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Cluster Algebras and Maximal Green Sequences for Closed Surfaces

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CLUSTER ALGEBRAS AND MAXIMAL GREEN
SEQUENCES FOR CLOSED SURFACES

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by

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Table of Contents

Acknowledgments	ii
Abstract	iv
Chapter 1: Introduction	1
Chapter 2: A brief survey of cluster algebras	6
2.1 Quivers and Mutation	6
2.2 Cluster Algebras from Quivers	9
2.3 Some Preliminary Properties of Cluster Algebras	12
2.3.1 Classification of finite type cluster algebras	13
Chapter 3: Cluster algebras from surfaces	15
3.1 Mutation of triangulations	19
3.1.1 Tagged Triangulations	20
3.1.2 Why study cluster algebras from surfaces?	22
Chapter 4: Maximal green sequences	24
4.1 Definitions for maximal green sequences	24
Chapter 5: Maximal green sequences for cluster algebras from surfaces	27
5.1 Constructing the Triangulation T_n	28
5.2 Statement and proof of main result	30
5.2.1 The cycle lemma	31
5.2.2 Proof of the main theorem	38
Chapter 6: Closed surfaces with at least 3 punctures	51
6.1 Constructing the Triangulation	51
6.2 Statement and Proof of Main Result	52
6.3 Survey and Future Interests	60
References	62
Vita	64

Abstract

Given a marked surface (S, M) we can add arcs to the surface to create a triangulation, T , of that surface. For each triangulation, T , we can associate a cluster algebra. In this paper we will consider orientable surfaces of genus n with two interior marked points and no boundary component. We will construct a specific triangulation of this surface which yields a quiver. Then in the sense of work by Keller we will produce a *maximal green sequence* for this quiver. Since all finite mutation type cluster algebras can be associated to a surface, with some rare exceptions, this work along with previous work by others seeks to establish a base case in answering the question of whether a given finite mutation type cluster algebra exhibits a maximal green sequence. In this paper we will provide a triangulation for orientable surfaces of genus n with an arbitrary number interior marked points (called punctures) whose corresponding quiver has a maximal green sequence.

Chapter 1

Introduction

Cluster algebras were invented by Fomin and Zelevinsky [8] in 2003. Within a very short period of time cluster algebras became an important tool in the study of phenomena in various areas of mathematics and mathematical physics. They play an important role in the study of Teichmüller theory, canonical bases, total positivity, Poisson Lie-groups, Calabi-Yau algebras, noncommutative Donaldson-Thomas invariants, scattering amplitudes, and representations of finite dimensional algebras. In this thesis we will address a brief but informative survey of cluster algebras, but for more information on the diverse scope of cluster algebras see the review paper by Williams [16].

After establishing cluster algebras we will turn our interests toward a specific sequence of cluster mutations, called maximal green sequences. The idea of maximal green sequences of cluster mutations was introduced by Keller in [12]. He explored important applications of this notion, by utilizing it in the explicit computation of noncommutative Donaldson-Thomas invariants of triangulated categories which were introduced by Kontsevich and Soibelman in [13]. Additionally, Alim, et al worked with this notion in connection with the computation of spectra of BPS states [1]. Very recently this notion also played a key role in the Gross-Hacking-Keel-Kontsevich [11] proof of the full Fock-Goncharov conjecture for large classes of cluster algebras.

The problem of existence of maximal green sequences of cluster mutations is difficult due to the iterative nature of the choices of mutations. This means that exhaustive methods are not always effective when searching for a maximal green

sequence. In spite of this difficulty there has been a vast amount of progress made in the area. Brüstle, Dupont, and Perótin proved the existence of maximal green sequences for cluster algebras of finite type in [2]. Alim et al. showed that cluster algebras from surfaces with nonempty boundary have a maximal green sequence [1]. Yakimov proved the existence of maximal green sequences for the Berenstein-Fomin-Zelevinsky cluster algebras on all double Bruhat cells in Kac-Moody groups in [17]. Also, Garver and Musiker constructed maximal green sequences for all type A quivers in [9]. One important aspect to note is that the existence of a maximal green sequence is dependent on the quiver and not the mutation class. Muller showed this in [15] by producing two mutation equivalent quivers in which one exhibits a maximal green sequence and the other does not. This means that the choice of initial quiver is extremely important, as there may not even be a maximal green sequence if you make the wrong choice. For this paper that will mean making a strategic decision for our initial triangulation.

In general a cluster algebra can be constructed from any orientable surface by looking at the possible triangulations of that surface. This construction is introduced by Gekhtman, Shapiro, and Vainshtein in [10] and in a more general setting by Fock and Goncharov in [5]. This construction is extremely important because any cluster algebra of finite mutation type can be realized as a cluster algebra which arises from a surface following this construction, with a few exceptions. The complete list of exceptions can be found in [3] and [4]. An important problem in cluster algebras is then to prove the existence or non-existence of maximal green sequences for each cluster algebra which arises from the triangulation of a surface. This paper will prove the existence of maximal green sequences for an infinite family of cluster algebras which arise this way. This family is of interest because at the moment there is little known about maximal green sequences which arise from sur-

faces without boundary components. The quivers produced contain a large number of cycles and this creates many difficulties when addressing the existence of green sequences. For a more in depth look into the procedure of creating a cluster algebra from a triangulated surface see the work by Fomin, Shapiro, and Thurston [6].

In this thesis we prove the existence of a maximal green sequence for cluster algebras which arise from triangulations of the closed marked surface of genus n , which has at least 2 punctures. We start by addressing the surfaces of arbitrary genus with exactly two punctures. This is an infinite family of cluster algebras for which we explicitly find a maximal green sequence. In general, the more cycles present in a quiver, the more difficult it is to construct a maximal green sequence. As mentioned above, by addressing surfaces without boundary components, we are addressing a class of quivers which contain many cycles. We will start with a surface of genus n . We then construct a specific triangulation of this surface, T_n . This triangulation is chosen to contain a large amount of symmetry, which will play an integral part in our main proof, and help reduce the impact of the presence of a large number of cycles. The construction of this triangulation will be discussed in subsequent sections. After constructing the triangulation, we look at the quiver Q_{T_n} it correlates to. We take advantage of the symmetry of this quiver, by breaking it into smaller parts. This cluster algebra contains a large n cycle with identical subquivers attached to each vertex. We construct a green sequence for the cycle, which leaves the attached subquivers unaffected. We can then apply a green sequence to the subquivers which will minimally effect the vertices on the cycle. Various mutations are then done to correct these minimal effects. We want to emphasize that the ability to correct these effects is directly related to the choice of triangulation. By creating subquivers of a certain structure we can guarantee that they will not be drastically affected by the sequence of mutations applied to the

interconnecting cycle. The combining of these sequences will result in a maximal green sequence for the quiver Q_{T_n} . In essence, we are creating separate maximal green sequences for each "piece" of the quiver and then creating a procedure for gluing these sequences together. The details of the proof are presented in the later sections of this thesis.

We will then prove the existence of a maximal green sequence for cluster algebras which arise from surfaces with empty boundary component and at least three punctures. We essentially use the twice punctured surfaces as a base case and show how to modify the triangulation as you add more punctures. The details of this triangulation appear in chapter 5. This triangulation again creates a large amount of symmetry in the quiver which we will utilize to break the quiver into smaller sub quivers. After the construction of the triangulation we look at the associated quiver which we denote Q_n^p , where p is the number of punctures on the surface. By adding punctures to the structure we add a ladder structure which we denote P_p . Increasing the number of punctures lengthens the ladder, but its connection to the rest of the quiver is unaffected. After setting up the precise triangulation we give the proof of our main result. The proof is done by inducting on the number of punctures. We will construct a maximal green sequence for the subquiver P_p and then show how to utilize that sequence to get a maximal green sequence for the larger quiver Q_n^p .

This work completes the classification of which cluster algebras arising from surfaces have a maximal green sequence. It is known that a maximal green sequence cannot be constructed for a cluster algebra from a surface with empty boundary component and only one puncture [14]. With our results and the results of those mentioned above, given any marked surface with empty boundary component we now know whether there exists a triangulation whose corresponding quiver has a

maximal green sequence. In general combining this with the results on cluster algebras arising from surfaces with nonempty boundary component, given any marked surface the question will be answered as to whether we can construct a cluster algebra with a maximal green sequence which corresponds to that surface. Below is a table which gives the story of surfaces and maximal green sequences so far. I have highlighted in blue how the results of this thesis will complete the missing cases in the theory.

Surface Classification	Number of MGS	Source
Surface with boundary	exists Δ with MGS	Alim et al.
Closed surface with 1 puncture	no MGS	S. Ladkani
Sphere with $p \geq 4$	exists Δ with MGS	Alim et al.
Torus with $p \geq 2$ punctures	exists Δ with MGS	Alim et al.
Closed surface $g \geq 2$ and $p = 2$	exists Δ with MGS	Bucher
Closed surface $g \geq 2$ and $p \geq 3$	exists Δ with MGS	Bucher and Mills

FIGURE 1.1. Surface classifications and the existence of triangulations which exhibit a maximal green sequence

Chapter 2

A brief survey of cluster algebras

The main object that this thesis will look at are cluster algebras, so the first thing to do is establish the basic definitions and get a feel for what these objects are and some of their key properties. This chapter will first give a heuristic way of thinking about cluster algebras, then the formal definitions, and then some (hopefully) tangible examples of cluster algebras.

We will start by first looking at defining a cluster algebra from a graph called a quiver. There is a matrix definition that we will give later on, that extends the idea to a slightly more general class of cluster algebras, but doing it in this order should make the material a bit more accessible. The main goal of cluster algebras as laid out by Fomin and Zelevinsky [6] is to create a combinatorial framework that will allow us to understand a subtle and complicated underlying algebraic structure. We will first introduce quivers and quiver mutation, which is the governing combinatorial structure and then address the concepts of labeled seeds which connects the algebraic structure. In general the definitions that are given in this sections are following those of Lauren Williams in [16].

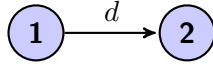
2.1 Quivers and Mutation

Definition 2.1 (*Quiver*). A *quiver* is an oriented graph given by a set of vertices Q_0 , a set of arrows Q_1 , and two maps $s : Q_1 \rightarrow Q_0$ and $t : Q_1 \rightarrow Q_0$ taking an arrow to its source and target, respectively. In addition a quiver does not contain loops or 2-cycles.

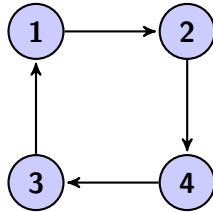
One thing to note is that you are allowed to have parallel edges in which case we will draw a single edge and label it with its multiplicity in Q_1 . In general, the

sets Q_0 and Q_1 may be infinite sets. Though in this work we will primarily deal with quivers in which both sets are finite in which case we say that we are dealing with a *finite quiver*. Below are a few basic examples of quivers.

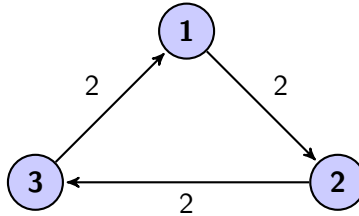
Example 2.2 (*Kronecker Quiver*).



Example 2.3 (*Oriented Cycle of Length Four*).



Example 2.4 (*Dreaded Torus*).



We will now define quiver mutation. This is an interesting operation that takes one quiver and mutates it “at a vertex” to produce a new quiver. It can sometimes help to think of mutation as pushing a button on the quiver and changing it into a new quiver.

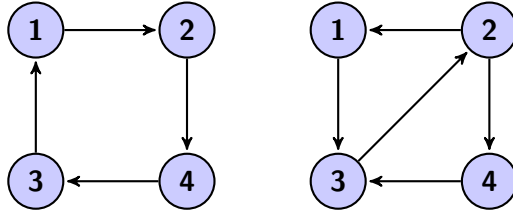
Definition 2.5 (*Quiver Mutation*). The *mutation* of a quiver Q at a vertex k is denoted μ_k , and produces a new quiver $\mu_k(Q)$. The vertices of $\mu_k(Q)$ are the same vertices from Q . The arrows of the new quiver are obtained by performing the following 3 steps:

1. For every 2-path $i \rightarrow k \rightarrow j$, adjoin a new arrow $i \rightarrow j$.

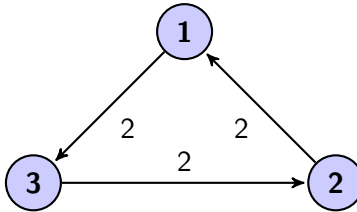
2. Reverse the direction of all arrows incident to k .
3. Delete any 2-cycles created during the first two steps.

Let's look at a few examples of quiver mutations:

Example 2.6 (*Oriented Cycle of Length 4*).



Example 2.7 (*Dreaded Torus*).



In general we say that two quivers Q and Q' are *mutation equivalent* if they can be obtained from each other by a sequence of mutations. One interesting thing to note right away is that mutation is an involution, or in other words $\mu_k^2(Q) = Q$. In general all the information that is encoded in the quiver, can also be encoded in an adjacency matrix. This gives a slightly more general way of defining mutation, and though it will not be commonly used in this thesis, it is likely useful to give the reader both definitions.

Definition 2.8 (*signed adjacency matrix*). Let Q be a finite quiver with no loops or 2-cycles and whose vertices are labeled $1, 2, \dots, m$. Then we may encode Q by an $m \times m$ skew-symmetric *exchange matrix* $B(Q) = b_{ij}$ where $b_{ij} = -b_{ji} = l$ whenever there are l arrows from vertex i to vertex j . We call $B(Q)$ the *signed adjacency matrix* of the quiver.

Definition 2.9 (*Matrix Mutation*). The mutated *signed adjacency matrix* is given by $B(\mu_k(Q)) = (b'_{ij})$ and is again an $m \times m$ skew-symmetric matrix, whose entries are give by:

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + b_{ik}b_{kj} & \text{if } b_{ik} > 0 \text{ and } b_{kj} > 0 \\ b_{ij} - b_{ik}b_{kj} & \text{if } b_{ik} < 0 \text{ and } b_{kj} < 0 \\ b_{ij} & \text{otherwise} \end{cases}$$

The two definitions of mutation coincide [6], and we will utilize either of the definitions when convenient. The main advantage of using the matrix definition of mutation is that it can extend to skew-symmetrizable matrices, and hence we can talk about mutation in a more general situation. The interpretation of this that is sometimes useful is thinking of a vertex as “seeing a different number of arrows, than the vertex that is sending the arrows”. In general one can develop a new type of combinatorics to encode this, but I will not give the details of this now. In this thesis we deal exclusively with cluster algebras associated to skew-symmetric matrices, but it is worth noting that there is an analogue theory for much of what we do that only requires a skew-symmetrizable matrix.

2.2 Cluster Algebras from Quivers

Now that we have the definition of quiver mutation, we need to establish the algebraic structure connected to a quiver. Unlike a more traditional approach to algebras, a cluster algebra is not given by a list of generators and relations, instead it is defined by something called labeled seeds.

Definition 2.10 (*Labeled Seeds*). Choose $m \geq n$ positive integers. Let \mathcal{F} be an ambient field of rational functions in n independent variables over $\mathbb{Q}(x_{n+1}, \dots, x_m)$. A labeled seed in \mathcal{F} is a pair (\mathbf{x}, Q) , where

- $\mathbf{x} = (x_1, \dots, x_m)$ forms a free generating set for \mathcal{F} , and
- Q is a quiver on vertices $1, 2, \dots, n, n+1, \dots, m$, whose vertices $1, 2, \dots, n$ are called mutable and whose vertices $n+1, \dots, m$ are called frozen.

We refer to \mathbf{x} as the (labeled) *extended cluster* of a labeled seed (\mathbf{x}, Q) . The variables $\{x_1, \dots, x_n\}$ are called *cluster variables*, and the variables $c = \{x_{n+1}, \dots, x_m\}$ are called the *frozen* or *coefficient variables*.

We can see that essentially a seed is a two pieces of information: a quiver Q and then the labeled cluster \mathbf{x} . We already know what mutation does to the quiver, so now we must understand what mutation does to the labeled cluster.

