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# A MULTIDIMENSIONAL RUIN PROBLEM

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*Dedicated to Professor K. R. Parthasarathy on the occasion of his 75th birthday*

Abstract. We consider the ruin problem for an insurance network modelled in terms of the Skorokhod problem in an orthant, where the interaction among the companies is only through the risk reducing treaty. In the case of a Cramer-Lundberg type network, we indicate a connection between the ruin probability and boundary value problems for the infinitesimal generator $L$; here $L$ is a first order integro- partial differential operator.

## 1. Introduction

Insurance models have been studied ever since probability theory was recognized as a major tool for mathematical modelling. However only one dimensional models have been extensively investigated till recently. Since ruin probability is considered an important theoretical measure of the health of an insurance company, ruin problems have a central role in the one dimensional set up. [9, 21, 14] have excellent accounts of these.

Barring a few exceptions, multidimensional models have attracted attention only in the last decade or so; see [1, 2, 3, 4, 5] and references therein. Unlike the one dimensional situation, there is no canonical way of defining "ruin" in the models discussed in these works. For example, in the two dimensional case, 3 reasonable candidates are: exit from the first quadrant (signifying ruin of at least one company), entry into the third quadrant (signifying simultaneous ruin of both the companies), and exit from the half space $\{x_1 + x_2 \geq 0\}$ (signifying combined surplus is negative).

A multidimensional model where the surplus of each company is required to be nonnegative has been studied in [16, 17, 19] in terms of the Skorokhod problem of probability theory; so the $d$-dimensional orthant is the state space, where $d \geq 2$. In the present work we consider the ruin problem for a simple set up under such a model. The interaction among the companies of the network considered here is only through the risk reducing treaty. For the renewal risk network in Section 2, we argue that the reflected process $Z(\cdot)$ in the orthant hitting the state $\vec{0}$ is a natural definition of "ruin" of the network. We show that, under net profit conditions, the

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ruin probability can be approximated by hitting probabilities of a sequence of open sets around $\vec{0}$. In Section 3 we specialize to Cramer-Lundberg type networks so that the natural Markovian framework is available. The infinitesimal generator is a first order integro-partial differential operator $L$; the integral operator in $L$ involves the linear complementarity problem of operations research. Hitting probabilities of open sets alluded to earlier turn out to be discontinuous solutions to certain "boundary value problems" for $L$. Using this connection we are able to find the ruin probability in a simple two dimensional example.

## 2. A Renewal Risk Network

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, P)$ be a filtered probability space; unless otherwise stated, all stochastic processes are $\{\mathcal{F}_t\}-$adapted with r.c.l.l. sample paths. For our purposes, it is enough to consider the simple version of Skorokhod problem in an orthant as described below. For more information see [11, 7, 13, 15, 19, 20] and references therein.

Let $d \geq 2$. We consider a renewal risk network of $d$ insurance companies such that the dynamics of the companies are independent in the absence of risk reducing treaty. For $1 \leq i \leq d$, the surplus of Company $i$, in the absence of risk reducing treaty is given by

$$H_i(t) \quad = \quad z_i + c_i t - \sum_{\ell=1}^{N_i(t)} X_\ell^{(i)} = z_i + c_i t - S_i(t), \ t \geq 0, \tag{2.1}$$

where $z_i \geq 0$ is the initial capital, $c_i$ is the constant premium rate, and $S_i(\cdot)$ represents the total claim amount process.

We assume the following:

**(A1)** (i) $c_i > 0$ for each $i$; (ii) $\{N_i(t) : t \geq 0\}, \{X_\ell^{(k)} : \ell \geq 1\}, 1 \leq i, k \leq d$ are independent families of random variables; (iii) each $N_i(\cdot)$ is a renewal counting process having only finitely many jumps on any finite time interval; (iv) each $X_\ell^{(k)}$ is a positive random variable; (v) for fixed $k$, $X_\ell^{(k)}, \ell \geq 1$ are i.i.d. random variables.

**(A2)** $R = ((R_{ij}))$ is a constant $(d \times d)$ real matrix with $R_{ii} = 1$ for all $i$. Set $W = ((W_{ij}))$ with $W_{ii} = 0,\ W_{ij} = |R_{ij}|, j \neq i,\ 1 \leq i, j \leq d$. The spectral radius of $W$ is assumed to be strictly less than 1.

Let $A_\ell^{(i)}, \ell \geq 1$ denote the interarrival times of $N_i$ for $1 \leq i \leq d$; for fixed $i$, these are i.i.d. random variables, and are assumed to be strictly positive. So $H_1(\cdot), \ldots, H_d(\cdot)$ are $d$ independent Sparre Andersen (or renewal risk) processes. Write $S(\cdot) = (S_1(\cdot), \ldots, S_d(\cdot)),\ H(\cdot) = (H_1(\cdot), \ldots, H_d(\cdot))$.

Let $G = \{x \in \mathbb{R}^d : x_i > 0, 1 \leq i \leq d\}$ denote the $d-$dimensional positive orthant, and $\bar{G}$ its closure. Also $\vec{0}$ shall denote zero vector in $\mathbb{R}^d$.

Let $\{\mathcal{F}_t\}-$adapted processes $\{Y(t) = (Y_1(t), \ldots, Y_d(t)) : t \geq 0\}, \{Z(t) = (Z_1(t), \ldots, Z_d(t)) : t \geq 0\}$ satisfy the following.

**(S0)** $Y(0) = \vec{0}, Z(0) = (z_1, \cdots, z_d)$.

**(S1)** For $1 \leq i \leq d$ the *Skorokhod equation* holds, that is,

$$Z_i(t) \quad = \quad z_i + c_i t - S_i(t) + Y_i(t) + \sum_{j \neq i} R_{ij} Y_j(t), \ t \geq 0; \tag{2.2}$$

or equivalently in vector notation

$$Z(t) \quad = \quad H(t) + R \cdot (Y(t) - Y(0), \ t \geq 0. \tag{2.3}$$

**(S2)** $Z_i(t) \geq 0, \ t \geq 0, 1 \leq i \leq d$; so $Z(\cdot)$ is a $\bar{G}-$valued process.

**(S3)** For $1 \leq i \leq d, \ Y_i(\cdot)$ is a nondecreasing process and $Y_i(\cdot)$ can increase only when $Z_i(\cdot) = 0$; that is,

$$Y_i(t) - Y_i(s) = \int_{(s,t]} \mathbf{1}_{\{0\}}(Z_i(r)) dY_i(r), \ t \geq s \geq 0. \tag{2.4}$$

Note that (S2) is a *constraint,* while (2.4) is a *minimality condition.* A pair $Y(\cdot), Z(\cdot)$ of processes satisfying (S0)-(S3) is called a solution to the Skorokhod problem with drift $(c_1, \cdots, c_d)$, and reflection matrix $R$, initial value $(z_1, \cdots, z_d)$, and given stochastic data $S(\cdot)$, or simply a solution to $\mathrm{SP}(c, R; S(\cdot), z)$. As $R$ satisfies (A2) the processes $Y(\cdot), Z(\cdot)$ can be obtained by solving the corresponding deterministic Skorokhod problem path by path; see [11, 20, 15]; $Z$ is called the reflected/ regulated part and $Y$ is called the pushing part. We tacitly assume here that $R_{ij}$ and even $c_i$ have been negotiated and agreed upon by the companies.

