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BELL'S INEQUALITY VIOLATIONS: RELATION WITH DE FINETTI'S COHERENCE PRINCIPLE AND INFERENTIAL ANALYSIS OF EXPERIMENTAL DATA

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Dedicated to Professor K. R. Parthasarathy on the occasion of his 75th birthday

Abstract. It is often believed that de Finetti's coherence principle naturally leads, in the finite case, to the Kolmogorov's probability theory of random phenomena, which then implies Bell's inequality. Thus, not only a violation of Bell's inequality looks paradoxical in the Kolmogorovian framework, but it should violate also de Finetti's coherence principle. Firstly, we show that this is not the case: the typical theoretical violations of Bell's inequality in quantum physics are in agreement with de Finetti's coherence principle. Secondly, we look for statistical evidence of such violations: we consider the experimental data of measurements of polarization of photons, performed to verify empirically violations of Bell's inequality, and, on the basis of the estimated violation, we test the null hypothesis of Kolmogorovianity for the observed phenomenon. By standard inferential techniques we compute the p-value for the test and get a clear strong conclusion against the Kolmogorovian hypothesis.

1. Introduction

Three decades ago classical probability theory, studied by mathematicians in the classical axiomatic framework introduced by A.N. Kolmogorov [14], was already an established mathematical discipline with several deep results and a wide range of applications.

Nevertheless, quantum mechanics used for its statistical predictions a completely different mathematical model due to J. von Neumann [19] unifying the treatment of classical and quantum randomness in an elegant algebraic formalism. Functional analytic and operator theoretic tools of the von Neumann model were well developed, but its deeper probabilistic aspects (independence, conditioning, limit theorems, stochastic processes etc.) had not yet been investigated.

The first studies on quantum central limit theorems (W. von Waldenfels [12], R.L. Hudson [8]) and quantum Markov chains (L. Accardi [1]) in the seventies arose the interest of classical probabilists for the emerging quantum probability. As a

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result, the field grew quickly at the beginning of the eighties with several contributions which, after three decades, still go on finding new and fruitful applications in different fields of mathematics and quantum physics. Among them, quantum stochastic calculus developed by R.L. Hudson and K.R. Parthasarathy[13] was perhaps the most influential development, bringing new ideas and methods also to neighboring fields as classical probability (P.-A. Meyer [16], K.R. Parthasarathy [18]) and quantum physics, in particular in the study of quantum optical models and in the mathematical theory of quantum continual measurements (A. Barchielli and M. Gregoratti [5]).

The conceptual breakthrough marking the birth of quantum probability as a model for the laws of chance - not as mere mathematical (noncommutative) generalization of a classical theory - was L. Accardi's [2] discovery of statistical invariants: identities or inequalities on probabilities of events or expectations of random variables that can be observed in some experiment. Measurements turned out to violate some of these inequalities that are natural consequences of elementary rules of classical (Kolmogorovian) probability; L. Accardi explained this fact as an experimental evidence of the existence of several different (non-Kolmogorovian) models for chance, just as Euclidean and non-Euclidean geometry are different models for space. Thus quantum probability became a necessary consequence of experimental results.

The Kolmogorovian model, however, is very popular and it is still a common belief among statisticians and probabilists that all random phenomena can be fitted in its classical framework, where events and random variables of an experiment always live in a common probability space. Events can be represented by a family of subsets of a certain sample space with some structure, typically closed by logical operations, and random variables are functions on the sample space with their joint distributions determined by a given probability (measure) leading one to the construction of a Kolmogorovian model.

Quantum physics, however, provides examples of pairs of events that are not simultaneously observable leading to several counterintuitive obstructions. Indeed, it is impossible to say whether both occurred or not and, moreover, trying to fit them in a common probability space leads to paradoxes (see [4] for a detailed discussion). These problems, arising from the very basic and fundamental probabilistic notions, were pointed out by B. de Finetti [9]: "there exist real problems which arise in various connections with the notion of the 'verifiability' of an event; a notion which is often vague and elusive. Strictly speaking, the phrase itself is an unfortunate one because verifiability is the essential characteristic of the definition of an event (to speak of an 'unverifiable event' is like saying 'bald with long hair'). It is necessary, however, to recognize that there are various degrees and shades of meaning attached to the notion of verifiability. [...] The most precise and important, however, is that which arises in theoretical physics in connection with observability and complementarity. It seems strange that a question of such overwhelming interest, both conceptually and practically (and concerning the most unexpected and deep forms of application of probability theory to the natural sciences), should be considered, by and large, only by physicists and philosophers, whereas it is virtually ignored in treatments of the calculus of probability. We agree that it is a new element, whose introduction upsets the existing framework, making it something of a hybrid. We see no reason, however, to prefer tinkering about with bogus innovations rather than enriching the existing structure by incorporating stimulating refinements (disruptive though they may be)."

Quantum physics provides examples of random variables, related to the same random phenomenon, that can not be simultaneously observed. Trying to represent them on a common probability space, assuming then that they have a joint distribution, leads quickly to intriguing contradictions.