Definition 2.11 (*Seed Mutation*). Let (\mathbf{x}, Q) be a labeled seed in \mathcal{F} , and let $k \in \{1, 2, \dots, n\}$. The seed mutation μ_k in direction k transforms (\mathbf{x}, Q) into the labeled seed $\mu_k(\mathbf{x}, Q) = (\mathbf{x}', \mu_k(Q))$, where the cluster $\mathbf{x}' = (x'_1, x'_2, \dots, x'_m)$ is defined as follows: $x'_j = x_j$ for $j \neq k$, whereas $x'_k \in \mathcal{F}$ is determined by the exchange relation

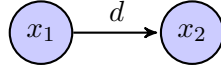
$$x'_k x_k = \prod_{\substack{a \in Q_1 \\ s(\alpha)=k}} x_{t(\alpha)} + \prod_{\substack{a \in Q_1 \\ t(\alpha)=k}} x_{s(\alpha)}$$

The important thing to notice is that the only cluster variable that is changed by mutation in the direction k is the variable x_k . Also one can check that in fact mutation on the labeled cluster is also a involution.

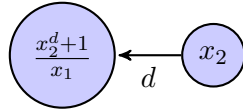
Definition 2.12 (*Cluster Pattern*). Consider the n -regular tree \mathbb{T}_n whose edges are labeled by the numbers $1, \dots, n$ so that the n edges emanating from each vertex receive different labels. A cluster pattern is an assignment of a labeled seed

$\Sigma_t = (\mathbf{x}_t, Q_t)$ to every vertex $t \in \mathbb{T}_n$, such that the seeds assigned to the endpoints of any edge $t \xrightarrow{k} t'$ are obtained from each other by the seed mutation in direction k . The components of \mathbf{x}_t are written as $\mathbf{x}_t = (x_{1:t}, \dots, x_{n:1})$.

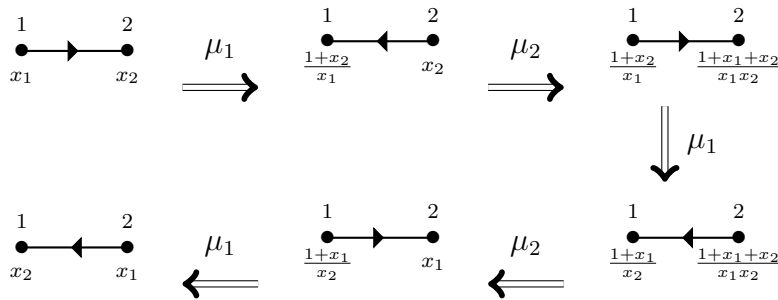
Example 2.13. If we look at the example of the Kronecker quiver from above we can see what happens when we mutate in the direction 1:



Now by applying the mutation μ_1 you get the following quiver with labeled seed



Example 2.14. Another nice hands on example to look at is if we look at the type A_2 quiver (which is also the Kronecker quiver with $d = 1$). In this example we work out all of the different labeled seeds which can arise.



Now that we have a mild handle on mutation, we are ready to define the actual cluster algebra. Essentially a cluster algebra is an algebra that lives between the polynomial ring, $\mathbb{Z}[\mathbf{x}]$, and the ring of rational functions, $\mathbb{Z}(\mathbf{x})$.

Definition 2.15 (Cluster Algebra). Given a cluster pattern, we denote

$$\chi = \bigcup_{t \in \mathbb{T}_n} \mathbf{x}_t = \{x_{i,t} : t \in \mathbb{T}_n, 1 \leq i \leq n\},$$

the union of clusters of all the seeds in the pattern. The elements $x_{i,t} \in \chi$ are called cluster variables. The cluster algebra \mathcal{A} associated with a given pattern is the $\mathbb{Z}[c]$ -subalgebra of the ambient field \mathcal{F} generated by all cluster variables: $\mathcal{A} = \mathbb{Z}[c][\chi]$. We denote $\mathcal{A} = \mathcal{A}(\mathbf{x}, Q)$, where (\mathbf{x}, Q) is any seed in the underlying cluster pattern. In this generality, \mathcal{A} is called a cluster algebra from a quiver, or a skew-symmetric cluster algebra of geometric type. We say that \mathcal{A} has rank n because each cluster contains n cluster variables.

2.3 Some Preliminary Properties of Cluster Algebras

At first it seems like we are arbitrarily defining these cluster algebras, but they carry with them many interesting intrinsic properties. The first of these interesting properties is called the *Laurent phenomenon*. As we mentioned earlier a good way of thinking of a cluster algebra, is an algebra that “lives between the ring of rational functions on n variables and the polynomial functions on n variables. The Laurent phenomenon states that these are not just rational functions but Laurent polynomials. If we think about what that means, it means that every element of \mathcal{A} is in fact not just a rational function in $\mathbb{Z}(\mathbf{x})$ but is Laurent polynomial in the variables $\{x_1, \dots, x_n\}$. This is by no means obvious when looking at the definition of \mathcal{A} and was proven by Fomin and Zelevinsky in [6].

Theorem 2.16 (*Laurent Phenomenon*). The cluster algebra \mathcal{A} associated to a seed $\Sigma = (\mathbf{x}, B)$ is contained in the Laurent polynomial ring $\mathbb{Z}[\mathbf{x}^{\pm 1}]$.

Taking this a step further if you look at the cluster variables which show up in our example above you may notice that they not only are Laurent polynomials in \mathbf{x} , but in fact they are also minus-free rational functions. This particular phenomenon is known as the positivity conjecture. The conjecture is that in fact the cluster variables are always minus-free.

Conjecture 2.17 (*Positivity Conjecture*). For any cluster algebra \mathcal{A} , any seed Σ , and any cluster variable x , the Laurent polynomial $[x]_{\Sigma}^{\mathcal{A}}$ has coefficients which are nonnegative integer linear combinations in the field.

A lot of work has been done on the positivity conjecture and for many cases of cluster algebras it is known to be true, but in general the problem still remains open for a general cluster algebra.

2.3.1 Classification of finite type cluster algebras

When dealing with any mathematical object, the question of classification often comes up. How can we classify cluster algebras up to mutation equivalence? In general there is no answer to this question, but two very interesting classes of cluster algebras can be classified: *finite type cluster algebras* and *finite-mutation type cluster algebras*. I bring up the classification of these two classes of cluster algebras because this thesis will work to show the existence of a certain property for all cluster algebras of finite mutation type and will directly utilize their classification.

Definition 2.18 (*Finite type cluster algebra*). A cluster algebra is of finite type if it has only finitely many seeds in the cluster pattern. That is to say that there are only a finite number of labeled seeds which are mutation equivalent to your initial seed.

Definition 2.19 (*Finite mutation type cluster algebra*). A cluster algebra is of finite mutation type if there are only a finite number of quivers which are mutation equivalent to your initial seed. That is to say that you may have different labels on your seeds, but only a finite number of adjacency matrices appear.

The finite type cluster algebras have a remarkable connection to classical classification of an algebraic object: simple Lie algebras. As it turns out finite type cluster algebras are classified by Dynkin diagrams.

Theorem 2.20 ([6]). The cluster algebra \mathcal{A} is of finite type if and only if it has a seed (\mathbf{x}, B) such that the quiver associated to B is an orientation of a finite type Dynkin diagram.

The finite mutation type cluster algebras however have a slightly more complicated classification. They are classified by *marked surfaces*. In the next chapter we will give a brief description of how you can associated a cluster algebra to a marked surface.

Chapter 3

Cluster algebras from surfaces

This chapter will introduce one way of getting a cluster algebra associated to a marked surface. This chapter is designed to give a working understanding of the area and for more detail the reader should visit the source material by Fomin, Shapiro, and Thurston [6]. We discussed in the previous chapter how to construct a cluster algebra from a quiver: in this chapter we define the “cluster algebra from a surface” by showing how one gets a quiver from the surface.

Definition 3.1 (Bordered surface with marked points). Let \mathbf{S} be a connected oriented 2-dimensional Riemann surface with boundary. Fix a finite set \mathbf{M} of marked points in the closure of \mathbf{S} . Marked points in the interior of \mathbf{S} are called punctures. The marked surface is denoted (\mathbf{S}, \mathbf{M}) .

In general one way to construct a quiver from a marked surface is to look at the possible triangulations of this space. A good way of thinking about these triangulations is to think about drawing the surface with the marked points, \mathbf{M} , and then drawing paths on the surface which start and end at points in \mathbf{M} until cutting along these paths would create a set of disconnected triangles. We will make all of this formal with a few definitions, but it is useful to have an overview first of what a triangulation is. In this thesis we will exclude a small set of surfaces which make it impossible to construct triangulations or make the theory of triangulations uninteresting:

- a sphere with one or two punctures;
- an unpunctured or once-punctured monogon;

- an unpunctured digon;
- an unpunctured triangle; or
- a sphere with three punctures.

We also require that \mathbf{M} be nonempty. Up to homeomorphism we can see that a marked surface is classified up to homeomorphism by

- the genus g of the surface;
- the number of boundary components denoted b ;
- the number of marked points on each boundary components;
- the number of punctures denoted p .

Definition 3.2 (Arc). An arc in (\mathbf{S}, \mathbf{M}) is the isotopy class of a curve in \mathbf{S} connecting 2 marked points such that

- the curve does not have self-intersections, except possibly coinciding end points
- the interior of the curve is disjoint from \mathbf{M} and $\delta\mathbf{S}$
- the curve does not cut out an unpunctured monogon or an unpunctured bigon, i.e. the curve is not contractible to a puncture and is not homotopic to a curve in $\delta\mathbf{S}$.

In general the collection of arcs for a given surface is infinite, and in fact it is only finite under very restrictive circumstances.

Theorem 3.3 ([6]). The set of arcs in (\mathbf{S}, \mathbf{M}) is finite if and only if (\mathbf{S}, \mathbf{M}) is an unpunctured or once-punctured polygon.

Definition 3.4. Two arcs are said to be *compatible* if the isotopy classes contain curves which do not intersect, except possibly at the end points. We say that a maximal collection of distinct compatible arcs is an ideal triangulation (or triangulation for short) of the marked surface. The arcs of the triangulation cut the surface \mathbf{S} into ideal triangles. The three sides of the triangle do not have to be distinct, and in the case where a triangle has an a repeated arc we say that it is a *self-folded triangle*.

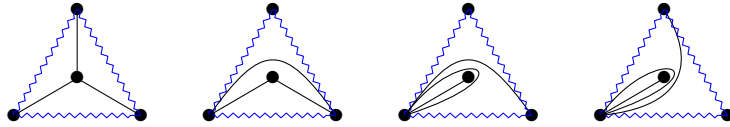


FIGURE 3.1. Triangulations of the once punctured 3-gon

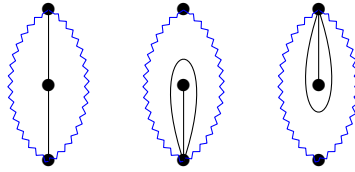


FIGURE 3.2. Triangulations of the once punctured 2-gon

The number n of arcs in a triangulation only depends on the surface (\mathbf{S}, \mathbf{M}) and is called the *rank* of the (\mathbf{S}, \mathbf{M}) .

Proposition 3.5. Each ideal triangulation consists of

$$n = 6g + 3b + 3p + c - 6$$

arcs, where g is the genus of (S) , b is the number of boundary components, p is the number of punctures, and c is the number of marked points on the boundary.

Now that we have the basic definitions of marked surfaces and triangulations on those surfaces, we will summarize the procedure given in [6] and [5] which produces

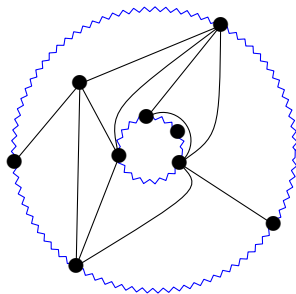


FIGURE 3.3. Triangulation of marked annulus with different types of triangles involving boundary arcs and interior arcs.

a quiver from the given triangulation. Given a triangulation T of (\mathbf{S}, \mathbf{M}) , choose a bijection

$$\phi : A(T) \rightarrow \{1, \dots, n\}.$$

The exchange quiver Q_T of the triangulation T is defined as follows:

1. For every triangle $\delta \in T(\mathcal{T})$ that is not self-folded add an arrow $[\alpha] \rightarrow [\beta]$ in each of the following cases:
 - α and β are sides of Δ , and β follows α in the clockwise order;
 - β is a radius of a self-folded triangle with a loop α , and α and γ are sides of Δ such that γ follows α in the clockwise order;
 - α is a radius of a self-folded triangle with a loop γ , and β and γ are sides of Δ such that β follows γ in the clockwise order.
2. Remove the arrows in a maximal set of pairwise disjoint 2-cycles.

The cluster algebra $\mathcal{A}(\mathbf{S}, \mathbf{M})$ associated to the bordered surface with marked points, (\mathbf{S}, \mathbf{M}) is defined to be $\mathcal{A}(\mathbf{x}, Q_{\mathcal{T}})$ for every triangulation \mathcal{T} of (\mathbf{S}, \mathbf{M}) . The different triangulations are related by a sequence of flips of arcs, and the corresponding quivers $Q_{\mathcal{T}}$ are related by mutation. This is proven in [6]. Each tri-

angulation of (\mathbf{S}, \mathbf{M}) defines a seed of $\mathcal{A}(\mathbf{S}, \mathbf{M})$ if you give the appropriate ordering in the case of each bijection.

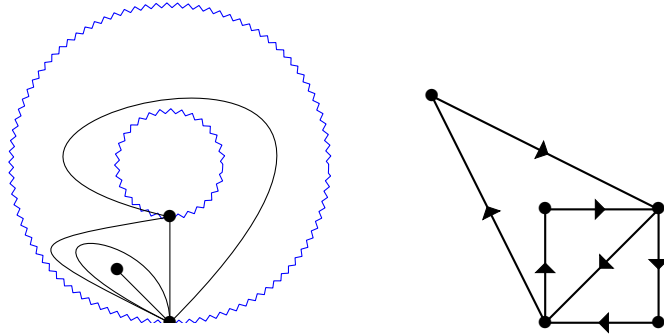


FIGURE 3.4. Example of a triangulation and the associated quiver.

3.1 Mutation of triangulations

Now that we have established a way of constructing a quiver given a triangulation of a marked surface, the natural question is what does mutation correspond to. Mutation corresponds to “flips” of arcs in the triangulation. This means deleting an arc and replacing it with the only non-isotopic arc that still produces a proper triangulation. This corresponds directly to mutation at the vertex in the quiver which corresponds to this arc.

Proposition 3.6 (Fomin, Shapiro, and Thurston [6]). Suppose that an ideal triangulation T' is obtained from T by a flip replacing an arc labeled k . (The labeling of all other arcs remains unchanged.) Then $B(T') = \mu_k(B(T))$.

And more generally, this means that the mutation class of the quiver associated to a surface depends only on the surface (\mathbf{S}, \mathbf{M}) and not on the choice of triangulation. That is to say that we can get from any triangulation to another by a sequence of “flips” corresponding to mutations on the level of the quiver. This is a direct result of the proposition by Fomin, Shapiro, and Thurston.

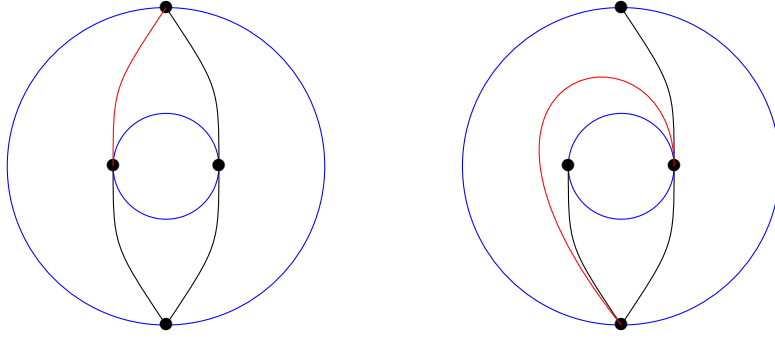


FIGURE 3.5. Example of mutation on the annulus with two marked points on each boundary.

From the correlation between mutation and “flipping diagonals”, one might notice a bit of a problem. Specifically what happens if we flip at the radius of a self-folded triangle. This procedure does not make sense, but on the associated quiver $Q_{\mathcal{T}}$ we should be able to mutate at this vertex. This is fixed by adding additional combinatorial data to the triangulation of the surface.