In a special case, the above set up describes the joint dynamics of the $d$ companies operating under a risk reducing treaty. Accordingly, if Company $i$ needs at some instant of time an amount $dy_i$ *to avert ruin,* then for $j \neq i$, Company $j$ is required to give a preassigned fraction $|R_{ji}| dy_i$. Suppose $R_{ji} \leq 0, j \neq i$ and $\sum_{j \neq i} |R_{ji}| \leq 1, \ 1 \leq i \leq d$. The shortfall $\big(1 - \sum_{j \neq i} |R_{ji}|\big) dy_i$ has to be procured by Company $i$ from "external" sources. The objective of the treaty is to keep the surplus of each company nonnegative. The rationale is: Because of mutual obligations, internal borrowing carries softer repayment terms. With each company trying to minimize its repayment liability, the set up leads naturally to a $d$-person dynamic game with state space constraints. Using the "sample path analysis", it is shown in [16] that the pushing part $Y(\cdot)$ of the above Skorokhod problem provides a (unique) Nash equilibrium. Thus under optimality, a company can borrow, invoking the treaty, only when its reserve is zero/ it is in the red, and the amount borrowed should be just enough to keep it afloat. In such a case, for $1 \leq i \leq d, t > 0$, $Y_i(t)$ represents the optimal cumulative amount obtained by Company $i$ from internal and external sources during $[0, t]$ specifically for the purpose of averting ruin, while $Z_i(t)$ is the optimal current surplus of Company $i$. See [16, 17, 19] for details and extensions. In (S0)-(S3) note that the only interaction among the $d$ companies is through the risk reducing treaty.

For $x \in \mathbb{R}^d$ we shall denote $|x| = \sum_{i=1}^{d} |x_i|$; in particular, if $z \in \bar{G}$ then $|z| = z_1 + \cdots + z_d$.

Recall that in one dimensional insurance models, "ruin" is the event that the surplus of the company is strictly negative; see [21, 9, 14]. In multidimensional models with risk reducing treaty as above, by definition, surplus is nonnegative. So we need an appropriate notion of ruin. To motivate we begin with a class of examples.

**Example 2.1.** Let the set up be as above. In addition assume the following:

**(E)** There is $1 \leq k \leq d$ such that $(R^{-1})_{ik} > 0$ for all $i$, and the support of $X_1^{(k)}$ is $[0, \infty)$. Also let $R_{ij} \leq 0, i \neq j$; so $R = I - W$ with all entries of $W$ being nonnegative.

By a permutation of indices, if necessary, in condition (E) we may take $k = 1$. So all the entries in the first column of $R^{-1}$ are strictly positive, and claim size $X_1^{(1)}$ can take arbitrarily large values with positive probability. As spectral radius of $W$ is less than one, note that $R^{-1} = (I - W)^{-1} = I + W + W^2 + \cdots$ is a matrix with nonnegative entries.

Now suppose $A_1^{(i)} > A_1^{(1)}$ for each $i = 2, \ldots, d$; this means that the first claim arrival in the network concerns Company 1, and by our assumption this is an event of positive probability. Note that $S(t) = \vec{0}, Y(t) = \vec{0}, Z(t) = H(t) = z + tc$ for $t < A_1^{(1)}$, where $z = (z_1, \ldots, z_d), c = (c_1, \ldots, c_d)$. Observe that

$$\begin{aligned} Z(A_1^{(1)}) &= Z(A_1^{(1)}-) - [S(A_1^{(1)}) - S(A_1^{(1)}-)] + R[Y(A_1^{(1)}) - Y(A_1^{(1)}-)] \\ &= z + (A_1^{(1)})c - \bar{X}_1 + RY(A_1^{(1)}), \end{aligned}$$

where $\bar{X}_1 = (X_1^{(1)}, 0, \ldots, 0)$. As $R = I - W$ is invertible, it now follows that $Z(A_1^{(1)}) = \vec{0}$ if and only if $Y(A_1^{(1)}) = (I - W)^{-1}[\bar{X}_1 - z - (A_1^{(1)})c]$. Since we need $Y_i(A_1^{(1)}) \geq 0$ for each $i$, it is required that

$$R^{-1}\bar{X}_1 \geq R^{-1}[z + (A_1^{(1)})c],$$

where the inequality has to be satisfied componentwise. This will be so if

$$X_1^{(1)} \geq (\beta/\alpha) \sum_{j=1}^{d} (z_j + A_1^{(1)} c_j),$$

where $\alpha = \min\{(R^{-1})_{i1} : 1 \leq i \leq d\}$, $\beta = \max\{(R^{-1})_{ij} : 1 \leq i, j \leq d\}$. By condition (E), $\alpha > 0$. Since $A_1^{(i)}, 1 \leq i \leq d, X_1^{(1)}$ are independent random variables we have

$$\begin{aligned} & P(Z(t) = \vec{0} \text{ for some } 0 < t < \infty) \\ \geq \quad & P(A_1^{(\ell)} > A_1^{(1)}, \text{ for } 2 \leq \ell \leq d,\ Z(A_1^{(1)}) = \vec{0}) \\ \geq \quad & P(A_1^{(\ell)} > A_1^{(1)}, \text{ for } 2 \leq \ell \leq d,\ X_1^{(1)} \geq (\beta/\alpha) \sum_{j=1}^{d} (z_j + A_1^{(1)} c_j)) \\ = \quad & \int_0^\infty (\prod_{\ell=2}^{d} P(A_1^{(\ell)} > t)) \cdot P(X_1^{(1)} \geq (\beta/\alpha) \sum_{j=1}^{d} (z_j + tc_j)) dP(A_1^{(1)})^{-1}(t) \\ > \quad & 0. \end{aligned} \tag{2.5}$$

In the penultimate step in the above, we have conditioned w.r.t. $A_1^{(1)}$. Thus the $d$-dimensional regulated process $Z(\cdot)$ can hit the state $\vec{0}$ in finite time with positive probability.

In view of the preceding discussion, we can define *ruin* as the event that $Z(t) = \vec{0}$ for some $0 < t < \infty$, that is, surplus of all the companies in the network is zero simultaneously. The class of examples above indicates that ruin of the network can happen with positive probability. It may be noted that "ruin time" will be defined as the first hitting time of state $\vec{0}$, and not as the first entrance time; see (2.8) below.

Note that the spectral radius condition in (A2) implies that our framework is an "open system", in the sense that borrowing from external sources is allowed. Therefore, even when the surplus of each company is zero, by taking help from "outside", the system can come out of that state; thus, $\vec{0}$ is not an absorbing state. The next paragraph reinforces this further.

