T})} \\
&= e^{-i\Im(z_{\xi,T})}W(\xi,T) \qquad (3.20)
\end{aligned}$$

which proves (3.17). To prove Eq.(3.18) notice that, from Eq. (3.17) and using the fact that

$$z_{-\xi,-T}=\overline{z_{\xi,T}},\ v_{\xi,T}=u_{-\xi,-T},$$

one deduces that

$$\begin{aligned}
\Big(W(\xi,T)\Big)^* &= e^{-i\Im(z_{\xi,T})}\Big(W_E(u_{\xi,T}, e^{iT})\Big)^* \\
&= e^{i\Im(\overline{z_{\xi,T}})}W_E\Big(-(e^{iT})^*u_{\xi,T}, (e^{iT})^*\Big) \\
&= e^{i\Im(\overline{z_{\xi,T}})}W_E\Big(-e^{-iT}u_{\xi,T}, e^{-iT}\Big) \\
&= e^{i\Im(\overline{z_{\xi,T}})}W_E\Big(v_{\xi,T}, e^{-iT}\Big) \\
&= e^{i\Im(z_{-\xi,-T})}W_E\Big(u_{-\xi,-T}, e^{-iT}\Big) \\
&= W(-\xi,-T).
\end{aligned}$$

□

Remark 3.9. A more direct proof of (3.17) can be obtained, noting that the action of the standard Weyl operator is given by

$$W_E(u_{\xi,T}, e^{iT})\psi_x = e^{-\frac{1}{2}\|u_{\xi,T}\|^2-\langle u_{\xi,T}, e^{iT}x\rangle}\psi_{u_{\xi,T}+e^{iT}x}$$

and that

$$\langle u_{\xi,T}, e^{iT}x\rangle = \langle e^{-iT}u_{\xi,T}, x\rangle = -\langle v_{\xi,T}, x\rangle.$$

Therefore, from Eq. (3.19) and (3.14), one obtains

$$W_E(u_{\xi,T}, e^{iT})\psi_x = e^{\Re(z_{\xi,T})+\langle v_{\xi,T},x\rangle}\psi_{u_{\xi,T}+e^{iT}x} = e^{-i\Im(z_{\xi,T})}W(\xi,T)\psi_x.$$

Theorem 3.10. *The operator valued function* $t \longmapsto W(t\xi, tT)$ *is a strongly continuous one-parameter unitary group with generator* $G(\xi, T)$ *which is the closure of*

$$H(\xi, T) := A^+(\xi) + A^-(\xi) + \Lambda(T). \tag{3.21}$$

To prove the above theorem, we need the following lemmata.

Lemma 3.11. *In the notation (3.12), let*

$$u_t := u_{t\xi,tT} \quad ; \; v_t := v_{t\xi,tT} = -e^{-itT}u_t \quad ; \; z_t := z_{t\xi,tT} \quad , \; t \in \mathbb{R}, \tag{3.22}$$

then the following relations hold

$$u_s + e^{isT}u_t = u_{s+t}, \tag{3.23}$$

$$v_t + e^{-itT}v_s = v_{s+t}, \tag{3.24}$$

$$z_s + z_t + \langle v_s, u_t\rangle = z_{s+t}. \tag{3.25}$$

Proof. Define

$$f_s(z) := se_1(isz).$$

Then f_s is analytic in z and

$$f_s(z) + e^{isz}f_t(z) = s\frac{e^{isz}-1}{isz} + te^{isz}\frac{e^{itz}-1}{itz} = (s+t)e_1(i(s+t)z) = f_{s+t}(z).$$

While $u_t = if_t(T)\xi$ for all $t \in \mathbb{R}$, then

$$u_s + e^{isT}u_t = i\big(f_s(T) + e^{isT}f_t(T)\big)\xi = if_{s+t}(T)\xi = u_{s+t} \tag{3.26}$$

which proves (3.23). From the definition of v_t in (3.22) and from (3.23), one has

$$\begin{aligned} v_t + e^{-itT}v_s &= -\big(e^{-itT}u_t + e^{-itT}e^{-isT}u_s\big) = -e^{-i(s+t)T}\big(e^{isT}u_t + u_s\big) \\ &= -e^{-i(s+t)T}u_{t+s} = v_{s+t} \end{aligned}$$

then (3.24) is proved. To prove (3.25), denote

$$g_{s,t}(z) = s^2e_2(isz) + t^2e_2(itz) + ste_1(isz)e_1(itz).$$

Clearly $g_{s,t}$ is analytic and it is not difficult to see that

$$g_{s,t}(z) = (s+t)^2e_2(i(s+t)z)$$

then

$$\begin{aligned} z_s + z_t + \langle v_s, u_t\rangle &= -\langle s\xi, e_2(isT)s\xi\rangle - \langle t\xi, e_2(itT)t\xi\rangle \\ &+ \langle -ie_1(-isT)s\xi, ie_1(itT)t\xi\rangle \\ &= -\langle \xi, s^2e_2(isT)\xi\rangle - \langle \xi, t^2e_2(itT)\xi\rangle \\ &- \langle \xi, ste_1(isT)e_1(itT)\xi\rangle \\ &= -\langle \xi, \big(s^2e_2(isT) + t^2e_2(itT) + ste_1(isT)e_1(itT)\big)\xi\rangle \\ &= -\langle \xi, g_{s,t}(T)\xi\rangle \\ &= -\langle \xi, (s+t)^2e_2(i(s+t)T)\xi\rangle \\ &= -\langle (s+t)\xi, e_2(i(s+t)T)(s+t)\xi\rangle \\ &= z_{s+t} \end{aligned}$$

hence (3.25) is proved. $\square$

Lemma 3.12. *Let u_t, v_t, z_t are as in Lemma (3.11), then*

$$\lim_{t\to 0} u_t = \lim_{t\to 0} v_t = \lim_{t\to 0} z_t = 0 \tag{3.27}$$

and, for all $x, y \in \mathcal{H}$, the function

$$t \in \mathbb{R} \mapsto h(t) := z_t + \langle v_t, x\rangle + \langle y, u_t\rangle + \langle y, e^{itT}x\rangle$$

is derivable at $t = 0$ and

$$h'(0) = i(\langle \xi, x\rangle + \langle y, \xi\rangle + \langle Ty, x\rangle). \tag{3.28}$$

Proof. We have

$$\|u_t\| =| t | \|e_1(itT)\xi\| \leq| t | \|\xi\| e_1\big(| t | \|T\|\big) \longrightarrow 0, \text{ as } t \to 0$$

and

$$\|v_t\| = \| - e^{-itT}u_t\| = \|u_t\| \longrightarrow 0, \text{ as } t \to 0$$

and

$$| -t^2\langle \xi, e_2(itT)\xi\rangle |\leq t^2\|\xi\|^2 e_2\big(| t | \|T\|\big) \longrightarrow 0, \text{ as } t \to 0.$$

For the second part, note that $h(0) = \langle y, x\rangle$, then

$$\begin{aligned}
\frac{h(t)-h(0)}{t} &= \frac{1}{t}\big(- t^2\langle \xi, e_2(itT)\xi\rangle + it\langle e_1(-itT)\xi, x\rangle + it\langle y, e_1(itT)\xi\rangle \\
&+ \langle y, e^{itT}x\rangle - \langle y, x\rangle\big) \\
&= -t\langle \xi, e_2(itT)\xi\rangle + i\langle e_1(-itT)\xi, x\rangle + i\langle y, e_1(itT)\xi\rangle \\
&+ \langle y, \frac{e^{itT}-1}{t}x\rangle.
\end{aligned}$$

Taking the limit as $t \to 0$, one obtains h is derivable at $t = 0$ and Eq.(3.28) holds. ☐

Proof. (of Theorem 3.10).
We have already shown that $W(\xi, T)$ is unitary, for all $\xi \in \mathcal{H}, T \in \mathcal{B}_s(\mathcal{H})$, so is for $W(t\xi, tT), t \in \mathbb{R}$. In the following we prove that:
-(a) $W(s\xi, sT)W(t\xi, tT) = W((s+t)\xi, (s+t)T), s, t \in \mathbb{R}$;
-(b) the map $t \longmapsto W(t\xi, tT)$ is strongly continuous.
For simplicity we use the notation:

$$W(t) := W(t\xi, tT) = \Gamma\big(u_t, e^{itT}, v_t, z_t\big),\ t \in \mathbb{R} \tag{3.29}$$

then (a) follows from Lemma (3.11) because

$$\begin{aligned}
W(s)W(t) &= \Gamma\Big(u_s, e^{isT}, v_s, z_s\Big)\Gamma\Big(u_t, e^{itT}, v_t, z_t\Big) \\
&= \Gamma\Big(u_s + e^{isT}u_t, e^{isT}e^{itT}, v_t + e^{-itT}v_s, z_s + z_t + \langle v_s, u_t\rangle\Big) \\
&= \Gamma\Big(u_s + e^{isT}u_t, e^{i(s+t)T}, v_t + e^{-itT}v_s, z_s + z_t + \langle v_s, u_t\rangle\Big) \\
&= \Gamma\Big(u_{s+t}, e^{i(s+t)T}, v_{s+t}, z_{s+t}\Big) \\
&= W(s+t).
\end{aligned}$$

By the group property (a), the strong continuity is reduced to time zero. We prove that, for all $\psi \in \Gamma(\mathcal{H})$,

$$\lim_{t\to 0} \|W(t)\psi - \psi\| = 0.$$

Let $\psi = \psi_x, x \in \mathcal{H}$, then

$$\begin{aligned}
\|W(t)\psi - \psi\|^2 &= \langle W(t)\psi_x - \psi_x, W(t)\psi_x - \psi_x\rangle \\
&= \|W(t)\psi_x\|^2 - 2\Re\Big(\langle \psi_x, W(t)\psi_x\rangle\Big) + \|\psi_x\|^2 \\
&= 2\|\psi_x\|^2 - 2\Re\Big(e^{z_t+\langle v_t,x\rangle}\langle \psi_x, \psi_{u_t+e^{itT}x}\rangle\Big) \\
&= 2e^{\|x\|^2} - 2\Re\Big(e^{z_t+\langle v_t,x\rangle}e^{\langle x,u_t\rangle+\langle x,e^{itT}x\rangle}\Big).
\end{aligned}$$

