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STOCHASTIC ANALYSIS OF BACKWARD TIDAL DYNAMICS EQUATION

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Abstract. The backward stochastic tidal dynamics equations, a system of coupled backward stochastic differential equations, in bounded domains are studied in this paper. Under suitable projections and truncations, a priori estimates are obtained, which enable us to establish the uniformly boundedness of an adapted solution to the system. Such regularity does not usually hold for stochastic differential equations. The well-posedness of the projected system is given by means of the contraction property of the elevation component. The existence of solutions are proved by utilizing the Galerkin approximation scheme and the monotonicity properties for bounded terminal conditions. The uniqueness and continuity of solutions with respect to terminal conditions are also provided.

1. Introduction

Tide, the alternate rising and falling of the sea levels, is the result of the combination of the rotation of the Earth, and the gravitational attraction exerted at different parts of the Earth by the Moon and the Sun. The study of ocean tides trace back to the early seventeenth century. The first attempt of a theoretical explanation of ocean tides was given by Galileo Galilei[6] in 1632. Although not very successful, it inspired many successors to further the studies in this field, including Johannes Kepler and Isaac Newton[25]. The latter established a scientific formulation in 1687, which pointed out the role of the lunar and solar gravitational effect on ocean tides. Later Maclaurin[20] used Newton's theory of fluxion and took into account Earth's rotational effects on motion, Euler discovered that the horizontal component of the tidal force, as opposed to the vertical component, is the main driving force of ocean tides that causes the wavelike progression of high tide, and Jean le Rond d'Alembert observed tidal equations for the atmosphere which did not include rotation. A major break through of the mathematical formulation of ocean tides is accredited to Laplace[14], who introduced a system of three linear partial differential equations for the horizontal components of ocean velocity, and the vertical displacement of the ocean surface in 1775. His work remains the basis of tidal computation to this day, and was followed up by Thomson and Tait[33], and Poincaré[28], among others. The former applied systematic harmonic analysis to tidal analysis and rewrote Laplace's equations in terms of vorticity.

In the first half of the nineteenth century, researchers further expanded the studies of ocean tides. Encouraging developments include Arctic tides, and geophysical tides such as tidal motion in the atmosphere and in the Earth's molten core. In the second

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half of the nineteenth century, quantum advances in computers, satellite technology and numerical methods of solving PDEs made it possible to solve Laplace tidal equations with realistic boundary conditions and depth functions. In this article, we consider the tidal dynamics equations constructed by Marchuk and Kagan[21, 22]. The existence and uniqueness of the tide equations in forward case have been shown by Ipatova[9], Marchuk and Kagan[21], and Manna, Menaldi and Sritharan [23].

The backward version of stochastic tidal dynamics equations is, to our best knowledge, new. It appears as an inverse problem wherein the velocity profile and elevation component at a time T are observed and given, and the noise coefficient has to be ascertained from the given terminal data. Such a motivation arises naturally when one understands the importance of inverse problems in partial differential equations (see J. L. Lions [15, 16]). Since the problem of specifying the function of boundary condition on the liquid boundary, and the problem of specifying the tide-generating forces must be solved simultaneously with the tide theory equation system, Agoshkov[2], among other authors, has considered tidal dynamics models as inverse problems. Some studies of backward stochastic analysis on fluid dynamics has been put forth in our previous work [31]. Linear backward stochastic differential equations were introduced by Bismut in 1973 ([3]), and the systematic study of general backward stochastic differential equations (BSDEs for short) were put forward first by Pardoux and Peng[27], Ma, Protter, Yong, Zhou, and several other authors in a finite-dimensional setting. Ma and Yong[19] have studied linear degenerate backward stochastic differential equations motivated by stochastic control theory. Later, Hu, Ma and Yong [8] considered the semi-linear equations as well. Backward stochastic partial differential equations were shown to arise naturally in stochastic versions of the Black-Scholes formula by Ma, Protter and Yong [17, 18]. A nice introduction to backward stochastic differential equations is presented in the book by Yong and Zhou [34], with various applications.

The usual method of proving existence and uniqueness of solutions by fixed point arguments do not apply to the stochastic system on hand since the drift coefficient in the backward stochastic tidal flow is nonlinear, non-Lipschitz and unbounded. However, the drift coefficient is monotone on bounded $\mathbb{L}^4(G)$ balls in $\mathbb{H}_0^1(G)$, which was first observed by Manna, Menaldi and Sritharan [23]. One may also refer to Menaldi and Sritharan [24] for more information. The Galerkin approximation scheme is employed in the proof of existence and uniqueness of solutions to the system. To this end, a priori estimates of finite-dimensional projected systems are studied, and uniformly boundedness of adapted solutions are established. Such regularity does not usually hold for stochastic differential equations. The well-posedness of the projected system is also given by means of the contraction property of the elevation component. In order to establish the monotonicity property of the drift term, a truncation of $R(x)$, the depth of the calm sea, is introduced. Then the generalized Minty-Browder technique is used in this paper to prove the existence of solutions to the tidal dynamics system. The proof of the uniqueness and continuity of solutions are wrought by establishing the closeness of solutions of the system via monotonicity arguments.

The structure of the paper is as follows. The functional setup of the paper is introduced and several frequently used inequalities are listed in section 2. Some a priori estimates for the solutions of the projected system are given under different assumptions on the terminal conditions and external force in section 3. Section 4 is devoted to well-posedness of the projected system. The existence of solutions of the tidal dynamics equations under suitable assumptions is shown by Minty-Browder monotonicity argument in section 5. The uniqueness and continuity of the solution under the assumption that terminal condition is uniformly bounded in H^1 sense are given in section 6.

2. Formulation of the Problem

Let us consider the time interval $[0, T]$, and let G, the horizontal ocean basin where tides are induced, be a bounded domain in $\mathbb{R}^2$ with smooth boundary conditions. The boundary contour ∂G is composed of two disconnected parts, a solid part of Γ_1 coinciding with the edge of the continental and island shelves, and an open boundary Γ_2. Let us assume that sea water is incompressible and the vertical velocities are small compared with the horizontal velocities. Thus we are able to exclude acoustic waves. Also long waves, including tidal waves, are stood out from the family of gravitational oscillations. Furthermore, to reduce computational difficulties, we assume that the Earth is absolutely rigid, and the gravitational field of the Earth is not affected by movements of ocean tides. Also the effect of the atmospheric tides on the ocean tides and the effect of curvature of the surface of the Earth on horizontal turbulent friction are ignored. Under these commonly used assumptions, we are able to adopt the following tide dynamics model:

$$\begin{cases} \frac{\partial \mathbf{w}}{\partial t} + lk \times \mathbf{w} = -g\nabla\zeta - \frac{r}{R}|\mathbf{w}|\mathbf{w} + \kappa_h \triangle \mathbf{w} + \mathbf{g}; \\ \frac{\partial \zeta}{\partial t} + \nabla \cdot (R\mathbf{w}) = 0; \\ \mathbf{w} = \mathbf{w}^0 \quad \text{on } [0, T] \times \partial G; \\ \mathbf{w}^0 = 0 \ \text{ on } \Gamma_1, \quad \text{and} \quad \int_{\Gamma_2} \mathbf{w}^0 d\Gamma_2 = 0, \end{cases} \tag{2.1}$$

where $\mathbf{w}$, the horizontal transport vector, is the averaged integral of the velocity vector over the vertical axis, $l = 2\rho\cos\theta$ is the Coriolis parameter, where ρ is the angular velocity of the Earth rotation and θ is the colatitude, k is an unit vector oriented vertically upward, g is the free fall acceleration, r is the bottom friction factor, κ_h is the horizontal turbulent viscosity coefficient, $\mathbf{g}$ is the external force vector, and ζ is the displacement of the free surface with respect to the ocean floor. The function $\mathbf{w}^0$ is an known function on the boundary. The restriction $\mathbf{w}^0|_{\Gamma_1} = 0$ is the no-slip condition on the shoreline, and $\int_{\Gamma_2} \mathbf{w}^0 d\Gamma_2 = 0$ follows from the mass conservation law. Here R is the vertical scale of motion, i.e., the depth of the calm sea. Let us assume that R is a continuously differentiable function of x, so that $\inf_{x\in G}\{R(x)\} \geq C_0$ and $\sup_{x\in G}\{R(x) + |\nabla R(x)|\} \leq C_1$ for some positive constants C_0 and C_1.

In order to simplify the non-homogeneous boundary value problem to a homogeneous Dirichlet boundary value problem, we set

$$\mathbf{u}(t, x) = \mathbf{w}(t, x) - \mathbf{w}^0(t, x),$$

and

$$\xi(t, x) = \zeta(t, x) + \int_0^t \nabla \cdot \Big(R(x)\mathbf{w}^0(s, x)\Big) ds.$$

Let us denote by $\mathbf{A}$ the matrix

$$\mathbf{A} = \begin{pmatrix} -\kappa_h\triangle & -2\rho\cos\theta \\ 2\rho\cos\theta & -\kappa_h\triangle \end{pmatrix},$$

and $\gamma(x) \triangleq \frac{r}{R(x)}$. Thus we are able to rewrite the tide dynamics model as

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} = -\mathbf{A}\mathbf{u} - \gamma|\mathbf{u} + \mathbf{w}^0|(\mathbf{u} + \mathbf{w}^0) - g\nabla\xi + \mathbf{f} \quad \text{on } [0,T]\times G; \\ \frac{\partial \xi}{\partial t} + \nabla\cdot(R\mathbf{u}) = 0; \\ \mathbf{u} = 0 \quad \text{on } [0,T]\times\partial G; \\ \mathbf{u} = \mathbf{u}_0 \text{ and } \xi = \xi_0 \quad \text{at } t = 0, \end{cases} \tag{2.2}$$

where

$$\begin{cases} \mathbf{f} = \mathbf{g} - \frac{\partial \mathbf{w}^0}{\partial t} + g\nabla\int_0^t \nabla\cdot(R\mathbf{w}^0)ds + \kappa_h\triangle\mathbf{w}^0 - lk\times\mathbf{w}^0; \\ \mathbf{u}_0(x) = \mathbf{w}_0(x) - \mathbf{w}^0(0,x); \\ \xi_0(x) = \zeta_0(x). \end{cases}$$

To unify the language, let us introduce the following definitions and notations.

Definition 2.1. Let A be an operator on a separable Hilbert space K with complete orthonormal system (CONS for short) $\{e_j\}_{j=1}^{\infty}$. If $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for any $x, y \in K$, then A^* is called the *adjoint* of A. If $A = A^*$, then A is called *self-adjoint*.

Definition 2.2. Let A be a linear operator from a separable Hilbert space K with CONS $\{e_j\}_{j=1}^{\infty}$ to a separable Hilbert space H.

(a) We denote by $L(K,H)$ the class of all bounded linear operators with the uniform operator norm $\|\cdot\|_L$.

(b) If $\|A\|_{L_1} = \sum_{k=1}^{\infty}\langle (A^*A)^{\frac{1}{2}}e_k, e_k\rangle_K < \infty$, then A is called a *trace class(nuclear) operator*. We denote by $L_1(K,H)$ the class of trace class operators equipped with norm $\|\cdot\|_{L_1}$.

(c) We also denote by $L_2(K,H)$ the class of *Hilbert-Schmidt operators* with norm $\|\cdot\|_{L_2}$ given by $\|A\|_{L_2} = (\sum_{k=1}^{\infty}\langle Ae_k, Ae_k\rangle_H)^{\frac{1}{2}}$. Sometimes $\|\cdot\|_{L_2}$ is also denoted by $\|\cdot\|_{H.S.}$

(d) Let $Q \in L_1(K,K)$ be self-adjoint and positive definite. Let K_0 be the Hilbert subspace of K with inner product

$$\langle f, g\rangle_{K_0} = \langle Q^{-\frac{1}{2}}f, Q^{-\frac{1}{2}}g\rangle_K,$$

and we denote $L_Q = L_2(K_0, H)$ with the inner product

$$\langle F, G\rangle_{L_Q} = tr(FQG^*) = tr(GQF^*), \qquad F, G \in L_Q.$$

Definition 2.3. A stochastic process $W(t)$ is called an *H-valued Q-Wiener process*, where Q is a trace class operator on H, if $W(t)$ satisfies the following:

(a) $W(t)$ has continuous sample paths in H-norm with $W(0) = 0$.

(b) $(W(t), h)$ has stationary independent increments for all $h \in H$.