Let $S(\cdot), H(\cdot)$ be as in (2.1). So $H_i(\cdot), 1 \leq i \leq d$ are independent renewal risk processes, and the interaction among the companies is only through the risk reducing treaty. Let $\{\tau_n : n \geq 1\}$ be the successive jump times of $S(\cdot) = (S_1(\cdot), \ldots, S_d(\cdot))$. As $X_\ell^{(k)}$ are strictly positive, clearly these are claim arrival times for companies in the network. Since each $N_i$ can have only finitely many jumps in any finite interval, it follows that $\tau_n \uparrow \infty$ a.s. For each $n$, on $(\tau_n, \tau_{n+1})$, note that each $H_i(\cdot)$ is strictly increasing, and hence each $Z_i(\cdot)$ is strictly increasing. So $Y_i(t) = Y_i(\tau_n), Z_i(t) > 0$, for $\tau_n < t < \tau_{n+1}$, $1 \leq i \leq d$. Hence $Z_i(\cdot)$ can possibly become zero only at claim arrival times $\tau_n$; of course, this can happen at claim arrival time of another company in the network. In particular, for initial value $Z(0) \neq \vec{0}$, the $d$-dimensional sample path $Z(\cdot, \omega)$ can reach state $\vec{0}$ only at a claim arrival time. Even if $Z(0) = \vec{0}$, on $[0, \tau_1)$ each component $Z_i(\cdot)$ is strictly increasing (since $c_i > 0$) and $Y_i(\cdot) = 0$. Therefore $\inf\{t > 0 : Z(t) = \vec{0}\} > 0$. Once again $Z(\cdot, \omega)$ can "hit" state $\vec{0}$ only at a claim arrival time. Thus our definition of ruin is quite reasonable, capturing some salient features of one dimensional analogue.

**Proposition 2.2.** *Assume (A1) and (A2). Assume also that the interarrival times $A_j^{(i)}$, and the claim sizes $X_\ell^{(k)}$ have finite expectations. Suppose for $1 \leq i \leq d$,*

$$c_i \quad > \quad \frac{1}{E(A_1^{(i)})} E(X_1^{(i)}) \tag{2.6}$$

*that is, each $H_i$ satisfies the* net profit condition. *Then $Z_i(t) \to +\infty$ a.s. as $t \uparrow \infty$ for each $1 \leq i \leq d$.*

*Proof.* Under the net profit condition, by the law of large numbers, we know that $H_i(t) \to +\infty$ a.s. for each $i$; see [21]. Hence for a.e. $\omega \in \Omega$, there is $t^{(i)}(\omega) > 0$ such that $H_i(t) > 0$ for all $t > t^{(i)}(\omega)$, for $1 \leq i \leq d$.

For fixed $\omega$, by solving the deterministic Skorokhod problem corresponding to $H(\cdot, \omega)$ and reflection matrix $R$ we can obtain $Y(\cdot, \omega), Z(\cdot, \omega)$. So by Proposition 3.2 of [15], $Y(t) \leq (I - W)^{-1} \vartheta(t)$, that is

$$Y_i(t) \leq ((I - W)^{-1} \vartheta)_i(t), \quad 1 \leq i \leq d, \;\; t \geq 0,$$

where $\vartheta(\cdot) = (\vartheta_1(\cdot), \ldots, \vartheta_d(\cdot))$, $\vartheta_i(t) = \sup_{0 \leq s \leq t} \max\{0, -H_i(s)\}, t \geq 0$. Consequently, for fixed $\omega, i$ observe that $\vartheta_i(t, \omega) = \vartheta_i(t^{(i)}(\omega), \omega)$ for $t \geq t^{(i)}(\omega)$. In particular $t \mapsto \vartheta_i(t, \omega)$ is a bounded function for any $i$, for fixed $\omega$. Hence it now follows that $t \mapsto Y_j(t, \omega)$ is a bounded function for each $j, \omega$. Since $Z_k(t) = H_k(t) + \sum_{j=1}^d R_{kj} Y_j(t)$, it is clear now that $Z_k(t) \to +\infty$ a.s. as $t \to \infty$. □

Now define

$$\varrho_r(\omega) = \inf\{t > 0 : |Z(t,\omega)| < r\}, \text{ for } r > 0 \tag{2.7}$$

$$\varsigma_0(\omega) = \inf\{t > 0 : |Z(t,\omega)| = 0\}. \tag{2.8}$$

Note that $\varsigma_0$ is the *ruin time.* Observe that the ruin time is the first hitting time of state $\vec{0}$.

**Theorem 2.3.** *Let the hypotheses be as in Proposition 2.2. Then for any $\vec{0} \neq z \in \bar{G}$,*

$$\begin{aligned} & P_z(\varsigma_0 < \infty) \doteq P(\varsigma_0 < \infty | Z(0) = z) \\ = & \lim_{r\downarrow 0} P(\varrho_r < \infty | Z(0) = z) \doteq \lim_{r\downarrow 0} P_z(\varrho_r < \infty). \end{aligned} \tag{2.9}$$

*In other words, for $z \neq 0$,*

$$P_z(Z(t) = \vec{0} \text{ for some } t > 0) = \lim_{r\downarrow 0} P_z(|Z(t)| < r \text{ for some } t > 0).$$

*Proof.* Let $z \neq \vec{0}$ be an arbitrary but fixed element of $\bar{G}$. It is enough to consider a sequence $r_n \downarrow 0$; also we may assume $0 < r_n < |z|$ for all $n$. Set $C_n = \{\omega : \varrho_{r_n}(\omega) < \infty\}$, $n \geq 1$, and $C_0 = \{\omega : \varsigma_0(\omega) < \infty\}$. We need to prove $P_z(C_0) = \lim_{n\to\infty} P_z(C_n)$. Clearly $C_n$ is a decreasing sequence of measurable sets. Let $C = \bigcap_{n\geq 1} C_n$. Note that $C_0 \subseteq C$.

Now we prove that

$$P_z(C \backslash C_0) = 0. \tag{2.10}$$

Indeed, let $\omega \in C$. Then $\varrho_{r_n}(\omega) < \infty$ for all $n \geq 1$. Also $\varrho_{r_n}(\omega)$ is a nondecreasing sequence in $[0,\infty)$. Since $Z_i(\cdot)$ is strictly increasing between claim arrivals, observe that $0 \leq |Z(\varrho_{r_n})| < r_n < |z|$ for all $n \geq 1$. Suppose $\varrho_{r_n}(\omega) \uparrow +\infty$. As $r_n \downarrow 0$ it is clear that $Z(\varrho_{r_n}(\omega)) \to \vec{0}$. This would contradict Proposition 2.2 (that is, transience to $+\infty$ in each coordinate) unless $P_z(\varrho_{r_n} \uparrow +\infty) = 0$. Hence $\lim_{n\to\infty} \varrho_{r_n} < \infty$ with probability one on $C$. So we may assume that $\varrho_{r_n}(\omega) \uparrow \varrho_\infty(\omega) < \infty$.

Let $M(\omega) = \{\varrho_{r_n}(\omega) : n \geq 1\}$. Between claim arrivals, each $Z_i$ is strictly increasing. Since $r_n$ is a decreasing sequence in $(0, |z|)$, it is not difficult to see that each element of $M(\omega)$ is a jump time of $S(\cdot,\omega)$. As $M(\omega) \subset [0, \varrho_\infty(\omega)]$, and as $S(\cdot,\omega)$ can have only finitely many jumps in a finite interval, it now follows that $M(\omega)$ has only finitely many distinct elements.