3.1.1 Tagged Triangulations

Definition 3.7 (Tagged arcs). Each arc γ in (\mathbf{S}, \mathbf{M}) has two ends obtained arbitrarily cutting γ into three pieces, then throwing out the middle one. We think of the two ends as locations near the endpoint to be used for labeling, or *tagging*, an arc. A tagged arc is an arc in which each end has been tagged in one of two ways, plain or notched, so that the following conditions are satisfied:

- the arc does not cut out a once-punctured monogon;
- an endpoint lying on the boundary is tagged plain; and
- both ends of a loop are tagged in the same way.

A *tagged arc* is an arc on (S, M) whose ends are marked (tagged) in 2 possible ways, *plain* or *notched*, so that the following conditions are satisfied:

- the arc does not cut out a once-punctured monogon;

- an endpoint lying on the boundary is tagged plain;
- if the arc is a loop, its endpoints are tagged in the same way.

The set of tagged arcs of (S, M) will be denoted by $A_{\times}(S, M)$.

Two tagged arcs α and β are called *compatible* if the plain arcs $\bar{\alpha}$ and $\bar{\beta}$, obtained from α and β by forgetting the taggings, are compatible and satisfy the following:

- if $\bar{\alpha} = \bar{\beta}$, then at least one end of α and β is tagged in the same way;
- if $\bar{\alpha} \neq \bar{\beta}$, but α and β have a common end point, then their taggings at the other end are the same.

A *tagged triangulation* of (S, M) is a maximal collection of distinct pairwise compatible tagged arcs. Each tagged triangulation \mathcal{T} gives rise to an ordinary triangulation \mathcal{T}° in the following way. The *signature* of a puncture x , with respect to a tagged triangulation \mathcal{T} , is defined by

$$\delta_{\mathcal{T}}(x) = \begin{cases} 1, & \text{if all tagged arcs of } \mathcal{T}, \text{ containing } x, \text{ are tagged plain at } x \\ -1, & \text{if all tagged arcs of } \mathcal{T}, \text{ containing } x, \text{ are tagged notched at } x \\ 0, & \text{otherwise.} \end{cases}$$

The definition of tagged triangulation easily implies that in the third case there are precisely 2 arcs of \mathcal{T} containing x , α and β , such that $\bar{\alpha} = \bar{\beta}$ and the taggings of α and β at x are different, while at the other end are the same. To each tagged triangulation \mathcal{T} , one associates an ordinary triangulation \mathcal{T}° by performing the two operations:

- replace all notched ends of arcs at the punctures with $\delta_{\mathcal{T}}(x) = -1$ by plain ones;

- for each puncture x with $\delta_{\mathcal{T}}(x) = 0$, we will have two arcs α and β containing x which will satisfy $\bar{\alpha} = \bar{\beta}$ and have different taggings at x and the same taggings at the other endpoint; replace the arc β notched at x with a loop based at the other end point of β and closely wrapping around β .

The set of tagged arcs of \mathcal{T} will be denoted by $A_{\bowtie}(\mathcal{T})$. There is an obvious bijection $A_{\bowtie}(\mathcal{T}) \xrightarrow{\cong} A(\mathcal{T}^\circ)$. The vertices of the exchange quiver $\mathcal{Q}_{\mathcal{T}}$ will be indexed by $A_{\bowtie}(\mathcal{T})$; the vertex corresponding to $\alpha \in A_{\bowtie}(\mathcal{T})$ will be denoted by $[\alpha]$. The edge set of $\mathcal{Q}_{\mathcal{T}}$ is defined by

$$\mathcal{Q}_{\mathcal{T}} := \mathcal{Q}_{\mathcal{T}^\circ}$$

in the above bijection.

We have an embedding $\tau: A(S, M) \hookrightarrow A_{\bowtie}(S, M)$. The map sends every arc that is cutting a once-punctured monogon to the radius of the corresponding self-folded triangle notched at the puncture of the monogon, and is the identity otherwise. This way, each ordinary triangulation \mathcal{T} gives rise to a tagged one $\tau(\mathcal{T})$ such that $(\tau(\mathcal{T}))^\circ = \mathcal{T}$ and $\mathcal{Q}_{\tau(\mathcal{T})} = \mathcal{Q}_{\mathcal{T}}$ under the identification between $A(\mathcal{T})$ and $A_{\bowtie}(\tau(\mathcal{T}))$.

It was proved in [6, 7] that cluster variables of $\mathcal{A}(S, M)$ are indexed by $A(S, M)$ if (S, M) is a once-punctured closed surface, and by $A_{\bowtie}(S, M)$ otherwise. That is to say that we can now fully think of mutation at γ as the deletion of the tagged arc γ and then replacing it with the only other compatible tagged arc which results in a triangulation.

3.1.2 Why study cluster algebras from surfaces?

We discussed in the previous chapter that finite-type cluster algebras were classified by type A and D Dynkin quivers along with a short finite list of exceptional cases. In fact one connection to surfaces is that these Dynkin quivers correspond to triangulations of unpunctured and once-punctured discs. A natural inclination is

to try and generalize this to an arbitrary surface, and then ask what cluster algebras are classified by “coming from surfaces”.

Definition 3.8. A cluster algebra $\mathcal{A}(\mathbf{x}, \mathbf{Q})$ is said to be of finite mutation type if there are only finitely many quivers mutation, Q' , which are mutation equivalent to Q .

This is a weaker condition than a cluster algebra of finite type, as it does not say that there are only a finite number of mutation equivalent seeds, simply a finite number of mutation equivalent quivers. The cluster algebra given by the Kronecker quiver with $d \neq 1$ is an example of a finite mutation type cluster algebra which is not finite type.

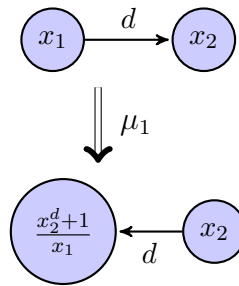


FIGURE 3.6. Cluster algebra for the Kronecker quiver

In general cluster algebras of finite type are classified by tagged triangulations of marked surfaces with a finite list of exceptions.

Theorem 3.9 (Fomin, Shapiro, and Thurston [6]). The quivers of finite mutation type are as follows:

- The Kronecker quiver of $d \geq 0$.
- A quiver arising from a triangulation of a marked surface with boundary.
- A quiver mutation equivalent to an orientation of E_6 , E_7 , or E_8 .
- a quiver mutation equivalent to one of those found in the below figure.

Chapter 4

Maximal green sequences

As mentioned in the introduction the main focus of this thesis is to produce maximal green sequences for quivers which previously had not been known to exhibit maximal green sequences. The existence of such a sequence has ramifications and connections in many areas of math. As mentioned in the introduction chapter, they play roles in string-theory, category theory, and representation theory just to name a few. Their origin is due to Bernhard Keller [12] and were originally studied to compute Donaldson-Thomas invariants. In this chapter we give the basic definitions that are necessary to talk about maximal green sequences, and then in the following two chapters we will present our primary results on green sequences, which is the construction of maximal green sequences for a specific quiver associated to closed surfaces of arbitrary genus with 2 or more punctures.

4.1 Definitions for maximal green sequences

We will follow the notation laid out by Brüstle, Dupont, and Perótin [2].

In this paper we will be concerned with a process called mutation. Mutation is a process of obtaining a new ice quiver from an existing one.

The quivers which are studied in throughout this thesis have a very specific set of frozen vertices. We will be looking at what are referred to as the framed and coframed quivers associated to Q .

Definition 4.1. The **framed quiver** associated with Q is the quiver \hat{Q} such that:

$$\begin{aligned}\hat{Q}_0 &= Q_0 \sqcup \{i' \mid i \in Q_0\} \\ \hat{Q}_1 &= Q_1 \sqcup \{i \rightarrow i' \mid i \in Q_0\}\end{aligned}$$

The **coframed quiver** associated with Q is the quiver \check{Q} such that:

$$\check{Q}_0 = Q_0 \sqcup \{i' \mid i \in Q_0\}$$

$$\check{Q}_1 = Q_1 \sqcup \{i' \rightarrow i \mid i \in Q_0\}$$

Both quivers \hat{Q} and \check{Q} are quivers in the same sense as our original definition, whose frozen vertices are commonly written as \hat{Q}'_0 and \check{Q}'_0 . Next we will talk about what it means for a vertex to be green or red.

Definition 4.2. Let $R \in Mut(\hat{Q})$. A non-frozen vertex $i \in R_0$ is called **green** if

$$\{j' \in \check{Q}'_0 \mid \exists j' \rightarrow i \in R_1\} = \emptyset.$$

It is called **red** if

$$\{j' \in \check{Q}'_0 \mid \exists j' \leftarrow i \in R_1\} = \emptyset.$$

It was shown in [2] that every non-frozen vertex in R_0 is either red or green. This idea motivates our work in this paper. It arises as a question of green sequences.

Definition 4.3. A **green sequence** for Q is a sequence $\mathbf{i} = \{i_1, \dots, i_l\} \subset Q_0$ such that i_1 is green in \hat{Q} and for any $2 \leq k \leq l$, the vertex i_k is green in $\mu_{i_{k-1}} \circ \dots \circ \mu_{i_1}(\hat{Q})$. The integer l is called the length of the sequence \mathbf{i} and is denoted by $l(\mathbf{i})$.

A green sequence \mathbf{i} is called maximal if every non-frozen vertex in $\mu_{\mathbf{i}}(\hat{Q})$ is red where $\mu_{\mathbf{i}} = \mu_{i_l} \circ \dots \circ \mu_{i_1}$. We denote the set of all maximal green sequences for Q by

$$\text{green}(Q) = \{\mathbf{i} \mid \mathbf{i} \text{ is a maximal green sequence for } Q\}.$$

The main purpose of this study was to find green sequences. In essence what we want to show is that $\text{green}(Q) \neq \emptyset$ for each quiver, Q , in this family. The following chapter will prove the existence of a maximal green sequence for a specific quiver

which is associated to a closed surface with exactly two punctures. Then in the subsequent chapter the proof will be generalized to include an arbitrary amount of additional punctures on the surface.

Chapter 5

Maximal green sequences for cluster algebras from surfaces

In this chapter we will present the statement of our first main result. That is we show how to construct a maximal green sequence for a triangulation corresponding to a closed surface with 2 punctures. In general since the existence of a maximal green sequence has nice ramifications on the underlying cluster algebra, it would be nice to know when we can find such a sequence. Recall from chapter 3 that finite mutation type cluster algebras are classified by surfaces with a few exceptional cases, and so by attacking this problem we are addressing the existence of maximal green sequences for a very important class of cluster algebras. This infinite family is of interest because at the moment there is little known about maximal green sequences which arise from surfaces without boundary components. Below is again the chart of what is currently known regarding maximal greens sequences and cluster algebras from surfaces.

Surface Classification	Number of MGS	Source
Surface with boundary	exists Δ with MGS	Alim et al.
Closed surface with 1 puncture	no MGS	S. Ladkani
Sphere with $p \geq 4$	exists Δ with MGS	Alim et al.
Torus with $p \geq 2$ punctures	exists Δ with MGS	Alim et al.
Closed surface $g \geq 2$ and $p = 2$	exists Δ with MGS	Bucher
Closed surface $g \geq 2$ and $p \geq 3$	exists Δ with MGS	Bucher and Mills

FIGURE 5.1. Surface classifications and the existence of triangulations which exhibit a maximal green sequence

You'll notice that in this list the only missing surfaces where it is unknown whether such a triangulation exists are the two cases which will be presented in this thesis. The way this chapter is outlined is that we will first present a the construction of the triangulation whose associated cluster algebra will yield a

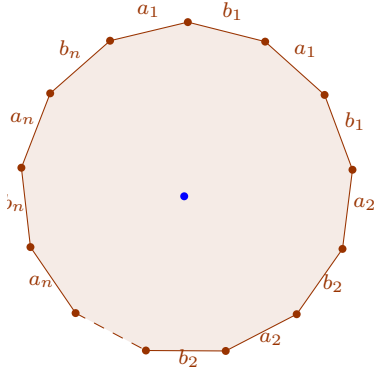
maximal green sequence and then we will prove that the presented sequence is in fact green. We take advantage of the symmetry of this quiver, by breaking it into smaller parts. This cluster algebra contains a large n cycle with identical subquivers attached to each vertex. We construct a green sequence for the cycle, which leaves the attached subquivers unaffected. We can then apply a green sequence to the subquivers which will minimally effect the vertices on the cycle. Various mutations are then done to correct these minimal effects. We want to emphasize that the ability to correct these effects is directly related to the choice of triangulation. By creating subquivers of a certain structure we can guarantee that they will not be drastically affected by the sequence of mutations applied to the interconnecting cycle. The combining of these sequences will result in a maximal green sequence for the quiver Q_{T_n} . In essence, we are creating separate maximal green sequences for each "piece" of the quiver and then creating a procedure for gluing these sequences together.

5.1 Constructing the Triangulation T_n

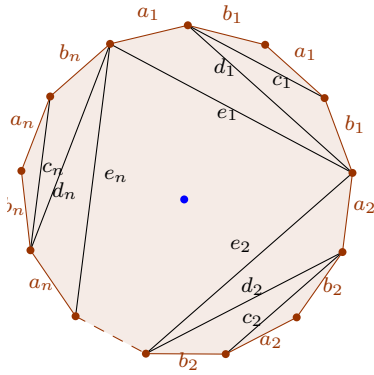
In work by Fomin, Shapiro, and Thurston [6] there is a very precise description of how you can associate a quiver Q to a triangulated surface. This is exactly the procedure summarized in chapter 3. The surfaces that we will be discussing in this chapter are twice punctured surfaces of genus n . We will find a specific triangulation on those surfaces which we will denote T_n . By following the techniques outlined in [6] from there we will form the associated quiver which we will denote by Q_{T_n} .

Start by letting (S, M) be a surface of genus n with two interior marked points. Now we will construct the desired triangulation T_n for the marked surface (S, M) . We start by drawing (S, M) as the identification space below.

After we have created the identification space we want to add additional arcs to create a triangulation of this space. At the moment the set of arcs we will be using



in our triangulation are $a_1, b_1, a_2, b_2, \dots, a_n, b_n$. The additional arcs we wish to add can be seen in the diagram below.



Now we will finish our triangulation by adding a wheel pattern to the center puncture. The arcs added will be labeled as below and there will be n edges added.

Now we have completed our desired triangulation T of the surface (S, M) . The arcs which are required are

$$\{a_1, b_1, c_1, d_1, e_1, f_1, a_2, b_2, c_2, d_2, e_2, f_2, \dots, a_n, b_n, c_n, d_n, e_n, f_n\}.$$

Now following the procedure from [6] we can construct the quiver Q_{T_n} , for the above triangulation. If you are unfamiliar with this procedure, the important thing to note is that each arc of the surface is associated to exactly one vertex in the quiver. In the diagram below the green vertices are given the same label as the associated arcs. We then create the framed quiver by adding the blue frozen vertices.