Using Lemma (3.11), one has

$$\lim_{t\to 0}\langle v_t, x\rangle = \lim_{t\to 0}\langle x, u_t\rangle = \lim_{t\to 0} z_t = 0\ ;\ \lim_{t\to 0} e^{itT}x = x$$

then

$$\lim_{t\to 0} \|W(t)\psi - \psi\|^2 = 0.$$

This limit can be extended by linearity to all $\psi \in \mathcal{E}$. Finally an arbitrary element ψ of $\Gamma(\mathcal{H})$ is a limit of a sequence $(\psi_n)_n \in \mathcal{E}$. It follows that, for all $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$\|\psi_{n_0} - \psi\| \leq \epsilon/4.$$

This gives

$$\begin{aligned}
\|W(t)\psi - \psi\| &= \|W(t)(\psi - \psi_{n_0}) + (W(t)\psi_{n_0} - \psi_{n_0}) + (\psi_{n_0} - \psi)\| \\
&\leq \|W(t)(\psi - \psi_{n_0})\| + \|W(t)\psi_{n_0} - \psi_{n_0}\| + \|\psi_{n_0} - \psi\| \\
&= 2\|\psi_{n_0} - \psi\| + \|W(t)\psi_{n_0} - \psi_{n_0}\| \\
&\leq \frac{\epsilon}{2} + \|W(t)\psi_{n_0} - \psi_{n_0}\|.
\end{aligned}$$

But we have already shown that $\lim_{t\to 0} \|W(t)\psi_{n_0} - \psi_{n_0}\| = 0$. Therefore for t sufficiently small we have

$$\|W(t)\psi_{n_0} - \psi_{n_0}\| \leq \frac{\epsilon}{2}.$$

This gives

$$\|W(t)\psi - \psi\| \leq \epsilon$$

hence $\lim_{t\to 0} W(t)\psi = \psi$ for all $\psi \in \Gamma(\mathcal{H})$ which proves the property (b). By Stone's theorem, there is a self-adjoint operator $G(\xi, T)$, such that

$$W(t) = e^{itG(\xi,T)}.$$

Let us prove that, in the notation (3.21)

$$H(\xi, T) \subset G(\xi, T). \tag{3.30}$$

Let $x, y \in \mathcal{H}$, then

$$\begin{aligned} \langle \psi_y, W(t)\psi_x \rangle &= e^{z_t + \langle v_t, x\rangle} \langle \psi_y, \psi_{u_t + e^{itT}x} \rangle \\ &= e^{z_t + \langle v_t, x\rangle} e^{\langle y, u_t + e^{itT}x\rangle} \\ &= e^{z_t + \langle v_t, x\rangle + \langle y, u_t\rangle + \langle y, e^{itT}x\rangle} \\ &= e^{h(t)}, \end{aligned} \tag{3.31}$$

where $h(t)$ is as in Lemma (3.12).

Taking the derivative of the left hand side of (3.31) at time zero, one obtains

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0}\Big(\langle \psi_y, W(t)\psi_x \rangle\Big) &= \langle \psi_y, \frac{d}{dt}\Big|_{t=0} W(t)\psi_x \rangle \\ &= \langle \psi_y, \frac{d}{dt}\Big|_{t=0} e^{itG(\xi,T)}\psi_x \rangle \\ &= \langle \psi_y, iG(\xi,T)\psi_x \rangle. \end{aligned} \tag{3.32}$$

On the other hand the derivative of the right hand side of (3.31) gives

$$\frac{d}{dt}\Big|_{t=0} e^{h(t)} = h'(0)e^{h(0)} = i(\langle \xi, x\rangle + \langle y, \xi\rangle + \langle Ty, x\rangle)e^{\langle y,x\rangle}. \tag{3.33}$$

But we have

$$\begin{aligned} \langle \psi_y, iH(\xi,T)\psi_x \rangle &= \langle \psi_y, i(A^+(\xi) + A^-(\xi) + \Lambda(T))\psi_x \rangle \\ &= i\big(\langle A^-(\xi)\psi_y, \psi_x\rangle + \langle \psi_y, A^-(\xi)\psi_x\rangle \\ &+ \langle A^-(Tx)\psi_y, \psi_x\rangle\big) \\ &= i\big(\langle y, \xi\rangle + \langle \xi, x\rangle + \langle y, Tx\rangle\big)\langle \psi_y, \psi_x\rangle \\ &= i\big(\langle y, \xi\rangle + \langle \xi, x\rangle + \langle y, Tx\rangle\big)e^{\langle y,x\rangle}. \end{aligned} \tag{3.34}$$

Combining Eqs.(3.31)–(3.34), one obtains

$$\langle \psi_y, iH(\xi,T)\psi_x \rangle = \langle \psi_y, iG(\xi,T)\psi_x \rangle \tag{3.35}$$

this gives (3.30) which ends the proof. □

Lemma 3.13. *Let* $\mathbf{D}$ *be the set of self-adjoint bounded operators on* $\mathcal{H}$ *with spectrum in* $[-\pi, \pi[$

$$\mathbf{D} := \{T \in \mathcal{B}_s(\mathcal{H});\ \sigma(T) \subset [-\pi, \pi[\}$$

Then for all unitary operator $U \in \mathcal{U}(\mathcal{H})$ *there exists a unique operator* $T_\pi \in \mathbf{D}$ *such that*

$$U = e^{iT_\pi}.$$

Proof. Existence. Let U be a unitary operator. Since U is normal, the Von Neumann algebra generated by U is abelian hence, by a theorem of Von Neumann (see [17]), it is generated by a single self-adjoint operator T. Denote $F : \mathbb{R} \to [-\pi, \pi[$ the map defined by

$$F(x) := (x \bmod 2\pi) \qquad ; \qquad x \in \mathbb{R}.$$

Clearly F is a measurable, bounded, real valued function on $\mathbb{R}$. Therefore $T_\pi := F(T)$, defined by the spectral theorem, is bounded, self-adjoint and by construction satisfies

$$e^{iT} = e^{iT_\pi} = U.$$

Moreover, denoting $T = \int xE_T(dx)$ the spectral decomposition of T, one has

$$\begin{aligned}\|T_\pi\|^2 &= \int_{\mathbb{R}} | F(x) |^2 \, dE_T(x) \\ &\leq \pi^2 \int_{\mathbb{R}} dE_T(x) \\ &= \pi^2.\end{aligned}$$

Uniqueness. Let $T_1 \in \mathbf{D}$ be an operator such that $e^{iT_1} = e^{iT_\pi} = U$. Then the spectra of T_1 and T_π can differ only by multiples of 2π. Since both T_1 and T_π are in $\mathbf{D}$, this implies that $T_1 = F(T_1) = F(T_\pi) = T_\pi$. □

Theorem 3.14. *For all $(\xi_j, T_j) \in \mathcal{H} \times \mathcal{B}_s(\mathcal{H})$; $j = 1, 2$, the generalized Weyl relations hold*

$$W(\xi_1, T_1)W(\xi_2, T_2) = e^{i\gamma_{\xi,T}} W(\xi, T) \tag{3.36}$$

where the pair $(\xi, T) \in \mathcal{H} \times \mathbf{D}$ and $\gamma_{\xi,T} \in \mathbb{R}$ are uniquely determined by the relations

$$e^{iT_1} e^{iT_2} = e^{iT}, \tag{3.37}$$

$$e_1(iT)\xi = e_1(iT_1)\xi_1 + e^{iT_1} e_1(iT_2)\xi_2, \tag{3.38}$$

$$\begin{aligned}\gamma_{\xi,T} &= i\Big(\langle \xi_1, e_2(iT_1)\xi_1\rangle + \langle \xi_2, e_2(iT_2)\xi_2\rangle - \langle \xi, e_2(iT)\xi\rangle \\ &+ \langle e_1(-iT_1)\xi_1, e_1(iT_2)\xi_2\rangle\Big).\end{aligned} \tag{3.39}$$

Remark 3.15. The reality of $\gamma_{\xi,T}$, given by Eq. (3.39), i.e.

$$S := -(i\gamma_{\xi,T} + \overline{i\gamma_{\xi,T}}) = 0$$

can be directly checked as follows: From Eqs. (3.7) and (3.8), one has, for all self-adjoint bounded operator T on $\mathcal{H}$,

$$\big(e_1(iT)\big)^* = e_1(-iT) \quad ; \quad \big(e_2(iT)\big)^* = e_2(-iT).$$

Then

$$\overline{\langle \xi, e_2(iT)\xi\rangle} = \langle e_2(iT)\xi, \xi\rangle = \langle \xi, e_2(-iT)\xi\rangle \ \ \forall\, T \in \mathcal{B}_s(\mathcal{H}),\ \xi \in \mathcal{H}.$$