(c) $W(t)$ is a Gaussian process with mean zero and covariance operator Q, i.e.

$$E(W(t), g)(W(s), h) = (t\wedge s)(Qg, h) \quad \text{for all } g, h \in H.$$

Let $\mathbb{L}^2(G)$ and $\mathbb{H}_0^1(G)$ be standard Sobolev spaces with norms

$$\|\mathbf{u}\|_{\mathbb{L}^2}^2 \triangleq \int_G |\mathbf{u}|^2 dx$$

and

$$\|\mathbf{u}\|^2_{\mathbb{H}^1_0} \triangleq \int_G |\nabla \mathbf{u}|^2 dx,$$

respectively. Denote $\mathbb{H}^{-1}(G)$ the dual space of $\mathbb{H}^1_0(G)$. Denote $(\cdot,\cdot)$ the inner product of $\mathbb{L}^2(G)$, $(\cdot,\cdot)_{\mathbb{H}^1_0}$ the inner product of $\mathbb{H}^1_0(G)$, and $\langle\cdot,\cdot\rangle$ the duality paring between $\mathbb{H}^1_0(G)$ and $\mathbb{H}^{-1}(G)$. Let $\|\cdot\|_{\mathbb{L}^2}$ be the norm of $\mathbb{L}^2$ and $\|\cdot\|_{\mathbb{H}^1_0}$ be the norm of $\mathbb{H}^1_0(G)$. Similarly, we can define the norms, inner products of $L^2(G)$, $H^1_0(G)$ and $H^{-1}(G)$. It is clear that

$$\mathbb{H}^1_0(G) \subset \mathbb{L}^2(G) \subset \mathbb{H}^{-1}(G) \quad \text{and} \quad H^1_0(G) \subset L^2(G) \subset H^{-1}(G)$$

are Gelfand triples, and for any $\mathbf{x} \in \mathbb{L}^2(G)$ and $\mathbf{y} \in \mathbb{H}^1_0(G)$, there exists $\mathbf{x}' \in \mathbb{H}^{-1}(G)$, such that $(\mathbf{x},\mathbf{y}) = \langle \mathbf{x}',\mathbf{y}\rangle$. The mapping $\mathbf{x} \mapsto \mathbf{x}'$ is linear, injective, compact and continuous. A similar result holds for $L^2(G)$, $H^1_0(G)$ and $H^{-1}(G)$.

Remark 2.4. (i) Let $\mathbb{Q}$ be a trace class operator on $\mathbb{L}^2(G)$. Let $\{\mathbf{e}_j\}_{j=1}^\infty \in \mathbb{L}^2(G) \cap \mathbb{H}^1_0(G) \cap \mathbb{L}^4(G)$ be a CONS in $\mathbb{L}^2(G)$ such that there exists a nondecreasing sequence of positive numbers $\{\lambda_j\}_{j=1}^\infty$, $\lim_{j\to\infty} \lambda_j = \infty$ and $-\triangle \mathbf{e}_j = \lambda_j \mathbf{e}_j$ for all j. Let $\mathbb{Q}\mathbf{e}_k = q_k \mathbf{e}_k$ with $\sum_{k=1}^\infty q_k < \infty$, and $\{b^k(t)\}$ be a sequence of iid Brownian motions in $\mathbb{R}$. Then the $\mathbb{L}^2(G)$-valued $\mathbb{Q}$-Wiener process is taken as $\mathbb{W}(t) = \sum_{k=1}^\infty \sqrt{q_k} b^k(t)\mathbf{e}_k$.

(ii) Let Q be a trace class operator on $L^2(G)$. Similarly, we can define a complete orthonormal system $\{e_j\}_{j=1}^\infty$, a nondecreasing sequence of positive numbers $\{\rho_j\}_{j=1}^\infty$ such that $-\triangle e_j = \rho_j e_j$, and positive numbers q'_j such that $Qe_j = q'_j e_j$ and $\sum_{j=1}^\infty q'_j < \infty$. Let $W(t) = \sum_{j=1}^\infty \sqrt{q'_j} b^j(t) e_j$. Then $W(t)$ is an $L^2(G)$-valued Q-Wiener process.

Thus according to Definition 2.2 and 2.3, $L_{\mathbb{Q}}$, the space of linear operators $\mathbf{E}$ such that $\mathbf{E}\mathbb{Q}^{\frac{1}{2}}$ is a Hilbert-Schmidt operator from $\mathbb{L}^2(G)$ to $\mathbb{L}^2(G)$, is well-defined, and so is L_Q. In this paper we consider a filtered complete probability space $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}_{t\geq 0})$, where $\{\mathcal{F}_t\}$ is the natural filtration of $\{\mathbb{W}(t)\}$ and $\{W(t)\}$, augmented by all the P-null sets of $\mathcal{F}$. Introducing randomness to system (2.2), and suppose the terminal value of the tide is given, one can construct the following backward stochastic tidal dynamics equations:

$$\begin{cases} \frac{\partial \mathbf{u}(t)}{\partial t} = -\mathbf{A}\mathbf{u}(t) - \gamma|\mathbf{u}(t) + \mathbf{w}^0(t)|(\mathbf{u}(t) + \mathbf{w}^0(t)) - g\nabla\xi(t) + \mathbf{f}(t) + \mathbf{Z}(t)\frac{d\mathbb{W}(t)}{dt}; \\ \frac{\partial \xi(t)}{\partial t} + \nabla\cdot(R\mathbf{u}(t)) = Z(t)\frac{dW(t)}{dt}; \\ \mathbf{u}(T) = \phi \text{ and } \xi(T) = \psi, \end{cases} \tag{2.3}$$

where $\mathbf{f} \in L^2(0,T;\mathbb{H}^{-1})$, $\phi \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{L}^2(G))$ and $\psi \in L^2_{\mathcal{F}_T}(\Omega; L^2(G))$.

Definition 2.5. A quaternion of $\mathcal{F}_t$-Adapted processes $(\mathbf{u}, \mathbf{Z}, \xi, Z)$ is called a *solution* of backward tidal dynamics equation (2.3) if it satisfies the integral form of the system

$$\begin{cases} \mathbf{u}(t) = \phi + \int_t^T \Big\{\mathbf{A}\mathbf{u}(s) + \gamma|\mathbf{u}(s) + \mathbf{w}^0(s)|(\mathbf{u}(s) + \mathbf{w}^0(s)) + g\nabla\xi(s) - \mathbf{f}(s)\Big\}ds \\ \qquad\quad - \int_t^T \mathbf{Z}(s)d\mathbb{W}(s); \\ \xi(t) = \psi + \int_t^T \nabla\cdot(R\mathbf{u}(s))ds - \int_t^T Z(s)dW(s); \end{cases}$$

P-a.s., and the following holds:

(a) $\mathbf{u} \in L^2_{\mathcal{F}}(\Omega; L^\infty(0,T;\mathbb{L}^2(G))) \cap L^2_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{H}^1_0(G)))$;

(b) $\mathbf{Z} \in L^2_{\mathcal{F}}(\Omega; L^2(0,T;L_{\mathbb{Q}}))$;

(c) $\xi \in L^2_{\mathcal{F}}(\Omega; L^\infty(0,T;L^2(G))) \cap L^2_{\mathcal{F}}(\Omega; L^2(0,T;H^1_0(G)))$;
(d) $Z \in L^2_{\mathcal{F}}(\Omega; L^2(0,T;L_Q))$.

We list below some commonly used results and omit some of the proofs. Readers may refer to Adams[1], Kesavan[11], Ladyzhenskaya[12], Manna, Menaldi and Sritharan[23], and Temam[32] for more details.

Lemma 2.6. *For any real-valued, compact supported smooth functions* $\mathbf{x}$ *and* $\mathbf{y}$ *in* $\mathbb{R}^2$*, the following holds:*

$$||\mathbf{x}\mathbf{y}||^2_{\mathbb{L}^2} \le ||\mathbf{x}\partial_1\mathbf{x}||_{\mathbb{L}^1}||\mathbf{y}\partial_2\mathbf{y}||_{\mathbb{L}^1},$$

$$||\mathbf{x}||^4_{\mathbb{L}^4} \le 2||\mathbf{x}||^2_{\mathbb{L}^2}||\nabla\mathbf{x}||^2_{\mathbb{L}^2}.$$

Lemma 2.7. *Let X be a normed linear space. Let O be an open subset of X, and K be a convex subset of O. Let $J : O \to \mathbb{R}$ be twice differentiable in O. Then J is convex if and only if, for all u and $v \in K$,*

$$J''(v;u,u) = \frac{d^2}{d\theta d\alpha}J(v+\theta u+\alpha u)|_{\theta,\alpha=0} \ge 0.$$

Lemma 2.8. *Denote* $\mathbf{B}(\mathbf{u}) \triangleq \gamma|\mathbf{u}+\mathbf{w}^0|(\mathbf{u}+\mathbf{w}^0)$*. Then* $\mathbf{B}(\cdot)$ *is a continuous operator from* $\mathbb{L}^4(G)$ *into* $\mathbb{L}^2(G)$*, and for all* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{L}^4(G)$*,*

$$\langle \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{u}-\mathbf{v}\rangle \ge 0.$$

Lemma 2.9. (a) *For any* $\mathbf{u}$ *and* $\mathbf{v} \in \mathbb{H}^1_0(G)$*, and* $\mathbf{u}$ *has a smooth second derivative,*

$$(\mathbf{A}\mathbf{u},\mathbf{u}) = \kappa_h||\mathbf{u}||^2_{\mathbb{H}^1_0}$$

and

$$(\mathbf{A}\mathbf{u},\mathbf{v}) \le C_2||\mathbf{u}||_{\mathbb{H}^1_0}||\mathbf{v}||_{\mathbb{H}^1_0}$$

for some constant $C_2 = \kappa_h + 2\rho\cos\theta$*.*

(b) *For any* $\mathbf{u}$ *and* $\mathbf{w}^0 \in \mathbb{L}^4(G)$*,*

$$||\mathbf{B}(\mathbf{u})||_{\mathbb{L}^2} \le C_3||\mathbf{u}||_{\mathbb{L}^4},$$

where $C_3 = \sup_{x\in G}\gamma(x)$*.*

(c) *For any* $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}^0 \in \mathbb{L}^4(G)$*,*

$$||\mathbf{B}(\mathbf{u})-\mathbf{B}(\mathbf{v})||_{\mathbb{L}^2} \le C_3\Big\{||\mathbf{u}||_{\mathbb{L}^4} + ||\mathbf{v}||_{\mathbb{L}^4}\Big\}||\mathbf{u}-\mathbf{v}||_{\mathbb{L}^4},$$

and

$$|\langle \mathbf{B}(\mathbf{u})-\mathbf{B}(\mathbf{v}),\mathbf{u}-\mathbf{v}\rangle| \le C_4\Big\{||\mathbf{u}||^2_{\mathbb{L}^4} + ||\mathbf{v}||^2_{\mathbb{L}^4}\Big\}||\mathbf{u}-\mathbf{v}||_{\mathbb{L}^2}.$$

(d) *For any* $\mathbf{u}, \mathbf{v} \in \mathbb{H}^1_0(G)$ *and* $\mathbf{w}^0 \in \mathbb{L}^4(G)$*,*

$$|\langle \mathbf{B}(\mathbf{u})-\mathbf{B}(\mathbf{v}),\mathbf{u}-\mathbf{v}\rangle| \le C_5\Big\{||\mathbf{u}||_{\mathbb{L}^4} + ||\mathbf{v}||_{\mathbb{L}^4}\Big\}||\mathbf{u}-\mathbf{v}||^{\frac{3}{2}}_{\mathbb{L}^2}||\mathbf{u}-\mathbf{v}||^{\frac{1}{2}}_{\mathbb{H}^1_0}.$$

3. A Priori Estimates

In this section we are going to show some a priori estimates for a projected system. These projections are useful for the Galerkin approximation scheme employed in Section 5 and Section 6. For any $N \in \mathbb{N}$, let

$$\mathbb{L}^2{}_N(G) \triangleq \mathrm{span}\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_N\}$$

be the N-dimensional subspace of $\mathbb{L}^2(G)$. Likewise, we can define $\mathbb{H}^1_{0N}(G)$, $\mathbb{H}^{-1}{}_N(G)$, $L^2_N(G)$, $H^1_{0N}(G)$ and $H^{-1}_N(G)$. Note that since $\{\mathbf{e}_j\}_{j=1}^{\infty} \in \mathbb{L}^2(G) \cap \mathbb{H}^1_0(G) \cap \mathbb{L}^4(G)$, we have

$$\mathbb{L}^2{}_N(G) = \mathbb{H}^1_{0N}(G) = \mathbb{H}^{-1}{}_N(G).$$

Similarly, we have

$$L^2_N(G) = H^1_{0N}(G) = H^{-1}_N(G).$$

Let P_N be the orthogonal projection from $\mathbb{L}^2(G)$ to $\mathbb{L}^2{}_N(G)$. Let

$$\mathbb{W}^N(t) \triangleq P_N \mathbb{W}(t) \quad \text{and} \quad W^N(t) \triangleq P_N W(t).$$

Note that by Remark 2.4, $\mathbb{W}^N(t) = \sum_{i=1}^N \sqrt{q_i} b^i(t)\mathbf{e}_i$ and $W^N(t) = \sum_{i=1}^N \sqrt{q_i'} b^i(t) e_i$. Let $\{\mathcal{F}_t^N\}$ be the natural filtration of $\{\mathbb{W}^N(t)\}$ and $\{W^N(t)\}$, and we introduce the following projections:

$$\mathbf{f}^N(t) \triangleq P_N \mathbf{f}(t), \phi^N \triangleq E(P_N\phi|\mathcal{F}_T^N) \text{ and } \psi^N \triangleq E(P_N\psi|\mathcal{F}_T^N).$$

The projected backward tide dynamics system is given by

$$\begin{cases} \frac{\partial \mathbf{u}^N(t)}{\partial t} = -\mathbf{A}\mathbf{u}^N(t) - \mathbf{B}^N(\mathbf{u}^N(t)) - g\nabla\xi^N(t) + \mathbf{f}^N(t) + \mathbf{Z}^N(t)d\mathbb{W}^N(t); \\ \frac{\partial \xi^N(t)}{\partial t} + \nabla\cdot(R^N\mathbf{u}^N(t)) = Z^N(t)dW^N(t); \\ \mathbf{u}^N(T) = \phi^N \text{ and } \xi^N(T) = \psi^N, \end{cases} \tag{3.1}$$

where $\mathbf{B}^N(\mathbf{u}) \triangleq \gamma^N|\mathbf{u} + \mathbf{w}^{0N}|(\mathbf{u} + \mathbf{w}^{0N})$ for all $\mathbf{u} \in \mathbb{L}^4(G)$.