From the preceding paragraph we get $\varrho_\infty(\omega) = \varrho_{r_k}(\omega)$ for some integer $k$. So it follows that $Z(\varrho_{r_k}(\omega)) = \vec{0}$. Therefore $\omega \in C_0$, and hence (2.10) follows.

By (2.10) we have $P_z(C_0) = P_z(C) = \lim_{n\to\infty} P_z(C_n)$. □

**Note:** The above result is not true for general stochastic processes. Let $B(t) = (B_1(t), B_2(t)), t \geq 0$ be a standard two dimensional Brownian motion. Then $(|B_1(t)|, |B_2(t)|), t \geq 0$ is equivalent in law to the reflected Brownian motion in the nonnegative quadrant with normal reflection at the boundary. As $B(\cdot)$ is neighbourhood recurrent, that property carries over to the normally reflected process. Hence $\varrho_r < \infty$ a.s. $P_z$ for every $z \neq 0$. However, as the two dimensional Brownian motion does not hit a point, $\varsigma_0 = \infty$ a.s. Thus l.h.s. of (2.9) is 0 while r.h.s. is 1.

## 3. A Cramer-Lundberg Network

In this section we shall deal with the Cramer-Lundberg set up; to be more precise, in addition to the hypotheses in Proposition 2.2 and Theorem 2.3, we shall assume that the claim number processes $N_i(\cdot), 1 \leq i \leq d$ are independent homogeneous Poisson processes. This is because we will need the Markovian framework. As the coefficients are constants, note that the reflected/ regulated process $\{Z(t) : t \geq 0\}$ itself is strong Markov; see [11, 15]. The corresponding semigroup of operators is given by

$$T_t f(z) \quad = \quad E_z[f(Z(t))] \doteq E[f(Z(t))|Z(0) = z], \; z \in \bar{G}, t \geq 0, \tag{3.1}$$

for any bounded measurable function $f$ on $\bar{G}$.

For fixed $r > 0$, set

$$g_r(z) \quad = \quad P_z(\varrho_r < \infty), \; z \in \bar{G}. \tag{3.2}$$

By Theorem 2.3 note that for $z \neq \vec{0}$

$$P_z(\varsigma_0 < \infty) \quad = \quad \lim_{r \downarrow 0} g_r(z). \tag{3.3}$$

If $|z| < r$ it is clear that $g_r(z) = 1$. As each $Z_i(\cdot)$ is strictly increasing on $[0, \tau_1)$, and because of transience we heuristically expect $g_r(z) < 1$ if $|z| \geq r$. Thus the function $z \mapsto g_r(z)$ may be discontinuous along $|z| = r$. (cf. Recall that in the classical one dimensional Cramer-Lundberg model, ruin probability, as a function of the initial capital, has a discontinuity at 0; thanks to the net profit condition, survival probability is strictly positive even with zero initial capital; see [18].)

Let $f$ be a bounded measurable function on $\bar{G}$. Suppose there is bounded measurable function $h$ on $\bar{G}$ such that

$$h(z) \quad = \quad \lim_{t \downarrow 0} \frac{1}{t}[T_t f(z) - f(z)], \; z \in \bar{G},$$

where the limit is pointwise. Then we write $h = Lf$, and $f \in \mathcal{D}(L)$; we call $L$ the *weak infinitesimal generator* of $\{T_t\}$, and $\mathcal{D}(L)$ is the domain of the infinitesimal generator $L$; see the last paragraph on p.50, vol.I of [8]. In [19] we have shown that $C_b^2$-functions satisfying certain boundary conditions are in $\mathcal{D}(L)$. However we may need to identify also some functions having certain limited discontinuity as perhaps members of $\mathcal{D}(L)$. The account below is an attempt at that; see Chapter 6 of [18] for one dimensional analogue.

Fix $r > 0$. Denote by $\mathcal{D}_{0,r}$ the collection of bounded measurable functions $f : \bar{G} \to \mathbb{R}$ having the following properties:

(i) $f$ is constant on $\bar{G} \cap \{z : |z| < r\}$;

(ii) if $z \in \bar{G}$ is a discontinuity point of $f$ then $|z| = r$;

(iii) $f$ restricted to $\bar{G} \cap \{z : |z| \geq r\}$ is continuous; in particular, if $z \in \bar{G}, |z| = r$, and $x \to z, |x| \geq r$, then $f(x) \to f(z)$.

Next, denote by $\mathcal{D}_{1,r}$ the collection of all functions $f$ satisfying the following:

(a) $f \in \mathcal{D}_{0,r}$;

(b) $(D_+f)_i \in \mathcal{D}_{0,r}$ for all $i$, where $D_+f(\cdot) = ((D_+f)_1(\cdot), \ldots, (D_+f)_d(\cdot))$ is given by

$$\begin{aligned} D_+f(z) &= \nabla f(z), \text{ if } |z| > r, \\ &= (\frac{\partial^+ f}{\partial z_1}(z), \cdots, \frac{\partial^+ f}{\partial z_d}(z)), \text{ if } |z| = r, \\ &= \vec{0}, \text{ if } |z| < r; \end{aligned}$$

here $(\partial^+ f(\cdot))/(\partial z_i)$ denotes the right derivative w.r.t. $z_i$.

If $f \in \mathcal{D}_{1,r}$ note that $f(\cdot)$ is continuously differentiable on $\{|z| > r\}$; and if $c_j > 0, 1 \leq j \leq d$, then

$$\frac{\partial^+ f}{\partial \vec{c}}(z) = \sum_{i=1}^{d} c_i \frac{\partial^+ f}{\partial z_i}(z)$$

is well defined at any $z \in \bar{G}$ with $|z| = r$; that is, "right derivative" of $f$ in the direction $\vec{c}$ exists at $|z| = r$.

We need some notation for stating the next result. Let $v, x \in \mathbb{R}^d$ and $R$ be reflection matrix satisfying (A2). Let $q, p \in \mathbb{R}^d$ be the solution to the following *linear complementarity problem (LCP)* associated with $x, v \in \mathbb{R}^d$, that is,

$$\begin{aligned} &p \geq 0, \ (q - v) \geq 0, \text{ both componentwise}, \\ &p = x + R \cdot (q - v), \text{ and } \langle p, (q - v) \rangle = 0. \end{aligned} \tag{3.4}$$

We denote $q := \Phi(v, x), p := \Psi(v, x)$ and say that $\Phi(v, x), \Psi(v, x)$ is the solution to LCP$(v, x; R)$. Because of the spectral radius condition in (A2) the above LCP can be uniquely solved; see [6]. If the sample path is identically a constant $x \in \mathbb{R}^d$, then the corresponding deterministic Skorokhod problem in the orthant is the same as LCP$(0, x; R)$. A constructive method, using contraction mapping, of solving the above is given in [22] as well as in Step 1 of the proof of Theorem 4.1 of [15]. So $\Phi, \Psi$ are continuous functions. The relevance here is that at a jump time of $S(\cdot)$, solution to the Skorokhod problem is propagated in terms of LCP.