From Eq. (3.39), one has

$$\begin{aligned}S &= \Big(\langle \xi_1, e_2(iT_1)\xi_1\rangle + \langle \xi_2, e_2(iT_2)\xi_2\rangle - \langle \xi, e_2(iT)\xi\rangle \\ &+ \langle e_1(-iT_1)\xi_1, e_1(iT_2)\xi_2\rangle\Big) + \Big(\langle \xi_1, e_2(-iT_1)\xi_1\rangle + \langle \xi_2, e_2(-iT_2)\xi_2\rangle \\ &- \langle \xi, e_2(-iT)\xi\rangle + \langle e_1(iT_2)\xi_2, e_1(-iT_1)\xi_1\rangle\Big) \\ &= \langle \xi_1, \big(e_2(iT_1) + e_2(-iT_1)\big)\xi_1\rangle + \langle \xi_2, \big(e_2(iT_2) + e_2(-iT_2)\big)\xi_2\rangle \\ &- \langle \xi, \big(e_2(iT) + e_2(-iT)\big)\xi\rangle + \langle e_1(-iT_1)\xi_1, e_1(iT_2)\xi_2\rangle \\ &+ \langle e_1(iT_2)\xi_2, e_1(-iT_1)\xi_1\rangle.\end{aligned} \tag{3.40}$$

Using Eqs. (3.9) and (3.10), Eq. (3.40) becomes

$$\begin{aligned} S &= \langle \xi_1, e_1(-iT_1)e_1(iT_1)\xi_1\rangle + \langle \xi_2, e_1(-iT_2)e_1(iT_2)\xi_2\rangle - \langle \xi, e_1(-iT)e_1(iT)\xi\rangle \\ &+ \langle e^{-iT_1}e_1(iT_1)\xi_1, e_1(iT_2)\xi_2\rangle + \langle e_1(iT_2)\xi_2, e^{-iT_1}e_1(iT_1)\xi_1\rangle \\ &= \langle e_1(iT_1)\xi_1, e_1(iT_1)\xi_1\rangle + \langle e_1(iT_2)\xi_2, e_1(iT_2)\xi_2\rangle - \langle e_1(iT)\xi, e_1(iT)\xi\rangle \\ &+ \langle e_1(iT_1)\xi_1, e^{iT_1}e_1(iT_2)\xi_2\rangle + \langle e^{iT_1}e_1(iT_2)\xi_2, e_1(iT_1)\xi_1\rangle. \end{aligned} \tag{3.41}$$

Denote

$$\eta = e_1(iT)\xi, \;\; \eta_j = e_1(iT_j)\xi_j\,;\; j = 1, 2. \tag{3.42}$$

Then Eq.(3.38) becomes

$$\eta = \eta_1 + e^{iT_1}\eta_2 \tag{3.43}$$

and from equation (3.41), one obtains

$$\begin{aligned} S &= \|\eta_1\|^2 + \|\eta_2\|^2 - \|\eta\|^2 + \langle \eta_1, e^{iT_1}\eta_2\rangle + \langle e^{iT_1}\eta_2, \eta_1\rangle \\ &= \|\eta_1\|^2 + \|\eta_2\|^2 - \|\eta_1 + e^{iT_1}\eta_2\|^2 + \langle \eta_1, e^{iT_1}\eta_2\rangle + \langle e^{iT_1}\eta_2, \eta_1\rangle \\ &= \|\eta_1\|^2 + \|\eta_2\|^2 - \big(\|\eta_1\|^2 + \|e^{iT_1}\eta_2\|^2 + \langle \eta_1, e^{iT_1}\eta_2\rangle + \langle e^{iT_1}\eta_2, \eta_1\rangle\big) \\ &+ \langle \eta_1, e^{iT_1}\eta_2\rangle + \langle e^{iT_1}\eta_2, \eta_1\rangle \\ &= 0. \end{aligned}$$

Proof. (of Theorem 3.14).
Let $(\xi_j, T_j) \in \mathcal{H} \times \mathcal{B}_s(\mathcal{H})$; $j = 1, 2$.
Using the notations of Eq. (3.12) in Lemma (3.6), one obtains

$$W(\xi_1, T_1)W(\xi_2, T_2) = \Gamma(u_{\xi_1,T_1}, e^{iT_1}, v_{\xi_1,T_1}, z_{\xi_1,T_1})\Gamma(u_{\xi_2,T_2}, e^{iT_2}, v_{\xi_2,T_2}, z_{\xi_2,T_2}). \tag{3.44}$$

Denote

$$u_j = u_{\xi_j,T_j} \quad ; \quad v_j = v_{\xi_j,T_j} \quad ; \quad z_j = z_{\xi_j,T_j}\,; \;\; j = 1, 2.$$

Using (3.3), Eq. (3.44) becomes

$$W(\xi_1, T_1)W(\xi_2, T_2) = \Gamma\big(u_1 + e^{iT_1}u_2, e^{iT_1}e^{iT_2}, v_2 + e^{-iT_2}v_1, z_1 + z_2 + \langle v_1, u_2\rangle\big). \tag{3.45}$$

While $e^{iT_1}e^{iT_2}$ is unitary then by Lemma (3.13) there exists a unique operator $T \in \mathbf{D}$ such that Eq. (3.37) holds. But it is well known (see [16]) that the Bernoulli series $\sum_{n\geq 0} \frac{b_n}{n!}x^n$ is convergent with radius equal to 2π and with sum

$$e_{-1}(x) := \frac{x}{e^x - 1} = \frac{1}{e_1(x)} \quad ; \quad |\, x \,| < 2\pi.$$

It follows that for all operator $T \in \mathcal{B}(\mathcal{H})$ with norm $\|T\| < 2\pi$, $e_1(T)$ is invertible in $\mathcal{B}(\mathcal{H})$ with inverse equal to $e_{-1}(T)$. Since $\sigma(iT) \subset i[-\pi, \pi[$, then iT is of norm less than 2π. From these prescriptions $e_1(iT)$ is invertible with inverse $e_{-1}(iT)$. Denote ξ, the unique vector of $\mathcal{H}$ defined by

$$\xi := e_{-1}(iT)\Big(e_1(iT_1)\xi_1 + e^{iT_1}e_1(iT_2)\xi_2\Big)$$

so that Eq. (3.38) holds and

$$u_1 + e^{iT_1}u_2 = i\big(e_1(iT_1)\xi_1 + e^{iT_1}e_1(iT_2)\xi_2\Big) = ie_1(iT)\xi =: u_{\xi,T}. \tag{3.46}$$

On the other hand, we have

$$\begin{aligned} v_2 + e^{-iT_2} v_1 &= -i\Big(e_1(-iT_2)\xi_2 + e^{-iT_2} e_1(-iT_1)\xi_1\Big) \\ &= -i\Big(e^{-iT_2} e_1(iT_2)\xi_2 + e^{-iT_2} e^{-iT_1} e_1(iT_1)\xi_1\Big) \\ &= -ie^{-iT_2} e^{-iT_1}\Big(e^{iT_1} e_1(iT_2)\xi_2 + e_1(iT_1)\xi_1\Big) \\ &= -i(e^{iT_1} e^{iT_2})^{-1} e_1(iT)\xi \\ &= -i(e^{iT})^{-1} e_1(iT)\xi \\ &= -ie^{-iT} e_1(iT)\xi \\ &= -ie_1(-iT)\xi \\ &= : v_{\xi,T}. \end{aligned} \tag{3.47}$$

Let $\gamma_{\xi,T} \in \mathbb{C}$ defined by

$$i\gamma_{\xi,T} := z_1 + z_2 + \langle v_1, u_2\rangle - z_{\xi,T}, \tag{3.48}$$

where $z_{\xi,T}$ be as in (3.12). Combining (3.45) with Eqs.(3.46)–(3.48), we deduce that

$$\begin{aligned} W(\xi_1, T_1)W(\xi_2, T_2) &= \Gamma(u_{\xi,T}, e^{iT}, v_{\xi,T}, z_{\xi,T})e^{i\gamma_{\xi,T}} \\ &= e^{i\gamma_{\xi,T}} W(\xi, T). \end{aligned} \tag{3.49}$$

$\square$

Remark 3.16. The set of all generalized Weyl operators

$$\mathcal{W} := \{W(\xi, T) \quad ; \ \xi \in \mathcal{H}, T \in \mathcal{B}_s(\mathcal{H})\}$$

is not linearly independent. In fact, for example, $W(0, -\pi 1) = W(0, \pi 1)$.

In the following we will look for a domain $\mathfrak{D}$, of pairs of the form (ξ, T) with the property that the associated Weyl operators $W(\xi, T)$ are linearly independent and which is maximal with respect to this property. To this goal define the equivalence relation $\sim$, on $\mathcal{H} \times \mathcal{B}_s(\mathcal{H})$, by

$$(\xi_1, T_1) \sim (\xi_2, T_2) \iff \text{there exists } \alpha \in \mathbb{C} \quad ; \text{ such that } W(\xi_2, T_2) = \alpha W(\xi_1, T_1)$$

(in this case necessarily $|\alpha| = 1$). Denote by $\big(\mathcal{H} \times \mathcal{B}_s(\mathcal{H})\big)/\sim$, the set of equivalence classes for this relation. For all equivalence class, we can choose only one representant (ξ, T) in some domain $\mathfrak{D} \subset \mathcal{H} \times \mathcal{B}_s(\mathcal{H})$, hence we construct a bijection between $\mathfrak{D}$ and $\big(\mathcal{H} \times \mathcal{B}_s(\mathcal{H})\big)/\sim$. Note that several choices of domain $\mathfrak{D}$ are possible. Intuitively there is a some domain $\mathfrak{D}_0$ which is chosen in a natural way. This domain will be called the principal domain of generalized Weyl operators, but here we will not discuss this choice. (see [9] for an example of principal domain of quadratic Weyl operator). In the spirit to deal with some properties of generalized Weyl operators, such as independence linearity, we assume, in the following, that a such domain $\mathfrak{D}$ is fixed. Clearly, from above prescriptions, that the linear span of $\mathcal{W}$, coincides with the linear span of the set of $W(\xi, T)$, of which (ξ, T) in $\mathfrak{D}$. Note also that this space is in fact a $*$–algebra and this comes from theorems (3.8) and (3.14).