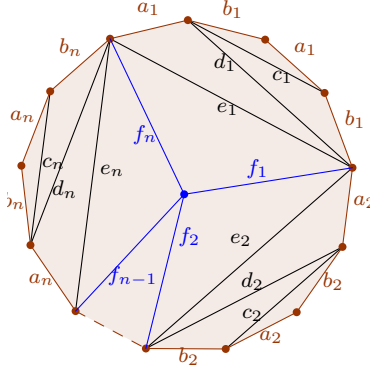
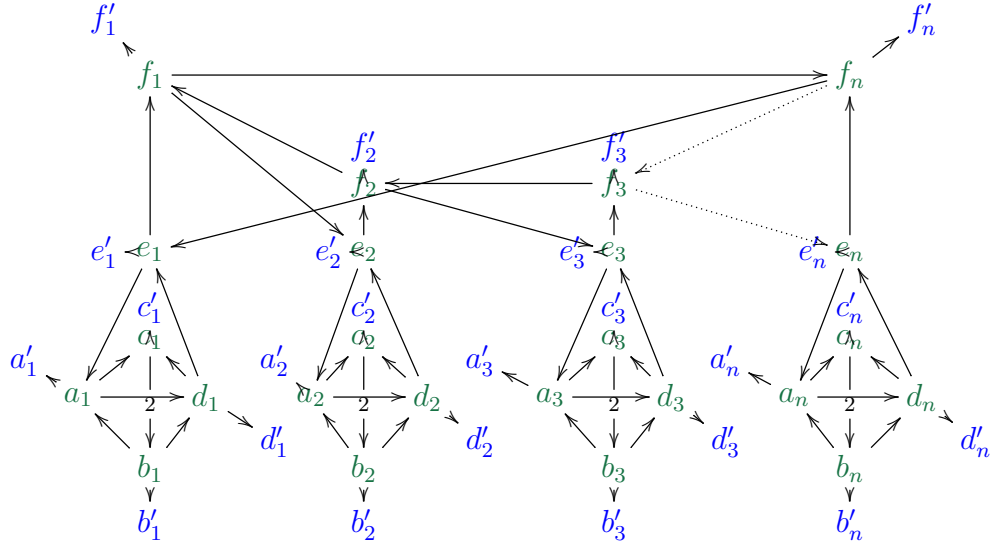


FIGURE 5.2. The triangulation of the n -torus with two punctures



5.2 Statement and proof of main result

In this chapter we will establish a maximal green sequence for the quiver Q_{T_n} constructed above. Our main result is the following:

Theorem 5.1. The quiver Q_{T_n} has a maximal green sequence of

$$(f_n, f_{n-1}, \dots, f_1, f_3, f_4, \dots, f_n, \sigma_n, \sigma_{n-1}, \dots, \sigma_1, \\ f_3, f_4, \dots, f_n, f_2, f_1, f_n, f_{n-1}, \dots, f_3, \tau_n, \tau_{n-1}, \dots, \tau_1)$$

where $\sigma_i := (e_i, d_i, b_i, c_i, a_i, b_i, d_i, e_i, c_i, a_i, b_i)$ and $\tau_i := (e_i, b_i, a_i, c_i, e_i, d_i, b_i, a_i, e_i)$.

We will look at the quiver Q_{T_n} , and try and break it down into smaller subquivers. The first subquiver of Q_{T_n} we will consider is the oriented n -cycle, C , which consists of vertices $C_0 = \{f_1, f_2, f_3, \dots, f_n\}$ and arrows $C_1 = \{f_i \rightarrow f_{i-1} | 1 \leq i \leq n \text{ with } f_0 = f_n\}$.

5.2.1 The cycle lemma

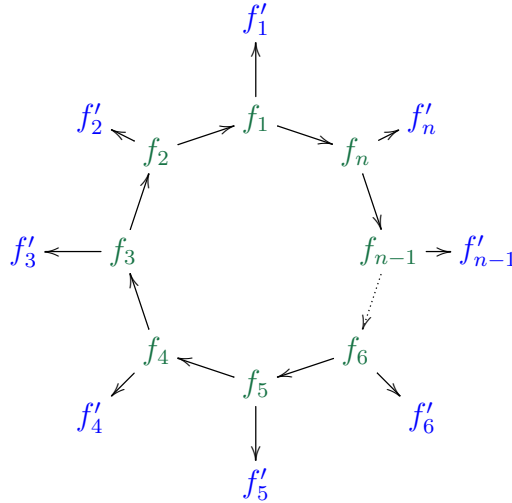


FIGURE 5.3. Quiver cycle of length n

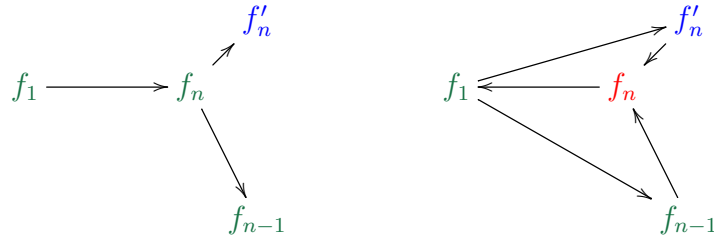
Lemma 5.2 (Cycle lemma). The sequence $(f_n, f_{n-1}, f_{n-2}, \dots, f_1, f_3, f_4, \dots, f_n)$ is a maximal green sequence for the subquiver C .

Proof. First we must check that each mutation which occurs in the sequence occurs at a green vertex. In [2] Lemma 2.16 shows that if a vertex k is green in the quiver Q , then vertex k is green in the quiver $\mu_j(Q)$ as long as $k \neq j$. Therefore every mutation in the sequence must occur at a green vertex until its second appearance in the sequence. In our case the first n mutations must occur at green vertices.

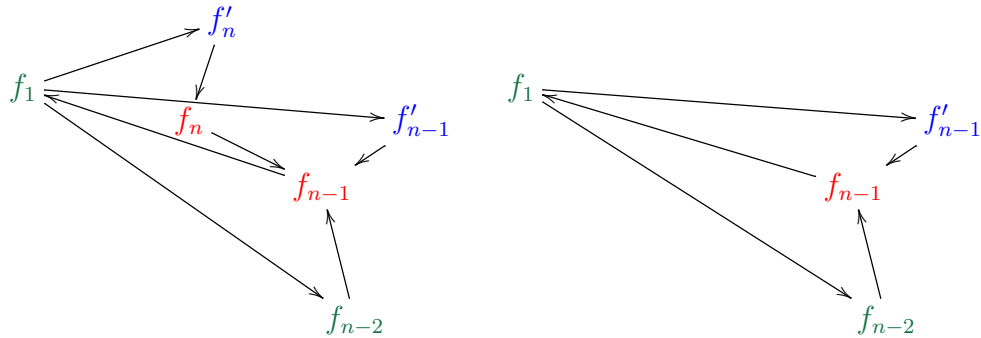
In order to understand why the other mutations occur at green vertices it is important to recall from [2] that each vertex is either green or red at every mutation step of the sequence. Therefore in order to show that a vertex, f_k , is green we must

find one arrow $f_k \rightarrow f'_j$ for some $f_j \in C_0$ and all other arrows between f_k and frozen vertices should have f_k as a source as well.

Let us start by considering the quiver, \hat{C} , before we have done any mutations. Consider the vertex f_n ; it is involved in the following arrows: $f_n \rightarrow f_{n-1}$, $f_1 \rightarrow f_n$, and $f_n \rightarrow f'_n$. The only arrow with target f_n is the arrow $f_1 \rightarrow f_n$. In the picture below you see what occurs after our first mutation μ_{f_n} .

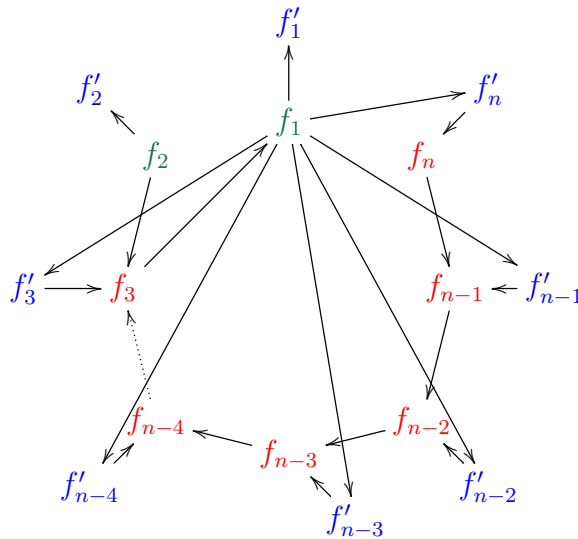


At this point the only arrow with target f_{n-1} is the arrow $f_1 \rightarrow f_{n-1}$. The next mutation is at f_{n-1} . After completing this mutation we end up with the diagram below on the left. If we focus only on the subquiver where we delete the vertices f_n and f'_n we obtain the diagram below on the right. It gives us the same diagram that resulted from our mutation at f_n , but we have shifted the index down one.

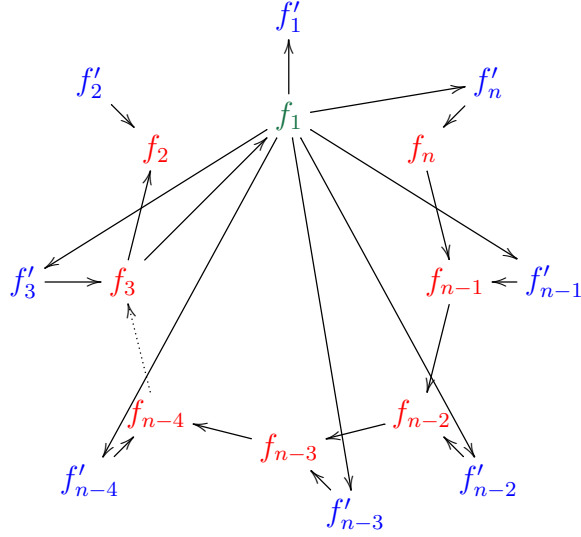


The most important thing to notice is that each previously mutated vertex remains red, while the only arrows created between the frozen vertices, $\{f'_j\}$, and

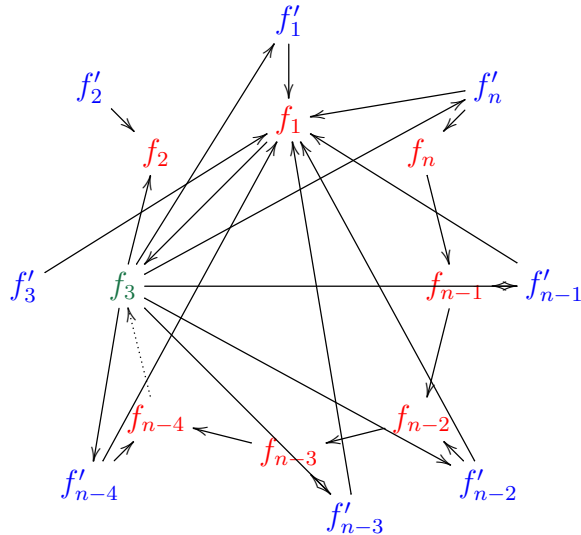
the mutable vertices, $\{f_j\}$, are the arrows $\{f_1 \rightarrow f'_j\}$. The other important thing to make note of is that the mutation $\mu_{f_{n-1}}$ deletes the arrow $f_1 \leftarrow f_n$ which was created by the mutation before it. Since f_n is not adjacent to f_{n-2} , and the resulting diagram we get by removing $\{f_n, f'_n\}$ is the same as the previous diagram but with an index shift, we know that this pattern will continue to hold for the mutations μ_{n-2}, \dots, μ_3 . The resulting quiver $\mu_{f_3} \circ \mu_{f_4} \circ \dots \circ \mu_{f_n}(Q)$ will be the following:



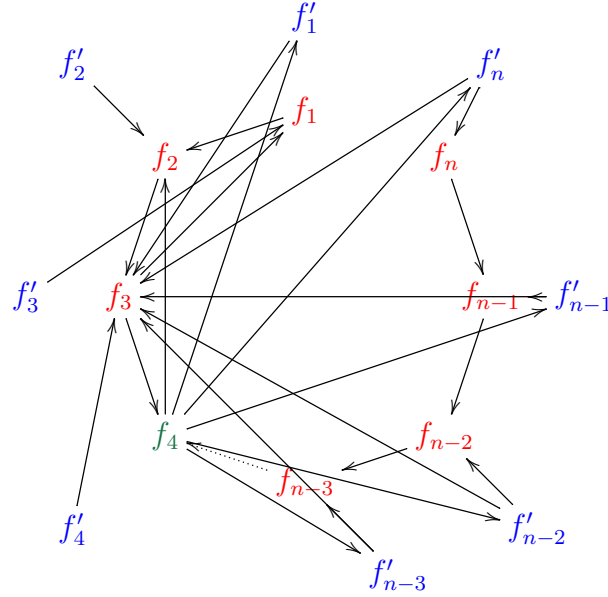
We can see by looking at the above picture that when we perform the next mutation, no additional arrows will be created from step (1) of the mutation process. Hence the only impact on the quiver will be the reversing of arrows which are incident to the vertex f_2 .



Now if we consider the current state of the quiver, there is only one vertex which is green, f_1 . We notice that the only arrow with target f_1 is the arrow $f_3 \rightarrow f_1$. Therefore step (1) of the mutation μ_{f_1} will only create arrows with source f_3 . Hence the only possible vertex which could shift from red to green is f_3 , and in fact f_3 will become green. The result of the mutation will be creating the arrows $\{f_3 \rightarrow f'_j \mid j = n, n-1, n-2, \dots, 4 \text{ and } j = 1\}$. It will also delete the arrow $f'_3 \rightarrow f_3$.

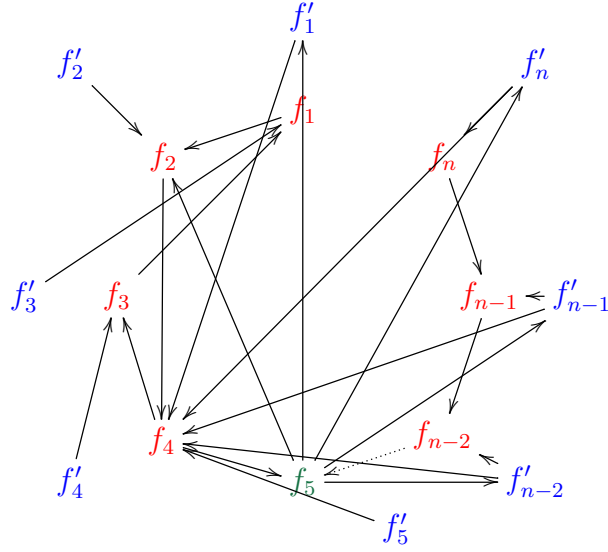


We now are forced to mutate at our only green vertex in the quiver. Step (1) of μ_{f_3} creates the arrows $\{f_1 \rightarrow f'_i \mid i = 4, 5, \dots, n, \text{ and } i = 1\}$, but step (3) will delete these arrows since the arrows $\{f_1 \leftarrow f'_i \mid i = 4, 5, \dots, n, \text{ and } i = 1\}$ already exist in our current state of the quiver. Meaning the vertex f_1 will remain red. The only other arrow whose target is f_3 is $f_4 \rightarrow f_3$, so the only vertex which could possibly turn from red to green is f_4 and this will occur. Step (1) of the mutation μ_{f_3} will create the arrows $\{f_4 \rightarrow f'_i \mid i = 4, 5, 6, \dots, n \text{ and } i = 1\}$, but step (3) will delete the arrow $f_4 \rightarrow f'_4$, because the arrow $f'_4 \rightarrow f_4$ is already in the quiver prior to this mutation.

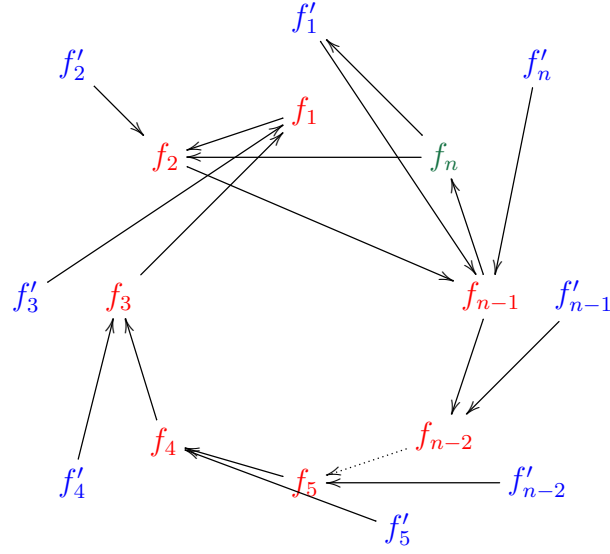


Our next mutation is then forced to be μ_{f_4} because it is the only green vertex in the quiver. The only arrows with target f_4 , are the arrows $f_3 \rightarrow f_4$ and $f_5 \rightarrow f_4$. First let us consider the arrows created with source f_3 . Step (1) of the mutation process will create the arrows $\{f_3 \rightarrow f'_i \mid i = 5, 6, 7, \dots, n \text{ and } i = 1\}$. It also creates the arrow $f_3 \rightarrow f_2$. All of these arrows will be deleted by step (3) of the mutation process. This means that no new outgoing arrows are created with source f_3 , therefore f_3 remains a red vertex after mutation. Now we consider the arrows

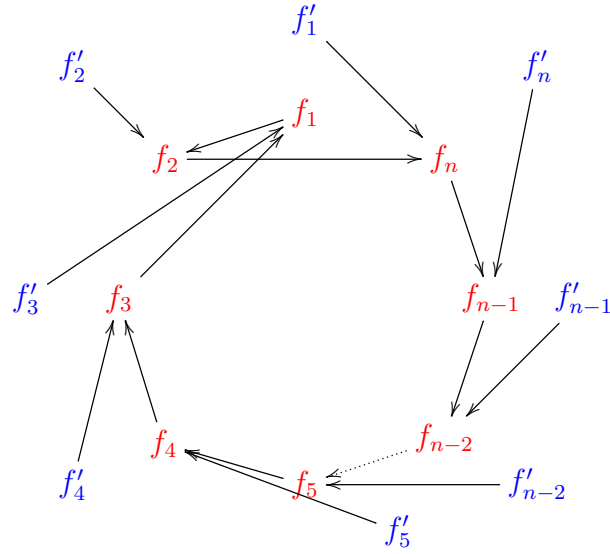
with source f_5 , which are created by the mutation μ_{f_4} . The arrows created are $\{f_5 \rightarrow f'_i \mid i = 5, 6, 7, \dots, n \text{ and } i = 1\}$, but the arrow $f_5 \rightarrow f'_5$ is deleted by step (3) of the mutation process.