Proposition 3.1. *Suppose that the terminal conditions satisfy $\phi \in L^\infty_{\mathcal{F}_T}(\Omega; \mathbb{L}^2(G))$, $\psi \in L^\infty_{\mathcal{F}_T}(\Omega; L^2(G))$, and the external force $\mathbf{f} \in L^2(0,T;\mathbb{H}^{-1}(G))$. Then for any solution of system* (3.1)*, the following is true:*

$$(\mathbf{u}^N, \mathbf{Z}^N) \in \left\{L^\infty_{\mathcal{F}}([0,T]\times\Omega; \mathbb{L}^2(G)) \cap L^2_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{H}^1_0(G)))\right\} \times L^2_{\mathcal{F}}(\Omega; L^2(0,T;L_{\mathbb{Q}})),$$

$$(\xi^N, Z^N) \in \left\{L^\infty_{\mathcal{F}}([0,T]\times\Omega; L^2(G)) \cap L^2_{\mathcal{F}}(\Omega; L^2(0,T;H^1_0(G)))\right\} \times L^2_{\mathcal{F}}(\Omega; L^2(0,T;L_Q)).$$

Proof. An application of the Itô formula to $\|\mathbf{u}^N(t)\|^2_{\mathbb{L}^2}$ yields

$$\begin{aligned} &\|\mathbf{u}^N(t)\|^2_{\mathbb{L}^2} + \int_t^T \|\mathbf{Z}^N(s)\|^2_{L_{\mathbb{Q}}} ds \\ =&\|\phi^N\|^2_{\mathbb{L}^2} + 2\int_t^T \langle \mathbf{A}\mathbf{u}^N(s) + \mathbf{B}^N(\mathbf{u}^N(s)) + g\nabla\xi^N(s) - \mathbf{f}^N(s), \mathbf{u}^N(s)\rangle ds \\ &- 2\int_t^T \langle \mathbf{Z}^N(s)d\mathbb{W}^N(s), \mathbf{u}^N(s)\rangle. \end{aligned} \tag{3.2}$$

By Lemma 2.6 and 2.9, we have

$$2\langle \mathbf{A}\mathbf{u}^N(s), \mathbf{u}^N(s)\rangle = 2(\mathbf{A}\mathbf{u}^N(s), \mathbf{u}^N(s)) = 2\kappa_h\|\mathbf{u}^N(s)\|^2_{\mathbb{H}^1_0}, \tag{3.3}$$

$$\begin{aligned}
&2\langle \mathbf{B}^N(\mathbf{u}^N(s)), \mathbf{u}^N(s)\rangle \\
&\quad \le 2C_3\|\mathbf{u}^N(s)\|_{\mathbb{L}^4}\|\mathbf{u}^N(s)\|_{\mathbb{L}^2} \\
&\quad \le 4C_3\|\mathbf{u}^N(s)\|_{\mathbb{L}^2}^{\frac{3}{2}}\|\mathbf{u}^N(s)\|_{\mathbb{H}_0^1}^{\frac{1}{2}} \\
&\quad \le 3C_3\|\mathbf{u}^N(s)\|_{\mathbb{L}^2}^2 + C_3\|\mathbf{u}^N(s)\|_{\mathbb{H}_0^1}^2,
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&2\langle g\nabla \xi^N(s), \mathbf{u}^N(s)\rangle \\
&\quad = -2g\langle \xi^N(s), \nabla\cdot \mathbf{u}^N(s)\rangle \\
&\quad \le 2g\|\xi^N(s)\|_{L^2}\|\nabla\cdot \mathbf{u}^N(s)\|_{L^2} \\
&\quad \le g\|\xi^N(s)\|_{L^2}^2 + g\|\mathbf{u}^N(s)\|_{\mathbb{H}_0^1}^2
\end{aligned} \tag{3.5}$$

and

$$2\langle -\mathbf{f}^N(s), \mathbf{u}^N(s)\rangle \le \|\mathbf{f}^N(s)\|_{\mathbb{H}^{-1}}^2 + \|\mathbf{u}^N(s)\|_{\mathbb{H}_0^1}^2. \tag{3.6}$$

Thus (3.2) becomes

$$\begin{aligned}
&\|\mathbf{u}^N(t)\|_{\mathbb{L}^2}^2 + \int_t^T \|\mathbf{Z}^N(s)\|_{L_{\mathbb{Q}}}^2 ds \\
&\le \|\phi^N\|_{\mathbb{L}^2}^2 + \int_t^T \Big\{(2\kappa_h + C_3 + g + 1)\|\mathbf{u}^N(s)\|_{\mathbb{H}_0^1}^2 + 3C_3\|\mathbf{u}^N(s)\|_{\mathbb{L}^2}^2 + g\|\xi^N(s)\|_{L^2}^2 \\
&\qquad + \|\mathbf{f}^N(s)\|_{\mathbb{H}^{-1}}^2\Big\} ds - 2\int_t^T \langle \mathbf{Z}^N(s) d\mathbb{W}^N(s), \mathbf{u}^N(s)\rangle.
\end{aligned} \tag{3.7}$$

Applying the Itô formula to $\|\xi^N(s)\|_{L^2}^2$ to get

$$\begin{aligned}
&\|\xi^N(t)\|_{L^2}^2 + \int_t^T \|Z^N(s)\|_{L_Q}^2 ds \\
&= \|\psi^N\|_{L^2}^2 + \int_t^T 2\langle \nabla\cdot (R^N\mathbf{u}^N(s)), \xi^N(s)\rangle ds - \int_t^T 2\langle Z^N(s) dW^N(s), \xi^N(s)\rangle.
\end{aligned} \tag{3.8}$$

The term

$$\begin{aligned}
&2\langle \nabla\cdot (R^N\mathbf{u}^N(s)), \xi^N(s)\rangle \\
&= 2\langle R^N\nabla\cdot \mathbf{u}^N(s), \xi^N(s)\rangle + 2\langle \mathbf{u}^N(s)\nabla R^N, \xi^N(s)\rangle \\
&\le 2\|R^N\|_{L^\infty}\|\mathbf{u}^N(s)\|_{\mathbb{H}_0^1}\|\xi^N(s)\|_{L^2} + 2\|\mathbf{u}^N(s)\|_{\mathbb{L}^2}\|\nabla R^N\|_{\mathbb{L}^2}\|\xi^N(s)\|_{L^2} \\
&\le C_1\Big\{\|\mathbf{u}^N(s)\|_{\mathbb{L}^2}^2 + \|\mathbf{u}^N(s)\|_{\mathbb{H}_0^1}^2 + 2\|\xi^N(s)\|_{L^2}^2\Big\}.
\end{aligned} \tag{3.9}$$

Thus substituting (3.9) into (3.8), and adding up (3.7) and (3.8), one gets

$$\begin{aligned}
&E^{\mathcal{F}_r}\|\mathbf{u}^N(t)\|_{\mathbb{L}^2}^2 + E^{\mathcal{F}_r}\|\xi^N(t)\|_{L^2}^2 + E^{\mathcal{F}_r}\int_t^T \|\mathbf{Z}^N(s)\|_{L_{\mathbb{Q}}}^2 ds + E^{\mathcal{F}_r}\int_t^T \|Z^N(s)\|_{L_Q}^2 ds \\
&\le E^{\mathcal{F}_r}\|\phi^N\|_{\mathbb{L}^2}^2 + E^{\mathcal{F}_r}\|\psi^N\|_{L^2}^2 + E^{\mathcal{F}_r}\int_t^T \Big\{(2\kappa_h + C_3 + g + 1 + C_1)\|\mathbf{u}^N(s)\|_{\mathbb{H}_0^1}^2 \\
&\quad + (3C_3 + C_1)\|\mathbf{u}^N(s)\|_{\mathbb{L}^2}^2 + (g+2)\|\xi^N(s)\|_{L^2}^2 + \|\mathbf{f}^N(s)\|_{\mathbb{H}^{-1}}^2\Big\} ds
\end{aligned} \tag{3.10}$$

for $0 \leq r \leq t$, P-a.s. Since

$$\begin{aligned}&\|\mathbf{u}^N(s)\|^2_{\mathbb{H}^1_0}\\ =&\langle -\triangle \mathbf{u}^N(s), \mathbf{u}^N(s)\rangle\\ =&\sum_{i=1}^{N}\langle \lambda_i \mathbf{e}_i, \mathbf{u}^N(s)\rangle\\ \leq&\lambda_N \|\mathbf{u}^N(s)\|^2_{\mathbb{L}^2},\end{aligned}$$

where λ_i, as stated in Section 2, is the eigenvalue of $-\triangle$ with respect to $\mathbf{e_i}$. Thus equation (3.10) becomes

$$\begin{aligned}&E^{\mathcal{F}_r}\|\mathbf{u}^N(t)\|^2_{\mathbb{L}^2} + E^{\mathcal{F}_r}\|\xi^N(t)\|^2_{L^2} + E^{\mathcal{F}_r}\int_t^T \Big\{\|\mathbf{u}^N(s)\|^2_{\mathbb{H}^1_0} + \|\xi^N(s)\|^2_{H^1_0}\Big\}ds\\ &+ E^{\mathcal{F}_r}\int_t^T \|\mathbf{Z}^N(s)\|^2_{L_{\mathbb{Q}}} ds + E^{\mathcal{F}_r}\int_t^T \|Z^N(s)\|^2_{L_Q} ds \qquad (3.11)\\ \leq& E^{\mathcal{F}_r}\|\phi^N\|^2_{\mathbb{L}^2} + E^{\mathcal{F}_r}\|\psi^N\|^2_{L^2} + E^{\mathcal{F}_r}\int_t^T \Big\{K(N)\|\mathbf{u}^N(s)\|^2_{\mathbb{L}^2} + K(N)\|\xi^N(s)\|^2_{L^2} + \|\mathbf{f}^N(s)\|^2_{\mathbb{H}^{-1}}\Big\}ds,\end{aligned}$$

P-a.s., where $K(N)$ is a constant depending on N only. By means of the Gronwall inequality and letting $r = t$, one obtains

$$\begin{aligned}&\sup_{t\in[0,T]}\Big\{\|\mathbf{u}^N(t)\|^2_{\mathbb{L}^2} + \|\xi^N(t)\|^2_{L^2}\Big\} + E\int_0^T \Big\{\|\mathbf{u}^N(s)\|^2_{\mathbb{H}^1_0} + \|\xi^N(s)\|^2_{H^1_0}\Big\}ds\\ &+ E\int_0^T \|\mathbf{Z}^N(s)\|^2_{L_{\mathbb{Q}}} ds + E\int_0^T \|Z^N(s)\|^2_{L_Q} ds\\ \leq& K(N)\Big\{\sup_{t\in[0,T]} E^{\mathcal{F}_t}\|\phi^N\|^2_{\mathbb{L}^2} + \sup_{t\in[0,T]} E^{\mathcal{F}_t}\|\psi^N\|^2_{L^2} + \int_0^T \|\mathbf{f}^N(s)\|^2_{\mathbb{H}^{-1}} ds\Big\}, \qquad (3.12)\end{aligned}$$