**Theorem 3.1.** *In addition to the hypotheses of Proposition 2.2 and Theorem 2.3, assume that $N_i(\cdot), 1 \leq i \leq d$ are independent homogeneous Poisson processes with respective rates $\lambda_i$. For $1 \leq i \leq d$, write $\bar{X}_i = (0, \cdots, 0, X_1^{(i)}, 0, \cdots, 0)$ with all the coordinates except the $i$-th coordinate being zero. Let $L$ be the weak infinitesimal generator of the strong Markov process $\{Z(t) : t \geq 0\}$ as described above. Fix $r > 0$. Let $f(\cdot) \in \mathcal{D}_{1,r}$. Then $f \in \mathcal{D}(L)$ and*

$$\begin{aligned} Lf(z) &= \frac{\partial^+ f}{\partial \vec{c}}(z) + \sum_{i=1}^{d} \lambda_i E[f(\Psi(0, z - \bar{X}_i)) - f(z)] \\ &= \sum_{i=1}^{d} c_i \frac{\partial^+ f}{\partial z_i}(z) + \sum_{i=1}^{d} \lambda_i E[f(\Psi(0, z - \bar{X}_i)) - f(z)], \end{aligned} \tag{3.5}$$

*for $z \in \bar{G}$, where $\Psi(\cdot, \cdot)$ as in (3.4).*

*Proof.* Note that r.h.s. of (3.5) is a well defined bounded function on $\bar{G}$. We shall give below a proof of the theorem when $|z| \geq r$. The case $|z| < r$ is much simpler;

in the latter case it is not difficult to show that both sides of (3.5) are 0. Moreover, as the proof has similarities with that of Theorem 4.3 of [19], we shall highlight only the aspects where there are differences.

Let $z \in \bar{G}$ with $|z| \geq r$. Let $N(\cdot) = N_1(\cdot) + \cdots + N_d(\cdot)$ as in the proof of Theorem 4.3 of [19]. Clearly $N(\cdot)$ is a homogeneous Poisson process with rate $\lambda = \lambda_1 + \cdots + \lambda_d$. Note that jump times $\tau_n$ of $S(\cdot)$ are precisely the jump times of $N(\cdot)$. Note that

$$\begin{aligned} T_t f(z) - f(z) = &= \sum_{k=0}^{\infty} E_z\big[1_{\{k\}}(N(t)) \cdot \{f(Z(t)) - f(z)\}\big] \\ &= E_z\big[1_{\{0\}}(N(t)) \cdot \{f(Z(t)) - f(z)\}\big] \\ &\quad + E_z\big[1_{\{1\}}(N(t)) \cdot \{f(Z(t)) - f(z)\}\big] \\ &\quad + E_z\big[1_{\{[2,\infty)\}}(N(t)) \cdot \{f(Z(t)) - f(z)\}\big] \\ &\doteq H_0(t) + H_1(t) + H_2(t), \end{aligned} \tag{3.6}$$

giving the analogue of (4.10) of [19]. Since $f(\cdot)$ is bounded and $N(\cdot)$ is a Poisson process, it is clear that $H_2(t) = o(t),\ t \downarrow 0$.

On the set $\{N(t) = 0\}$, note that $Z_i(s) = z_i + c_i s, 0 \leq s \leq t, 1 \leq i \leq d$. Consequently

$$\begin{aligned} \frac{1}{t} H_0(t) &= \frac{1}{t} E_z[I_{\{0\}}(N(t)) \cdot \{f(z + tc) - f(z)\}] \\ &= \frac{1}{t}[f(z + tc) - f(z)] P(N(t) = 0) \\ &\to \frac{\partial^+ f}{\partial \vec{c}}(z) = \sum_{i=1}^{d} c_i \frac{\partial^+ f}{\partial z_i}(z),\ t \downarrow 0. \end{aligned} \tag{3.7}$$

This is the analogue of (4.11) in [19].

As $S(\cdot)$ and $N(\cdot)$ have the same jump times, we can write

$$\begin{aligned} H_1(t) &= E_z\big[1_{\{\tau_1 \leq t < \tau_2\}} \cdot \{f(Z(t)) - f(z)\}\big] \\ &= E_z\big[1_{\{\tau_1 \leq t < \tau_2\}} \cdot \{f(Z(t)) - f(Z(\tau_1))\}\big] \\ &\quad + E_z\big[1_{\{\tau_1 \leq t < \tau_2\}} \cdot \{f(Z(\tau_1)) - f(z)\}\big] \\ &\doteq H_{11}(t) + H_{12}(t). \end{aligned} \tag{3.8}$$

Using the arguments in the derivation of (3.7), strong Markov property, and proceeding as in the derivation of (4.15) in [19], it can be proved that

$$\frac{1}{t} H_{11}(t) = \frac{1}{t} E_z[I_{\{\tau_1 \leq t < \tau_2\}} \cdot (f(Z(t)) - f(Z(\tau_1)))] \quad \to \quad 0,\ t \downarrow 0. \tag{3.9}$$

Since $f$ may have discontinuity at $|x| = r$, we now need to deviate from the proof of Theorem 4.3 of [19]. On the set $\{N_i(t) = 1, N_j(t) = 0, j \neq i\}$, note that $\tau_1 = \tau_1^{(i)}$ = first claim arrival time for Company $i$, $Y(\tau_1-) = 0$ and $Z(\tau_1) =$

$\Psi(0, [z + \tau_1^{(i)} c - \bar{X}_i])$. So, denoting by $e_i$ the $i$-th unit vector, we have

$$\begin{aligned} H_{12}(t) &\doteq E_z[I_{\{\tau_1 \le t < \tau_2\}} \cdot \{f(Z(\tau_1)) - f(z)\}] \\ &= \sum_{i=1}^{d} E_z[(\prod_{j \neq i} I_{\{0\}}(N_j(t))) I_{\{1\}}(N_i(t)) \cdot \{f(Z(\tau_1)) - f(z)\}] \\ &= \sum_{i=1}^{d} E_z\left[I_{\{e_i\}}(N(t)) \cdot \{f(\Psi(0, [z + \tau_1^{(i)} c - \bar{X}_i])) - f(z)\}\right]. \end{aligned} \tag{3.10}$$

Fix $1 \le i \le d$. Note that $\{N_j(\cdot), j \neq i\}, \{N_i(\cdot), \tau_1^{(i)}\}, \{\bar{X}_i\}$ are independent families. We also know that $P(\tau_1^{(i)} \in ds | N_i(t) = 1)$ is the uniform distribution on $[0, t]$. Hence using a conditioning argument we have

$$\begin{aligned} &\frac{1}{t} E_z\left[I_{\{e_i\}}(N(t)) \cdot \{f(\Psi(0, [z + \tau_1^{(i)} c - \bar{X}_i])) - f(z)\}\right] \\ &= \lambda_i e^{-\lambda t} \frac{1}{t} \int_0^t E_z[f(\Psi(0, z + sc - \bar{X}_i)) - f(z)] ds. \end{aligned}$$