If we continue this pattern what we are seeing is that by mutating at f_i we are deleting all of the currently existing arrows $\{f'_j \rightarrow f_{i-1} \mid j \neq i\}$ and we are creating the arrows $\{f_{i+1} \rightarrow f'_j \mid j = i + 2, i + 3, \dots, n \text{ and } j = 1\}$. This means at each mutation step, μ_{f_i} , the only vertex which will turn green is f_{i+1} . Essentially we are transferring all the outgoing arrows from the vertex f_i to the vertex f_{i+1} . This process continues for each mutation in the sequence until the last mutation step. Lets look at the quiver right before this mutation step, $\mu_{f_{n-1}} \circ \mu_{f_{n-2}} \circ \dots \circ \mu_{f_3} \circ \mu_{f_1} \circ \mu_{f_2} \circ \dots \circ \mu_{f_n}(Q)$.



At this point step (1) of the final mutation in the sequence, will create only the arrows $f_{n-1} \rightarrow f'_1$ and $f_{n-1} \rightarrow f_2$, both of which will be deleted by step (3) of the mutation. Therefore no vertex which is red can become green, meaning that all of the vertices are red. Hence, the sequence of mutations is a maximal green sequence.

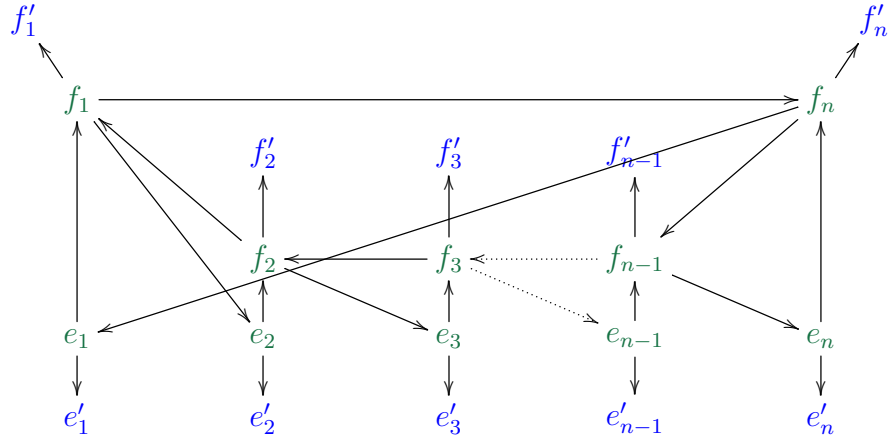


□

The important thing to notice about this sequence is that we pick a starting point and mutate in direction of the cycle until we hit the end of the cycle. At this point we run the mutation sequence backwards from the ending point, but we skip the first two steps of the mutation. We will make use of this sequence again later on in the proof.

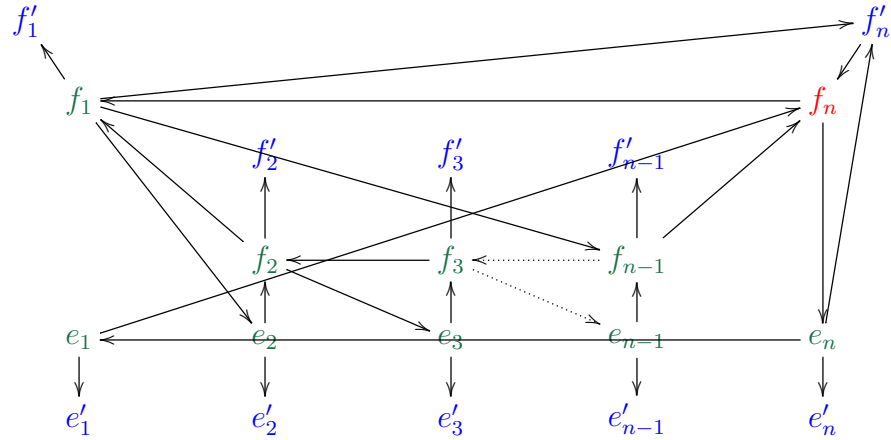
5.2.2 Proof of the main theorem

Now we must consider what this portion of the sequence does to the rest of the quiver Q_{T_n} . Mutation is a local property which only affects adjacent vertices, and since this mutation sequence only involves the vertices $\{f_i\}$ the only vertices that can be affected by the sequence are the vertices $\{f_i\} \cup \{e_i\}$. From the lemma we know that the first part of our sequence, $(f_n, f_{n-1}, \dots, f_1, f_3, f_4, \dots, f_n)$, is green and that after performing this sequence of mutations all of the vertices f_i for $1 \leq i \leq n$, will be red. We must now look at what effect the sequence of mutations has on $\{e_i\}$. So we will look at a diagram of the quiver with the vertices $\{e_i \mid 1 = 1, 2, \dots, n\}$ drawn in.

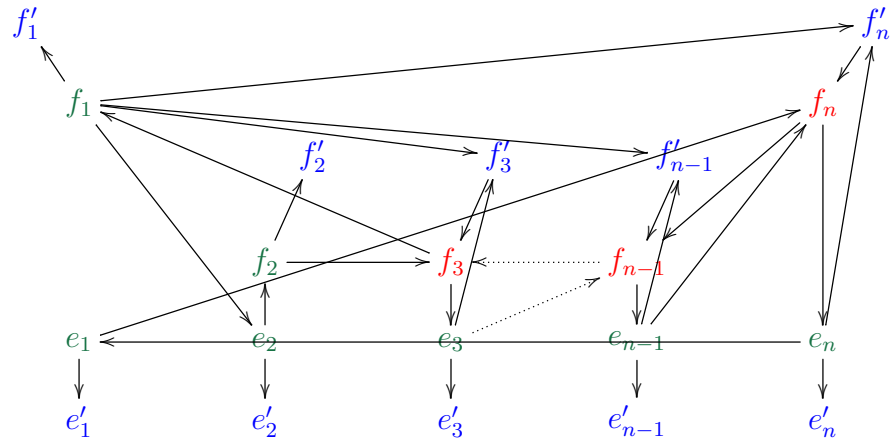


We see that the initial mutation μ_{f_n} will result in creating the arrows $e_n \rightarrow f'_1$ and $e_n \rightarrow f'_n$. It will delete the arrow $f_{n-1} \rightarrow e_n$. This leaves the vertex e_n not adjacent to any vertex f_i , for any $i \neq n$. Meaning that since our sequence consists

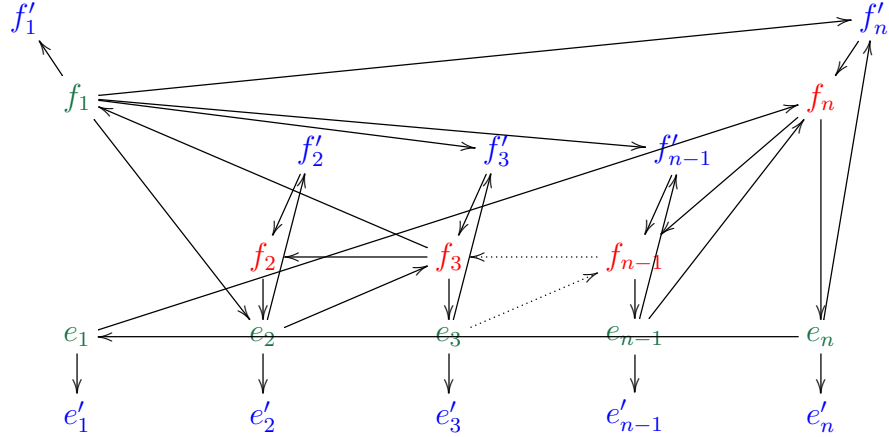
only of mutations at the vertices f_i until the vertex f_n is mutated at we cannot create new arrows involving e_n .



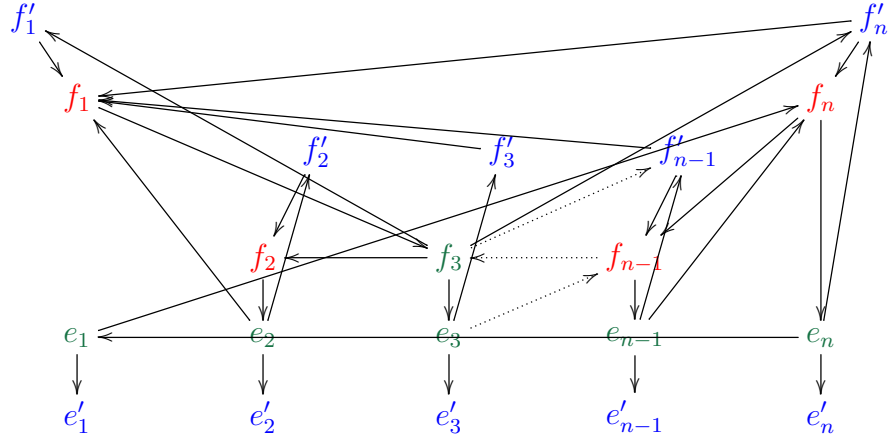
The next mutation $\mu_{f_{n-1}}$ will create the arrows $e_{n-1} \rightarrow f'_{n-1}$ and $e_{n-1} \rightarrow f_n$. It will also delete the arrow $f_{n-2} \rightarrow e_{n-1}$. In general the mutation step μ_i will create the arrows $e_i \rightarrow f'_i$ and $e_i \rightarrow f_{i+1}$, while deleting the arrow $f_{i-1} \rightarrow e_i$. This pattern holds until we have arrived at the quiver $\mu_{f_3} \circ \mu_{f_4} \circ \dots \circ \mu_{f_n}(Q_{T_n})$.



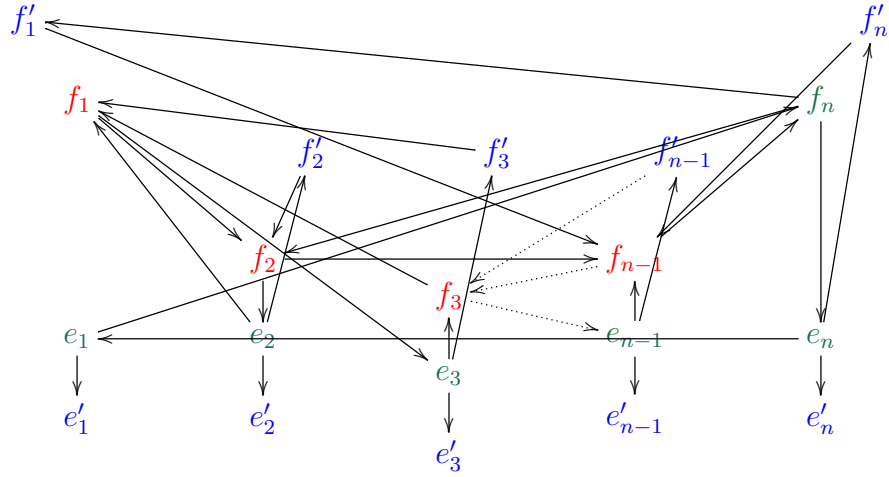
Now we see that at this stage in the mutation sequence we do not have the arrow $f_2 \rightarrow f_1$, and so our next mutation μ_{f_2} will only create the edges $e_2 \rightarrow f_3$ and $e_2 \rightarrow f'_2$.



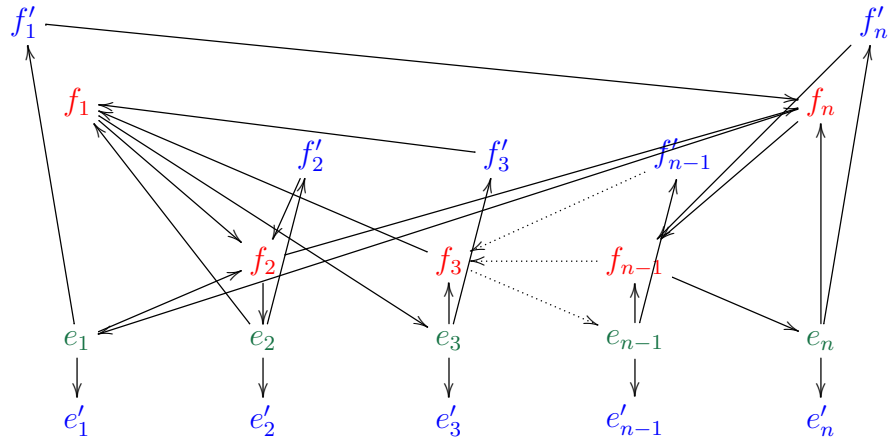
Next, we look at what occurs when we perform the mutation μ_{f_1} . Step (1) of this mutation will create the arrow $e_2 \leftarrow f_3$, but step (3) will delete this arrow because the quiver already has the arrow $e_2 \rightarrow f_3$.



Now we notice that at this stage the vertex e_2 is not adjacent to any vertices that will be mutated during the remainder of our sequence. Therefore its current arrows will not be affected by the sequence. As we continue performing the mutations of this occurs for each e_i for $i = 2, 3, \dots, n$. More specifically, after the mutation μ_{f_i} the arrows incident to the vertex e_i will be fixed for the remainder of the mutations in the sequence. This pattern continues until we have the quiver, $\mu_{f_{n-1}} \circ \mu_{f_{n-2}} \circ \dots \circ \mu_{f_3} \circ \mu_{f_1} \circ \dots \circ \mu_{f_n}(Q_{T_n})$.

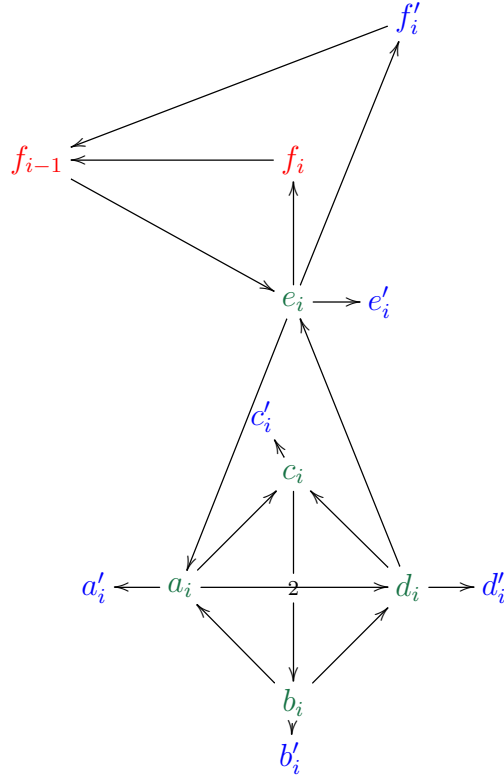


Now if we look at the final mutation μ_{f_n} , the net result from the mutation will be creating the arrow $f_{n-1} \rightarrow e_n$ and deleting this arrow $e_n \rightarrow e_1$. It will also create the arrow $e_1 \rightarrow f'_1$. The end result is a quiver that up to permuting the vertices f_1 and f_2 , we have the same structure that we had prior to the sequence with the following exceptions: the arrows $\{e_i \rightarrow f'_i \mid 1 \leq i \leq n\}$ are now in the quiver and the vertices f_i are all red instead of green.