P-a.s., which completes the proof. □

Proposition 3.2. *Suppose that the terminal conditions satisfy $\phi \in L^n_{\mathcal{F}_T}(\Omega; \mathbb{L}^2(G))$, $\psi \in L^n_{\mathcal{F}_T}(\Omega; L^2(G))$, and the external force $\mathbf{f} \in L^2(0, T; \mathbb{H}^{-1}(G))$, for all $n \in \mathbb{N}$ and $n \geq 2$. The following is true for any solution of system* (3.1)*:*

$$\begin{aligned}(\mathbf{u}^N, \mathbf{Z}^N) &\in \Big\{L^\infty(0, T; L^n_{\mathcal{F}}(\Omega; \mathbb{L}^2(G))) \cap L^n_{\mathcal{F}}(\Omega; L^n(0, T; \mathbb{H}^1_0(G)))\Big\} \times L^2_{\mathcal{F}}(\Omega; L^2(0, T; L_{\mathbb{Q}})),\\ (\xi^N, Z^N) &\in \Big\{L^\infty(0, T; L^n_{\mathcal{F}}(\Omega; L^2(G))) \cap L^n_{\mathcal{F}}(\Omega; L^n(0, T; H^1_0(G)))\Big\} \times L^2_{\mathcal{F}}(\Omega; L^2(0, T; L_Q)).\end{aligned}$$

Proof. First of all, the case when $n = 2$ can be proved by applying the Gronwall inequality to equation (3.11) in Proposition 3.1, taking the expectation, and then taking supremum over the time interval $[0, T]$. Secondly, suppose the proposition holds for all $2 \leq m \leq n-1$. Let us show that the proposition is still true for $m = n$. An application of the Itô formula yields

$$\begin{aligned}&\|\mathbf{u}^N(t)\|^n_{\mathbb{L}^2} + \frac{n^2 - n}{2}\int_t^T \|\mathbf{u}^N(s)\|^{n-2}_{\mathbb{L}^2}\|\mathbf{Z}^N(s)\|^2_{L_{\mathbb{Q}}} ds\\ =&\|\phi^N\|^n_{\mathbb{L}^2} + n\int_t^T \|\mathbf{u}^N(s)\|^{n-2}_{\mathbb{L}^2}\langle \mathbf{A}\mathbf{u}^N(s) + \mathbf{B}^N(\mathbf{u}^N(s)) + g\nabla\xi^N(s) - \mathbf{f}^N(s), \mathbf{u}^N(s)\rangle ds\end{aligned}$$

$$- n \int_t^T \|\mathbf{u}^N(s)\|_{\mathbb{L}^2}^{n-2} \langle \mathbf{Z}^N(s) d\mathbb{W}^N(s), \mathbf{u}^N(s)\rangle.$$

Taking the expectation on both sides, one obtains

$$\begin{aligned}
&E\|\mathbf{u}^N(t)\|_{\mathbb{L}^2}^n + E\int_t^T \|\mathbf{u}^N(s)\|_{\mathbb{H}_0^1}^n ds + \frac{n^2-n}{2} E\int_t^T \|\mathbf{u}^N(s)\|_{\mathbb{L}^2}^{n-2} \|\mathbf{Z}^N(s)\|_{L_{\mathbb{Q}}}^2 ds \\
=&E\|\phi^N\|_{\mathbb{L}^2}^n + nE\int_t^T \|\mathbf{u}^N(s)\|_{\mathbb{L}^2}^{n-2} \langle \mathbf{A}\mathbf{u}^N(s) + \mathbf{B}^N(\mathbf{u}^N(s)) + g\nabla\xi^N(s)\rangle ds \\
&- nE\int_t^T \|\mathbf{u}^N(s)\|_{\mathbb{L}^2}^{n-2} \|\mathbf{u}^N(s)\|_{\mathbb{H}_0^1} \|\mathbf{f}^N(s)\|_{\mathbb{H}^{-1}} ds + E\int_t^T \|\mathbf{u}^N(s)\|_{\mathbb{H}_0^1}^n ds \\
\le&E\|\phi^N\|_{\mathbb{L}^2}^n + K(n,N)E\int_t^T \|\mathbf{u}^N(s)\|_{\mathbb{L}^2}^n ds + K(n,N)E\int_t^T \|\xi^N(s)\|_{L^2}^n ds \\
&+ n\sqrt{\lambda_N}\int_t^T \|\mathbf{f}^N(s)\|_{\mathbb{H}^{-1}} E\|\mathbf{u}^N(s)\|_{\mathbb{L}^2}^{n-1} ds \\
\le&E\|\phi^N\|_{\mathbb{L}^2}^n + K(n,N)E\int_t^T \|\mathbf{u}^N(s)\|_{\mathbb{L}^2}^n ds + K(n,N)E\int_t^T \|\xi^N(s)\|_{L^2}^n ds \\
&+ n\sqrt{\lambda_N} \sup_{s\in[0,T]} E\|\mathbf{u}^N(s)\|_{\mathbb{L}^2}^{n-1} \int_t^T \|\mathbf{f}^N(s)\|_{\mathbb{H}^{-1}} ds. \qquad (3.13)
\end{aligned}$$

where $K(n,N)$ is a constant depending on n and N only. Similar, one can show that

$$\begin{aligned}
&E\|\xi^N(t)\|_{L^2}^n + E\int_t^T \|\xi^N(s)\|_{H_0^1}^n ds + \frac{n^2-n}{2} E\int_t^T \|\xi^N(s)\|_{L^2}^{n-2} \|Z^N(s)\|_{L_Q}^2 ds \\
\le&E\|\psi^N\|_{L^2}^n + K(n,N)E\int_t^T \left\{\|\mathbf{u}^N(s)\|_{\mathbb{L}^2}^n + \|\xi^N(s)\|_{L^2}^n\right\} ds. \qquad (3.14)
\end{aligned}$$

Adding up (3.13) and (3.14) to get

$$\begin{aligned}
&E\|\mathbf{u}^N(t)\|_{\mathbb{L}^2}^n + E\|\xi^N(t)\|_{L^2}^n + E\int_t^T \|\mathbf{u}^N(s)\|_{\mathbb{H}_0^1}^n ds + E\int_t^T \|\xi^N(s)\|_{H_0^1}^n ds \\
\le&E\|\phi^N\|_{\mathbb{L}^2}^n + E\|\psi^N\|_{L^2}^n + K(n,N)E\int_t^T \left\{\|\mathbf{u}^N(s)\|_{\mathbb{L}^2}^n + \|\xi^N(s)\|_{L^2}^n\right\} ds \\
&+ K(n,N)\int_t^T \|\mathbf{f}^N(s)\|_{\mathbb{H}^{-1}}^n ds,
\end{aligned}$$

which completes the proof after an application of the Gronwall inequality. □

4. Well-posedness of the Projected System

In this section, we are going to show the well-posedness of the projected system (3.1). In order to do so, we need to truncate the system. For every $M \in \mathbb{N}$, let L_M to be the Lipschitz C^∞ function given as follows:

$$L_M(\|\mathbf{u}\|_{\mathbb{H}_0^1}) = \begin{cases} 1 & \text{if } \|\mathbf{u}\|_{\mathbb{H}_0^1} < M \\ 0 & \text{if } \|\mathbf{u}\|_{\mathbb{H}_0^1} > M+1 \\ 0 \le L_M(\|\mathbf{u}\|_{\mathbb{H}_0^1}) \le 1 & \text{otherwise} \end{cases}$$

Proposition 4.1. $\|L_M(\|\mathbf{x}\|_{\mathbb{H}_0^1})\mathbf{B}^N(\mathbf{x}) - L_M(\|\mathbf{y}\|_{\mathbb{H}_0^1})\mathbf{B}^N(\mathbf{y})\|_{\mathbb{L}^2} \leq C(N,M)\|\mathbf{x}-\mathbf{y}\|_{\mathbb{H}_0^1}$ *for any* $\mathbf{x},\mathbf{y} \in \mathbb{L}^2{}_N(G)$ *and* $M \in \mathbb{N}$, *where* $C(N,M)$ *is a constant depending on* N, M *and* G *only.*

Proof. Let $\mathbf{x}$ and $\mathbf{y}$ be any two elements in $\mathbb{L}^2{}_N(G)$. Without lose of generality, we assume that $\|\mathbf{x}\|_{\mathbb{H}_0^1} \leq \|\mathbf{y}\|_{\mathbb{H}_0^1}$, and let us discuss it in the following 3 cases:

Case I. $\|\mathbf{x}\|_{\mathbb{H}_0^1} > M+1$.

By the definition of L_M, $\|L_M(\|\mathbf{x}\|_{\mathbb{H}_0^1})\mathbf{B}^N(\mathbf{x}) - L_M(\|\mathbf{y}\|_{\mathbb{H}_0^1})\mathbf{B}^N(\mathbf{y})\|^2_{\mathbb{L}^2} = 0 \leq \|\mathbf{x}-\mathbf{y}\|^2_{\mathbb{H}_0^1}$. Thus we see that $L_M\mathbf{B}^N$ is Lipschitz.

Case II. $\|\mathbf{y}\|_{\mathbb{H}_0^1} \leq M+1$.

It is clear that

$$
\begin{aligned}
&\|L_M(\|\mathbf{x}\|_{\mathbb{H}_0^1})\mathbf{B}^N(\mathbf{x}) - L_M(\|\mathbf{y}\|_{\mathbb{H}_0^1})\mathbf{B}^N(\mathbf{y})\|^2_{\mathbb{L}^2}\\
&= \sum_{i=1}^N |\langle L_M(\|\mathbf{x}\|_{\mathbb{H}_0^1})\mathbf{B}^N(\mathbf{x}) - L_M(\|\mathbf{y}\|_{\mathbb{H}_0^1})\mathbf{B}^N(\mathbf{y}), \mathbf{e}_i\rangle|^2\\
&= \sum_{i=1}^N |\langle L_M(\|\mathbf{y}\|_{\mathbb{H}_0^1})\mathbf{B}^N(\mathbf{x}) - L_M(\|\mathbf{y}\|_{\mathbb{H}_0^1})\mathbf{B}^N(\mathbf{y}) + L_M(\|\mathbf{x}\|_{\mathbb{H}_0^1})\mathbf{B}^N(\mathbf{x}) - L_M(\|\mathbf{y}\|_{\mathbb{H}_0^1})\mathbf{B}^N(\mathbf{x}), \mathbf{e}_i\rangle|^2\\
&\leq 2\sum_{i=1}^N |L_M(\|\mathbf{y}\|_{\mathbb{H}_0^1})\langle \mathbf{B}^N(\mathbf{x}) - \mathbf{B}^N(\mathbf{y}), \mathbf{e}_i\rangle|^2 + 2\sum_{i=1}^N |\langle \mathbf{B}^N(\mathbf{x}), \mathbf{e}_i\rangle|^2 |L_M(\|\mathbf{x}\|_{\mathbb{H}_0^1}) - L_M(\|\mathbf{y}\|_{\mathbb{H}_0^1})|^2\\
&\leq 2\sum_{i=1}^N L_M^2(\|\mathbf{y}\|_{\mathbb{H}_0^1})|\langle \mathbf{B}^N(\mathbf{x}) - \mathbf{B}^N(\mathbf{y}), \mathbf{e}_i\rangle|^2 + 2C_M^2\sum_{i=1}^N |\langle \mathbf{B}^N(\mathbf{x}), \mathbf{e}_i\rangle|^2 \|\mathbf{x}-\mathbf{y}\|^2_{\mathbb{H}_0^1},
\end{aligned}
\tag{4.1}
$$

where C_M is Lipschitz coefficient of L_M. By Lemma 2.6, Lemma 2.9, and the Poincaré inequality, one has

$$
\begin{aligned}
&\sum_{i=1}^N |\langle \mathbf{B}^N(\mathbf{x}) - \mathbf{B}^N(\mathbf{y}), \mathbf{e}_i\rangle|^2\\
&= \sum_{i=1}^N \|\mathbf{B}^N(\mathbf{x}) - \mathbf{B}^N(\mathbf{y})\|^2_{\mathbb{L}^2}\\
&\leq 2C_3^2\Big[\|\mathbf{x}\|^2_{\mathbb{L}^4} + \|\mathbf{y}\|^2_{\mathbb{L}^4}\Big]\|\mathbf{x}-\mathbf{y}\|^2_{\mathbb{L}^4}\\
&\leq 4C_3^2 C_G\Big[\|\mathbf{x}\|^2_{\mathbb{H}_0^1} + \|\mathbf{y}\|^2_{\mathbb{H}_0^1}\Big]\|\mathbf{x}-\mathbf{y}\|^2_{\mathbb{H}_0^1},
\end{aligned}
$$

where C_G is a constant depending on G only. Also

$$
\sum_{i=1}^N |\langle \mathbf{B}^N(\mathbf{x}), \mathbf{e}_i\rangle|^2 \leq C_2^2\|\mathbf{x}\|^2_{\mathbb{L}^4} \leq C_2^2 C_G\|\mathbf{x}\|^2_{\mathbb{H}_0^1}.
$$

Thus (4.1) becomes

$$
\begin{aligned}
&\|L_M(\|\mathbf{x}\|_{\mathbb{H}_0^1})\mathbf{B}^N(\mathbf{x}) - L_M(\|\mathbf{y}\|_{\mathbb{H}_0^1})\mathbf{B}^N(\mathbf{y})\|^2_{\mathbb{L}^2}\\
&\leq \Big\{8C_3^2 C_G L_M^2(\|\mathbf{y}\|_{\mathbb{H}_0^1})\Big[\|\mathbf{x}\|^2_{\mathbb{H}_0^1} + \|\mathbf{y}\|^2_{\mathbb{H}_0^1}\Big] + 2C_M^2 C_2^2 C_G\|\mathbf{x}\|^2_{\mathbb{H}_0^1}\Big\}\|\mathbf{x}-\mathbf{y}\|^2_{\mathbb{H}_0^1}.
\end{aligned}
$$

Since $\|\mathbf{x}\|_{\mathbb{H}_0^1}$ and $\|\mathbf{y}\|_{\mathbb{H}_0^1}$ are all bounded by $M+1$, $\|L_M(\|\mathbf{x}\|_{\mathbb{H}_0^1})\mathbf{B}^N(\mathbf{x}) - L_M(\|\mathbf{y}\|_{\mathbb{H}_0^1})\mathbf{B}^N(\mathbf{y})\|^2_{\mathbb{L}^2} \leq C(N,M)\|\mathbf{x}-\mathbf{y}\|^2_{\mathbb{H}_0^1}$, where $C(N,M)$ is only related to N, M, and G.