Definition 3.17. The C^*–subalgebra of $\mathcal{B}\big(\Gamma(\mathcal{H})\big)$, generated by $\mathcal{W}$ is called *oscillator Weyl algebra* over $\mathcal{H}$, we denote it by $\mathcal{W}_g(\mathcal{H})$.

Lemma 3.18. *For all* $(\xi_j, T_j) \in \mathfrak{D};\ j = 1, 2,$

$$\Big((\xi_1, T_1) = (\xi_2, T_2)\Big) \Longleftrightarrow \Big(u_{\xi_1,T_1} = u_{\xi_2,T_2} \text{ and } e^{iT_1} = e^{iT_2}\Big).$$

Proof. Let $(\xi_j, T_j) \in \mathfrak{D};\ j = 1, 2$ such that

$$u_{\xi_1,T_1} = u_{\xi_2,T_2} \quad ; \quad e^{iT_1} = e^{iT_2}.$$

Then

$$\begin{aligned} v_{\xi_2,T_2} : &= -ie_1(-iT_2)\xi_2 = -e^{-iT_2}\big(ie_1(iT_2)\xi_2\big) = -e^{-iT_2}u_{\xi_2,T_2} = -e^{-iT_1}u_{\xi_1,T_1} \\ &= v_{\xi_1,T_1} \end{aligned}$$

this gives

$$\begin{aligned} W(\xi_2, T_2) : &= \Gamma(u_{\xi_2,T_2}, e^{iT_2}, v_{\xi_2,T_2}, z_{\xi_2,T_2}) = \Gamma(u_{\xi_1,T_1}, e^{iT_1}, v_{\xi_1,T_1}, z_{\xi_2,T_2}) \\ &= e^{z_{\xi_2,T_2} - z_{\xi_1,T_1}}\Gamma(u_{\xi_1,T_1}, e^{iT_1}, v_{\xi_1,T_1}, z_{\xi_1,T_1}) \\ &= e^{z_{\xi_2,T_2} - z_{\xi_1,T_1}}W(\xi_1, T_1) \end{aligned}$$

which implies $(\xi_2, T_2) \sim (\xi_1, T_1)$ but $(\xi_1, T_1), (\xi_2, T_2) \in \mathfrak{D}$, then $(\xi_1, T_1) = (\xi_2, T_2)$. □

Lemma 3.19. *Let* $j = 1, ..., m,\ m \in \mathbb{N}^*$ *and* $(\xi_j, T_j) \in \mathfrak{D}$ *satisfying*

$$(\xi_j, T_j) \neq (\xi_k, T_k)\ \forall j, k = 1, ..., m,\ j \neq k. \tag{3.50}$$

Denote

$$f_j : \mathcal{H} \ni y \longmapsto f_j(y) = u_{\xi_j,T_j} + e^{iT_j}y$$

then, there exists $x \in \mathcal{H}$ *separating maps* f_j,

$$(i.e.\ \ f_j(x) \neq f_k(x)\ \ \forall\, j, k = 1, ..., m,\ j \neq k.)$$

Proof. From condition (3.50) and Lemma (3.18), we deduce that for $j, k = 1, ..., m$, $j \neq k$,

$$u_{\xi_j,T_j} \neq u_{\xi_k,T_k} \qquad or \qquad e^{iT_j} \neq e^{iT_k}. \tag{3.51}$$

Denoting

$$J = \{(j, k) \in \{1, ..., m\}^2\ , E_{j,k} := e^{iT_j} - e^{iT_k} \neq 0\}$$

-a) If $J = \emptyset$, then for all $j, k = 1, ..., m$, one has $e^{iT_j} = e^{iT_k}$ then from (3.51), one has $(u_{\xi_j,T_j} \neq u_{\xi_k,T_k}\ \forall\, j \neq k)$, then $f_j(x) \neq f_k(x)\, \forall j \neq k\ \forall\, x \in \mathcal{H}$. Hence all x of $\mathcal{H}$ separates maps f_j. -b) If $J \neq \emptyset$, we consider

$$\mathcal{F} = \bigcup_{(j,k)\in J} \ker(E_{j,k}).$$

-b.i) Let us prove, in a first step, that $\mathcal{F} \neq \mathcal{H}$. By contradiction, assuming that $\mathcal{F} = \mathcal{H}$. While the reunion of finite subspaces is a vectorial space only if, one among them contains all the others, then there exists some $(j_0, k_0) \in J$ such that

$$\mathcal{H} = \bigcup_{(j,k)\in J} \ker(E_{j,k}) = \ker(E_{j_0,k_0})$$

This gives

$$e^{iT_{j_0}} = e^{iT_{k_0}}$$

then by definition of J, $(j_0, k_0) \notin J$ absurd. -b.ii) In a second step, assuming by contradiction, that

$$\forall\, x \in \mathcal{H}\backslash\mathcal{F},\ \exists\,(j,k) \in J,\ f_j(x) = f_k(x). \qquad (\mathfrak{H})$$

Fix $x \in \mathcal{H}\backslash\mathcal{F}$ and consider $x_n = \frac{x}{n} \in \mathcal{H}\backslash\mathcal{F}, n \geq 1$. (it is easily checked that $\frac{x}{n} \in \mathcal{H}\backslash\mathcal{F}$). By hypothesis $(\mathfrak{H})$, there exists (for x_n) a pair $(j_n, k_n) \in J$ such that

$$f_{j_n}(x_n) = f_{k_n}(x_n). \tag{3.52}$$

While J is a finite set then, the sequence (j_n, k_n) has a stationary subsequence, what we take the same notation. Then for n sufficiently big, $(j_n, k_n) = (j_0, k_0) \in J$ and Eq. (3.52), becomes

$$f_{j_0}(x_n) = f_{k_0}(x_n)$$

which gives

$$u_{\xi_{j_0},T_{j_0}} - u_{\xi_{k_0},T_{k_0}} = (e^{iT_{k_0}} - e^{iT_{j_0}})x_n. \tag{3.53}$$

Taking the limit as $n \longrightarrow +\infty$, one has

$$u_{\xi_{j_0},T_{j_0}} - u_{\xi_{k_0},T_{k_0}} = 0$$

and Eq. (3.53) becomes

$$(E_{j_0,k_0})x_n := (e^{iT_{k_0}} - e^{iT_{j_0}})x_n = 0.$$

Then $x_n \in \ker(E_{j_0,k_0})$ and also $x = nx_n \in \ker(E_{j_0,k_0}) \subset \mathcal{F}$ which is not true by definition of x, (i.e. absurd). From above prescriptions, we conclude that there exists $x_0 \in \mathcal{H}\backslash\mathcal{F}$, separating maps f_j corresponding to all pair $(j,k) \in J$. The case for which the pair $(j,k) \notin J$, is similar to the one for which $J = \emptyset$. In Conclusion, x_0 separates maps corresponding to all pair $(j,k), j,k = 1,...,m,\ j \neq k$. □

Proposition 3.20. *The set of generalized Weyl operators*

$$\mathcal{W}(\mathfrak{D}) := \{W(\xi,T), \quad (\xi,T) \in \mathfrak{D}\}$$

is linearly independent.

Proof. Let $(\xi_j, T_j) \in \mathfrak{D},\ \alpha_j \in \mathbb{C},\ j = 1,...,m,\ m \geq 1$, satisfying conditions (3.50) of Lemma (3.19) and

$$\sum_{j=1}^{m} \alpha_j W(\xi_j, T_j) = 0. \tag{3.54}$$

By Lemma (3.19), there exists $x_0 \in \mathcal{H}$ separating maps f_j. Applying Eq. (3.54) to exponential vector ψ_{x_0}, one obtains

$$\sum_{j=1}^{m} \alpha_j e^{z_{\xi_j,T_j} + <v_{\xi_j,T_j}, x_0>} \psi_{f_j(x_0)} = 0 \tag{3.55}$$

Since the exponential vectors are linearly independent, then

$$\alpha_j e^{z_{\xi_j,T_j} + <v_{\xi_j,T_j}, x_0>} = 0$$

which implies $\alpha_j = 0$ for all $j = 1,...,m$. □

4. Lie Algebra Time Shift

4.1. Lie algebra time shift as $*$–homomorphism of Lie algebras. Let $\mathcal{L}$ be a complex $*$–Lie algebra. Let

$$\{X^+_\alpha, X^-_\alpha, X^0_\beta \quad ; \quad \alpha \in F \quad ; \beta \in F_0\}$$

where F, F_0 are disjoint sets, be set of generators of $\mathcal{L}$ satisfying the following conditions:

$$(X^0_\beta)^* = X^0_\beta \quad \forall \beta \in F_0,$$
$$(X^+_\alpha)^* = X^-_\alpha \quad \forall \alpha \in F.$$

We assume that there exists a single central element, denoted 1_0, among the generators and that it is of X^0-type (i.e. self-adjoint). We will denote $C^\gamma_{\alpha,\beta}(\varepsilon, \varepsilon', \varepsilon")$ the structure constants of $\mathcal{L}$ with respect to the generators (X^ε_α), i.e. with $\alpha, \beta \in F \cup F_0, \varepsilon, \varepsilon', \varepsilon'' \in \{+, -, 0\}$, and assuming summation over repeated indices:

$$[X^\varepsilon_\alpha, X^{\varepsilon'}_\beta] = \sum_{\gamma \in F_0} C^\gamma_{\alpha,\beta}(\varepsilon, \varepsilon', 0) X^0_\gamma + \sum_{\gamma \in F} C^\gamma_{\alpha,\beta}(\varepsilon, \varepsilon', +) X^+_\gamma + \sum_{\gamma \in F} C^\gamma_{\alpha,\beta}(\varepsilon, \varepsilon', -) X^-_\gamma.$$

In the following we will consider only locally finite Lie algebra, i.e. such that, for any pair $\alpha, \beta \in F \cup F_0$ only a finite number of structure constants $C^\gamma_{\alpha,\beta}(\varepsilon, \varepsilon', \varepsilon")$ are different from zero.