This concludes what we need to consider from the initial part of the sequence. Now we must look at the additional pieces which are attached to the bottom of the quiver. We will call these subquivers H_i , and define it as $(H_i)_0 = \{a_i, b_i, c_i, d_i, e_i, a'_i, b'_i, c'_i, d'_i, e'_i\}$ and $(H_i)_1 = \{\text{all arrows between elements of } (H_i)_0\}$. Below is a diagram of H_i af-

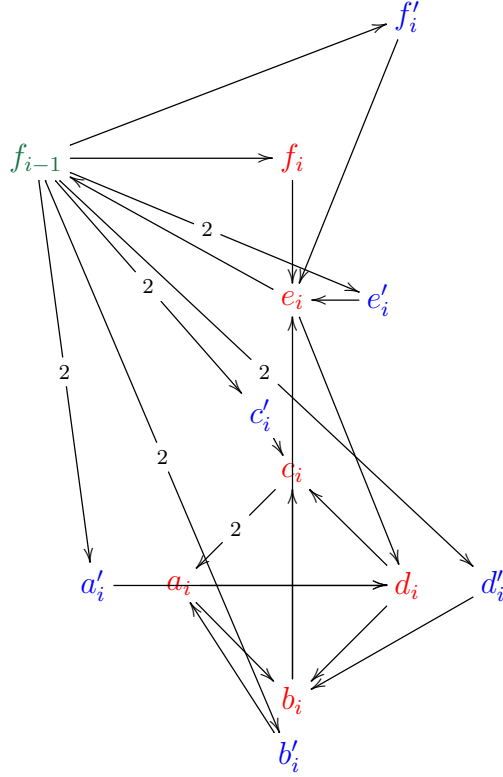
ter performing the mutation sequence above. We have included in the diagram the vertices which H_i is adjacent to as well, though they are not part of H_i .



The next part of our maximal green sequence will be mutation sequences that occur only on the vertices of the H_i , specifically we will consider what happens when we apply the σ_i for each i .

We look at what occurs when we perform the mutation sequence σ_i . Since mutation is a local condition it will only effect the vertices shown in the above diagram, $(H_i)_0 \cup \{f_i, f_{i-1}, f'_i\}$. In addition it is important to note that only arrows between these vertices can be affected by the mutation sequence. Therefore the mutation sequences σ_i and σ_j will not interact with each other.

By computation we can check the result of performing the sequence of mutations σ_i on the subquiver H_i since it is a finite number of steps. These computations were checked using the java applet developed by Keller.

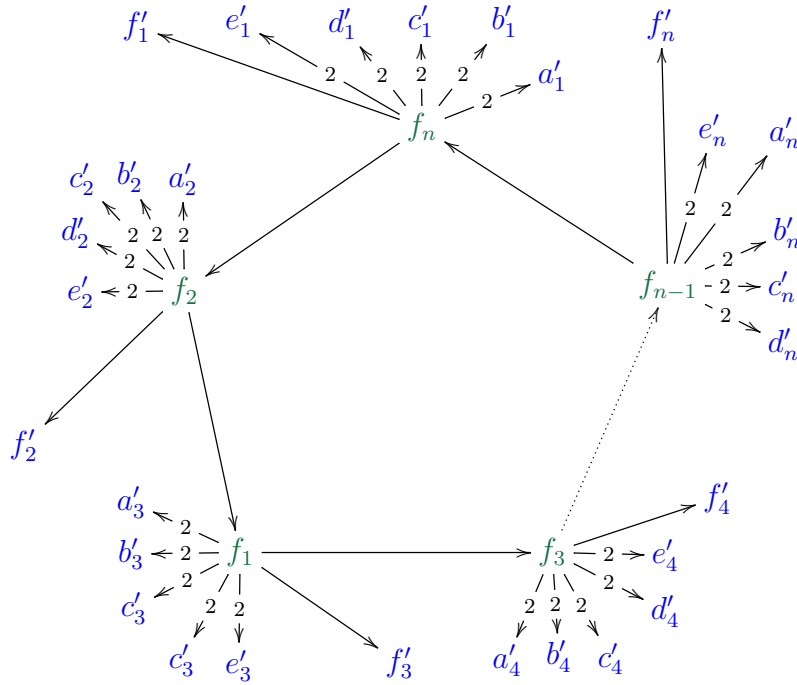


Notice that each sequence σ_i results in the vertex f_{i-1} becoming a green vertex. Therefore the only green vertices in the quiver after performing all of the sequences, σ_i , are the vertices $\{f_1, f_2, f_3, \dots, f_n\}$.

Another important aspect of the current state of the quiver is that there are no arrows $f_{i-1} \rightarrow a_i$, $f_{i-1} \rightarrow b_i$, $f_{i-1} \rightarrow c_i$, $f_{i-1} \rightarrow d_i$, or arrows in the opposite directions. Therefore when we perform the next portion of the mutation sequence, $(f_n, f_{n-1}, \dots, f_3, f_1, f_2, \dots, f_n)$, since all of the mutations occur at the vertices $\{f_i\}$ we will not introduce new arrows involving the vertices $\{a_i, b_i, c_i, d_i\}$.

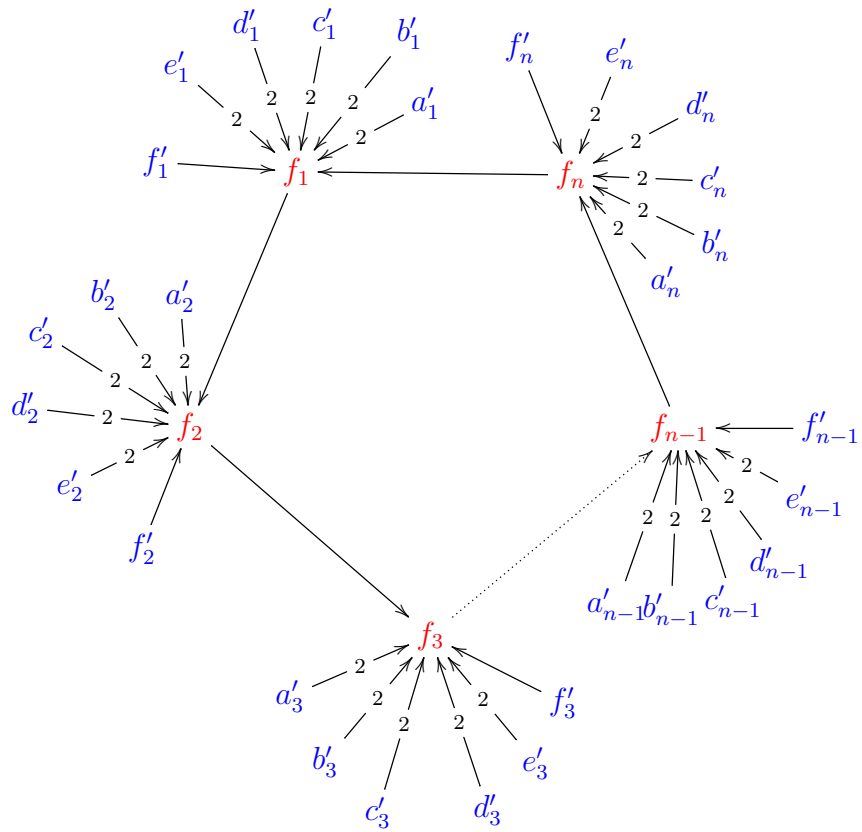
Below is a diagram of the current state of the quiver, $\mu_{\sigma_1} \circ \mu_{\sigma_2} \cdots \mu_{\sigma_n} \circ \mu_{f_n} \circ \mu_{f_{n-1}} \circ \cdots \circ \mu_{f_3} \circ \mu_{f_1} \circ \cdots \circ \mu_{f_n}(Q_{T_n})$, in which we have omitted all the vertices

except for the $\{f_i\}$.

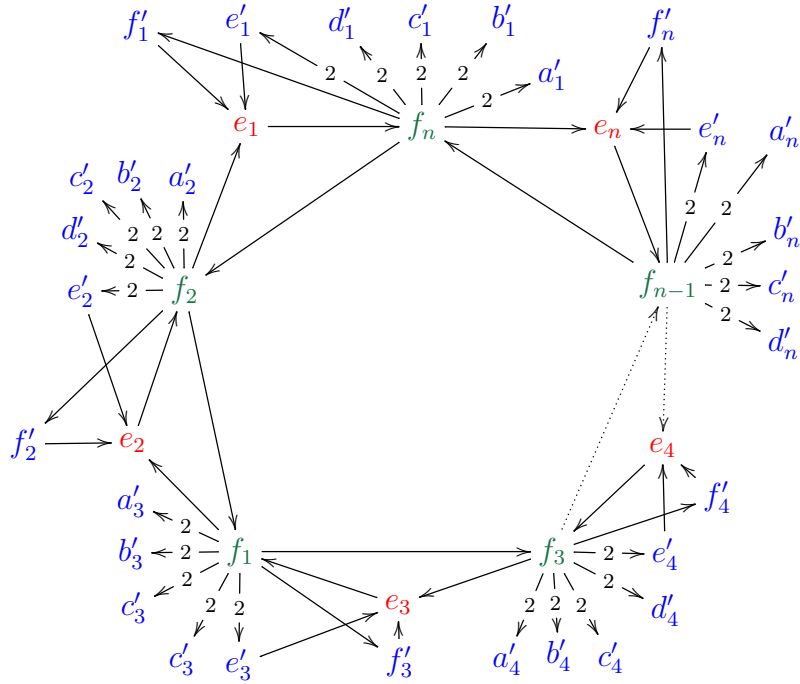


We notice that this diagram is the same as the diagram from Lemma 6.4, with some minor alterations. First, the cycle has a reversed orientation. We now have attached multiple frozen vertices to each f_i and we have permuted the vertices f_1 and f_2 . Additionally the indices of the frozen vertices do not match the indices of the mutable vertex they are adjacent to. The important aspect is that we can utilize the same sequence of mutations that we used before to turn all of these vertices red, by adjusting for the new ordering of the vertices $\{f_i\}$.

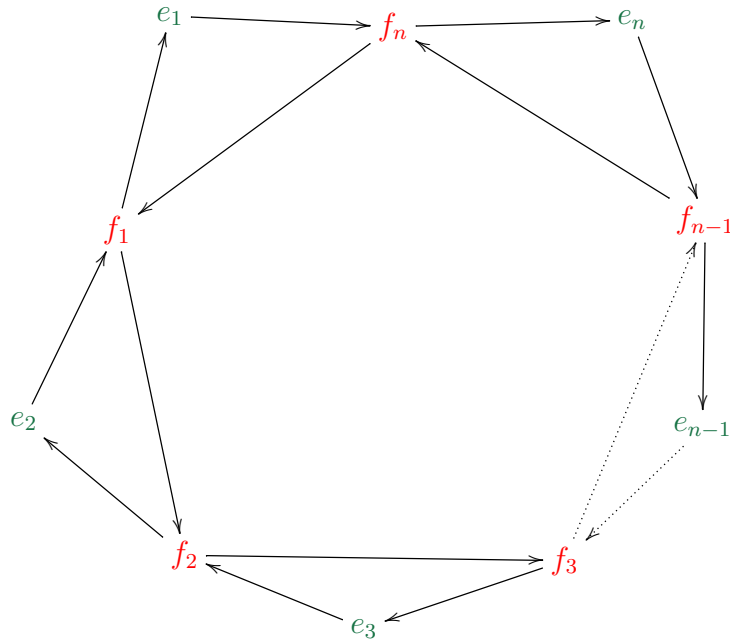
We choose a starting point and then mutate in the direction of the cycle, until we reach the end of the cycle, in which case we turn around and run the sequence in reverse, but skipping the first two steps of the sequence. The sequence of mutations we use is the following $(f_3, f_4, \dots, f_n, f_2, f_1, f_n, f_{n-1}, \dots, f_3)$. This sequence is chosen because the result will permute the vertices f_1 and f_2 , undoing the permutation from performing the sequence in Lemm 6.4. The resulting quiver after performing the sequence of mutations is shown below.



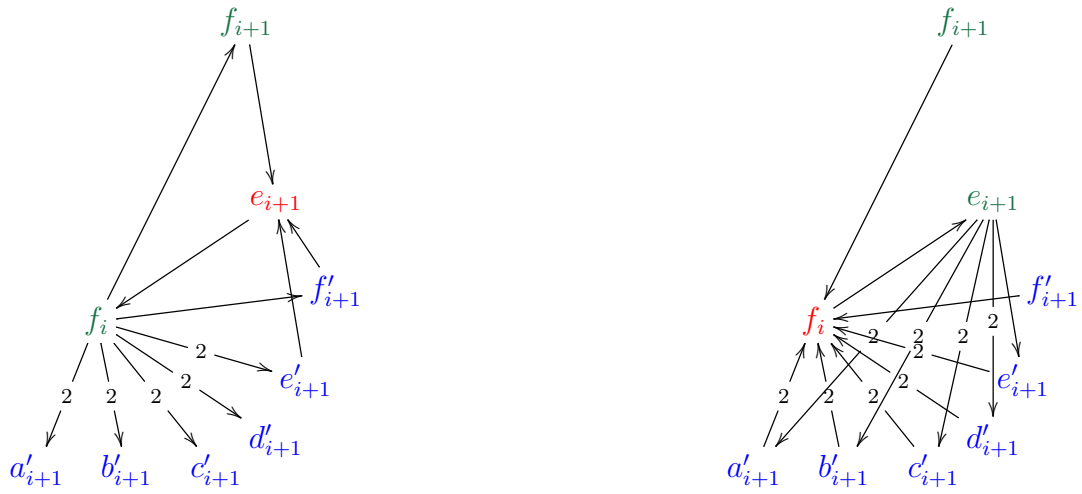
To understand how this sequence will impact the other vertices of the quiver it is important to note that the only vertices which connect to the $\{f_i\}$ at this stage of the quiver are the vertices $\{e_i\}$ and the frozen vertices. Below is a diagram depicting the quiver and the vertices which are adjacent to the $\{f_i\}$.



First, we notice that the subquiver including only the vertices $\{f_i\} \cup \{e_i\}$ is exactly the same quiver as the quiver we started with before we did any mutations (with a change of orientation). Therefore since this sequence of mutations is the same as before with an adjustment for this change of orientation we can see that it will have the same effect on the vertices $\{e_i\}$, in terms of creating arrows between the vertices $\{f_i\}$ and $\{e_i\}$. Therefore like before it will not effect the arrows $e_i \rightarrow f_{i-1}$ and $f_n \rightarrow e_n$ except for the fact that the vertices f_1 and f_2 are permuted by this sequence of mutations. Below is a diagram of the final result, with the frozen variables removed to make it easier to see the end result.

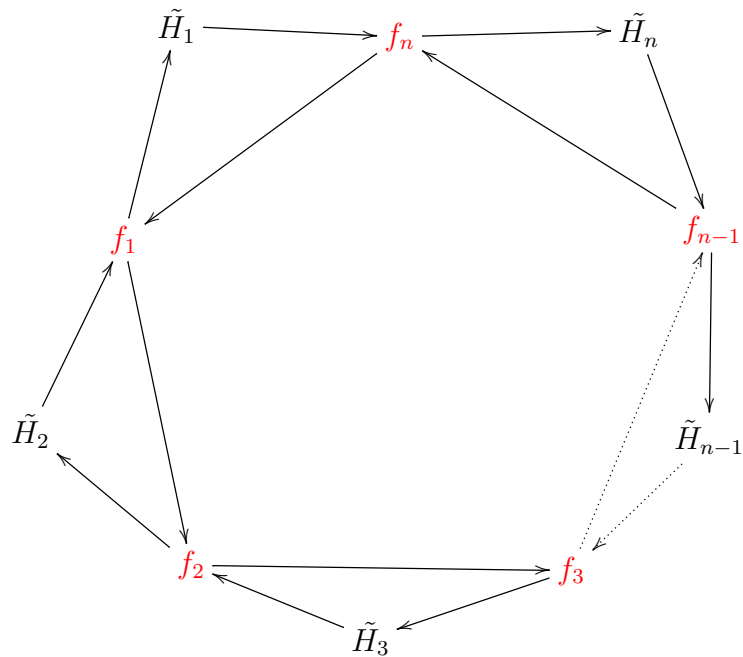


Now the only thing left to do is keep track of the arrows created between the frozen vertices and the vertices $\{e_i\}$ as this sequence of mutations is performed. At each initial step of the mutation, μ_{f_i} , the vertex e_{i+1} will gain arrows to each frozen vertex incident to f_i . Also important, is that the arrow $e_{i+1} \rightarrow f_{i+1}$ is deleted from this mutation. This means that no additional arrows between e_{i+1} and the frozen vertices will be created during this mutation sequence. Below is a diagram showing this interaction before and after the mutation μ_{f_i} .