Case III. $||\mathbf{y}||_{\mathbb{H}_0^1} > M+1$ and $||\mathbf{x}||_{\mathbb{H}_0^1} \le M+1$.

Then by the definition of L_M, $L_M(||\mathbf{y}||_{\mathbb{H}_0^1}) = 0$. Thus

$$||L_M(||\mathbf{x}||_{\mathbb{H}_0^1})\mathbf{B}^N(\mathbf{x}) - L_M(||\mathbf{y}||_{\mathbb{H}_0^1})\mathbf{B}^N(\mathbf{y})||^2_{\mathbb{L}^2} \le 2C_M^2 C_2^2 C_G ||\mathbf{x}||^2_{\mathbb{H}_0^1} ||\mathbf{x}-\mathbf{y}||^2_{\mathbb{H}_0^1}.$$

Thus we have shown that

$$||L_M(||\mathbf{x}||_{\mathbb{H}_0^1})\mathbf{B}^N(\mathbf{x}) - L_M(||\mathbf{y}||_{\mathbb{H}_0^1})\mathbf{B}^N(\mathbf{y})||_{\mathbb{L}^2} \le C(N,M)||\mathbf{x}-\mathbf{y}||_{\mathbb{H}_0^1},$$

where $C(N,M)$ is a constant which is only related to N, M and G. □

Let us state without proof an useful result from Yong and Zhou [34].

Proposition 4.2. *For any* $(y,z) \in \mathbb{R}^k \times \mathbb{R}^{k\times m}$, *assume that* $h(t,y,z) : [0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times m}\times\Omega \to \mathbb{R}^k$ *is* $\{\mathcal{F}_t\}_{t\ge 0}$*-adapted with* $h(\cdot,0,0) \in L^2_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{R}^k))$. *Moreover, there exists an* $L > 0$, *such that*

$$|h(t,y,z) - h(t,\bar{y},\bar{z})| \le L\{|y-\bar{y}| + |z-\bar{z}|\}$$

$\forall t \in [0,T]$, $y, \bar{y} \in \mathbb{R}^k$ *and* $z, \bar{z} \in \mathbb{R}^{k\times m}$ *P-a.s. For any given* $\xi \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^k)$, *the BSDE*

$$\begin{cases} dY(t) = h(t, Y(t), Z(t))dt + Z(t)dW(t), \quad t\in[0,T),\ a.s. \\ Y(T) = \xi, \end{cases} \tag{4.2}$$

admits a unique adapted solution $(Y(\cdot), Z(\cdot)) \in \mathcal{M}[0,T]$, *where*

$$\mathcal{M}[0,T] = L^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R})) \times L^2_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{R}))$$

and it is equipped with the norm

$$||Y(\cdot), Z(\cdot)||_{\mathcal{M}[0,T]} = \{E(\sup_{0\le t\le T}|Y(t)|^2) + E\int_0^T |Z(t)|^2 dt\}^{\frac{1}{2}}.$$

Now we are able to prove the main result of this section.

Theorem 4.3. *System* (3.1) *admits a unique adapted solution* $(\mathbf{u}^N, \mathbf{Z}^N, \xi^N, Z^N)$ *in*

$$\Big\{L^\infty_{\mathcal{F}}([0,T]\times\Omega;\mathbb{L}^2(G)) \cap L^2_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{H}_0^1(G)))\Big\} \times L^2_{\mathcal{F}}(\Omega; L^2(0,T;L_{\mathbb{Q}}))$$
$$\times\Big\{L^\infty_{\mathcal{F}}([0,T]\times\Omega;L^2(G)) \cap L^2_{\mathcal{F}}(\Omega; L^2(0,T;H_0^1(G)))\Big\} \times L^2_{\mathcal{F}}(\Omega; L^2(0,T;L_Q)),$$

provided that the terminal conditions satisfy $\phi \in L^\infty_{\mathcal{F}_T}(\Omega;\mathbb{L}^2(G))$, $\psi \in L^\infty_{\mathcal{F}_T}(\Omega;L^2(G))$, *and the external force* $\mathbf{f} \in L^2(0,T;\mathbb{H}^{-1}(G))$.

Proof. First of all, for any $M \in \mathbb{R}$, let us define a truncated system as follows:

$$\begin{cases} \frac{\partial \mathbf{u}^{NM}(t)}{\partial t} = -\mathbf{A}\mathbf{u}^{NM}(t) - L_M(||\mathbf{u}^{NM}(t)||_{\mathbb{H}_0^1})\mathbf{B}^N(\mathbf{u}^{NM}(t)) - g\nabla\xi^{NM}(t) + \mathbf{f}^N(t) \\ \qquad\qquad +\mathbf{Z}^{NM}(t)d\mathbb{W}^N(t); \\ \frac{\partial \xi^{NM}(t)}{\partial t} + \nabla\cdot(R^N\mathbf{u}^{NM}(t)) = Z^{NM}(t)dW^N(t); \\ \mathbf{u}^{NM}(T) = \phi^N \text{ and } \xi^{NM}(T) = \psi^N. \end{cases} \tag{4.3}$$

From Lemma 4.1, it is clear that all coefficients of the above system are Lipschitz continuous for fixed N and M. Let us fix $\zeta(t) \in L^\infty_{\mathcal{F}}([0,T]\times\Omega; L^2_N(G)) \cap L^2_{\mathcal{F}}(\Omega; L^2(0,T;H^1_{0N}(G)))$. Consider

$$\begin{cases} \frac{\partial \mathbf{u}^{NM}(t)}{\partial t} = -\mathbf{A}\mathbf{u}^{NM}(t) - L_M(||\mathbf{u}^{NM}(t)||_{\mathbb{H}_0^1})\mathbf{B}^N(\mathbf{u}^{NM}(t)) - g\nabla\zeta(t) + \mathbf{f}^N(t) + \mathbf{Z}^{NM}(t)d\mathbb{W}^N(t); \\ \mathbf{u}^{NM}(T) = \phi^N. \end{cases} \tag{4.4}$$

Let us map system (4.4) to $\mathbb{R}^N$. It is obviously that the image of system is equivalent to system (4.4). Since the coefficients in the image system are Lipschitz, Proposition 4.2 guarantees the existence of a unique adapted solution of the image system. By the equivalence between two systems, we claim that system (4.4) admits a unique adapted solution $(\mathbf{u}^{NM}, \mathbf{Z}^{NM})$. Clearly, for this $\mathbf{u}^{NM}$, the following system

$$\begin{cases} \frac{\partial \xi^{NM}(t)}{\partial t} + \nabla \cdot (R^N \mathbf{u}^{NM}(t)) = Z^{NM}(t) dW^N(t); \\ \xi^{NM}(T) = \psi^N \end{cases} \tag{4.5}$$

admits a unique adapted solution (ξ^{NM}, Z^{NM}). Hence we can define an operator Φ, such that $\Phi(\zeta) \triangleq \xi^{NM}$. We would like to show that Φ is a contraction mapping. For any ζ_1 and $\zeta_2 \in L^\infty_{\mathcal{F}}([0,T]\times\Omega; L^2_N(G)) \cap L^2_{\mathcal{F}}(\Omega; L^2(0,T; H^1_{0N}(G)))$, let $\Phi(\zeta_1) = \xi_1^{NM}$ and $\Phi(\zeta_2) = \xi_2^{NM}$. Denote

$$\hat{\mathbf{u}} \triangleq \mathbf{u}_1^{NM} - \mathbf{u}_2^{NM}, \quad \hat{\xi} \triangleq \xi_1^{NM} - \xi_2^{NM}, \quad \hat{\zeta} \triangleq \zeta_1 - \zeta_2,$$
$$\hat{\mathbf{Z}} \triangleq \mathbf{Z}_1^{NM} - \mathbf{Z}_2^{NM}, \quad \hat{Z} \triangleq Z_1^{NM} - Z_2^{NM}.$$

Similar to the proof of Proposition 4.1, one can verify that

$$\begin{aligned} &|\langle L_M(\|\mathbf{u}_1^{NM}(t)\|_{\mathbb{H}^1_0})\mathbf{B}^N(\mathbf{u}_1^{NM}(t)) - L_M(\|\mathbf{u}_2^{NM}(t)\|_{\mathbb{H}^1_0})\mathbf{B}^N(\mathbf{u}_2^{NM}(t)), \hat{\mathbf{u}}(t)\rangle| \\ \le& C(M, C_G, C_3)\|\hat{\mathbf{u}}(t)\|^2_{\mathbb{H}^1_0}, \end{aligned}$$

where $C(M, C_G, C_3)$ is a constant depending on M, C_G, C_3 only. Let η be a positive number such that

$$\eta > \max(\frac{\rho_N}{2}, \frac{2\kappa_h\lambda_N + C(M, C_G, C_3)\lambda_N + \lambda_N}{2}).$$

Applying the Itô formula to $\|\hat{\xi}(t)\|^2_{L^2} e^{2\eta t}$ to get

$$\begin{aligned} &\|\hat{\xi}(t)\|^2_{L^2} e^{2\eta t} + \int_t^T \|\hat{Z}(s)\|^2_{L_Q} e^{2\eta s} ds \\ =& \int_t^T \Big\{ -2\eta\|\hat{\xi}(s)\|^2_{L^2} + 2\langle \nabla \cdot (R^N \hat{\mathbf{u}}(s)), \hat{\xi}(s)\rangle \Big\} e^{2\eta s} ds - \int_t^T 2e^{2\eta s}\langle \hat{Z}(s) dW^N(s), \hat{\xi}(s)\rangle \\ \le& \int_t^T \Big\{ -2\eta\|\hat{\xi}(s)\|^2_{L^2} + 2C_1(1+C_G)\|\hat{\mathbf{u}}(s)\|_{\mathbb{H}^1_0}\|\hat{\xi}(s)\|_{L^2} \Big\} e^{2\eta s} ds \\ &- \int_t^T 2e^{2\eta s}\langle \hat{Z}(s) dW^N(s), \hat{\xi}(s)\rangle, \end{aligned}$$

where the estimates are obtained similar to (3.9). Thus for any $0 \le r \le t$,

$$\begin{aligned} &E^{\mathcal{F}_r}\|\hat{\xi}(t)\|^2_{L^2} e^{2\eta t} + E^{\mathcal{F}_r}\int_t^T \|\hat{\xi}(s)\|^2_{H^1_0} e^{2\eta s} ds + E^{\mathcal{F}_r}\int_t^T \|\hat{Z}(s)\|^2_{L_Q} e^{2\eta s} ds \\ \le& E^{\mathcal{F}_r}\int_t^T \Big\{ -2\eta\|\hat{\xi}(s)\|^2_{L^2} + \|\hat{\xi}(s)\|^2_{H^1_0} + 2C_1(1+C_G)\|\hat{\mathbf{u}}(s)\|_{\mathbb{H}^1_0}\|\hat{\xi}(s)\|_{L^2} \Big\} e^{2\eta s} ds \\ \le& E^{\mathcal{F}_r}\int_t^T \Big\{ (-2\eta + \rho_N)\|\hat{\xi}(s)\|^2_{L^2} + 2C_1(1+C_G)\|\hat{\mathbf{u}}(s)\|_{\mathbb{H}^1_0}\|\hat{\xi}(s)\|_{L^2} \Big\} e^{2\eta s} ds \\ \le& E^{\mathcal{F}_r}\int_t^T e^{2\eta s}\frac{C_1^2(1+C_G)^2}{2\eta - \rho_N}\|\hat{\mathbf{u}}(s)\|^2_{\mathbb{H}^1_0} ds, \end{aligned} \tag{4.6}$$