By Theorem 4.1 (comparison theorem) in [15], if $\hat{\xi} \ge \xi$ componentwise, then $\Psi(0, \hat{\xi}) \ge \Psi(0, \xi)$. And as $\Psi$ is continuous, it follows that $\Psi(0, z + sc - \bar{X}_i(\omega)) \downarrow \Psi(0, z - \bar{X}_i(\omega)), 0 \le s \le t$ as $t \downarrow 0$, for every $\omega$. By our assumption on $f$, note that $f$ is continuous on $\bar{G} \cap \{|\xi| \ge r\}$ as well as on $\bar{G} \cap \{|\xi| < r\}$. Therefore it follows that $f(\Psi(0, z + sc - \bar{X}_i(\omega))) \to f(\Psi(0, z - \bar{X}_i(\omega))), 0 \le s \le t$ as $t \downarrow 0$. In view of (3.10) it is now easily seen that

$$\lim_{t \downarrow 0} \frac{1}{t} H_{12}(t) = \sum_{i=1}^{d} \lambda_i E_z[f(\Psi(0, z - \bar{X}_i)) - f(z)]. \tag{3.11}$$

Now (3.6)–(3.11), together with details as in the proof of Theorem 4.3 in [19], yield the required result (3.5). □

*Remark* 3.2. There is an interesting difference between Theorem 4.3 of [19] and the above result. In [19] $f(\cdot)$ is assumed to satisfy the boundary condition (4.6) on $\partial G$, while there has been no such requirement in the preceding theorem. In the former situation, because of the presence of diffusion, the regulated process $Z(\cdot)$ can possibly hit the boundary $\partial G$ even before $\tau_1$. So the second term on r.h.s. of (4.5) in [19] has to be appropriately dealt with to identify the differential part of the generator in Theorem 4.3 of [19]; the alluded boundary condition is an easy/obvious way of doing this. Thanks to $c > 0$, and the absence of diffusion, such a problem does not arise in the latter case while deriving (3.5).

With $r > 0$ fixed, let $f \in \mathcal{D}_{1,r}$. By Theorem 3.1 and a result on p.162 of [10], it follows that $f(Z(t)) - f(z) - \int_0^t Lf(Z(s)) ds, t \ge 0$ is a $P_z$-martingale w.r.t. the filtration generated by the process $Z(\cdot)$, for any starting point $z \in \bar{G}$. Using this observation we have the following result.

**Corollary 3.3.** *Let the notation and hypotheses be as in Theorem 3.1. Let $r > 0$ be fixed and $f \in \mathcal{D}_{1,r}$ satisfy the following: (i) $Lf(\cdot) = 0$ on $\bar{G}$; (ii) $f(z) = 1$, if $|z| < r$; (iii) $f(z) \to 0$ as $|z| \to \infty$. Then $f(z) = P_z(\varrho_r < \infty), z \in \bar{G}$.*

*Proof.* It is enough to consider $z \in \bar{G}$ with $|z| \geq r$. Fix such a $z$. For $k > |z|$ let $\eta_k = \inf\{t \geq 0 : |Z(t)| \geq k\}$. By Proposition 2.2, $\eta_k < \infty$ a.s.$P_z$, for every $k$. Using the observation above, the optional sampling theorem, and condition (i) we get

$$\begin{aligned} f(z) &= E_z[f(Z(t \wedge \eta_k \wedge \varrho_r)) - \int_0^{t\wedge\eta_k\wedge\varrho_r} Lf(Z(s))ds] \\ &= E_z[f(Z(t \wedge \eta_k \wedge \varrho_r))], \end{aligned}$$

for every $t > 0, k > |z|$. By condition (ii) note that $f(Z(\varrho_r)) = 1$. So letting $t \uparrow \infty$ in the above

$$f(z) = P_z(\varrho_r < \eta_k) + E_z[I_{\{\varrho_r > \eta_k\}} \cdot f(Z(\eta_k))]. \tag{3.12}$$

As the process does not explode in finite time, note that $\eta_k \uparrow +\infty$ with probability one as $k \to \infty$. Consequently, again by Proposition 2.2, note that $|Z(\eta_k)| \to \infty$ a.s. $P_z$ as $k \to \infty$. Hence, letting $k \to \infty$ in (3.12), and invoking condition (iii) we get the desired result. □

*Remark* 3.4. The infinitesimal generator $L$ in Theorem 3.1 is a first order integro-partial differential operator. With $r > 0$ fixed, note that $Lf(z) = 0,\ |z| < r$ for any $f \in \mathcal{D}_{1,r}$. Denote $Q_r = \{z \in \bar{G} : |z| > r\}$. By (2.7) it is clear that $\varrho_r$ is the first exit time from $\bar{Q}_r$; it is also the first hitting time of $\{|z| < r\}$. Corollary 3.3 concerns a solution $f \in \mathcal{D}_{1,r}$ to the equation $Lf(z) = 0,\ z \in \bar{Q}_r$, with the boundary conditions (ii), (iii). The probabilistic formulation implicit here corresponds to exit time from $\bar{Q}_r$, rather than exit time from $Q_r$. This is appropriate in our context, because we expect $f$ to be continuous when restricted to $\bar{Q}_r$, but $f$ can be discontinuous from "below" at $|z| = r$. Since the integral operator in $L$ is nonlocal, "boundary values" need to be specified for $|z| < r$ as in (ii); of course, the condition (iii) at infinity is a natural one. (cf. It is known that for the first boundary value problem for second order strictly elliptic PDE's in nice domains, the two probabilistic formulations are equivalent. However, in the case of degenerate elliptic operators the two need not coincide; and the formulation in terms of exit time from the closure, if it makes sense, has certain advantages; see Section 5 of [23].)

Theorem 2.3 and Corollary 3.3 suggest a way of obtaining ruin probability. We are able to get an exact expression in the following two dimensional example.

**Example 3.5.** Let $d = 2, c_1, c_2 > 0$; let $R$ be a $2 \times 2$ matrix with $R_{11} = R_{22} = 1, R_{12} = 0, R_{21} = (-1)$. Let $N_1(\cdot), N_2(\cdot)$ be independent Poisson processes with respective rates $\lambda_1, \lambda_2$. Let $X_\ell^{(1)}, \ell \geq 1$ be i.i.d. random variables having an $\mathrm{Exp}(\mu_1)$ distribution, where $\mu_1 > 0$. Let $X_\ell^{(2)} \equiv 0$ for all $\ell$. So $S_2(\cdot) \equiv 0$ while $S_1(\cdot)$ is a nontrivial compound Poisson process. In the absence of risk reducing treaty, note that $H_1(t) = z_1 + c_1 t - \sum_{i=1}^{d} X_i^{(1)}$, $H_2(t) = z_2 + c_2 t$, for $t \geq 0$. According to the risk reducing treaty, if Company 1 needs money to avert ruin, the entire amount is given by Company 2; if Company 2 needs money to avert ruin, it has to get the amount from external sources. The joint dynamics of the companies under the treaty is governed by Skorokhod problem corresponding to $H(\cdot) = (H_1(\cdot), H_2(\cdot))$

and reflection matrix $R$. So

$$\begin{aligned} Z_1(t) &= z_1 + c_1 t - S_1(t) + Y_1(t) \\ Z_2(t) &= z_2 + c_2 t + Y_2(t) - Y_1(t), \end{aligned}$$

for $t \geq 0$; $Z_1(\cdot), Z_2(\cdot)$ are nonnegative; $Y_i(0) = 0$, $Y_i$ is nondecreasing, and can increase only when $Z_i(\cdot) = 0$, for $i = 1, 2$. In this set up, Company 2 is like a "reinsurer" only, and has no primary insurance business of its own. However, Company 2 provides money to Company 1 only when a demand is made to avert ruin, and the amount is just enough to keep afloat. It is simple to check that condition (E) of Example 2.1 holds with $k = 1$. By Example 2.1, for any starting point $z = (z_1, z_2) \neq (0, 0)$ probability of ruin in finite time is positive. Assume that $c_1\mu_1 - \lambda_1 > 0$ so that (2.6) holds for $i = 1$; it trivially holds for $i = 2$. So by Proposition 2.2, it is clear that $Z_i(t) \to \infty$ as $t \to \infty, i = 1, 2$. Also note that Theorem 2.3 and Corollary 3.3 are applicable here. We use these to get the ruin probability.