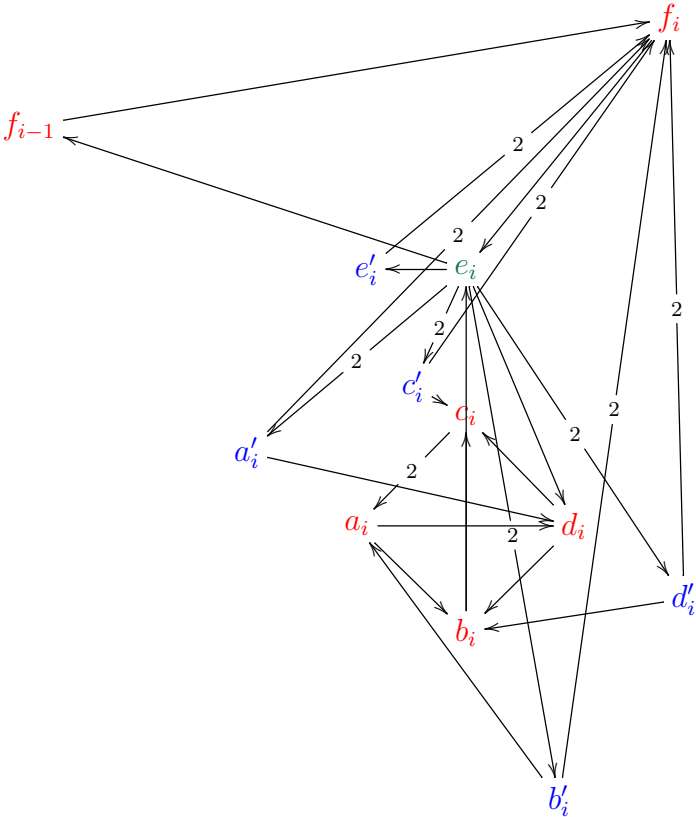


At this point there is only one part of the sequence left to consider: $(\tau_n, \tau_{n-1}, \dots, \tau_1)$.

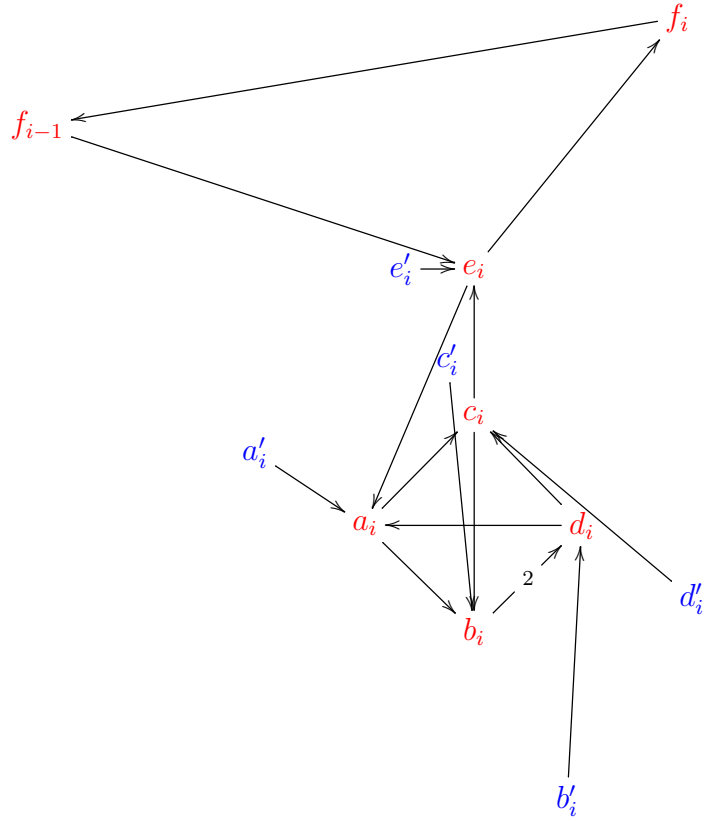
Before we do this let us look at the state of the current quiver. We have the cycle below with subquiver \tilde{H}_i attached to each vertex.



Below is a diagram of each \tilde{H}_i and how it connects to the large cycle:



The mutation sequence τ_i is a sequence only on the of \tilde{H}_i and hence will not effect the vertices of \tilde{H}_j with $j \neq i$. By computation we can check to see that each mutation sequence τ_i will turn the vertex e_i into a red vertex while leaving all of the other vertices red as well. The end result can be checked by using the Keller mutation applet and is shown below.



The vertices included in τ_i belong to \tilde{H}_i only and hence the performance of each sequence τ_i does not create any green vertices. It only turns the vertex e_i from a green vertex to a red vertex. Therefore after completing each mutation sequence τ_i , every vertex in the quiver will be red. This means that the sequence of mutation which we performed was a maximal green sequence. Or in other words that,

$$(f_n, f_{n-1}, \dots, f_1, f_3, f_4, \dots, f_n, \sigma_n, \sigma_{n-1}, \dots, \sigma_1, f_3, f_4, \dots, f_n, f_2, f_1, f_n, f_{n-1}, \dots, f_3, \tau_n, \tau_{n-1}, \dots, \tau_1)$$

is a maximal green sequence for the quiver Q_{T_n} .

Chapter 6

Closed surfaces with at least 3 punctures

This chapter will look following up the result from chapter 5, by generalizing the sequence to include an arbitrary number of additional punctures. The chapter will be outlined in a similar fashion to chapter 5 in that it will first present the triangulation that we will use, and then show that the proposed sequence is in fact a maximal green sequence.

6.1 Constructing the Triangulation

Following the work done in the previous chapter we construct a quiver Q_n^p associated to the genus n surface with no boundary and $p \geq 3$ punctures. The triangulation of the surface with two punctures that was given in the previous chapter (see Figure 6.1) gives rise to a very natural generalization. When $p \geq 3$ we can remove the arc f_n and replace it with a $(p - 2)$ -times punctured digon. We then triangulate the digon as shown on the right of Figure 6.2. The final triangulation is also given in Figure 6.2.

From this triangulation we can construct the quiver Q_n^p . The quiver associated to the 3-torus with 7 punctures is given on the left of Figure 6.3. Also we define the quiver P^{p-3} for $p \geq 3$ to be the full subquiver of Q_n^p consisting of the vertices $\{g_0, g_1^1, g_2^1, g_3^1, \dots, g_1^{p-3}, g_2^{p-3}, g_3^{p-3}\}$. P^4 is given on the right of Figure 6.3. Note that by increasing the genus of the surface the cycle containing the f vertices gets longer, and more handles are added. Increasing the number of punctures will increase the number of rows in the P subquiver. The important thing to notice is that the fundamental shape of the quiver Q_n^p doesn't change.

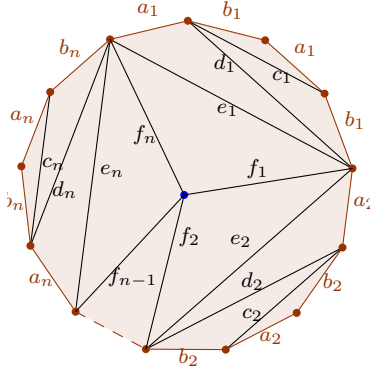


FIGURE 6.1. A triangulation of an n -torus with two punctures.

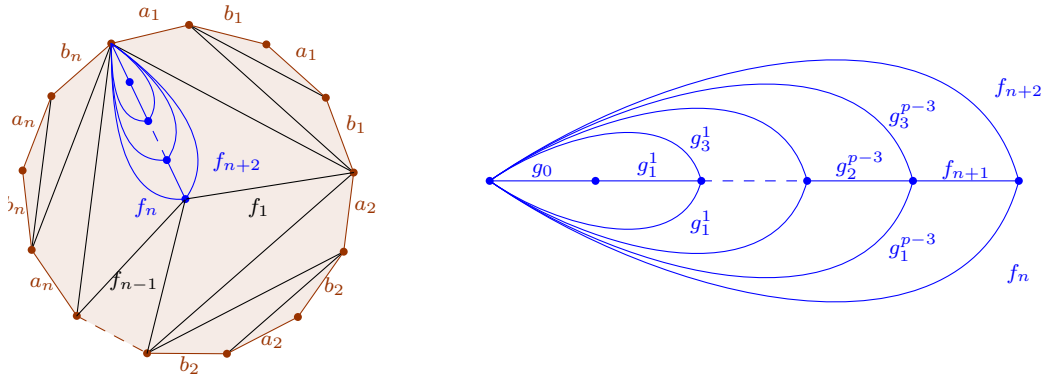


FIGURE 6.2. A triangulation of n -torus with p punctures.

6.2 Statement and Proof of Main Result

Theorem 6.1. Let Q_n^p be the quiver obtained from our triangulation of a genus n surface with no boundary and $p \geq 3$ punctures. Then Q_n^p has a maximal green sequence given by

$$f_{n+2}f_{n+1} \cdots f_1f_3f_4 \cdots f_{n+2}\sigma_n \cdots \sigma_1\alpha_0\alpha_1 \cdots \alpha_{p-3}f_{n+2}f_2f_1$$

$$f_3f_4 \cdots f_n f_{n-2}f_{n-3} \cdots f_3f_1f_2f_{n+2}f_{n+1}\beta_{p-3}\tau_1\tau_2 \cdots \tau_n,$$

where σ, τ, α , and β are defined as follows:

$$\sigma_i = e_id_ib_ic_ia_ib_id_ie_ia_ib_i, \quad \tau_i = e_ib_ia_ice_id_ib_ia_e_i,$$

$$\alpha_j = \begin{cases} g_0 & j = 0 \\ g_3^1 g_2^1 g_1^1 g_3^1 g_0 & j = 1 \\ g_3^2 g_2^2 g_1^1 g_1^2 g_3^2 g_1^1 g_0 & j = 2 \\ g_3^j g_2^j g_1^{j-2} g_1^{j-1} g_1^j g_3^j g_1^{j-2} g_3^{j-1} g_1^{j-3} g_3^{j-2} \cdots g_1^1 g_3^2 g_1^1 g_3^1 g_0 & j \geq 3 \end{cases}$$

$$\beta_j = \begin{cases} \emptyset & j = 0 \\ g_1^1 g_2^1 g_3^1 & j = 1 \\ g_1^1 g_1^2 g_1^1 g_3^2 g_2^1 g_3^1 & j = 2 \\ g_1^{n-1} g_1^j g_1^{n-1} g_3^n g_1^{n-2} g_3^{n-1} g_1^{n-3} \cdots g_3^3 g_1^1 g_3^2 g_1^1 g_3^1 & j \geq 3 \end{cases}$$

Remark 6.2. You'll notice that the sequence similarly resembles the sequence from the previous chapter, but has different adjusting sequences for the new structure that is added into the quiver. A brief comment about the methods might make the combinatorics easier to follow. Essentially what we have done in this chapter is construct a sequence that deals with the subgraph P_n and then we show how to attach this sequence into the chapter 5 sequence. Adding additional punctures into the surface only increases the depth of ladder type structure of P_n and hence by proving for an arbitrary depth P_n that we have a green sequence that appropriately effects the remainder of the quiver we can add an arbitrary number of punctures to our surface and still produce a maximal green sequence.

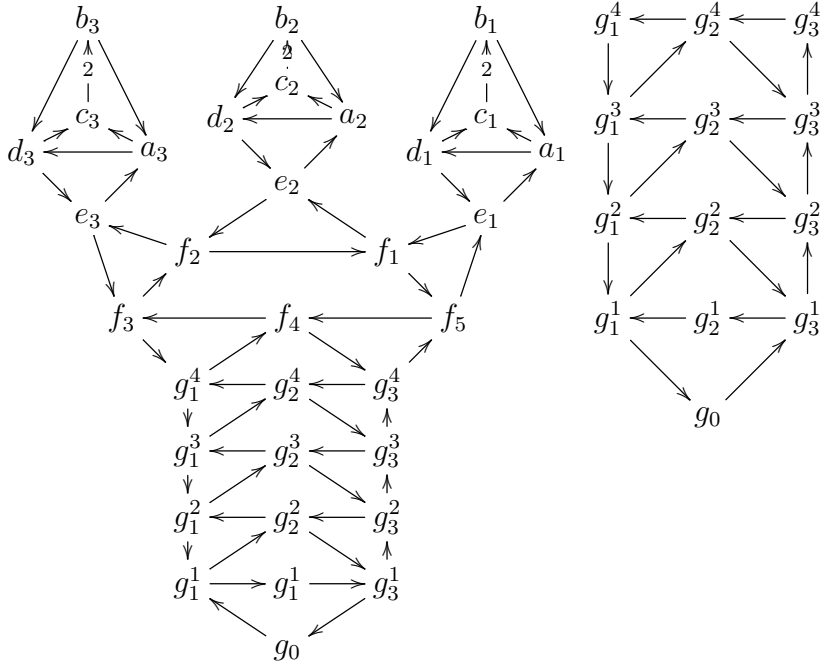
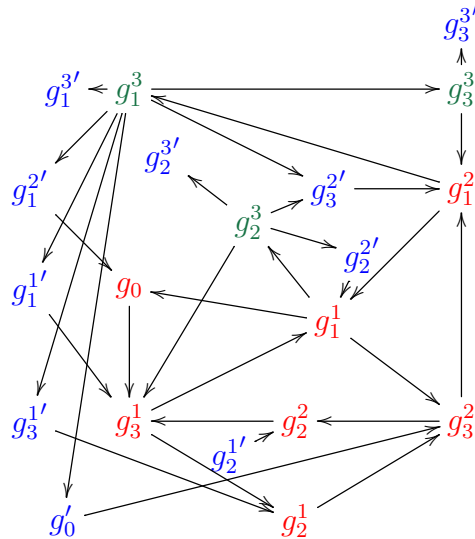


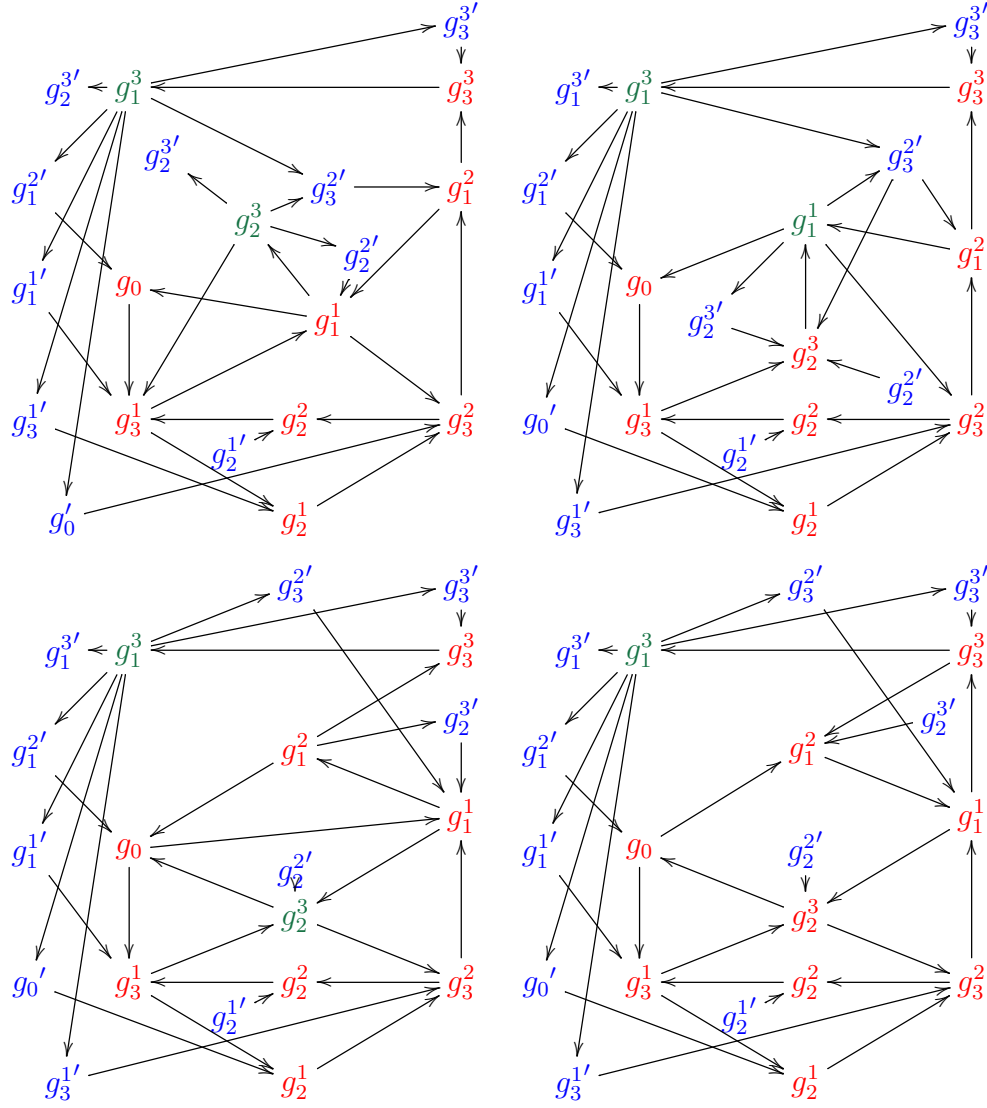
FIGURE 6.3. The quivers Q_3^7 (left) and P_4 (right).