The best known concern the Bell's inequality (2.1) which is an upper bound for a certain combination b of the mixed second moments of four random variables taking values ± 1; this can be at most 2, independently of their joint distribution. Anyway, it is violated both theoretically, by quantum mechanics, and experimentally, by measurements on some physical systems. Indeed, there are quantum systems with four random variables in which the theoretical value is $b = 2\sqrt{2}$. At the same time, several experiments, repeated several times with more and more accuracy, turn out to give statistical estimates of b bigger than 2.5.

There are now many versions of Bell's inequality. The original one appeared for the first time in 1964 in a paper by J.S. Bell [6] on the Einstein - Podolsky - Rosen paradox [10]. Since then it has been at the center of much active interest as its theoretical and experimental violation is discussed in connection with foundations and interpretations of quantum mechanics and limits of classical Kolmogorovian probability. In 1981 Accardi [2] first interpreted the violation of Bell's inequality as an evidence that the four random variables involved could not fit in the same classical probability space and, as a consequence, this proved the inadequacy of the classical Kolmogorov model for dealing with quantum mechanical randomness. This made also clear that the von Neumann model is indeed a wider axiomatic framework for the laws of chance and the parameter b can be looked at as a statistical invariant; when $b \leq 2$ the four random variables can be defined on the same classical probability space, when $2 < b \leq 2\sqrt{2}$ this is not the case. The name 'statistical invariant' is justified by the analogy with the sum of inner angles of a triangle allowing one to distinguish between Euclidean and non-Euclidean geometries.

The aim of the paper is not at all to review the subject of Bell's inequality violations. As an homage to the fundamental and deep contributions brought by K.R. Parthasarathy from classical to quantum probability, we re-examine the topic for an audience of (classical) probabilists and statisticians, without any need of notions of quantum mechanics, trying to attract some of their interest. We look at the subject from two different points of view, one theoretical / probabilistic, the other experimental / statistical, in order to convince the reader that violations of Bell's inequality are not paradoxical in principle and, moreover, can really happen in practice.

First, we explain why, even if violations of the Bell's inequality are incompatible with a Kolmogorovian interpretation of randomness, they do not necessarily clash with de Finetti's coherence principle. The reader will be convinced that such violations are not paradoxical, but that they simply reveal the need of a theory of randomness more general than Kolmogorov's one. In particular, the analysis of de Finetti's coherence principle and the examples of non Kolmogorovian random phenomena outside quantum physics should convince the reader that probability theories do not have to be necessarily Kolmogorovian and that generalizations deserve his interest.

Second, once removed objections of principle against non Kolmogorovian probabilities, we consider data from a celebrated physical experiment [20] in order to convince the reader that random phenomena really violating Bell's inequality do exist. This allows us to obtain two goals at the same time. A first goal is to analyze thoroughly the relationship between the random phenomenon under consideration on one side, which could be non Kolmogorovian, and the related observed data on the other side, which are anyway Kolmogorovian, in some sense. Thus, the second goal is to exploit this feature to analyze the experimental data by standard methods of statistical inference.

The data analysis is performed by a suitable hypothesis test (surprisingly not common at all in the physical literature): the null hypothesis corresponds to the Kolmogorovianity of the random phenomenon under consideration and the critical region is based on the estimated violation of the Bell's inequality. The asymptotic p-value of the data is computed, leading to a clear rejection of the null hypothesis.

2. Violation of Bell's Inequality vs Kolmogorov's Probability Theory

There exist various formulations of Bell's inequality. Let us state and prove a rather general form of the Clauser, Horne, Shimony, Holt version [7].

Theorem 2.1. *Let X_1, X_2, Y_1, Y_2 be random variables taking values ± 1 on a measurable space $(\Omega, \mathcal{F})$. Then the following inequality holds for every probability $\mathbb{P}$ on $\mathcal{F}$:*

$$b = \Big| \mathbb{E}\, X_1 Y_1 + \mathbb{E}\, X_1 Y_2 \Big| + \Big| \mathbb{E}\, X_2 Y_1 - \mathbb{E}\, X_2 Y_2 \Big| \leq 2. \tag{2.1}$$

Proof. Since $|Y_\ell| = 1$, the following equalities hold:

$$\Big| Y_1 - Y_2 \Big| = 1 - Y_1 Y_2, \qquad \Big| Y_1 + Y_2 \Big| = 1 + Y_1 Y_2.$$

Then, as $|X_k| = 1$,

$$\begin{aligned} b &= \Big| \mathbb{E}\, X_1 Y_1 + \mathbb{E}\, X_1 Y_2 \Big| + \Big| \mathbb{E}\, X_2 Y_1 - \mathbb{E}\, X_2 Y_2 \Big| \\ &= \Big| \mathbb{E}\, X_1 (Y_1 + Y_2) \Big| + \Big| \mathbb{E}\, X_2 (Y_1 - Y_2) \Big| \\ &\leq \mathbb{E}\left[|X_1|\, |Y_1 + Y_2| + |X_2|\, |Y_1 - Y_2| \right] \\ &= \mathbb{E}\left[1 + Y_1 Y_2 + 1 - Y_1 Y_2 \right] \\ &= 2. \end{aligned}$$

□

Thus *Bell's inequality* (2.1) necessarily holds under the only hypothesis that four random numbers taking values ± 1 exist in a same random experiment, independently of their joint distribution. When four random numbers can be modeled with a term $(\Omega, \mathcal{F}, \mathbb{P}, X_1, X_2, Y_1, Y_2)$, as in the hypothesis of Theorem 2.1, we say that they admit a common Kolmogorov's probability model. Note that we do not need σ-additivity of $\mathbb{P}$, but only additivity.