Definition 4.1. Let be given:
- a $*$–Lie algebra $\mathcal{L}$ with canonical set of generators

$$\{X^\varepsilon_\alpha, \varepsilon \in \{+, -, 0\}, \alpha \in F \cup F_0\}$$

with Lie-bracket as in the above definition
- a measurable space $(S, \mathcal{B}(S))$
- a $*$–sub-algebra $\mathcal{C} \subset L^\infty_\mathbb{C}(\mathcal{S}, \mathcal{B}(S))$.

The current algebra over S of $\{\mathcal{L}, X^\varepsilon_\alpha\}$, with test function algebra $\mathcal{C}$, is the $*$–Lie algebra $\mathcal{L}(S, \mathcal{C})$ defined as follows:
– as a vector space $\mathcal{L}(S, \mathcal{C})$ is the algebraic linear span of the family

$$\{X^\varepsilon_\alpha(f), f \in \mathcal{C}, \varepsilon \in \{+, -, 0\}, \alpha \in F_0 \cup F\}$$

and the generators are independent in the sense that

$$\sum_{\varepsilon,\alpha} X^\varepsilon_\alpha(f_{\varepsilon,\alpha}) = 0 \ ; \iff f_{\varepsilon,\alpha} = 0 \ \forall \varepsilon, \alpha$$

– the map $f \longmapsto X^\varepsilon_\alpha(f)$ is linear for $\varepsilon = +, 0$ and anti–linear for $\varepsilon = -$
– the Lie–brackets are defined by

$$[X^\varepsilon_\alpha(f), X^{\varepsilon'}_\beta(g)] = \sum_{\gamma,\varepsilon''} C^\gamma_{\alpha,\beta}(\varepsilon, \varepsilon', \varepsilon'') X^{\varepsilon''}_\gamma (f^{\varepsilon\varepsilon''} g^{\varepsilon'\varepsilon''})$$

where, for a test function f, we use the notation

$$f^\varepsilon = f \quad ; \quad \text{if } \varepsilon = +, 0 \quad ; \quad f^\varepsilon = \overline{f} \quad ; \quad \text{if } \varepsilon = -$$

and operations

$$\varepsilon\varepsilon' = \varepsilon'\varepsilon = \begin{cases} +, & \text{if } (\varepsilon, \varepsilon') \in \{(+, 0), (+, +), (0, 0), (-, -)\} \ ; \\ -, & \text{if } (\varepsilon, \varepsilon') \in \{(+, -), (0, -)\} \end{cases}$$

– the involution is defined by

$$(X_\alpha^\varepsilon(f))^* := X_\alpha^{\varepsilon^*}(f^{\varepsilon_\alpha})$$

with

$$+^* := - \ ; \ -^* := + \ ; \ 0^* := 0 \ ; \ \varepsilon_\alpha = +, \quad \text{if } \alpha \in F \ ; \ \varepsilon_\alpha = -, \quad \text{if } \alpha \in F_0.$$

Example 4.2. The one-mode Heisenberg Lie algebra $heis(1)$, is the complex $*$–Lie algebra generated as vector space by $a^+, a^-, 1$ with involution $1^* = 1 \quad ; \ (a^+)^* = a^-$ and Lie bracket

$$[a^-, a^+] = 1 \ ; \qquad [a^\pm, 1] = 0.$$

The current algebra of $heis(1)$ over $\mathbb{R}$ is the Lie algebra generated by the set

$$\{a^+(f), a^-(f), 1 \quad ; \ f \in \mathcal{C} := L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})\}$$

which are linearly independent in the sense that, $\forall f, g \in \mathcal{C}, \lambda \in \mathbb{C}$

$$a^+(f) + a^-(g) + \lambda 1 = 0 \Rightarrow f = g = 0, \lambda = 0.$$

The involution and the Lie bracket are given by

$$(a^+(f))^* = a^-(f)$$

and

$$[a^-(f), a^+(g)] = \langle f, g\rangle 1 \quad ; \ [a^+(f), a^+(g)] = [a^-(f), a^-(g)] = [a^\pm(f), 1] = 0.$$

Definition 4.3. A unitary representation of a Lie algebra $\mathcal{L}$ is a triple:

$$\{\mathcal{H}, \pi, \mathcal{D}\},$$

where: $\mathcal{D} \subseteq \mathcal{H}$ is a total subset of $\mathcal{L}$ which is a core for each $\pi(a)$ $(a \in \mathcal{L})$

$$\pi : \mathcal{L} \to \mathcal{L}^a(\mathcal{D}) \text{ (adjointable linear maps from lin-span}(\mathcal{D}) \to \mathcal{H})$$

$\pi(a)^+$, i.e. the $\mathcal{H}$–adjoint of $\pi(a)$ is defined on $\mathcal{D}$ and

$$\pi(a)^+ = \pi(a^*),$$

$$\pi([a, b]) = [\pi(a), \pi(b)] \quad \forall\, a, b \in \mathcal{L},$$

where the identity is meant weakly on $\mathcal{D}$.

Definition 4.4. Let $\mathcal{L}$ be a real or complex $*$–Lie algebra with center $Centre(\mathcal{L})$. A central decomposition of $\mathcal{L}$ is a direct sum of vector spaces

$$\mathcal{L} = Centre(\mathcal{L}) \oplus \mathcal{L}_0,$$

where $\mathcal{L}_0$ is a sub–Lie algebra of $\mathcal{L}$.

Remark 4.5. Let $\mathcal{L}$ and $\mathcal{G}$ be a real or complex $*$–Lie algebras and $\Phi : \mathcal{L} \longrightarrow \mathcal{G}$ a $*$–isomorphism of Lie algebras. Clearly if $\mathcal{L}$ has a central decomposition

$$\mathcal{L} = Centre(\mathcal{L}) \oplus \mathcal{L}_0,$$

then $\mathcal{G}$ has the central decomposition

$$\mathcal{G} = \Phi(Centre(\mathcal{L})) \oplus \Phi(\mathcal{L}_0).$$

In the following all Lie algebras are identified with their images under the corresponding unitary representations and these are omitted from the notations. We assume that all Lie algebras have a scalar center, i.e. with central decomposition

$$\mathcal{L} = \mathbb{C}1_0 \oplus \mathcal{L}_0.$$

If given a $*$–Lie algebra, denote by $\mathcal{L}_s$ the real subspace of $\mathcal{L}$, of self-adjoint non central elements, i.e. such that $\mathcal{L}_s \subset \mathcal{L}_0$ and $a^* = a$ for all $a \in \mathcal{L}_s$. For $I \subset \mathbb{R}$, denote by $\mathcal{C}_I$, the space of test functions in $\mathcal{C}$ with support in I.

Definition 4.6. Let be given:
- a complex $*$–Lie algebra $\mathcal{L}$ with scalar center $\mathbb{C}1_0$,
- a space of test functions $\mathcal{C}$,
- a current algebra $\mathcal{L}(\mathbb{R}, \mathcal{C})$ of $\mathcal{L}$ over $\mathbb{R}$.

A Lie algebra time shift is a family of homomorphisms of $*$–Lie algebras

$$\hat{j}_t : \mathcal{L} \longrightarrow \mathcal{L} \otimes 1 + 1_0 \otimes \mathcal{L}([0,t], \mathcal{C}_{[0,t]}) \tag{4.1}$$

with the following structure:

$$\hat{j}_t(X) := T(X) \otimes 1 + 1_0 \otimes T_{[0,t]}(X) \tag{4.2}$$

with the property that the exponential map exists and the map j_t, defined by

$$j_t(e^{iX}) := e^{i\hat{j}_t(X)} \qquad , \qquad X \in \mathcal{L}_s \tag{4.3}$$

is well defined and extends to a $*$–homomorphism of the C^*–algebra generated by the set

$$\{e^{iX}, X \in \mathcal{L}_s\}.$$

Note that in the above definition the map T must be a $*$–Lie algebra homomorphism of $\mathcal{L}$ and $T_{[0,t]}$ is a $*$–Lie algebra homomorphism from $\mathcal{L}$ to its current algebra over $[0,t]$. In the following we take $T = Id$, i.e. $T(X) = X$, for all $X \in \mathcal{L}$.

4.2. An example of Lie algebra time shift of $\mathcal{L}_{osc}(\mathcal{H})$. In this section we give an example of Lie algebra time shift of the oscillator Lie algebra.

Definition 4.7. The oscillator Lie algebra algebra over an Hilbert space $\mathcal{H}$, is the complex $*$–Lie algebra $\mathcal{L}_{osc}(\mathcal{H})$ generated as vector space by

$$A^+(u), A^-(u), \Lambda(T), 1_0 \text{ (the identity on } \mathcal{H}), \quad u \in \mathcal{H} \setminus \{0\}, \quad T \in \mathcal{B}(\mathcal{H}) \setminus \{0\}$$

which are linearly independent and satisfying the commutation relations from (2.1) to (2.4).