Fix $r > 0$. To carry out the above program, we need to find $f \in \mathcal{D}_{1,r}$ satisfying (i), (ii), (iii) of Corollary 3.3, where

$$Lf(z) \quad = \quad c_1 \frac{\partial^+ f}{\partial z_1}(z) + c_2 \frac{\partial^+ f}{\partial z_2}(z) + \lambda_1 E[f(\Psi(0, z - \bar{X}_1)) - f(z)]. \qquad (3.13)$$

In (3.13) we have used the fact that $\Psi(0, z - \bar{X}_2) = \Psi(0, z) = z,\ z \in \bar{G}$.

With $R$ as above, for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, the solution pair $\Phi(0, \xi) = (\hat{y}_1, \hat{y}_2)$, $\Psi(0, \xi) = (\hat{z}_1, \hat{z}_2)$ to LCP$(0, \xi; R)$ is given by $\hat{y}_1 = \max\{0, -\xi_1\}, \hat{z}_1 = \xi_1 + \hat{y}_1$, $\hat{y}_2 = \max\{0, -[\xi_2 - \hat{y}_1]\}$, $\hat{z}_2 = \xi_2 + \hat{y}_2 - \hat{y}_1$. Consequently, for any $z = (z_1, z_2) \in \bar{G}, \omega \in \Omega$, it follows that

$$\begin{aligned} \Psi(\vec{0}, z - \bar{X}_1(\omega)) &= ((z_1 - X_1^{(1)}(\omega)), z_2),\ \text{if}\ X_1^{(1)}(\omega) \leq z_1, \\ &= (0, z_1 + z_2 - X_1^{(1)}(\omega)),\ \text{if}\ z_1 \leq X_1^{(1)}(\omega) \leq z_1 + z_2, \\ &= (0, 0)\ \text{if}\ z_1 + z_2 \leq X_1^{(1)}(\omega), \end{aligned}$$

and hence

$$\begin{aligned} |\Psi(\vec{0}, z - \bar{X}_1(\omega))| &= z_1 + z_2 - X_1^{(1)}(\omega),\ \text{if}\ X_1^{(1)}(\omega) \leq z_1 + z_2, \\ &= 0,\ \text{if}\ X_1^{(1)}(\omega) \geq z_1 + z_2. \end{aligned} \qquad (3.14)$$

Note that $|\Psi(0, z - \bar{X}_1)| \leq z_1 + z_2$. Hence it follows that $E_z[f(\Psi(0, z - \bar{X}_1)) - f(z)] = 0$ if $|z| < r$.

We try to find $f \in \mathcal{D}_{1,r}$ of the form $f(z_1, z_2) = \varphi(z_1 + z_2)$ where $\varphi$ is an appropriate function on $[0, \infty)$. Since $f(z) = 1$ if $z_1 + z_2 < r$, by (3.14) we get

$$\begin{aligned} & E_z[f(\Psi(0, z - \bar{X}_1))] = E_z[\varphi(|\Psi(0, z - \bar{X}_1)|)] \\ &= E[1 \cdot I_{(z_1+z_2-r,\infty)}(X_1^{(1)})] + E[\varphi(z_1 + z_2 - X_1^{(1)}) \cdot I_{[0,z_1+z_2-r]}(X_1^{(1)})] \\ &= [1 - F_1(z_1 + z_2 - r)] + \int_{[0,z_1+z_2-r]} \varphi(z_1 + z_2 - \xi) dF_1(\xi), \end{aligned} \qquad (3.15)$$

where $F_1$ denotes the distribution function of $X_1$. In view of (3.13), (3.15) we need $\varphi(\cdot)$ restricted to $[r, \infty)$ to be a $C^1$-function, $\varphi(s) = 1, s < r$, $\lim_{s\to\infty} \varphi(s) = 0$, and that it satisfy the integro-differential equation

$$\begin{aligned}(c_1+c_2)\frac{d^+\varphi}{ds}(s) - \lambda_1\varphi(s) + \lambda_1[(1-F_1(s-r)) + \int_{[0,s-r]} \varphi(s-\xi)dF_1(\xi)] \\ = \ 0, \ s \geq r. \qquad (3.16)\end{aligned}$$

Of course, $\varphi$ may have a discontinuity at $s = r$.

To solve (3.16) we follow a procedure given in [12]; see also Section 6.1 of [18]. As $X_1^{(1)}$ has $\mathrm{Exp}(\mu_1)$ distribution, it can be seen that (3.16) is the same as

$$c\varphi'(s) - \lambda_1\varphi(s) + \lambda_1 e^{-\mu_1(s-r)} + \lambda_1 e^{-\mu_1 s}h(s) \ = \ 0, \ s \geq r, \qquad (3.17)$$

where $c = c_1 + c_2$, $\varphi' = \frac{d^+\varphi}{ds}$ and $h(s) = \int_r^s \varphi(\theta)\mu_1 e^{\mu_1\theta}d\theta$. Since $\varphi$ is required to be $C^1$ on $[r,\infty)$, and hence $h$ is differentiable on $(r,\infty)$, from (3.17) we see that $\varphi'$ is also differentiable on $(r,\infty)$. So differentiating (3.17), using the expression for $h'(\cdot)$, and using the ODE (3.17) to substitute for $h$, we get the second order ODE

$$c\varphi''(s) + (c\mu_1 - \lambda_1)\varphi'(s) \ = \ 0, \ s > r.$$

The general solution to the above ODE is

$$\varphi(s) \ = \ K_1 + K_2 \exp[-\frac{(c\mu_1 - \lambda_1)}{c}s], \ s \geq r,$$

where $K_1, K_2$ are arbitrary constants. By net profit condition, $c\mu_1 > c_1\mu_1 > \lambda_1$. Since $\lim_{s\to\infty}\varphi(s) = 0$ is a requirement, we see that $K_1 = 0$. Hence

$$\varphi(s) \ = \ K_2 \exp[-\frac{(c\mu_1 - \lambda_1)}{c}s], \ s \geq r. \qquad (3.18)$$

Using (3.18) one can compute $\varphi(r), \varphi'(r)$; also $h(r) = 0$. Substituting these into the ODE (3.17), we get $K_2 = \frac{\lambda_1}{c\mu_1}\exp[\frac{c\mu_1-\lambda_1}{c}r]$. Therefore

$$\varphi(s) \ = \ \frac{\lambda_1}{c\mu_1}\exp[-\frac{c\mu_1 - \lambda_1}{c}(s-r)], \ s \geq r. \qquad (3.19)$$

Also $\varphi(s) = 1, \ s < r$. It is clear that $\varphi$ has a discontinuity at $s = r$.