Lemma 6.3. P_n has a maximal green sequence given by $\alpha_0\alpha_1\cdots\alpha_n$.

Proof. The sequence is easily checked for $n = 0, 1, 2$. For $n=3$, apply $\alpha_0\alpha_1\alpha_2$ to P_3 . We know that this is a green sequence for the P_2 subquiver of P_3 . The current state of the quiver is given in the following diagram.

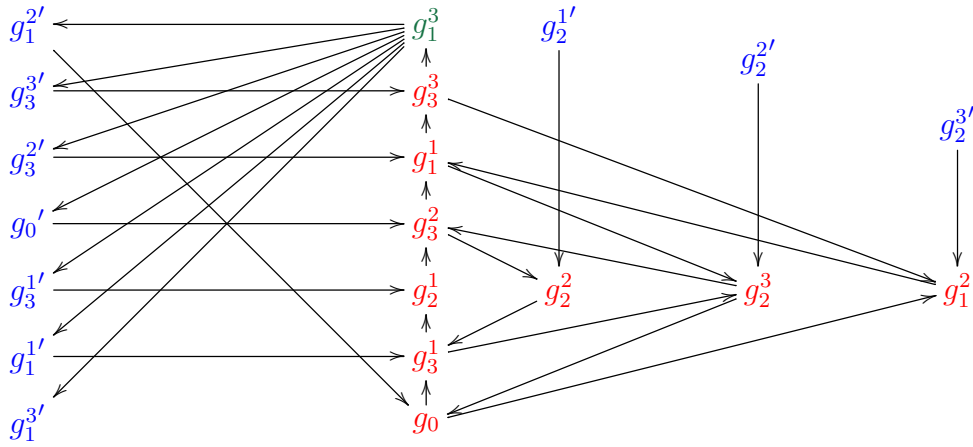


We now apply the first four mutations of α_3 to the quiver above.

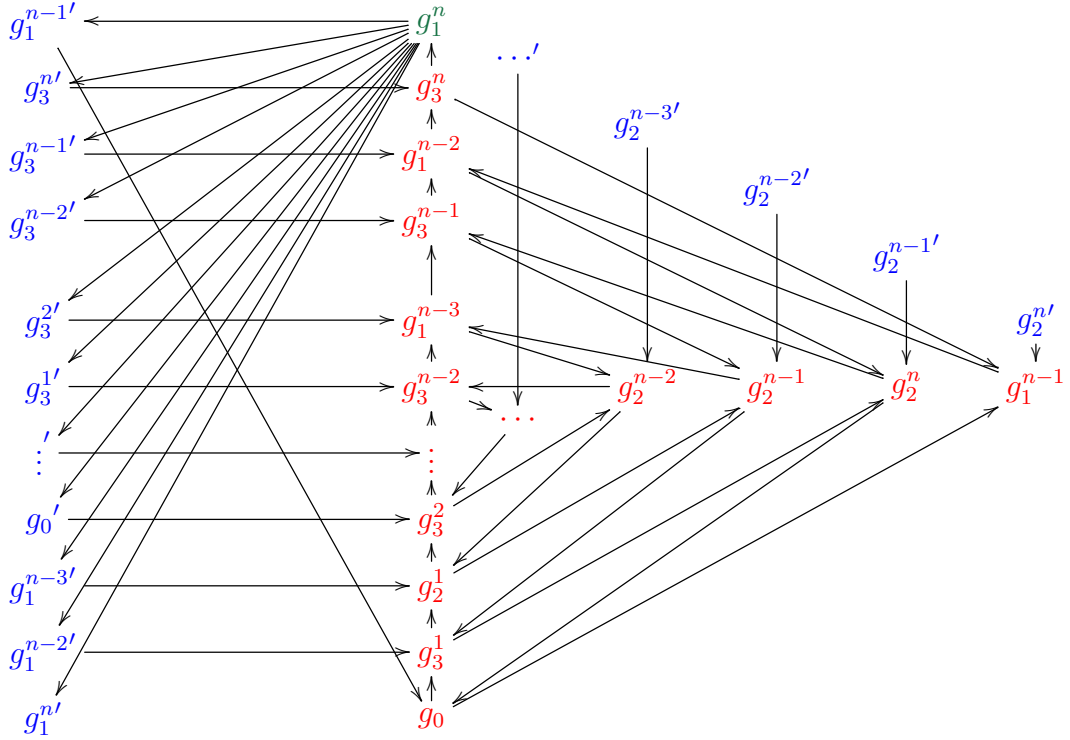


After these four mutations g_1^3 is the only remaining green vertex, and is the initial vertex in a 2-path through a frozen vertex for 6 vertices. However, the terminal vertex in these 2-paths form an equioriented affine subquiver with g_1^3 being the sink for this subquiver. The remaining mutations of α_3 is just the mutation along the vertices of this subquiver. Note that at each step through this part of the sequence there is a unique green vertex with a unique edge with head at mutable vertex and tail at the green vertex. Rearranging the vertices from our previous picture

we obtain the following picture from which it is easy to see that the remaining mutations give a maximal green sequence.



Thus $\alpha_0\alpha_1\alpha_2\alpha_3$ is a maximal green sequence for P_3 . Our claim follows from induction on n . Suppose for $1 \leq k < n$ $\alpha_0 \cdots \alpha_k$ gives a maximal green sequence for α_k . Note that Q_n has a subquiver of P_{n-1} for which $\alpha_0\alpha_{n-1}$ is a green sequence. Note that the local configuration of "top nine" vertices g_i^j $i = 1, 2, 3$ $j = n - 2, n - 1, n$ have the exact same configuration as the "top nine" vertices of P_3 and the first four mutations of α_n exactly mimic that of $n = 3$ case. (Possibly need to show this in a lemma.) Therefore we get that after the green sequence $\alpha_0 \cdots \alpha_{n-1}g_3^jg_2^jg_1^{j-2}g_1^{j-1}$ we have the quiver:



Where again it is easy to check the remaining mutations will give us a maximal green sequence for P_n . □

Lemma 6.4 (Cycle Lemma). If C is an oriented n -cycle with vertices labeled c_i $i = 1, \dots, n$, then $c_n c_{n-1} c_{n-2} \cdots c_1 c_3 c_4 \cdots c_n$ is a maximal green sequence for C .

Theorem 6.5 (Main Result Chapter 5). The quiver Q_n^2 has a maximal green sequence of

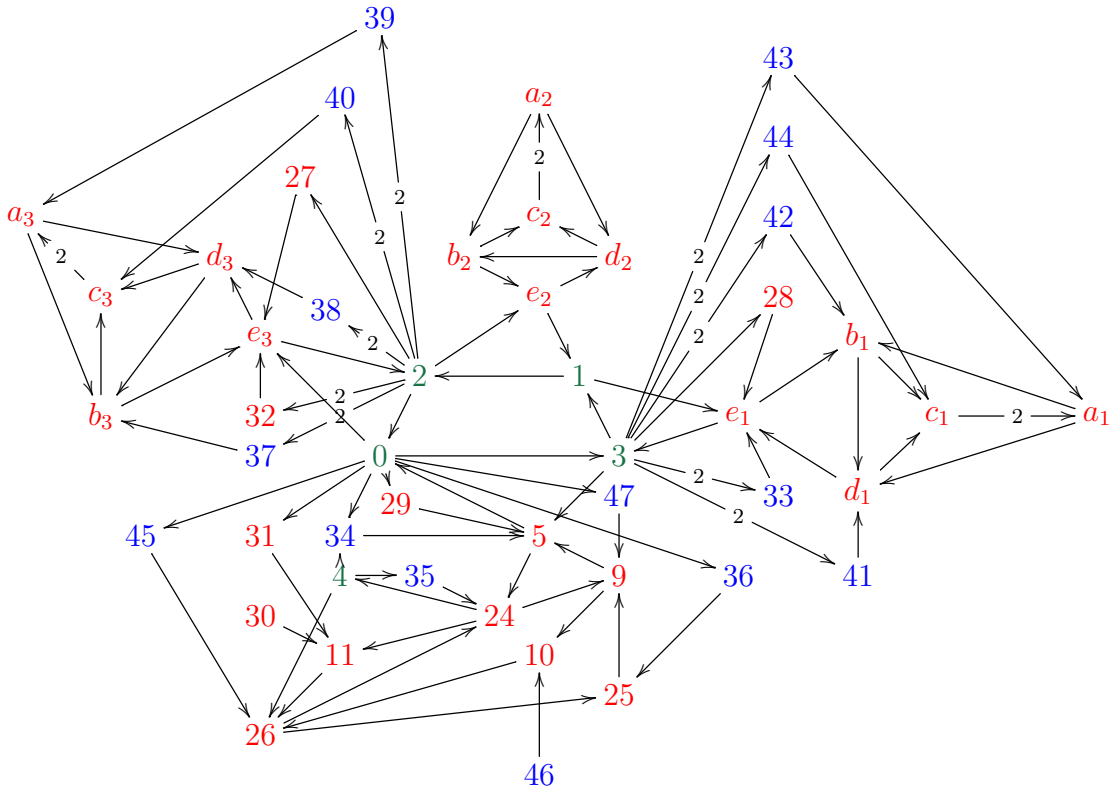
$$(f_n, f_{n-1}, \dots, f_1, f_3, f_4, \dots, f_n, \sigma_n, \sigma_{n-1}, \dots, \sigma_1, f_3, f_4, \dots$$

$$f_n, f_2, f_1, f_n, f_{n-1}, \dots, f_3, \tau_n, \tau_{n-1}, \dots, \tau_1.)$$

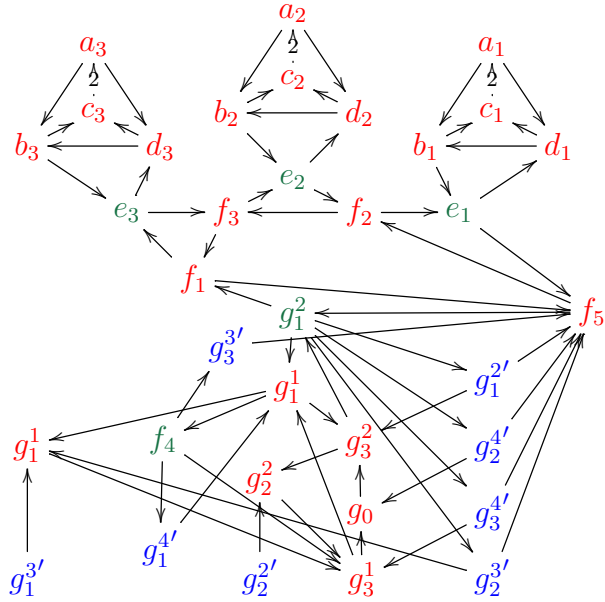
Proof of Theorem. By Lemma 6.3 and the proof of Theorem 6.5 in chapter 5 we know that

$$f_{n+2} f_{n+1} \cdots f_1 f_3 f_4 \cdots f_{n+2} \sigma_n \cdots \sigma_1 \alpha_0 \alpha_1 \cdots \alpha_{p-3}$$

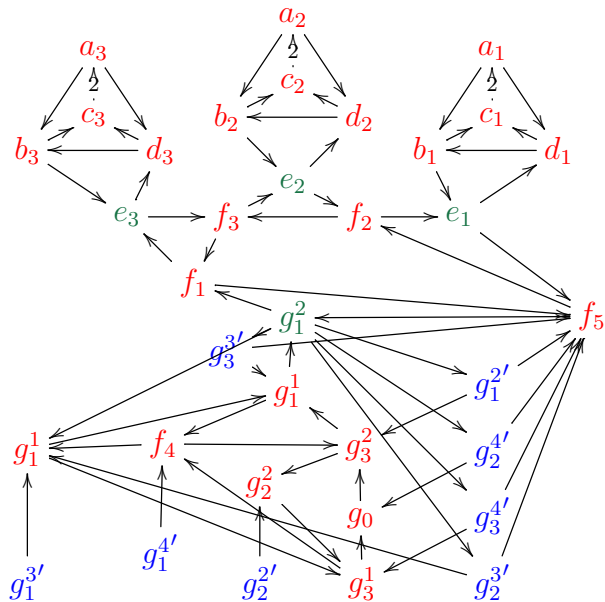
is a green sequence. After performing this mutation sequence we have the that all of the vertices are red except for the vertices $f_1 \cdots f_{n+2}$. Furthermore, all of these vertices except for f_{n+1} form an $(n + 1)$ -cycle .



By Lemma 6.4 our next section of the maximal green sequence is a green sequence for this cycle.



After mutating at f_{n+1} and g_1^{p-4} we have a similar situation as we did at the end of the proof of Lemma 6.3. The remaining vertices that we mutate along in the β subsequence form an equioriented affine subquiver. It is easy to follow that this is a green sequence that will make the P_{p-3} subquiver of Q_n^p red.



Finally, the only remaining green vertices are e_i for $i = 1, \dots, n$. By inspection of the local configuration of the frozen vertices we see that this is the exact same configuration as in the twice punctured case. Therefore it follows from the proof of Theorem 6.5. That τ_i is a green sequence for our quiver, and concluding the proof that our sequence is maximal. \square

6.3 Survey and Future Interests

As mentioned in the introduction, any cluster algebra of finite mutation type can be it's association to triangulated surfaces. Below is a table that shows which surfaces have an associated quiver which exhibits a maximal green sequence.

Surface Classification	Number of MGS	Source
Surface with boundary	exists Δ with MGS	Alim et al.
Closed surface with 1 puncture	no MGS	S. Ladkani
Sphere with $p \geq 4$	exists Δ with MGS	Alim et al.
Torus with $p \geq 2$ punctures	exists Δ with MGS	Alim et al.
Closed surface $g \geq 2$ and $p = 2$	exists Δ with MGS	Bucher
Closed surface $g \geq 2$ and $p \geq 3$	exists Δ with MGS	Bucher and Mills

There are still many open questions which should be raised. Muller in [15] gives an explicit quiver with a maximal green sequence and a mutation equivalent quiver which does not have a maximal green sequence. This example though cannot be associated to a surface, or in otherwords is not finite mutation type. It is the belief of the author that when dealing with cluster algebras that arise from surfaces that the existence of a maximal green sequence may in fact be a mutation invariant.

Conjecture 6.6. Let Q be a quiver associated to a surface. If Q exhibits a maximal green sequence, then any Q' mutation equivalent to Q also exhibits a maximal green sequence.

In general with this result and the results of others, much is known about the existence of maximal green sequences for cluster algebras of finite mutation type, but the question is still fairly open when discussing cluster algebras that are not associated to surfaces.

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Vita

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