P-a.s., where ρ_N is defined in Remark 2.4. An application of the Itô formula to $\|\hat{\mathbf{u}}(t)\|^2_{\mathbb{L}^2}e^{2\eta t}$ yields

$$\begin{aligned}
&\|\hat{\mathbf{u}}(t)\|^2_{\mathbb{L}^2}e^{2\eta t}+\int_t^T\|\hat{\mathbf{Z}}(s)\|^2_{L_{\mathbb{Q}}}e^{2\eta s}ds\\
&=\int_t^T\Big\{-2\eta\|\hat{\mathbf{u}}(s)\|^2_{\mathbb{L}^2}+2\langle\mathbf{A}\hat{\mathbf{u}}(s)+L_M(\|\mathbf{u}_1^{NM}(s)\|_{\mathbb{H}^1_0})\mathbf{B}^N(\mathbf{u}_1^{NM}(s))\\
&\qquad-L_M(\|\mathbf{u}_2^{NM}(s)\|_{\mathbb{H}^1_0})\mathbf{B}^N(\mathbf{u}_2^{NM}(s))+g\nabla\hat{\zeta}(s),\hat{\mathbf{u}}(s)\rangle\Big\}e^{2\eta s}ds\\
&\quad-\int_t^T2\langle\hat{\mathbf{Z}}(s)d\mathbb{W}^N(s),\hat{\mathbf{u}}(s)\rangle e^{2\eta s}\\
&\le\int_t^T\Big\{\Big(-2\eta+2\kappa_h\lambda_N+C(M,C_G,C_3)\lambda_N\Big)\|\hat{\mathbf{u}}(s)\|^2_{\mathbb{L}^2}+2g\sqrt{\lambda_N}\|\hat{\zeta}(s)\|_{L^2}\|\hat{\mathbf{u}}(s)\|_{\mathbb{L}^2}\Big\}e^{2\eta s}ds\\
&\quad-\int_t^T2\langle\hat{\mathbf{Z}}(s)d\mathbb{W}^N(s),\hat{\mathbf{u}}(s)\rangle e^{2\eta s}.
\end{aligned}$$

Thus for any $0\le r\le t$,

$$\begin{aligned}
&E^{\mathcal{F}_r}\|\hat{\mathbf{u}}(t)\|^2_{\mathbb{L}^2}e^{2\eta t}+E^{\mathcal{F}_r}\int_t^T\|\hat{\mathbf{u}}(s)\|^2_{\mathbb{H}^1_0}e^{2\eta s}ds+E^{\mathcal{F}_r}\int_t^T\|\hat{\mathbf{Z}}(s)\|^2_{L_{\mathbb{Q}}}e^{2\eta s}ds\\
&\le E^{\mathcal{F}_r}\int_t^T\Big\{\Big(-2\eta+2\kappa_h\lambda_N+C(M,C_G,C_3)\lambda_N+\lambda_N\Big)\|\hat{\mathbf{u}}(s)\|^2_{\mathbb{L}^2}\\
&\qquad+2g\sqrt{\lambda_N}\|\hat{\zeta}(s)\|_{L^2}\|\hat{\mathbf{u}}(s)\|_{\mathbb{L}^2}\Big\}e^{2\eta s}ds\\
&\le E^{\mathcal{F}_r}\int_t^T e^{2\eta s}\frac{g^2\lambda_N}{2\eta-2\kappa_h\lambda_N-C(M,C_G,C_3)\lambda_N-\lambda_N}\|\hat{\zeta}(s)\|^2_{L^2}ds,
\end{aligned}\tag{4.7}$$

P-a.s. Equations (4.6) and (4.7) imply

$$\begin{aligned}
&E^{\mathcal{F}_r}\|\hat{\xi}(t)\|^2_{L^2}e^{2\eta t}+E^{\mathcal{F}_r}\int_t^T\|\hat{\xi}(s)\|^2_{H^1_0}e^{2\eta s}ds+E^{\mathcal{F}_r}\int_t^T\|\hat{Z}(s)\|^2_{L_Q}e^{2\eta s}ds\\
&\le\frac{C_1^2(1+C_G)^2}{2\eta-\rho_N}\frac{g^2\lambda_N}{2\eta-2\kappa_h\lambda_N-C(M,C_G,C_3)\lambda_N-\lambda_N}E^{\mathcal{F}_r}\int_t^T e^{2\eta s}\|\hat{\zeta}(s)\|^2_{L^2}ds,
\end{aligned}$$

P-a.s. Hence we take η to be large enough such that

$$E^{\mathcal{F}_r}\int_t^T\|\hat{\xi}(s)\|^2_{H^1_0}e^{2\eta s}ds\le\frac{1}{2}E^{\mathcal{F}_r}\int_t^T e^{2\eta s}\|\hat{\zeta}(s)\|^2_{H^1_0}ds,$$

P-a.s. Taking the expectation and letting $t=0$, we see that Φ is a contraction mapping from $L^2_{\mathcal{F}}(\Omega;L^2(0,T;H^1_{0N}(G)))$ to $L^2_{\mathcal{F}}(\Omega;L^2(0,T;H^1_{0N}(G)))$. By the contraction mapping theorem, a unique adapted solution $(\mathbf{u}^{NM},\mathbf{Z}^{NM},\xi^{NM},Z^{NM})$ of (4.3) is guaranteed. As shown in Proposition 3.1, $\sup_{t\in[0,T]}\|\mathbf{u}^N(t)\|^2_{\mathbb{L}^2}\le K(N)$, where $K(N)$ is a constant associated with N only. Since for finite-dimensional spaces, the norms $\|\cdot\|_{\mathbb{L}^2}$ and $\|\cdot\|_{\mathbb{H}^1_0}$ are equivalent, we know that $\|\mathbf{u}^N(s)\|_{\mathbb{H}^1_0}$ is also uniformly bounded for every N. By the definition of the truncation L_M, it is clear that (3.1) and (4.3) are equivalent when M is large enough. Thus letting M approach infinity, the limit of the solution $(\mathbf{u}^{NM},\mathbf{Z}^{NM},\xi^{NM},Z^{NM})$ is the unique adapted solution of the projected system (3.1). The regularity of the solution can be obtained by Proposition 3.1. ◻

Continuity of the solution to the projected system (3.1) can also be obtained along similar lines of the proof of Theorem 6.1. We shall skip the proof and postpone it to Section 6. Thus the well-posedness of the projected system has been fully investigated.

5. Existence

In this section, we are going to show the existence of an adapted solution of system (2.3). The Galerkin approximation scheme and Minty-Browder technique will be employed. In order to assure an uniform bound on a priori estimates, we make the following assumptions. Such an approach is commonly taken in the study of stochastic Euler equations by several authors so that a dissipative effect arises. Also they are standard hypotheses in the theory of stochastic PDEs in infinite dimensional spaces (see Chow [5], Kallianpur and Xiong [10], Prévôt and Röckner [29]).

(A.1) (Continuity): $\mathbf{f}$: $\mathbb{H}_0^1 \to \mathbb{H}^{-1}$ is a continuous operator;

(A.2) (Coercivity): There exist positive constants α and β, such that

$$\langle \mathbf{Au} - \mathbf{f}(\mathbf{u}), \mathbf{u}\rangle \le \alpha \|\mathbf{u}\|_{\mathbb{L}^2}^2 - \beta \|\mathbf{u}\|_{\mathbb{H}_0^1}^2;$$

$$\langle \mathbf{Au} - \mathbf{f}(\mathbf{u}), \mathbf{Au}\rangle \le \alpha \|\mathbf{u}\|_{\mathbb{H}_0^1}^2 - \beta \|\mathbf{Au}\|_{\mathbb{H}_0^1}^2;$$

(A.3) (Monotonicity): There exist $\nu > 1$ and $\alpha > 0$, such that for any $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{H}_0^1$, and $M \in \mathbb{N}$,

$$\langle \nu \mathbf{A}(\mathbf{u} - \mathbf{v}) - (\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{v})), R^M(\mathbf{u} - \mathbf{v})\rangle \le \alpha \|\mathbf{u} - \mathbf{v}\|_{\mathbb{L}^2}^2,$$

where R^M is the projection of R into $\mathbb{L}^2{}_M(G)$;

(A.4) (Linear growth): For any $\mathbf{u} \in \mathbb{H}_0^1$ and some positive constant α,

$$|\langle \mathbf{f}(\mathbf{u}), \mathbf{u}\rangle| \le \alpha \|\mathbf{u}\|^2.$$

The system (2.3) can now be written as

$$\begin{cases} \frac{\partial \mathbf{u}(t)}{\partial t} = -\mathbf{Au}(t) - \gamma|\mathbf{u}(t) + \mathbf{w}^0(t)|(\mathbf{u}(t) + \mathbf{w}^0(t)) - g\nabla\xi(t) + \mathbf{f}(\mathbf{u}(t)) + \mathbf{Z}(t)\frac{d\mathbb{W}(t)}{dt}; \\ \frac{\partial \xi(t)}{\partial t} + \nabla\cdot(R\mathbf{u}(t)) = Z(t)\frac{dW(t)}{dt}; \\ \mathbf{u}(T) = \phi \text{ and } \xi(T) = \psi, \end{cases} \tag{5.1}$$

and the corresponding projected system is

$$\begin{cases} \frac{\partial \mathbf{u}^N(t)}{\partial t} = -\mathbf{Au}^N(t) - \mathbf{B}^N(\mathbf{u}^N(t)) - g\nabla\xi^N(t) + \mathbf{f}^N(\mathbf{u}^N(t)) + \mathbf{Z}^N(t)d\mathbb{W}^N(t); \\ \frac{\partial \xi^N(t)}{\partial t} + \nabla\cdot(R^N\mathbf{u}^N(t)) = Z^N(t)dW^N(t); \\ \mathbf{u}^N(T) = \phi^N \text{ and } \xi^N(T) = \psi^N. \end{cases} \tag{5.2}$$

Under these assumptions, we are able to prove a very important monotonicity result, which is the essence of proof of the existence theorem.

Lemma 5.1. *For any* $\mathbf{u}, \mathbf{v} \in L^{\frac{4}{3}}([0,T]; \mathbb{L}^4(G)) \cap L^0(0,T; \mathbb{H}_0^1(G))$, *and* $M \in \mathbb{N}$, *define*

$$r(t) \triangleq \int_t^T \left\{2\alpha + \frac{3}{2^{\frac{5}{3}}}\Big(\frac{C_1^4 C_5^4}{(\nu-1)\kappa_h C_0}\Big)^{\frac{1}{3}}\left\{\|\mathbf{u}(s)\|_{\mathbb{L}^4} + \|\mathbf{v}(s)\|_{\mathbb{L}^4}\right\}^{\frac{4}{3}}\right\}ds.$$

Then

$$\langle \mathbf{A}(\mathbf{u} - \mathbf{v}) + \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}) - (\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{v})) + \frac{1}{2}\dot{r}(t)(\mathbf{u} - \mathbf{v}), R^M(\mathbf{u} - \mathbf{v})\rangle \le 0.$$

Proof. From the monotonicity assumption and Lemma 2.9, one has

$$\begin{aligned}
&\langle \mathbf{A}(\mathbf{u}-\mathbf{v})+\mathbf{B}(\mathbf{u})-\mathbf{B}(\mathbf{v})-(\mathbf{f}(\mathbf{u})-\mathbf{f}(\mathbf{v})), R^M(\mathbf{u}-\mathbf{v})\rangle \\
=&\langle \nu\mathbf{A}(\mathbf{u}-\mathbf{v})-(\mathbf{f}(\mathbf{u})-\mathbf{f}(\mathbf{v})), R^M(\mathbf{u}-\mathbf{v})\rangle+\langle \mathbf{B}(\mathbf{u})-\mathbf{B}(\mathbf{v}), R^M(\mathbf{u}-\mathbf{v})\rangle \\
&+\langle (1-\nu)\mathbf{A}(\mathbf{u}-\mathbf{v}), R^M(\mathbf{u}-\mathbf{v})\rangle \\
\leq&\alpha\|\mathbf{u}-\mathbf{v}\|^2_{\mathbb{L}^2}+C_1C_5\Big\{\|\mathbf{u}\|_{\mathbb{L}^4}+\|\mathbf{v}\|_{\mathbb{L}^4}\Big\}\|\mathbf{u}-\mathbf{v}\|^{\frac{3}{2}}_{\mathbb{L}^2}\|\mathbf{u}-\mathbf{v}\|^{\frac{1}{2}}_{\mathbb{H}^1_0}+(1-\nu)\kappa_h C_0\|\mathbf{u}-\mathbf{v}\|^2_{\mathbb{H}^1_0} \\
\leq&\alpha\|\mathbf{u}-\mathbf{v}\|^2_{\mathbb{L}^2}+\frac{3}{2^{\frac{8}{3}}}\Big(\frac{C_1^4C_5^4}{(\nu-1)\kappa_h C_0}\Big)^{\frac{1}{3}}\Big\{\|\mathbf{u}\|_{\mathbb{L}^4}+\|\mathbf{v}\|_{\mathbb{L}^4}\Big\}^{\frac{4}{3}}\|\mathbf{u}-\mathbf{v}\|^2_{\mathbb{L}^2} \\
=&-\frac{1}{2}\dot{r}(t)\|\mathbf{u}-\mathbf{v}\|^2_{\mathbb{L}^2},
\end{aligned}$$

which completes the proof. □

The coercivity assumption assures a uniform a priori estimate. The following a priori estimate is very useful for the Galerkin approximation which will be used in Theorem 5.3.