In any case b is a parameter, that we shall call *Bell's parameter*. It depends on the bivariate distributions of the four pairs X_k, Y_ℓ, $k, \ell = 1, 2$.

Consider now an experiment on a physical system consisting of two particles. On each particle we can measure two quantities of modulus 1, getting the results $X_1 = \pm 1$, $X_2 = \pm 1$ on the first and $Y_1 = \pm 1$, $Y_2 = \pm 1$ on the second. Typically, X_1, X_2, Y_1, Y_2 are random results whose distribution depends on the procedure used to prepare the pair of particles for the experiment, that is on the initial state of the pair of particles.

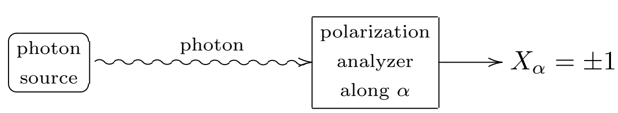
From a "classical point of view", they are the values of four quantities, which exist independently of some eventual constraint which could prevent us from measuring all of them simultaneously. Thus the four random results admit a common Kolmogorov's probability model, they do have a joint distribution and, whatever it is, their second mixed moments must satisfy Bell's inequality.

Nevertheless, there are experiments on elementary particles where "the classical point of view" is contradicted both theoretically, by quantum physics, and experimentally, by measured data. The most important is found by considering a pair of photons and, for each photon, measuring its polarization X_α (resp. Y_β) along two given angles α_1 and α_2 (resp. β_1 and β_2).

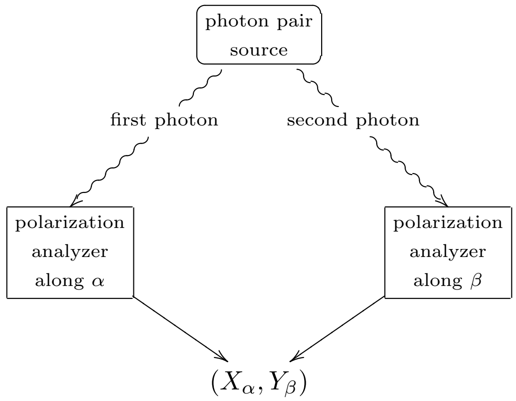
For the purpose of the paper we do not need to know what polarization is, but it suffices to keep in mind that:

(1) the polarization X_α of a photon is measured in the plane perpendicular to its speed, along a direction which is fixed by an angle α, $-\pi/2 \leq \alpha < \pi/2$;
(2) X_α is a random variable taking values ± 1;
(3) the distribution of X_α depends on the choice of α and on the photon initial state.

Moreover, only one angle α per measurement can be chosen: it is not physically possible to measure the polarization of a same photon along two different angles α_1 and α_2 simultaneously.



Therefore, given a photon pair, we can choose an angle for each photon and observe a pair X_α, Y_β. These are two random variables whose joint distribution $f_{(X_\alpha, Y_\beta)}$ depends on the choice of the angles α and β and on the preparation of the photon pair. Typically, if the preparation of two photons produces some relationship between their states (e.g. when they are emitted simultaneously by a common source), then X_α and Y_β turn out to be correlated.



In this situation, we shall fix four angles $\alpha_1, \alpha_2, \beta_1, \beta_2$ and consider

$$\begin{aligned} X_1 &= \text{polarization of the first photon along } \alpha_1, \\ X_2 &= \text{polarization of the first photon along } \alpha_2, \\ Y_1 &= \text{polarization of the second photon along } \beta_1, \\ Y_2 &= \text{polarization of the second photon along } \beta_2. \end{aligned} \tag{2.2}$$

For every repetition of the experiment, that is for every preparation of a photon pair, we can observe a pair X_k, Y_ℓ.

Quantum mechanics predicts that, if the photon pair is suitably prepared (Bell state, following the terminology in Quantum Information [17]), the random results X_α and Y_β have the joint distribution

$$f_{\alpha,\beta}(i,j) = \mathbb{P}(X_\alpha = i, Y_\beta = j) = \begin{cases} \frac{1}{2}\sin^2(\beta-\alpha), & \text{if } ij = +1, \\ \frac{1}{2}\cos^2(\beta-\alpha), & \text{if } ij = -1, \end{cases} \tag{2.3}$$

so that both X_α and Y_β are uniformly distributed, $\mathbb{E}\, X_\alpha = \mathbb{E}\, Y_\beta = 0$, and

$$\mathbb{E}\, X_\alpha Y_\beta = \mathrm{Cov}(X_\alpha, Y_\beta) = -\cos 2(\beta - \alpha). \tag{2.4}$$