In the definition (4.6), taking $\mathcal{L} = \mathcal{L}_{osc}(\mathcal{H})$ and sets of indices $F = \mathcal{H} \setminus \{0\}$, $\quad F_0 = \mathcal{B}(\mathcal{H})$. $\mathcal{L}_{osc}(\mathcal{H})$, is generated as vector space by the set

$$\{X_\alpha^\varepsilon \quad ; \ \alpha \in F \cup F_0 \quad ; \ \varepsilon = +, -, 0\},$$

where: If $\alpha = u \in F$, $\quad X_\alpha^\pm = A^\pm(u)$ and if $\alpha = T \in F_0 \setminus \{0\}$, $\quad X_\alpha^0 = \Lambda(T)$ and $X_0^0 = 1_0$. The current algebra, $\mathcal{L}_{osc}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, of $\mathcal{L}_{osc}(\mathcal{H})$ over $\mathbb{R}$ is given by the following -the space of test functions is $\mathcal{C} = L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, -for all $f \in \mathcal{C}$,

$$X_\alpha^\varepsilon(f) = \begin{cases} A^\pm(f \otimes u), & \text{if } \varepsilon = \pm \text{ and } \alpha = u \in F \ ; \\ \Lambda(M_f \otimes T), & \text{if } \varepsilon = 0 \text{ and } \alpha = T \in F_0 \setminus \{0\}; \\ (\int f(x)dx)1_0, & \text{if } \varepsilon = 0 \text{ and } \alpha = 0 \ , \end{cases}$$

where the operator M_f, is the multiplication by f. Since $f \otimes u \in \mathcal{H}' := L^2(\mathbb{R}) \otimes \mathcal{H} \equiv L^2(\mathbb{R}, \mathcal{H})$ and $M_f \otimes T$ acts on $\mathcal{H}'$, the representation space of this current algebra is

$$\Gamma(L^2(\mathbb{R}) \otimes \mathcal{H}) \equiv \Gamma(L^2(\mathbb{R}, \mathcal{H})).$$

If $f = \chi_{[0,t]}$, denote $X_t^{\varepsilon}(\alpha) := X_{\alpha}^{\varepsilon}(\chi_{[0,t]})$, then we have

$$A_t^+(u) = A^+(\chi_{[0,t]} \otimes u) \quad ; \; A_t^-(u) = A^-(\chi_{[0,t]} \otimes u) \quad ; \; \Lambda_t(T) = \Lambda(M_{\chi_{[0,t]}} \otimes T). \tag{4.4}$$

This shows clearly, with these notations, that the operators $A_t^{\pm}(u)$ and $\Lambda_t(T)$ are in the current algebra $\mathcal{L}_{osc}([0,t], \mathcal{C}_{[0,t]})$.

The Lie algebra time shift can be defined by its action on generators of $\mathcal{L}_{osc}(\mathcal{H})$ which takes the form:

$$\hat{j}_t(X_{\alpha}^{\varepsilon}) = X_{\alpha}^{\varepsilon} \otimes 1 + 1_0 \otimes \sum_{\gamma, \varepsilon'} X_t^{\varepsilon'}(\gamma). \tag{4.5}$$

Several choices of $*$–homomorphisms of Lie algebras are possible. In our case, we consider only a class of $*$–Lie algebra homomorphisms and we study in which cases they are a Lie algebra time shifts.

To this goal, let us fix:
- (i) a continuous unital $*$–homomorphism $\rho : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$,
- (ii) a surjective ρ–1–cocycle, i.e. a linear map $\delta : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{H}$ with the property

$$\delta(T'T) = \rho(T')\delta(T) \quad \forall T, T' \in \mathcal{B}(\mathcal{H}). \tag{4.6}$$

Then the cocycle property implies that the linear functional $L : \mathcal{B}(\mathcal{H}) \longrightarrow \mathbb{C}$, defined by

$$L(T) := \langle \delta(1), \delta(T) \rangle \qquad \forall\, T \in \mathcal{B}(\mathcal{H}) \tag{4.7}$$

is hermitian and satisfies:

$$L(TT') = \langle \delta(T^*), \delta(T') \rangle \qquad \forall\, T, T' \in \mathcal{B}(\mathcal{H}). \tag{4.8}$$

In fact we have

$$\begin{aligned} \overline{L(T)} &= \langle \delta(T), \delta(1) \rangle = \langle \rho(T)\delta(1), \delta(1) \rangle = \langle \delta(1), \rho(T^*)\delta(1) \rangle \\ &= \langle \delta(1), \delta(T^*) \rangle = L(T^*) \end{aligned}$$

and

$$\begin{aligned} L(TT') &= \langle \delta(1), \delta(TT') \rangle = \langle \delta(1), \rho(T)\delta(T') \rangle = \langle \rho(T^*)\delta(1), \delta(T') \rangle \\ &= \langle \delta(T^*), \delta(T') \rangle. \end{aligned}$$

Example 4.8. Let u be a fixed vector in $\mathcal{H}$. Taking $\rho(T) := T$ and $\delta(T) = \rho(T)u$, one finds $L(T) = \langle u, \rho(T)u \rangle$.

With the following choices:

$$\begin{aligned}
\hat{j}_t(A^+(\xi)) &:= A^+(\xi)\otimes 1 + 1_0\otimes\big(A_t^+(T_1\xi)+A_t^-(T_2\xi)\big), && (4.9)\\
\hat{j}_t(A^-(\xi)) &:= A^-(\xi)\otimes 1 + 1_0\otimes\big(A_t^+(T_2\xi)+A_t^-(T_1\xi)\big), && (4.10)\\
\hat{j}_t(\Lambda(T)) &:= \Lambda(T)\otimes 1 + 1_0\otimes\Big(A_t^+(\delta(T))+A_t^-(\delta(T^*))+\Lambda_t(\rho(T))\\
&\quad + tL(T)1\Big), && (4.11)\\
\hat{j}_t(1_0) &= 1_0\otimes 1 + 1_0\otimes R_t, && (4.12)
\end{aligned}$$

$\hat{j}_t$ will be a $\mathbb{R}$–linear $*$–map whenever T_1, T_2 are $\mathbb{R}$–linear operators on $\mathcal{H}$ and R_t is a self-adjoint operator acting on $\Gamma(L^2(\mathbb{R},\mathcal{H}))$.

Proposition 4.9. *The map $\hat{j}_t$, defined by the equations from (4.9) to (4.12), is a $*$–Lie algebra homomorphism if and only if $T_1 = T_2 = 0$ and $R_t = 0$ for all $t \ge 0$. In this case, denoting*

$$H_t(\xi,T) := A_t^+(\xi) + A_t^-(\xi) + \Lambda_t(T) \tag{4.13}$$

one has

$$\hat{j}_t\big(H(\xi,T)\big) = H(\xi,T)\otimes 1 + 1_0\otimes\Big(H_t\big(\delta(T),\rho(T)\big) + tL(T)1\Big)\ \forall(\xi,T)\in\mathcal{H}\times\mathcal{B}(\mathcal{H}). \tag{4.14}$$

Proof. The $*$–property is easily verified from the construction. The map $\hat{j}_t$ is a $*$–Lie algebra morphism if and only if for all $X, Y \in \{A^+(\xi), A^-(\xi), \Lambda(T), 1_0\}$, one has

$$\hat{j}_t([X,Y]) = [\hat{j}_t(X), \hat{j}_t(Y)],$$

this is equivalent to the following conditions:

$$\begin{aligned}
\langle\xi,\eta\rangle R_t &= t\big(\langle T_1\xi, T_1\eta\rangle - \langle T_2\eta, T_2\xi\rangle\big), && (4.15)\\
\langle T_2\xi, T_1\eta\rangle &= \langle T_2\eta, T_1\xi\rangle, && (4.16)\\
\rho(T)T_1\xi &= T_1\xi, && (4.17)\\
\rho(T^*)T_2\xi &= -T_2\xi, && (4.18)\\
\langle\delta(T^*), T_1\xi\rangle &= \langle T_2\xi, \delta(T)\rangle && (4.19)
\end{aligned}$$

for all $\xi,\eta\in\mathcal{H}$ and $T\in\mathcal{B}(\mathcal{H})$. Taking $T = 1$ (the identity of $\mathcal{B}(\mathcal{H})$) in (4.18), since $\rho(1) = 1$, it follows that $T_2 = 0$. Therefore (4.19) implies that also $T_1 = 0$ because δ is surjective. From this we see that (4.15) implies that $R_t = 0$ for all $t \ge 0$. Given the above, from the real linearity of $\hat{j}_t$ and the notation (4.13), one sees that Eq. (4.14) is the sum of (4.9), (4.10) and (4.11). □

Defining a map j_t on the set $\mathcal{W}(\mathfrak{D})$ by the following:

$$j_t\big(W(\xi,T)\big) = j_t\big(e^{iH(\xi,T)}\big) := e^{i\hat{j}_t(H(\xi,T))}. \tag{4.20}$$

Then

$$\begin{aligned}
j_t\big(W(\xi,T)\big) &= e^{i\big(H(\xi,T)\otimes 1\big)+i\Big(1_0\otimes\big(H_t(\delta(T),\rho(T))+tL(T)1\big)\Big)}\\
&= e^{iH(\xi,T)}\otimes e^{iH_t(\delta(T),\rho(T))}e^{itL(T)}\\
&= W(\xi,T)\otimes W_t\big(\delta(T),\rho(T)\big)e^{itL(T)}, && (4.21)
\end{aligned}$$

where we have used the notation

$$W_t(\xi,T) := e^{iH_t(\xi,T)}.$$

Note that from definition of $\mathfrak{D}$, the map j_t is well defined by Eq. (4.20). Since $\mathcal{W}(\mathfrak{D})$ is a linearly independent set, then j_t can be extended by linearity to whole the linear span of $\mathcal{W}(\mathfrak{D})$ and we have the following theorem.

Theorem 4.10. *The action of j_t given by (4.21) holds also for all $(\xi,T)\in\mathcal{H}\times\mathcal{B}_s(\mathcal{H})$. Moreover, j_t extends to a $*$–isomorphism from the C^*–algebra $\mathcal{W}_g(\mathcal{H})$ onto its image. Furthermore the map $\hat{j}_t$ is a Lie algebra time shift.*

For the proof of this theorem we need the following lemma.