Proposition 5.2. *(i) Suppose that the terminal conditions satisfy $\phi \in L^\infty_{\mathcal{F}_T}(\Omega;\mathbb{L}^2(G))$ and $\psi \in L^\infty_{\mathcal{F}_T}(\Omega;L^2(G))$. Then for any solution of system* (5.2)*, the following is true:*

$$\begin{aligned}
&(\mathbf{u}^N,\mathbf{Z}^N)\in\Big\{L^\infty_{\mathcal{F}}([0,T]\times\Omega;\mathbb{L}^2(G))\cap L^2_{\mathcal{F}}(\Omega;L^2(0,T;\mathbb{H}^1_0(G)))\Big\}\times L^2_{\mathcal{F}}(\Omega;L^2(0,T;L_{\mathbb{Q}})),\\
&(\xi^N,Z^N)\in L^\infty_{\mathcal{F}}([0,T]\times\Omega;L^2(G))\times L^2_{\mathcal{F}}(\Omega;L^2(0,T;L_Q)).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\sup_{t\in[0,T]}\|\mathbf{u}^N(t)\|^2_{\mathbb{L}^2}+E\int_0^T\|\mathbf{u}^N(s)\|^2_{\mathbb{H}^1_0}ds+\sup_{t\in[0,T]}\|\xi^N(t)\|^2_{L^2}\\
&\quad+E\int_0^T\|\mathbf{Z}^N(s)\|^2_{L_{\mathbb{Q}}}ds+E\int_0^T\|Z^N(s)\|^2_{L_Q}ds\leq K, \qquad (5.3)
\end{aligned}$$

P-a.s. for some constant K, independent of N.

(ii) Suppose the terminal conditions satisfy $\phi \in L^n_{\mathcal{F}_T}(\Omega;\mathbb{L}^2(G))$ and $\psi \in L^n_{\mathcal{F}_T}(\Omega;L^2(G))$ for all $n\in\mathbb{N}$ and $n\geq 2$. The following is true for any solution of system (5.2)*:*

$$\begin{aligned}
&(\mathbf{u}^N,\mathbf{Z}^N)\in\Big\{L^\infty(0,T;L^n_{\mathcal{F}}(\Omega;\mathbb{L}^2(G)))\cap L^2_{\mathcal{F}}(\Omega;L^2(0,T;\mathbb{H}^1_0(G)))\Big\}\times L^2_{\mathcal{F}}(\Omega;L^2(0,T;L_{\mathbb{Q}})),\\
&(\xi^N,Z^N)\in L^\infty(0,T;L^n_{\mathcal{F}}(\Omega;L^2(G)))\times L^2_{\mathcal{F}}(\Omega;L^2(0,T;L_Q)).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\sup_{t\in[0,T]}E\|\mathbf{u}^N(t)\|^n_{\mathbb{L}^2}+E\int_0^T\|\mathbf{u}^N(s)\|^{n-2}_{\mathbb{L}^2}\|\mathbf{u}^N(s)\|^2_{\mathbb{H}^1_0}ds+\sup_{t\in[0,T]}E\|\xi^N(t)\|^n_{L^2}\\
&\quad+E\int_0^T\|\mathbf{Z}^N(s)\|^n_{L_{\mathbb{Q}}}ds+E\int_0^T\|Z^N(s)\|^n_{L_Q}ds\leq K, \qquad (5.4)
\end{aligned}$$

for some constant K, independent of N.

(iii) Let the terminal conditions satisfy $\phi \in L^\infty_{\mathcal{F}_T}(\Omega;\mathbb{H}^1_0(G))$ and $\psi \in L^\infty_{\mathcal{F}_T}(\Omega;H^1_0(G))$. Then for any solution of system (5.2)*, the following is true:*

$$\begin{aligned}
&(\mathbf{u}^N,\mathbf{Z}^N)\in L^\infty_{\mathcal{F}}([0,T]\times\Omega;\mathbb{H}^1_0(G))\times L^2_{\mathcal{F}}(\Omega;L^2(0,T;L_{\mathbb{Q}})),\\
&(\xi^N,Z^N)\in L^\infty_{\mathcal{F}}([0,T]\times\Omega;H^1_0(G))\times L^2_{\mathcal{F}}(\Omega;L^2(0,T;L_Q)).
\end{aligned}$$

Moreover,

$$\sup_{t\in[0,T]} \|\mathbf{u}^N(t)\|^2_{\mathbb{H}^1_0} + \sup_{t\in[0,T]} \|\xi^N(t)\|^2_{H^1_0} + E\int_0^T \|\mathbf{Z}^N(s)\|^2_{L_{\mathbb{Q}}} ds + E\int_0^T \|Z^N(s)\|^2_{L_Q} ds \leq K, \tag{5.5}$$

P-a.s. for some constant K, independent of N.

Proof. The proof is very similar to Proposition 3.1, with few modifications of some estimates. It is clear that

$$2\langle \mathbf{B}^N(\mathbf{u}^N(s)), \mathbf{u}^N(s)\rangle \leq \sqrt[3]{\frac{27C_3^4}{2\beta}} \|\mathbf{u}^N(s)\|^2_{\mathbb{L}^2} + \frac{\beta}{4}\|\mathbf{u}^N(s)\|^2_{\mathbb{H}^1_0},$$

and

$$2\langle g\nabla\xi^N(s), \mathbf{u}^N(s)\rangle \leq \frac{4g^2}{\beta}\|\xi^N(s)\|^2_{L^2} + \frac{\beta}{4}\|\mathbf{u}^N(s)\|^2_{\mathbb{H}^1_0}.$$

Thus under part one of Assumption (A.2), we have

$$\begin{aligned}
&\|\mathbf{u}^N(t)\|^2_{\mathbb{L}^2} + \int_t^T \|\mathbf{Z}^N(s)\|^2_{L_{\mathbb{Q}}} ds + \frac{\beta}{2}\int_t^T \|\mathbf{u}^N(s)\|^2_{\mathbb{H}^1_0} ds \\
\leq &\|\phi^N\|^2_{\mathbb{L}^2} + \int_t^T \left\{\left(\alpha + \sqrt[3]{\frac{27C_3^4}{2\beta}}\right)\|\mathbf{u}^N(s)\|^2_{\mathbb{L}^2} + \frac{4g^2}{\beta}\|\xi^N(s)\|^2_{L^2}\right\} ds \\
&- 2\int_t^T \langle \mathbf{Z}^N(s) d\mathbb{W}^N(s), \mathbf{u}^N(s)\rangle.
\end{aligned}$$

Since

$$\begin{aligned}
&2\langle \nabla\cdot(R^N\mathbf{u}^N(s)), \xi^N(s)\rangle \\
\leq &C_1\|\mathbf{u}^N(s)\|^2_{\mathbb{L}^2} + \frac{\beta}{4}\|\mathbf{u}^N(s)\|^2_{\mathbb{H}^1_0} + \left(\frac{4C_1^2}{\beta} + C_1\right)\|\xi^N(s)\|^2_{L^2},
\end{aligned}$$

we have

$$\begin{aligned}
&E^{\mathcal{F}_r}\|\mathbf{u}^N(t)\|^2_{\mathbb{L}^2} + E^{\mathcal{F}_r}\|\xi^N(t)\|^2_{L^2} + \frac{\beta}{4}\int_t^T \|\mathbf{u}^N(s)\|^2_{\mathbb{H}^1_0} ds \\
&\quad + E^{\mathcal{F}_r}\int_t^T \|\mathbf{Z}^N(s)\|^2_{L_{\mathbb{Q}}} ds + E^{\mathcal{F}_r}\int_t^T \|Z^N(s)\|^2_{L_Q} ds \\
\leq &E^{\mathcal{F}_r}\|\phi^N\|^2_{\mathbb{L}^2} + E^{\mathcal{F}_r}\|\psi^N\|^2_{L^2} \\
&\quad + E^{\mathcal{F}_r}\int_t^T \left\{\left(\alpha + \sqrt[3]{\frac{27C_3^4}{2\beta}} + C_1\right)\|\mathbf{u}^N(s)\|^2_{\mathbb{L}^2} + \left(\frac{4g^2}{\beta} + \frac{4C_1^2}{\beta} + C_1\right)\|\xi^N(s)\|^2_{L^2}\right\} ds,
\end{aligned}$$

for $0 \leq r \leq t$, P-a.s. An application of the Gronwall inequality and letting $r = t$ completes the proof.

We skip the proof of part (ii) and (iii) since they are very similar to part (i) and the proof of Proposition 3.2. Note that the proof of (iii) uses the second half of the coercivity assumption. ◻

Under our assumptions, the well-posedness of system (5.2) can be obtained similarly to Theorem 4.3. We shall skip the proof. Now we are ready to present the main result of this paper.

Theorem 5.3. *Suppose that the terminal conditions satisfy* $\phi \in L^{\infty}_{\mathcal{F}_T}(\Omega; \mathbb{H}^1_0(G))$ *and* $\psi \in L^{\infty}_{\mathcal{F}_T}(\Omega; H^1_0(G))$. *Then there exists an adapted solution* $(\mathbf{u}, \mathbf{Z}, \xi, Z)$ *of system* (5.1), *such that*

$$(\mathbf{u}, \mathbf{Z}) \in L^{\infty}_{\mathcal{F}}([0,T]\times\Omega; \mathbb{H}^1_0(G)) \times L^2_{\mathcal{F}}(\Omega; L^2(0,T; L_{\mathbb{Q}})),$$
$$(\xi, Z) \in L^{\infty}_{\mathcal{F}}([0,T]\times\Omega; H^1_0(G)) \times L^2_{\mathcal{F}}(\Omega; L^2(0,T; L_{Q})).$$

Proof. For technical reasons, let us introduce a new system. For any $M_1 \in \mathbb{N}$, $M_1 \le N$, let R^{M_1} be the projection of R to $\mathbb{L}^2{}_{M_1}(G)$. Then clearly previous results on projected system (5.2) hold for

$$\begin{cases} \frac{\partial \mathbf{u}^{NM_1}(t)}{\partial t} = -\mathbf{A}\mathbf{u}^{NM_1}(t) - \mathbf{B}^N(\mathbf{u}^{NM_1}(t)) - g\nabla\xi^{NM_1}(t) + \mathbf{f}^N(\mathbf{u}^{NM_1}(t)) + \mathbf{Z}^{NM_1}(t)d\mathbb{W}^N(t); \\ \frac{\partial \xi^{NM_1}(t)}{\partial t} + \nabla\cdot(R^{M_1}\mathbf{u}^{NM_1}(t)) = Z^{NM_1}(t)dW^N(t); \\ \mathbf{u}^{NM_1}(T) = \phi^N \text{ and } \xi^{NM_1}(T) = \psi^N. \end{cases}$$