Thus, if the photon pair is prepared in the Bell state and if we choose

$$\alpha_1 = \pi/8, \quad \alpha_2 = 3\pi/8, \quad \beta_1 = \pi/4, \quad \beta_2 = 0, \tag{2.5}$$

then Bell's inequality is violated because

$$\begin{aligned} f_{\alpha_k,\beta_\ell}(i,j) &= \frac{1}{4}\left(1 - ij\frac{\sqrt{2}}{2}\right), \qquad \mathbb{E}\, X_k Y_\ell = -\frac{\sqrt{2}}{2}, \qquad (k,\ell) \neq (2,2), \\ f_{\alpha_2,\beta_2}(i,j) &= \frac{1}{4}\left(1 + ij\frac{\sqrt{2}}{2}\right), \qquad \mathbb{E}\, X_2 Y_2 = \frac{\sqrt{2}}{2}, \end{aligned} \tag{2.6}$$

so that

$$b = 2\sqrt{2}.$$

Therefore, the bivariate joint distributions (2.6) predicted by quantum mechanics are incompatible with the existence of a quadrivariate joint distribution for X_1, X_2, Y_1, Y_2, that is with the existence of the four polarizations X_1, X_2, Y_1, Y_2 at every replicate of the experiment, independently of the pair X_k, Y_ℓ that is actually measured. This is just the non Kolmogorovian character of the Bell's correlations (2.4) that was first pointed out by L. Accardi [2].

We could conclude that it does not make sense to assume the distributions (2.3) as they do not share a common Kolmogorovian model. Just as if one wanted to introduce three random variables X_1, X_2, X_3 with variances $\operatorname{Var} X_k = 1$ and covariances $\operatorname{Cov}(X_k, X_\ell) = -0.8$, $k \neq \ell$, so that the correlation matrix would have one negative eigenvalue.

Otherwise, we could conclude that, actually, there are only four different random experiments and four different Kolmogorov's probability models, one for each pair X_k, Y_ℓ. In this case however, the random variable X_1 in the pair (X_1, Y_1) would have no relation with the random variable X_1 in the pair (X_1, Y_2), even if they describe the same measurement on the first photon. Similarly, the four bivariate distributions $f_{(X_k, Y_\ell)}$ would remain without a clear relation among them, notwithstanding they originate from the same random situation (the same preparation of a photon pair). Then the only bound for the Bell parameter would be the trivial one: $b \leq 4$.

Thus the following questions are natural. Does it really make sense to assume the distributions (2.3)? Is it possible to introduce a unique model, where a unique object describes X_α, independently of the polarization measurement on the second photon, and where a unique object describes the randomness in the preparation of the photon pair and then generates the distributions $f_{(X_\alpha, Y_\beta)}$ of all the possible polarization measurements? Which relationships among the different $f_{(X_\alpha, Y_\beta)}$ would then be implied?

The next section answers affirmatively to the first question. Let us mention that quantum probability provides indeed a positive answer to the second question (see for example Accardi[2], Kümmerer and Maassen [15]). And, en passant, if we could consider all the possible initial states of the photon pair and all the possible choices of angles $\alpha_1, \alpha_2, \beta_1, \beta_2$, then we would get the *quantum Bell's inequality*: $b \leq 2\sqrt{2}$.

To conclude the section, let us stress that, in order to violate Bell's inequality, it is fundamental that X_α and its distribution do not depend on β and, analogously, that Y_β and its distribution do not depend on α. Otherwise, chosen the angles (2.5), we would get 8 random variables instead of 4, and so there would be no reason for Bell's inequality to hold. This assumption forbids any influence of the measurement over one photon on the measurement over the other one.

3. Violation of Bell's Inequality vs de Finetti's Coherence Principle

De Finetti's subjective approach to probability theory introduces the notion of probability and clarifies its meaning by means of the paradigm of bets and the notion of coherent evaluation [9].

Given a family $\mathcal{E}$ of events which could occur or not in a random experiment, the probability of $E \in \mathcal{E}$ is the price $\mathbb{P}(E)$ of a bet on E with payoff 1_E, that is 1 if E is observed to occur, 0 if E is observed not to occur. Chosen n events $E_1, \ldots, E_n \in \mathcal{E}$, a finite combination of bets on them, with amounts $c_i \, \mathbb{P}(E_i)$,

$c_i \neq 0$, determines the random total gain for the bank

$$G = \sum_{i=1}^{n} c_i \Big(\mathbb{P}(E_i) - 1_{E_i} \Big). \tag{3.1}$$

De Finetti's **coherence principle** states that the prices $\mathbb{P} : \mathcal{E} \to \mathbb{R}$ have to be fixed so that there is no combination of bets with surely positive (or surely negative) gain. That is, for every finite class $\{E_1, \ldots, E_n\}$ of events in $\mathcal{E}$ and every non vanishing $c_1, \ldots, c_n$, a probability $\mathbb{P}$ must give

$$\min G \leq 0 \leq \max G,$$

where the minimum and the maximum gain are computed with respect to the possible logical values of $E_1, \ldots, E_n$.