Now, we can put for $z = (z_1, z_2)$

$$f(z) \ = \ \frac{\lambda_1}{c\mu_1}\exp[-\frac{(c\mu_1 - \lambda_1)}{c}(z_1 + z_2 - r)], \ z_1 + z_2 \geq r, \qquad (3.20)$$

and $f(z) = 1$ if $z_1 + z_2 < r$. It is easy to verify that $f \in \mathcal{D}_{1,r}$ and that it satisfies (i), (ii), (iii) of Corollary 3.3. So by Corollary 3.3, $f(z) = P_z(\varrho_r < \infty), z \in \bar{G}$. It now also follows from Corollary 3.3 that $f$ is the unique element in $\mathcal{D}_{1,r}$ satisfying (i), (ii), (iii). Therefore by Theorem 2.3,

$$\begin{aligned} P_z(\varsigma_0 < \infty) &= \lim_{r\downarrow 0} P_z(\varrho_r < \infty) \\ &= \frac{\lambda_1}{c\mu_1}\exp[-\frac{(c\mu_1 - \lambda_1)}{c}(z_1 + z_2)], \ z \neq \vec{0}. \qquad (3.21)\end{aligned}$$

Next, we claim that (3.21) holds even for $z = \vec{0}$, that is,

$$P_0(\varsigma_0 < \infty) \ = \ \frac{\lambda_1}{c\mu_1}. \qquad (3.22)$$

For $r > 0$, let $\eta_r = \inf\{t \geq 0 : |Z(t)| = Z_1(t) + Z_2(t) = r\}$. Because of transience $\eta_r < \infty$ a.s. $P_0$ for each $r > 0$. As $Z(\cdot)$ is strictly increasing and right continuous in each coordinate on $[0, \tau_1)$, it follows that $P_0(\varsigma_0 < \infty) = P_0(0 < \varsigma_0 < \infty)$, and that $P_0(\eta_r \downarrow 0 \text{ as } r \downarrow 0) = 1$. Also note that $|Z(\eta_r)| = Z_1(\eta_r) + Z_2(\eta_r) = r$ a.s. $P_0$ as the jumps are negative. Therefore, using the strong Markov property and (3.21), we get

$$\begin{aligned} P_0(\varsigma_0 < \infty) &= P_0(0 < \varsigma_0 < \infty) = \lim_{r\downarrow 0} P_0(\eta_r < \varsigma_0 < \infty) \\ &= \lim_{r\downarrow 0} E_0[P_{Z(\eta_r)}(\varsigma_0 < \infty)] = \lim_{r\downarrow 0} E_0\{\frac{\lambda_1}{c\mu_1}\exp[-\frac{(c\mu_1 - \lambda_1)}{c}r]\} = \frac{\lambda_1}{c\mu_1}, \end{aligned}$$

proving (3.22).

Thus (3.21), (3.22) give ruin probability for the model.

We have used the norm $|x| = \sum_{i=1}^{d} |x_i|$ to make the analysis simple in the preceding example. Since balls are convex in any norm, and $\vec{c}$ points into the interior of the orthant, analogues of the above results hold in any norm.

## References


1. Avram, F., Palmowski, Z. and Pistorius, M.: A two-dimensional ruin problem on the positive quadrant. *Insurance: Mathematics and Economics* **42** (2008) 227–234.
2. Avram, F., Palmowski, Z. and Pistorius, M.: Exit problem of a two-dimensional risk process from the quadrant: exact and asymptotic results. *The Annals of Applied Probability* **18** (2008) 2421–2449.
3. Bauerle, N. and Grubel, R.: Multivariate counting processes: copulas and beyond. *ASTIN Bulletin* **35** (2005) 379–408.
4. Bauerle, N. and Grubel, R.: Multivariate risk processes with interacting intensities. *Advances in Applied Probability* **40** (2008) 578–601.
5. Collamore, J.: First passage times of general sequences of random vectors: a large deviations approach. *Stochastic Processes and Applications* **78** (1998) 97–130.
6. Cottle, R. W., Pang, J. S. and Stone, R.E.: *The Linear Complementarity Problem.* Academic Press, New York, 1992.
7. Dupuis, P. and Ishii, H.: SDE's with oblique reflection on nonsmooth domains. *The Annals of Probability* **21** (1993) 554–580.
8. Dynkin, E. B.: *Markov Processes.* vol. I. Springer-Verlag, New York, 1965.
9. Embrechts, P., Kluppelberg, C. and Mikosch, T.: *Modelling Extremal Events for Insurance and Finance.* Springer, Heidelberg, 1997.
10. Ethier, S. and Kurtz, T.: *Markov Processes: Characterization and Convergence.* Wiley, New York, 1986.
11. Harrison, J. M. and Reiman, M. I.: Reflected Brownian motion on an orthant. *The Annals of Probability* **9** (1981) 302–308.
12. Hipp, C.: Stochastic Control with Application in Insurance. In *Stochastic Methods in Finance,* pp. 127–164. Lecture Notes in Mathematics, No. 1856, Springer, Berlin, 2004.
13. Kruk, L., Lehoczky, J., Ramanan, K. and Shreve, S.: An Explicit Formula for the Skorokhod Map on $[0, a]$. *The Annals of Probability* **35** (2007) 1740–1768.
14. Mikosch, T.: *Non-Life Insurance Mathematics : an introduction with Stochastic Processes.* Springer (India), New Delhi, 2004.
15. Ramasubramanian, S.: A subsidy-surplus model and the Skorokhod problem in an orthant. *Mathematics of Operations Research* **25** (2000) 509–538.
16. Ramasubramanian, S.: An insurance network: Nash equilibrium. *Insurance: Mathematics and Economics* **38** (2006) 374–390.
17. Ramasubramanian, S.: A $d$ -person differential game with state space constraints. *Applied Mathematics and Optimization* **56** (2007) 312–342.

18. Ramasubramanian, S.: *Lectures on Insurance Models.* Hindustan Book Agency, New Delhi, 2009.
19. Ramasubramanian, S.: Multidimensional insurance model with risk reducing treaty. *Stochastic Models* **27** (2011) 363–387.
20. Reiman, M. I.: Open queueing networks in heavy traffic. *Mathematics of Operations Research* **9** (1984) 441–458.
21. Rolski, T., Schmidli, H., Schmidt, V. and Teugels, J. L.: *Stochastic Processes for Insurance and Finance.* Wiley, Chichester, 1999.
22. Shashiashvili, M. A.: A lemma of variational distance between maximal functions with application to the Skorokhod problem in a non-negative orthant with state-dependent reflection directions. *Stochastics and Stochastic Reports* **48** (1994) 161–194.
23. Stroock, D. W. and Varadhan, S. R. S.: On degenerate elliptic-parabolic operators of second order and their associated diffusions. *Communications on Pure and Applied Mathematics* **25** (1972) 651–713.

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