Lemma 4.11. *Let $f(z)=\sum_{n\geq 0}a_nz^n$ be an entire function and ρ and δ as in (4.6). Then we have:*
(i) For all $T\in\mathcal{B}(\mathcal{H})$,

$$f(\rho(T)) = \rho(f(T)). \tag{4.22}$$

-(ii) For all $T\in\mathcal{B}(\mathcal{H}),\xi\in\mathcal{H}$,

$$f(\chi_{[0,t]}\otimes T)(\chi_{[0,t]}\otimes\xi) = \chi_{[0,t]}\otimes f(T)\xi. \tag{4.23}$$

Proof. Note that from Lemma (3.4), $\rho(f(T))$ and $f(\rho(T))$ are well-defined. Moreover, since ρ is continuous,
-(i)

$$\begin{aligned} f(\rho(T)) &= \sum_{n\geq 0}a_n\big(\rho(T)\big)^n = \sum_{n\geq 0}a_n\rho(T^n)\\ \text{(continuity)} &= \rho\big(\sum_{n\geq 0}a_nT^n\big) = \rho(f(T)), \end{aligned}$$

-(ii)

$$\begin{aligned} f(\chi_{[0,t]}\otimes T)(\chi_{[0,t]}\otimes\xi) &= \sum_{n\geq 0}a_n\big(\chi_{[0,t]}\otimes T\big)^n(\chi_{[0,t]}\otimes\xi)\\ &= \sum_{n\geq 0}a_n\chi_{[0,t]}\otimes T^n\xi = \chi_{[0,t]}\otimes\sum_{n\geq 0}a_nT^n\xi\\ &= \chi_{[0,t]}\otimes f(T)\xi. \end{aligned}$$

□

Proof. (of Theorem 4.10).

Step I. In this Step we will prove the first part of Theorem. Let $(\xi,T)\in\mathfrak{D}$ then

$$\begin{aligned} j_t\Big(W(\xi,T)\Big) &= W(\xi,T)\otimes W_t\big(\delta(T),\rho(T)\big)e^{itL(T)}\\ &= W(\xi,T)\otimes\Gamma(U_t,e^{iT_t},V_t,Z_t)e^{itL(T)}, \end{aligned} \tag{4.24}$$

where

$$T_t=\chi_{[0,t]}\otimes\rho(T)\,,\ U_t=ie_1(iT_t)\Big(\chi_{[0,t]}\otimes\delta(T)\Big)\,,\ V_t=-e^{-iT_t}U_t$$

and

$$Z_t=-\langle\chi_{[0,t]}\otimes\delta(T),e_2(iT_t)\Big(\chi_{[0,t]}\otimes\delta(T)\Big)\rangle.$$

Using Eqs. (4.22) and (4.23) in Lemma (4.11), we get

$$e^{iT_t} = \big(1 \otimes 1\big) + \big(\chi_{[0,t]} \otimes \rho(e^{iT} - 1)\big), \tag{4.25}$$

$$\begin{aligned} U_t &= e_1\Big((\chi_{[0,t]} \otimes \rho(iT)\Big)\Big(\chi_{[0,t]} \otimes \delta(iT)\Big) = \chi_{[0,t]} \otimes e_1\big(\rho(iT)\big)\delta(iT) \\ &= \chi_{[0,t]} \otimes \rho\big(e_1(iT)\big)\delta(iT) = \chi_{[0,t]} \otimes \delta\big(e_1(iT)iT\big) \\ &= \chi_{[0,t]} \otimes \delta(e^{iT} - 1), \end{aligned} \tag{4.26}$$

$$\begin{aligned} V_t &= -e^{\chi_{[0,t]} \otimes \rho(-iT)}\Big(\chi_{[0,t]} \otimes \delta(e^{iT} - 1)\Big) = -\chi_{[0,t]} \otimes \rho\Big(e^{-iT}\Big)\delta(e^{iT} - 1) \\ &= -\chi_{[0,t]} \otimes \delta\Big(e^{-iT}(e^{iT} - 1)\Big) = \chi_{[0,t]} \otimes \delta(e^{-iT} - 1), \end{aligned} \tag{4.27}$$

$$\begin{aligned} Z_t &= -\langle \chi_{[0,t]} \otimes \delta(T), \chi_{[0,t]} \otimes \rho\big(e_2(iT)\big)\delta(T)\rangle = -t\langle \delta(T), \delta\big(e_2(iT)T\big)\rangle \\ &= tL\big(e_2(iT)(iT)^2\big) = tL(e^{iT} - iT - 1). \end{aligned} \tag{4.28}$$

Injecting right-hand sides of Eqs. (4.26)–(4.28) in (4.24), one obtains

$$j_t\Big(W(\xi, T)\Big) = W(\xi, T) \otimes \Gamma\Big(\chi_{[0,t]} \otimes \delta(e^{iT} - 1), e^{iT_t}, \chi_{[0,t]} \otimes \delta(e^{-iT} - 1), tL(e^{iT} - 1)\Big). \tag{4.29}$$

Note that from (4.25), arguments of exponential operator Γ in the right-hand side of the above equation dependent of e^{iT}. Now taking $(\xi', T') \in \mathcal{H} \times \mathcal{B}_s(\mathcal{H})$ then there exists a pair $(\xi, T) \in \mathfrak{D}$ such that $W(\xi', T') = e^{i\alpha}W(\xi, T)$, $\alpha \in \mathbb{R}$. But from definition of generalized Weyl operator given in Eq. (3.13) we get $e^{iT} = e^{iT'}$. Then by linearity of j_t we obtain

$$\begin{aligned} j_t\Big(W(\xi', T')\Big) &= e^{i\alpha} j_t\big(W(\xi, T)\big) \\ &= e^{i\alpha}W(\xi, T) \otimes \\ &\quad \Gamma\Big(\chi_{[0,t]} \otimes \delta(e^{iT} - 1), e^{iT_t}, \chi_{[0,t]} \otimes \delta(e^{-iT} - 1), tL(e^{iT} - 1)\Big) \\ &= W(\xi', T') \otimes \\ &\quad \Gamma\Big(\chi_{[0,t]} \otimes \delta(e^{iT'} - 1), e^{iT'_t}, \chi_{[0,t]} \otimes \delta(e^{-iT'} - 1), tL(e^{iT'} - 1)\Big) \\ &= W(\xi', T') \otimes W_t\big(\delta(T'), \rho(T')\big) \\ &= e^{i\hat{j}_t(H(\xi', T')} \end{aligned} \tag{4.30}$$

which proves that we can take the same expression of j_t for all $W(\xi, T)$.

Step II. Assuming, provisionally, that for all $\xi, \xi_1, \xi_2 \in \mathcal{H}$ and $T, T_1, T_2 \in \mathcal{B}_s(\mathcal{H})$, the following relations hold

$$\Big(j_t\big(W(\xi, T)\big)\Big)^* = j_t\Big(\big(W(\xi, T)\big)^*\Big) \tag{4.31}$$

and

$$j_t\Big(W(\xi_1, T_1)W(\xi_2, T_2)\Big) = j_t\Big(W(\xi_1, T_1)\Big)j_t\Big(W(\xi_2, T_2)\Big), \tag{4.32}$$

then j_t is a $*$–homomorphism from the $*$–algebra generated by the set of generalized Weyl operators onto $\mathcal{A}_{t]}$. In fact if

$$w = \sum_{1\le j\le n} \alpha_j W_j,$$

then

$$\begin{aligned}
(j_t(w))^* &= \Big(j_t\big(\sum_{1\le n}\alpha_j W_j\big)\Big)^* = \sum_{1\le j\le n}\bar{\alpha}_j\big(j_t(W_j)\big)^* = \sum_{1\le j\le n}\bar{\alpha}_j j_t(W_j^*) \\
&= j_t\Big(\sum_{1\le j\le n}\bar{\alpha}_j W_j^*\Big) = j_t\big(w^*\big). \qquad (4.33)
\end{aligned}$$

On the other hand, let $w_1 = \sum_{1\le j\le n}\alpha_j W_j$; $w_2 = \sum_{1\le j\le n}\beta_j W_j$, then

$$\begin{aligned}
j_t(w_1 w_2) &= j_t\Big(\sum_{1\le j,k\le n}\alpha_j\beta_k W_j W_k\Big) = \sum_{1\le j,k\le n}\alpha_j\beta_k j_t\big(W_j W_k\big) \\
&= \sum_{1\le j,k\le n}\alpha_j\beta_k j_t(W_j) j_t(W_k) = \sum_{1\le j\le n}\alpha_j j_t(W_j)\sum_{1\le k\le n}\beta_k j_t(W_k) \\
&= j_t(w_1) j_t(w_2). \qquad (4.34)
\end{aligned}$$

Eqs. (4.33) and (4.34) imply that j_t is a $*$–homomorphism. From the expression of j_t in equation (4.21), we see that it is an injective map. Since j_t is a $*$–homomorphism of $*$–algebra and identity preserving then it is automatically continuous. Hence it can be extended to the C^*–algebra generated by the set of Weyl operators $\mathcal{W}_g(\mathcal{H})$. Hence it is a $*$–isomorphism from $\mathcal{W}_g(\mathcal{H})$ onto its image.