The coherence principle does not uniquely determine $\mathbb{P}$, but it implies some of its properties:

(a) $0 \leq \mathbb{P}(E) \leq 1$ for every $E \in \mathcal{E}$,
(b) the probability of the certain event Ω is $\mathbb{P}(\Omega) = 1$ and the probability of the impossible event $\emptyset$ is $\mathbb{P}(\emptyset) = 0$,
(c) chosen n events $E_1, \ldots, E_n \in \mathcal{E}$ such that there exists the logic sum $\bigvee_{i=1}^{n} E_i \in \mathcal{E}$ and there exist the logic products $E_i \wedge E_j = \emptyset$ for every $i \neq j$, it holds $\mathbb{P}\left(\bigvee_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mathbb{P}(E_i)$.

These consequences are therefore necessary conditions for coherence, but, typically, they are not sufficient to guarantee that a function $\mathbb{P} : \mathcal{E} \to \mathbb{R}$ is coherent. Anyway this happens if the family of events $\mathcal{E}$ is a field of subsets of a given nonempty space Ω. Thus, a Kolmogorov's probability model satisfies the coherence principle, but de Finetti approach to probability theory leads only to finite additivity, not necessarily to σ-additivity. Nevertheless, when the events form a finite field $\mathcal{E}$, additivity and σ-additivity are equivalent, so that de Finetti's and Kolmogorov's approaches lead to the same mathematical model.

The assumption that $\mathcal{E}$ is a field, which is closely related with the notion of event, however, can not be always taken for granted, even in the finite case.

Basing the introduction of probability on the bet paradigm, de Finetti discusses deeply the notion of event and its essential feature of verifiability. By the end of a random experiment, every event should be assigned the value "true" (occurred) or "false" (not occurred), so to determine the gain of the gambler and of the bank.

In [9] verifiability is discussed in relation with precision, time, cost and number of partial verifications. It is also discussed in relation with conditioning: the verifiability of an event E can be conditional on the occurrence of another event H, so that a bet on E is won if H and E are true, lost if H is true and E false, annulled if H is false. For example, considered a piece of wood (given or randomly chosen from a given pile), the two events could be

$$E = \text{“The piece of wood burns in the fire in less than 15 minutes”},$$
$$H = \text{“The piece of wood is thrown in the fire”}.$$

Typically, H is "An observation is made to verify whether E occurs or not".

Furthermore, de Finetti discusses verifiability in relation with what he calls "indeterminism" / "observability" and "complementarity" in quantum mechanics. Indeed, similarly to conditioning, an event E regarding elementary particles typically is not simply true or false at the end of the experiment, but it can be also indeterminate, meaningless, if an appropriate measurement has not been performed. Moreover, two events E_1 and E_2 can be complementary in the sense that they can not be simultaneously verified, in a same experiment, so that one of them necessarily remains indeterminate.

De Finetti discusses the implications of indeterminism and complementarity on logic operations among events, shows similar behaviors outside quantum mechanics (for example, the behavior of a same object subjected to one or the other of two different destructive experiments), but he does not analyze the consequences with the coherence principle.

Of course, a bank could be asked to assign the prices $\mathbb{P}$ to a family $\mathcal{E}$ containing complementary events. Note that if two events E_1 and E_2 in $\mathcal{E}$ are complementary, then there exists no event $E_1 \wedge E_2$ as there is no way to verify it. Then $\mathcal{E}$ can not be a field and, consequently, a coherent probability on $\mathcal{E}$ does not have to be a Kolmogorovian probability. Moreover, some combinations of bets on events in $\mathcal{E}$ are not admissible: if a combination mixes two or more complementary events, at least one can not be verified, it remains indeterminate and the corresponding bet has to be nullified and so the whole combination.

Therefore we should specify the **coherence principle** as follows:

for every finite class $\{E_1, \ldots, E_n\}$ of *non complementary* events in $\mathcal{E}$ and every non vanishing $c_1, \ldots, c_n$, a probability $\mathbb{P}$ must give

$$\min G \leq 0 \leq \max G, \tag{3.2}$$

where the minimum and maximum gain are computed with respect to the possible logical values of $E_1, \ldots, E_n$.

For example, consider again the piece of wood mentioned above and the bets on

$$E_1 = \text{“The piece of wood burns in the fire in less than 15 minutes”},$$
$$E_2 = \text{“The piece of wood reaches the bottom of the swimming pool in less than 15 hours”}.$$

This is an example of complementary events outside quantum mechanics. These events can not be checked simultaneously, in a same random experiment, for the same piece of wood, so that there can be no combination of bets on E_1 and E_2 for the same piece of wood, and thus the coherence principle implies no relation between $\mathbb{P}(E_1)$ and $\mathbb{P}(E_2)$.

Polarization measurements produce just a similar situation. Given one photon pair, we can not bet on any combination of events regarding X_1, X_2, Y_1, Y_2, but only on events regarding a chosen pair X_k, Y_ℓ. Therefore, each bet involves only one of their bivariate distributions and, as each f_{α_k,β_ℓ} in (2.6) is a regular bivariate distribution, the prices (2.6) do not violate de Finetti's coherence principle. Thus it makes sense to assume distributions (2.3) for polarization measurements on a photon pair.

Let us stress that, when applying coherence principle, the logical relationships among the events in $\mathcal{E}$ are fundamental and play a double role because, first, they establish which combinations of bets are admissible and, second, they determine $\min G$ and $\max G$ in (3.2).

