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# SOME PROBLEMS INVOLVING AIRY FUNCTIONS

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*Dedicated to Professor K. R. Parthasarathy on the occasion of his 75th birthday*

Abstract. The Airy functions go back to the 19th century. But they have become important in many areas of mathematics, such as Analysis, Topology, Applied Mathematics, Probability, to mention a few. In recent years it has been noticed that they make sense over non-archimedean fields also. In this brief article I review some recent work on them and discuss several questions which remain unsolved.

## 1. Introduction

The Airy function was first introduced and studied [1] by Sir George Biddell Airy, mathematician and Astronomer Royal, in 1838. Its modern definition is given by

$$A(x) = \int_{-\infty}^{\infty} e^{i((1/3)y^3 - xy)} dy \ (x \in \mathbb{R}).$$

The integral is only conditionally convergent but becomes absolutely convergent if we move the path of integration into the complex plane. The function $A$ satisfies the linear differential equation

$$A''(x) + xA(x) = 0.$$

The function $A$ is, up to a multiplying constant, is the only solution to this equation having polynomial growth in $x$, and it extends to an entire function on $\mathbb{C}$.

In order to see what is the natural context to study such a function and its possible generalizations, it is necessary to reformulate its definition. The bounded function $e^{iy^3/3}$ can be viewed as a tempered distribution on $\mathbb{R}$ and so admits a Fourier transform; this Fourier transform turns out to be a function and is in fact the distribution $A(x)dx$. Thus, to generalize the concept of an Airy function the most natural method is to start with a real polynomial $P$ and define the Airy distribution associated to $P$ as the Fourier transform of the bounded function $e^{iP}$ (viewed as a tempered distribution), and study when the Fourier transform of this distribution is a function, namely of the form $A_P(x)dx$ with $A_P$ of polynomial growth; if this is the case we shall say that the polynomial $P$ has the *Airy property*, and call $A_P(x)$ the Airy function associated to $P$. Because the Airy functions are defined as Fourier transforms, these considerations make sense over any local field.

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## 2. Airy Functions over the Real Number Field

The importance of Airy functions was first seen from the work of Kontsevitch [4] who considered the real polynomial

$$P : X \mapsto \frac{1}{3}\mathrm{Tr}(X^3)$$

on the space of $n \times n$ hermitian matrices and showed that it has the Airy property and computed the function $A_P$. This was generalized by Fernandez and Varadarajan [2] where the real polynomials considered were the polynomials on the Lie algebra $\mathfrak{g}$ of a compact semi-simple Lie group invariant under the adjoint representation. It was proved in [2] that such a polynomial is Airy and has an extension to the complexification $\mathfrak{g}_{\mathbb{C}}$ as an entire function if and only if this is true of the restriction of the polynomial to a Cartan subalgebra. Moreover, in [2] the function $A_P$ was explicitly evaluated, giving a far-reaching generalization of the formulae of Kontsevitch.

The study of matrix Airy integrals by Kontsevitch was used by him to study certain problems involving the topology of the moduli space of curves. One problem suggested by the paper [2] is the following.

*Problem 1.* Whether the Airy integrals on the space of skew-symmetric matrices (Lie algebra of the orthogonal group $\mathrm{SO}(n)$) have a moduli-theoretic interpretation.

*Problem 2.* Whether the *Pfaffian* of skew symmetric real matrices of even order has the Airy property and whether the corresponding Airy integral has a moduli-theoretic significance.

The restriction theorem to a Cartan subalgebra established in [2] reduces the Airy problem to the Cartan subalgebra. There the functions of the *additive* form

$$(x_1, \ldots, x_n) \longmapsto p_1(x_1) + \cdots + p_n(x_n) \qquad (\deg(p_j) \geq 3)$$

are easily shown to be Airy, since all polynomials of degree $\geq 3$ in one variable are Airy. This is sufficient to give the Kontsevitch result and its generalizations when the polynomial is of the trace form $\mathrm{Tr}(X^r)$. However the general result, for a polynomial on an arbitrary real vector space, remains open.

Let me discuss this a little more. In [2] sufficient conditions are given for a polynomial $p$ on $\mathbb{R}^n$ to have the Airy property. They assume some sort of ellipticity for the polynomial and succeed in establishing the Airy property. The conditions define an *open set* in the space of polynomials of any fixed arbitrary degree, but unfortunately they do not reveal the deeper reasons why a polynomial should have the Airy property.