Step III. Here, we prove the $*$–property (4.31). We have

$$\begin{aligned}
\Big(j_t\big(W(\xi,T)\big)\Big)^* &= \Big(e^{i\hat{j}_t(H(\xi,T))}\Big)^* = e^{-i\left(\hat{j}_t(H(\xi,T))\right)^*} = e^{-i\hat{j}_t\left(\left(H(\xi,T)\right)^*\right)} \\
&= e^{-i\hat{j}_t\left(H(\xi,T)\right)} = j_t\big(e^{-iH(\xi,T)}\big) = j_t\Big(\big(W(\xi,T)\big)^*\Big).
\end{aligned}$$

Step IV. In this step we investigate to prove the multiplicative property (4.32). From Theorem (3.14) and linearity of j_t, we get

$$\begin{aligned}
j_t\Big(W(\xi_1,T_1)W(\xi_2,T_2)\Big) &= j_t\big(e^{i\gamma_{\xi,T}}W(\xi,T)\big) \\
&= e^{i\gamma_{\xi,T}} j_t\big(W(\xi,T)\big) \\
&= e^{i\gamma_{\xi,T}} W(\xi,T)\otimes\Gamma\big(U_t, e^{iT_t}, V_t, \zeta_t\big), \qquad (4.35)
\end{aligned}$$

where e^{iT_t}, U_t and V_t are as in Eqs. (4.25)–(4.27) and $\zeta_t = tL(e^{iT}-1)$. Similarly we obtain

$$\begin{aligned}
j_t\Big(W(\xi_1,T_1)\Big)j_t\Big(W(\xi_2,T_2)\Big) &= \Big(W(\xi_1,T_1)\otimes\Gamma\big(U_t^{(1)}, e^{iT_t^{(1)}}, V_t^{(1)}, \zeta_t^{(1)}\big)\Big) \\
&\times \Big(W(\xi_2,T_2)\otimes\Gamma\big(U_t^{(2)}, e^{iT_t^{(2)}}, V_t^{(2)}, \zeta_t^{(2)}\big)\Big) \\
&= e^{i\gamma_{\xi,T}}W(\xi,T)\otimes\Big[\Gamma\Big(U_t^{(1)}, e^{iT_t^{(1)}}, V_t^{(1)}, \zeta_t^{(1)}\Big) \\
&\times \Gamma\Big(U_t^{(2)}, e^{iT_t^{(2)}}, V_t^{(2)}, \zeta_t^{(2)}\Big)\Big],
\end{aligned}$$

where $U_t^{(j)}, e^{iT_t^{(j)}}, V_t^{(j)}$ and $\zeta^{(j)}$; $j = 1, 2$ are as U_t, e^{iT_t}, V_t and ζ_t respectively. Using Lemma (3.3), we get

$$j_t\Big(W(\xi_1, T_1)\Big) j_t\Big(W(\xi_2, T_2)\Big) = e^{i\gamma_{\xi,T}} W(\xi, T) \otimes \Gamma\big(U', e^{iT'}, V', \zeta'\big), \tag{4.36}$$

where

$$U' = U_t^{(1)} + e^{iT_t^{(1)}} U_t^{(2)} \quad ; \; e^{iT'} = e^{iT_t^{(1)}} e^{iT_t^{(2)}} \quad ; \; V' = V_t^{(2)} + e^{-iT_t^{(2)}} V_t^{(1)}$$

and

$$\zeta' = \zeta_t^{(1)} + \zeta_t^{(2)} + \langle V_t^{(1)}, U_t^{(2)} \rangle.$$

Then

$$\begin{aligned}
U' &= \chi_{[0,t]} \otimes \delta(e^{iT_1} - 1) + e^{\chi_{[0,t]} \otimes \rho(iT_1)} \Big(\chi_{[0,t]} \otimes \delta(e^{iT_2} - 1)\Big) \\
&= \chi_{[0,t]} \otimes \delta(e^{iT_1} - 1) + \chi_{[0,t]} \otimes e^{\rho(iT_1)} \delta(e^{iT_2} - 1) \\
&= \chi_{[0,t]} \otimes \Big(\delta(e^{iT_1} - 1) + \rho(e^{iT_1}) \delta(e^{iT_2} - 1)\Big) \\
&= \chi_{[0,t]} \otimes \Big(\delta(e^{iT_1} - 1) + \delta(e^{iT_1}(e^{iT_2} - 1)\Big) \\
&= \chi_{[0,t]} \otimes \delta(e^{iT_1} e^{iT_2} - 1) \\
&= \chi_{[0,t]} \otimes \delta(e^{iT} - 1) \\
&= U_t.
\end{aligned}$$

The same computations give $e^{iT'} = e^{iT_t}$, $V' = V_t$ and $\zeta' = \zeta_t$. Finally comparing Eqs. (4.35) and (4.36), one has (4.32). $\square$

5. The Generator of the Quantum Lévy Process

In this section, we use the notations of Section (2.2) and we consider the C^*-algebra $\mathcal{A}_0 = \mathcal{W}_g(\mathcal{H})$ and the stochastic process

$$j_t : \mathcal{A}_0 :\longrightarrow \mathcal{A}_{t]}$$

given on the generalized Weyl operators by Eq. (4.21). We have the following theorem:

Theorem 5.1. *The stochastic process j_t is a Markov cocycle.*

Proof. Clearly that $j_0\big(W(\xi, T)\big) = W(\xi, T) \otimes 1$ for all $(\xi, T) \in \mathcal{H} \otimes \mathcal{B}_s(\mathcal{H})$. Then by $*-$homomorphism properties we obtain $j_0(a) = a \otimes 1$ for all $a \in \mathcal{A}_0$. Let $\tilde{j}_t$ be as in Eq. (2.10) of Section (2.2). Then from (4.21), one has

$$j_t\big(W(\xi, T)\big) = W(\xi, T) \otimes e^{X_{[0,t]}},$$

where, in the notation (4.13):

$$X_{[0,t]} = i\Big(H_t(\delta(T), \rho(T)) + tL(T)1\Big).$$

Using Eqs. (2.7) and (2.8), one has

$$\begin{aligned}
\tilde{j}_s \circ u_s^\circ \circ j_t\Big(W(\xi,T)\Big) &= \tilde{j}_s \circ u_s^\circ\Big(W(\xi,T)\otimes e^{X_{[0,t]}}\Big)\\
&= \tilde{j}_s\Big(W(\xi,T)\otimes \Gamma(\theta_s)e^{X_{[0,t]}}\Gamma(\theta_s^*)\Big)\\
&= \tilde{j}_s\Big(W(\xi,T)\otimes 1_{[0,s]}\otimes e^{X_{[s,s+t]}}\Big)\\
&= \tilde{j}_s\Big(\big(W(\xi,T)\otimes 1\big)\big(1_0\otimes 1_{[0,s]}\otimes e^{X_{[s,s+t]}}\big)\Big)\\
&= \tilde{j}_s\Big(W(\xi,T)\otimes 1\Big)\tilde{j}_s\Big(1_0\otimes 1_{[0,s]}\otimes e^{X_{[s,s+t]}}\Big).
\end{aligned}$$

From Eq. (2.10), we get

$$\begin{aligned}
\tilde{j}_s \circ u_s^\circ \circ j_t\Big(W(\xi,T)\Big) &= j_s\Big(W(\xi,T)\Big)\Big(j_s(1_0)\otimes e^{X_{[s,s+t]}}\Big)\\
&= \Big(W(\xi,T)\otimes e^{X_{[0,s]}}\Big)\Big(1_0\otimes 1_{s]}\otimes e^{X_{[s,s+t]}}\Big)\\
&= W(\xi,T)\otimes e^{X_{[0,s]}}\otimes e^{X_{[s,s+t]}}\\
&= W(\xi,T)\otimes e^{X_{[0,s]}+X_{[s,s+t]}}\\
&= W(\xi,T)\otimes e^{X_{[0,s+t]}}\\
&= j_{s+t}\Big(W(\xi,T)\Big)
\end{aligned}$$

which proves the cocycle identity (2.11). □

Theorem 5.2. *Let P^t be the Markov semigroup associated to the process j_t via Eq. (2.12). Then its action on the generalized Weyl operators is given by*

$$P^t\big(W(\xi,T)\big) = e^{tL(e^{iT}-1)}W(\xi,T). \tag{5.1}$$

Moreover, $W(\xi,T)$ are eigenoperators of its generator G_0 so that

$$G_0\big(W(\xi,T)\big) = L(e^{iT}-1)W(\xi,T). \tag{5.2}$$

Proof. With help of Eqs. (2.12), (4.21), (2.9), (4.28) respectively, we get

$$\begin{aligned}
P^t\big(W(\xi,T)\big) &= E_{0]}j_t\big(W(\xi,T)\big) = E_{0]}\Big(W(\xi,T)\otimes W_t(\delta(T),\rho(T))e^{itL(T)}\Big)\\
&= \langle \Phi_{[0}, W_t(\delta(T),\rho(T))e^{itL(T)}\Phi_{[0}\rangle W(\xi,T)\\
&= e^{itL(T)}\langle \Phi_{[0}, W_t(\delta(T),\rho(T))\Phi_{[0}\rangle W(\xi,T)\\
&= e^{itL(T)}\langle \Phi_{[0}, e^{Z_t}\Phi_{[0}\rangle W(\xi,T)\\
&= e^{itL(T)+tL(e^{iT}-iT-1)}W(\xi,T)\\
&= e^{tL(e^{iT}-1)}W(\xi,T).
\end{aligned}$$

□

Remark 5.3. Taking the state L such that $L(T) = \langle u, Tu\rangle$, where u is a vector of $\mathcal{H}$ with norm equal to 1, Eq. (5.1) becomes

$$G_0\big(W(\xi,T)\big) = \langle u,(e^{iT}-1)u\rangle W(\xi,T) = \Big(\int_{\sigma(T)}(e^{ix}-1)\mu_{T,u}(dx)\Big)W(\xi,T), \tag{5.3}$$

where $\mu_{T,u}$ is the spectral measure of T associated to u. In the above equation, the eigenvalues of G_0

$$\eta := \int_{\sigma(T)} (e^{ix} - 1)\mu_{T,u}(dx),$$

are explicitly computed in terms of the Lévy-Khintchin factor of the underlying classical Lévy process and the generalized Weyl operators, associated to the oscillator Lie–algebra, are eigenoperators of the generator of the quantum Markov semigroups canonically associated to the quantum extension of the classical Lévy process induced by these Lie algebra shifts.

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