Of course, these logical relations go far beyond the set structure of $\mathcal{E}$. Take, for example, four jointly observable random variables X_1, X_2, Y_1, Y_2 taking values ± 1, and suppose that the bank decides to allow bets only on events regarding pairs X_k, Y_ℓ. Even if from a set-theoretical point of view the family $\mathcal{E}$ is the same as with polarization measurements, in this case the prices (2.6) are not coherent. Indeed, all the events

$$E_1 = (X_1 = Y_1), \quad E_2 = (X_1 = Y_2), \quad E_3 = (X_2 = Y_1), \quad E_4 = (X_2 = -Y_2),$$

belong to $\mathcal{E}$, according to (2.6) they all have probability $\mathbb{P}(E_i) = (2 - \sqrt{2})/4$, but, since now they can be jointly verified, a gambler can bet $c_i = 1$ on each E_i, producing the bank gain

$$G = 2 - \sqrt{2} - \sum_{i=1}^{4} 1_{E_i} < 0,$$

as at least one event must occur: if E_1, E_2 and E_3 are false, then E_4 is necessarily true.

On the contrary, in the case of polarization measurements, these events are complementary and so the logical relationships among them just forbid to consider this combination of bets, thus preserving the coherence principle also for prices (2.6).

As we know, the distributions (2.6) violate Bell's inequality, but there is no paradox. Indeed, the Bell parameter b mixes random variables which can not be simultaneously observed, i.e. it mixes expected values from different potential experiments on a same photon pair which can not be performed simultaneously. The paradox arises only if we ask for a common Kolmogorov's probability model for X_1, X_2, Y_1, Y_2. Since we can not observe them simultaneously, it is not absolutely necessary to ask for such a common model. This requirement typically arises from the idea that X_1, X_2, Y_1, Y_2 are the values of quantities existing independently of the measurement. The paradox is avoided by considering X_1, X_2, Y_1, Y_2 responses to some physical stimulation. The responses do not exist without the corresponding stimulation [4].

4. Non Kolmogorovian Random Phenomenon vs Kolmogorovian Data

Once we are convinced that violations of Bell's inequality are theoretically admissible, the main questions are: do polarization measurements on a photon pair really violate Bell's inequality? In other words, are they really incompatible with a Kolmogorovian description? Which kind of experimental data can we collect? How to analyze them?

Let us underline that Bell's inequality concerns the Bell parameter b, which depends on the distributions of the polarizations X_k, Y_ℓ, $k, \ell = 1, 2$. Thus, if we consider the problem of an experimental verification, then b and these distributions are considered unknown, and the problem turns out to be a genuine inferential problem.

We shall now analyze thoroughly the relationship between, on one hand, our random phenomenon possibly non Kolmogorovian (polarization measurements on a photon pair), and, on the other hand, the data which can be obtained in a physical experiment and then used for statistical inference.

Given a photon pair, we fix four angles $\alpha_1, \alpha_2, \beta_1, \beta_2$ and we consider again the four potential polarization measurements X_1, X_2, Y_1, Y_2 introduced in (2.2). As we have discussed, we can actually perform only one measurement per photon, so that we can observe only one of the four pairs of random variables

$$(X_k, Y_\ell) \sim f_{k\ell}, \qquad k, \ell = 1, 2. \tag{4.1}$$

The distributions $f_{k\ell}$ depend on the state of the photon pair. We want to make inferences about the quadruple $(f_{11}, f_{12}, f_{21}, f_{22})$.

According to quantum mechanics, if the photon pair is prepared in the Bell state and if $\alpha_1, \alpha_2, \beta_1, \beta_2$ are chosen as in (2.5), then the distributions $f_{11}, f_{12}, f_{21}, f_{22}$ are given by (2.6), they do violate Bell's inequality and they can not be the bivariate marginals of a joint distribution $f_{(X_1,X_2,Y_1,Y_2)}$.

We want to verify exactly if X_1, X_2, Y_1, Y_2 admit a common Kolmogorov's probability model, looking for a strong conclusion against it. Therefore we introduce the statistical hypotheses

$$\begin{aligned} H_0 &: f_{11}, f_{12}, f_{21}, f_{22} \text{ are bivariate marginals of a joint distr. } f_{(X_1,X_2,Y_1,Y_2)}, \\ H_1 &: f_{11}, f_{12}, f_{21}, f_{22} \text{ do not admit a joint distribution } f_{(X_1,X_2,Y_1,Y_2)}. \end{aligned} \tag{4.2}$$

On the basis of a unique measurement on a unique photon pair the inference would be quite hard. Thus, just as in common statistical practice, let us introduce a sample. The peculiar feature in this situation is the structure of the sample. First, we introduce N photon pairs, independent and identically prepared in the same state. Then, for each photon pair we chose a polarization measurement X_k, Y_ℓ. This produces a sample of independent bivariate random variables, consisting of four subgroups,

$$\begin{array}{lll} (X_1^{(i)}, Y_1^{(i)}), & i = 1, \dots, n_{11}, & \text{i.i.d.} \sim f_{11}, \\ (X_1^{(i)}, Y_2^{(i)}), & i = n_{11} + j, \quad j = 1, \dots, n_{12}, & \text{i.i.d.} \sim f_{12}, \\ (X_2^{(i)}, Y_1^{(i)}), & i = n_{11} + n_{12} + j, \quad j = 1, \dots, n_{21}, & \text{i.i.d.} \sim f_{21}, \\ (X_2^{(i)}, Y_2^{(i)}), & i = n_{11} + n_{12} + n_{21} + j, \quad j = 1, \dots, n_{22}, & \text{i.i.d.} \sim f_{22}, \end{array} \tag{4.3}$$

where $n_{11} + n_{12} + n_{21} + n_{22} = N$.