*Problem 3.* Find the definitive conditions on a real polynomial to have the Airy property.

## 3. Airy Functions Over a Local Field

The definition of the Airy property involves just the Fourier transform, and so makes sense when the base field is a local non-archimedean field. Let $K$ be a local compact non-discrete field and $V$ a finite dimensional vector space over

$K$. Let $p$ be a polynomial map from $V$ to $K$. Fix as usual a non-trivial additive character $\psi$ of $K$, using which we can define the Fourier transform of distributions defined over $V$. Indeed, let $\mathcal{SB}(V)$ be the space of *Schwarz-Bruhat functions* on $V$, namely the functions which are locally constant on $V$ and having a compact support. By a distribution on an open set $U \subset V$ we mean *any* linear function on the space $\mathcal{SB}(U)$ of elements of $\mathcal{SB}(V)$ whose supports are contained in $U$. If $f$ is a Schwarz-Bruhat function on $V$ its Fourier transform $\widehat{f}$ is the Schwarz-Bruhat function on $V$ defined by

$$\widehat{f}(x) = \int_V f(y)\psi(-(x,y))dy$$

where $(\cdot,\cdot)$ is a non-singular bilinear form $V \times V \longrightarrow K$. One knows the *inversion formula*

$$f(y) = \int_V \widehat{f}(x)\psi((x,y))dx$$

with the understanding that the Haar measure used in the definition of the Fourier transform is the *self-dual* one. If $T$ is any distribution on $V$, its Fourier transform $\widehat{T} = \mathcal{F}T$ is defined by

$$\widehat{T}(f) = T(\widehat{f}).$$

Let $h$ be a polynomial map of $V$ into $K$. Then the function $\psi\circ h : y \longmapsto \psi(h(y))$ is bounded and locally constant on $V$ and so defines a distribution on $V$. Its Fourier transform is called the *Airy distribution* associated to $h$. We say that $h$ has the *Airy property* if this Fourier transform is a locally integrable function on $V$ with a polynomial growth at infinity on $V$. It is proved in [3] that if $V = K^m$ and $h(y) = a_1y_1^n + \cdots + a_my_m^n$ with all $a_i \neq 0$, then $h$ has the Airy property, and in fact, the Airy distribution is locally constant on $V$ with essentially same growth at infinity as in the real case. One has to assume that either $K$ has characteristic 0 or that it has characteristic $p > 0$ and $p$ does not divide $n$. The exceptional cases are analyzed completely in [3] and need not concern us here.

*Problem 4.* Find less restrictive conditions on $h$ so that it has the Airy property.

It is natural to ask if one can generalize the results of Kontsevitch [4] and Fernandez-Varadarajan [2] to local fields. Some evidence in favor of this is presented in [3]. More precisely let $G$ be a compact Lie group over a local field $K$ and $\mathfrak{g}$ its Lie algebra. In [3] the case when $G = SO(3)$ is treated over *any* local field of characteristic $\neq 2$. Here it is proved that all the polynomials $\mathrm{Tr}(x^r)$ have the Airy property.

*Problem 5.* Whether the result in [3] can be extended for any compact Lie group defined over a non-archimedean local field $K$.

## 4. Concluding Remarks

It is hard to believe that K. R. Parthasarathy, or KRP as every one calls him, is 75. He has been, for close to fifty years, one of the most dynamic personalities in Indian mathematics, and has made profound and lasting contributions to a number of areas of contemporary mathematics and mathematical physics such as ergodic theory, information theory (classical and quantum), quantum stochastic analysis, coding theory, representation theory of Lie groups (both finite and infinite dimensional), and so on. During his long and stellar career of over fifty years, he has fulfilled, in an exemplary manner, many different roles: independent thinker and researcher, inspiring teacher, guide and role model to younger scientists, elder statesman, guru (benevolent but also occasionally stern!)to his many friends and students, and so on. I have known him for over a half century, as a collaborator, friend, and critic, and can say truthfully that my life had been made infinitely richer by my association with him. At our age, to quote Hermann Weyl, it is more pleasant to look back than look forward.

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