Let us stress that every polarization measurement $(X_k^{(i)}, Y_\ell^{(i)})$ is performed on a different photon pair and, because of the independence, there is no problem of compatibility between their marginal distributions, even if the different polarization measurements on a same photon pair could admit no common Kolmogorovian description. Thus, the sample (4.3) is an ordinary sample of independent bivariate random variables. In other words (see Gill [11]), as the sample (4.3) describes the random data from a realizable physical experiment, it is Kolmogorovian!

Moreover, the distribution of the sample (4.3) depends on the same quadruple of bivariate distributions $(f_{11}, f_{12}, f_{21}, f_{22})$ introduced in (4.1) for a single photon pair. Then the statistical model of the sample (4.3) depends on the range of possible values of $(f_{11}, f_{12}, f_{21}, f_{22})$ and an inference on the sample (4.3) produces an inference on the set of potential polarization measurements X_1, X_2, Y_1, Y_2 on a single photon pair. Thus, our statistical hypotheses H_0 and H_1 (4.2) can be seen as hypotheses on the Kolmogorovian sample (4.3) and standard inferential methods can be applied.

5. Inferential Analysis of Bell Test Experiments

Typically, Bell test experiments are performed to estimate the Bell parameter b, find an estimate bigger than 2, and so to conclude that no Kolmogorov's probability model can describe the observed phenomenon (polarization measurements on a photon pair, in our case).

This looks just like a test hypothesis reasoning, but, nevertheless, the data analysis in physical literature never goes far beyond a mere point estimate of b.

We want to afford the problem just as a hypothesis test problem for the null and alternative hypotheses H_0 and H_1 (4.2), to introduce a critical region based just on the estimate of b, and then to reject H_0 on the basis of the (asymptotic) p-value of physical data.

We consider a sample of observations as introduced in (4.3). Then the typical point estimator of b is the statistic

$$B = \left| \frac{1}{n_{11}} \sum_i X_1^{(i)} Y_1^{(i)} + \frac{1}{n_{12}} \sum_i X_1^{(i)} Y_2^{(i)} \right| + \left| \frac{1}{n_{21}} \sum_i X_2^{(i)} Y_1^{(i)} - \frac{1}{n_{22}} \sum_i X_2^{(i)} Y_2^{(i)} \right|. \tag{5.1}$$

Because of Theorem 2.1, we know that $H_0 \Rightarrow b \leq 2$ and that $b > 2 \Rightarrow H_1$. Therefore, in order to test H_0 vs H_1 on the basis of B, we introduce the critical region

$$B > s,$$

where $s > 2$. Of course, the size of the test is $\alpha = \sup_{H_0} \mathbb{P}(B > s)$ and, given a realization of the sample with estimate $\widehat{b}$ of b, the p-value of the data is

$$p = \sup_{H_0} \mathbb{P}(B > \widehat{b}).$$

In order to compute this p-value, let us introduce the probabilities

$$p_{k\ell} = \mathbb{P}(X_k = Y_\ell).$$

Then

$$\mathbb{E}[X_k Y_\ell] = 2p_{k\ell} - 1, \qquad \mathrm{Var}[X_k Y_\ell] = 4\, p_{k\ell}(1 - p_{k\ell}),$$
$$b = |2(p_{11} + p_{12}) - 2| + |2(p_{21} - p_{22})|.$$

Furthermore, the distribution of B, and thus the probability $\mathbb{P}(B > \widehat{b})$, depends only on $\mathbf{p} = (p_{11}, p_{12}, p_{21}, p_{22})$. The possible values of $\mathbf{p}$ depend on the statistical model for the sample, that is on the possible values of the quadruple of distributions $(f_{11}, f_{12}, f_{21}, f_{22})$. In particular, if one parametrizes the quadrivariate distributions $f_{(X_1,X_2,Y_1,Y_2)}$, it turns out that the values of $\mathbf{p}$ compatible with the null hypothesis H_0 are

$$p_{11} = \theta_1 + \theta_2 + \theta_3 + \theta_4, \qquad p_{12} = \theta_1 + \theta_2 + \theta_5 + \theta_6,$$
$$p_{21} = \theta_1 + \theta_3 + \theta_6 + \theta_7, \qquad p_{22} = \theta_1 + \theta_4 + \theta_5 + \theta_7,$$

with

$$\theta = (\theta_1, \dots, \theta_7) \in \Theta_0 = \left\{ \vartheta_i \geq 0 \; \forall i, \quad \sum_{i=1}^{7} \vartheta_i \leq 1 \right\}.$$

Thus, under the null hypothesis, $\mathbb{P}(B > \widehat{b})$ is a function of θ and

$$p = \sup_{H_0} \mathbb{P}(B > \widehat{b}) = \sup_{\theta \in \Theta_0} \mathbb{P}(B > \widehat{b}).$$

Then the asymptotic p-value can be easily computed. Every addendum in (5.1) is asymptotically normal by the Central Limit Theorem,

$$\frac{1}{n_{k\ell}} \sum_i X_k^{(i)} Y_\ell^{(i)} \sim AN\left(2p_{k\ell} - 1, \frac{4\, p_{k\ell}(1 - p_{k\ell})}{n_{k\ell}}\right);$$

thus, if $2(p_{11} + p_{12}) - 2 \neq 0$ and $2(p_{21} - p_{22}) \neq 0$, the Delta Method gives the asymptotic normality also of the estimator B,

$$B \sim AN\left(b, \sum_{k,\ell=1}^{2} \frac{4\, p_{k\ell}(1 - p_{k\ell})}{n_{k\ell}}\right); \tag{5.2}$$

therefore the asymptotic value of $\mathbb{P}(B > \widehat{b})$ is immediately found as a function of θ and the asymptotic p-value is got by a numerical computation of $\sup_{\theta \in \Theta_0}$.

Let us compute the asymptotic p-value of the data from the experiment performed on the 1st of May 1998 in Innsbruck by Gregor Weihs *et al.* (scan blue experiment) [20].

They performed polarization measurements on hundreds of thousands of photon pairs, prepared at least approximately in the Bell state, spanning a lot of angles α and β. For the first time they could avoid any possible influence of β on X_α and of α on Y_β, which is a fundamental condition to violate Bell's inequality, as discussed at the end of Sect. 2. Indeed, the two photons of each pair were spatially separated, before of the polarization measurements, and, moreover, the angles α and β were selected randomly and independently at the very last moment, so to exclude any mutual influence within the realm of Einstein relativity. The two photons of each pair were sent to two different experimental stations, each one registering the photon arrival time, the corresponding angle α (resp. β) of measurement and the corresponding result X_α (resp. Y_β). Because of the low efficiency of the apparata, a lot of photons were lost, and the arrival times are fundamental to couple the data of the same pairs: two photons belong to the same pair if they arrive "simultaneously". Following physical analysis of these data, we calculate coincidences with a time window of 4 ns (which, actually, is even smaller of the 6 ns window used by Weihs *et al.*) and we assume that grouping the results of the measurements on the basis of (α, β) gives independent random samples.

We analyze the data from the experiments scanblue1 – scanblue20. Here we can not find the angles (2.5), for which quantum mechanics foresees $b = 2\sqrt{2} \simeq 2.8284$ in the Bell state. Anyway we can analyze the data of polarization measurements performed along the angles $\alpha_1 = 3\pi/20,\ \alpha_2 = 2\pi/5,\ \beta_1 = \pi/4,\ \beta_2 = 0$,which also give a good theoretical Bell parameter, $b \simeq 2.7936$, bigger enough than 2 to hope for an evident experimental violation of Bell's inequality.

Experimental data consist of $N = 4099$ observations which are summarized by the following table. Each column corresponds to one subgroup of the sample, according to the value of (k, ℓ), that is of (α_k, β_ℓ). For each subgroup, the frequency of the possible values of (X_k, Y_ℓ) is registered, together with the total size $n_{k\ell}$ of the subgroup and the corresponding estimate of $\mathbb{E}\, X_k Y_\ell$. The last row gives the corresponding sample estimate of the Bell parameter b.

		$\alpha_k\,,\,\beta_\ell$			
		$3\pi/20\,,\,\pi/4$	$3\pi/20\,,\,0$	$2\pi/5\,,\,\pi/4$	$2\pi/5\,,\,0$
$X_k\,,\,Y_\ell$	$1\,,\,1$	55	138	187	486
	$1\,,\,-1$	541	338	412	43
	$-1\,,\,1$	540	431	220	19
	$-1\,,\,-1$	67	107	122	393
	$n_{k\ell}$	1203	1014	941	941
	$\sum_i X_k^{(i)} Y_\ell^{(i)} / n_{k\ell}$	-0.797174	-0.516765	-0.343252	0.868225
	B	2.525416			

Then we get the asymptotic p-value $= 7.4857 \cdot 10^{-8}$. Other data from the same experiments, related to the polarization measurements performed along the angles $\alpha_1 = \pi/10, 3\pi/20$ and $\alpha_2 = 2\pi/5, 7\pi/20$, give even smaller p-values.

Thus the statistical analysis leads to a clear rejection of the null hypothesis: given a photon pair, there is no common Kolmogorov's probability model $(\Omega, \mathcal{F}, \mathbb{P}, X_1, X_2, Y_1, Y_2)$ for the polarization measurements X_1, X_2, Y_1, Y_2.

Of course, to conclude against H_0 is not necessarily a point in favor of quantum probability. Not only the alternative hypothesis H_1 is wide, but one could also doubt some of the assumptions taken for granted both under H_0 and H_1.

Our aim was to introduce the core of the problem with violations of Bell's inequality, but the debate is very much wider and always vivid in the physical literature, notwithstanding the experimental results, as every assumption and conclusion is never taken for granted, and every possible fault in experimental setups, procedures and equipments is always under discussion.

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