Let the unique adapted solution be $(\mathbf{u}^{NM_1}, \mathbf{Z}^{NM_1}, \xi^{NM_1}, Z^{NM_1})$. First of all, let us establish several limits of convergent sequences. They are necessary when we perform the Galerkin approximation scheme. By Proposition 5.2, $\{\mathbf{u}^{NM_1}\}_{N=1}^{\infty}$, $\{\xi^{NM_1}\}_{N=1}^{\infty}$, $\{\mathbf{Z}^{NM_1}\}_{N=1}^{\infty}$ and $\{Z^{NM_1}\}_{N=1}^{\infty}$ are all uniformly bounded in respective spaces. Thus there exist $\mathbf{u}$, ξ, $\mathbf{Z}$, and Z, and a subsequence N_k, such that

$$\mathbf{u}^{N_kM_1} \xrightarrow{w} \mathbf{u} \qquad \text{in } L^2_{\mathcal{F}}(\Omega; L^2(0,T; \mathbb{H}^1_0(G))),$$
$$\xi^{N_kM_1} \xrightarrow{w} \xi \qquad \text{in } L^{\infty}_{\mathcal{F}}([0,T]\times\Omega; L^2(G)),$$
$$\mathbf{Z}^{N_kM_1} \xrightarrow{w} \mathbf{Z} \qquad \text{in } L^2_{\mathcal{F}}(\Omega; L^2(0,T; L_{\mathbb{Q}})),$$
$$Z^{N_kM_1} \xrightarrow{w} Z \qquad \text{in } L^2_{\mathcal{F}}(\Omega; L^2(0,T; L_{Q})).$$

Since $\mathbf{A}$ is a continuous mapping from $\mathbb{H}^1_0(G)$ to $\mathbb{H}^{-1}(G)$, we know that

$$||\mathbf{A}\mathbf{u}||_{\mathbb{H}^{-1}} \le C||\mathbf{u}||_{\mathbb{H}^1_0},$$

for all $\mathbf{u} \in \mathbb{H}^1_0(G)$ and some constant C. Thus combined with the assumptions on $\mathbf{f}$, one gets

$$\mathbf{A}\mathbf{u}^{N_kM_1} - \mathbf{f}^{N_k}(\mathbf{u}^{N_kM_1}) \xrightarrow{w} \mathbf{F}_1 \qquad \text{in } L^2_{\mathcal{F}}(\Omega; L^2(0,T; \mathbb{H}^{-1}(G))),$$

for some function $\mathbf{F}_1$ and some subsequence N_k. By Lemma 2.9,

$$||\mathbf{B}^N(\mathbf{u}^{NM_1}(t))||_{\mathbb{H}^{-1}} \le C_G||\mathbf{B}^N(\mathbf{u}^{NM_1}(t))||_{\mathbb{L}^2} \le C_GC_3||\mathbf{u}^{NM_1}(t)||_{\mathbb{L}^4} \le 2^{\frac{1}{4}}C_G^{\frac{3}{2}}C_3||\mathbf{u}^{NM_1}(t)||_{\mathbb{H}^1_0}.$$

Thus

$$\mathbf{B}^{N_k}(\mathbf{u}^{N_kM_1}) \xrightarrow{w} \mathbf{F}_2 \qquad \text{in } L^2_{\mathcal{F}}(\Omega; L^2(0,T; \mathbb{H}^{-1}(G))),$$

for some function $\mathbf{F}_2$ and some subsequence N_k. For every t, let us define

$$\mathcal{L}_t : L^2_{\mathcal{F}}(\Omega; L^2(0,T; L_{\mathbb{Q}})) \to L^2_{\mathcal{F}}(\Omega; L^2(0,T; \mathbb{H}^{-1}(G)))$$
$$J \mapsto \int_t^T J(s)d\mathbb{W}(s).$$

Clearly $\mathcal{L}_t$ is a bounded linear operator. Hence it maps weakly convergent sequences to weakly convergent sequences, and

$$\int_t^T \mathbf{Z}^{N_k M_1}(s)d\mathbb{W}^{N_k}(s) \xrightarrow{w} \int_t^T \mathbf{Z}(s)d\mathbb{W}(s) \qquad \text{in } L^2_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{H}^{-1}(G))).$$

Similarly, one can prove that

$$\int_t^T \Big\{\mathbf{A}\mathbf{u}^{N_k M_1}(s) - \mathbf{f}^{N_k}(\mathbf{u}^{N_k M_1}(s)) + \mathbf{B}^{N_k}(\mathbf{u}^{N_k M_1}(s))\Big\}ds \xrightarrow{w} \int_t^T \Big\{\mathbf{F}_1(s) + \mathbf{F}_2(s)\Big\}ds,$$

in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{H}^{-1}(G)))$ and

$$\int_t^T Z^{N_k M_1}(s)dW^{N_k}(s) \xrightarrow{w} \int_t^T Z(s)dW(s) \qquad \text{in } L^2_{\mathcal{F}}(\Omega; L^2(0,T;H^{-1}(G))).$$

One can also show that

$$\begin{aligned} \mathcal{L}_\xi : L^\infty_{\mathcal{F}}([0,T]\times\Omega; L^2(G)) &\to L^2_{\mathcal{F}}(\Omega; L^2(0,T;H^{-1}(G))) \\ \xi &\mapsto \int_t^T \nabla\xi(s)ds \end{aligned}$$

is a bounded linear operator. Since $\xi^{N_k M_1} \in L^\infty_{\mathcal{F}}([0,T]\times\Omega; L^2(G))$, we have

$$\int_t^T \nabla\xi^{N_k M_1}(s)ds \xrightarrow{w} \int_t^T \nabla\xi(s)ds \qquad \text{in } L^2_{\mathcal{F}}(\Omega; L^2(0,T;H^{-1}(G))).$$

Likewise, we have

$$\int_t^T \nabla\cdot(R^{M_1}\mathbf{u}^{N_k M_1}(s))ds \xrightarrow{w} \int_t^T \nabla\cdot(R^{M_1}\mathbf{u}(s))ds \qquad \text{in } L^2_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{H}^{-1}(G))).$$

Thus we have shown that

$$\mathbf{u}(t) = \phi + \int_t^T \Big\{\mathbf{F}_1(s) + \mathbf{F}_2(s) + g\nabla\xi(s)\Big\}ds - \int_t^T \mathbf{Z}(t)d\mathbb{W}(s), \tag{5.6}$$

and

$$\xi(t) = \psi + \int_t^T \nabla\cdot(R^{M_1}\mathbf{u}(s))ds - \int_t^T Z(s)dW(s) \tag{5.7}$$

hold P-a.s. For notational convenience, let us denote N_k by N again. For any $M_2 \le N$, let $\mathbf{v} \in L^\infty_{\mathcal{F}}([0,T]\times\Omega; \mathbb{H}^1_{0M_2}(G))$. Define

$$r(t) \triangleq \int_t^T \Big\{2\alpha + \frac{3}{2^{\frac{5}{3}}}\Big(\frac{C_1^4 C_5^4}{(\nu-1)\kappa_h C_0}\Big)^{\frac{1}{3}} K^{\frac{4}{3}}\Big\}ds,$$

where

$$K = \sup\Big\{\Big\{\sup_{(t,\omega)\in[0,T]\times\Omega} \|\mathbf{u}\|_{\mathbb{L}^4}\Big\} \cup \Big\{\sup_{(t,\omega)\in[0,T]\times\Omega} \|\mathbf{u}^{NM_1}\|_{\mathbb{L}^4}\Big\}_{N=1}^\infty\Big\} + \sup_{(t,\omega)\in[0,T]\times\Omega} \|\mathbf{v}\|_{\mathbb{L}^4}.$$

By Lemma 5.1, it is easy to see that

$$\begin{aligned} &\langle \mathbf{A}\mathbf{u}^{NM_1}(t) + \mathbf{B}^N(\mathbf{u}^{NM_1}(t)) - \mathbf{f}^N(\mathbf{u}^{NM_1}(t)) + \frac{1}{2}\dot{r}(t)\mathbf{u}^{NM_1}(t) \\ &- \mathbf{A}\mathbf{v}(t) - \mathbf{B}^N(\mathbf{v}(t)) + \mathbf{f}^N(\mathbf{v}(t)) - \frac{1}{2}\dot{r}(t)\mathbf{v}(t), R^{M_1}\mathbf{u}^{NM_1}(t) - R^{M_1}\mathbf{v}(t)\rangle \le 0. \end{aligned}$$

Integrating both sides and taking the expectation, one gets

$$
\begin{aligned}
&E\int_0^T e^{-r(s)}\langle \mathbf{A}\mathbf{u}^{NM_1}(s)+\mathbf{B}^N(\mathbf{u}^{NM_1}(s))-\mathbf{f}^N(\mathbf{u}^{NM_1}(s))+\frac{1}{2}\dot{r}(s)\mathbf{u}^{NM_1}(s),\\
&\qquad\qquad R^{M_1}\mathbf{u}^{NM_1}(s)-R^{M_1}\mathbf{v}(s)\rangle ds\\
\le& E\int_0^T e^{-r(s)}\langle \mathbf{A}\mathbf{v}(s)+\mathbf{B}^N(\mathbf{v}(s))-\mathbf{f}^N(\mathbf{v}(s))+\frac{1}{2}\dot{r}(s)\mathbf{v}(s),R^{M_1}\mathbf{u}^{NM_1}(s)-R^{M_1}\mathbf{v}(s)\rangle ds.
\end{aligned}
\tag{5.8}
$$

An application of the Itô formula to $e^{-r(s)}\|\sqrt{R^{M_1}}\mathbf{u}^{NM_1}(s)\|^2_{\mathbb{L}^2}$ yields

$$
\begin{aligned}
&E\|\sqrt{R^{M_1}}\phi^N\|^2_{\mathbb{L}^2}-Ee^{-r(0)}\|\sqrt{R^{M_1}}\mathbf{u}^{NM_1}(0)\|^2_{\mathbb{L}^2}+2E\int_0^T e^{-r(s)}\langle g\nabla\xi^{NM_1}(s),R^{M_1}\mathbf{u}^{NM_1}(s)\rangle ds\\
&\quad -E\int_0^T e^{-r(s)}\|\sqrt{R^{M_1}}\mathbf{Z}^{NM_1}(s)\|^2_{L_{\mathbb{Q}}}ds\\
&=-E\int_0^T e^{-r(s)}\dot{r}(s)\|\sqrt{R^{M_1}}\mathbf{u}^{NM_1}(s)\|^2_{\mathbb{L}^2}ds\\
&\quad -2E\int_0^T e^{-r(s)}\langle \mathbf{A}\mathbf{u}^{NM_1}(s)+\mathbf{B}^N(\mathbf{u}^{NM_1}(s))-\mathbf{f}^N(\mathbf{u}^{NM_1}(s)),R^{M_1}\mathbf{u}^{NM_1}(s)\rangle ds\\
&=-2E\int_0^T e^{-r(s)}\langle \mathbf{A}\mathbf{u}^{NM_1}(s)+\mathbf{B}^N(\mathbf{u}^{NM_1}(s))-\mathbf{f}^N(\mathbf{u}^{NM_1}(s))+\frac{1}{2}\dot{r}(s)\mathbf{u}^{NM_1}(s),\\
&\qquad\qquad R^{M_1}\mathbf{u}^{NM_1}(s)\rangle ds.
\end{aligned}
\tag{5.9}
$$

Applying the Itô formula to $\|\xi^{NM_1}(s)\|^2_{L^2}$ to get

$$
\begin{aligned}
&-E\int_0^T 2\langle g\nabla\cdot(R^{M_1}\mathbf{u}^{NM_1}(s)),\xi^{NM_1}(s)\rangle ds\\
&=E\int_0^T 2\langle g\nabla\xi^{NM_1}(s),R^{M_1}\mathbf{u}^{NM_1}(s)\rangle ds\\
&=gE\|\psi^N\|^2_{L^2}-gE\|\xi^{NM_1}(0)\|^2_{L^2}-gE\int_0^T\|Z^{NM_1}(s)\|^2_{L_Q}ds.
\end{aligned}
\tag{5.10}
$$

Substituting (5.10) into (5.9), one gets

$$
\begin{aligned}
&E\|\sqrt{R^{M_1}}\phi^N\|^2_{\mathbb{L}^2}-Ee^{-r(0)}\|\sqrt{R^{M_1}}\mathbf{u}^{NM_1}(0)\|^2_{\mathbb{L}^2}+gE\|\psi^N\|^2_{L^2}-gEe^{-r(0)}\|\xi^{NM_1}(0)\|^2_{L^2}\\
&\quad -gE\int_0^T e^{-r(s)}\|Z^{NM_1}(s)\|^2_{L_Q}ds-E\int_0^T e^{-r(s)}\|\sqrt{R^{M_1}}\mathbf{Z}^{NM_1}(s)\|^2_{L_{\mathbb{Q}}}ds\\
&=-2E\int_0^T e^{-r(s)}\langle \mathbf{A}\mathbf{u}^{NM_1}(s)+\mathbf{B}^N(\mathbf{u}^{NM_1}(s))-\mathbf{f}^N(\mathbf{u}^{NM_1}(s))+\frac{1}{2}\dot{r}(s)\mathbf{u}^{NM_1}(s),\\
&\qquad\qquad R^{M_1}\mathbf{u}^{NM_1}(s)\rangle ds.
\end{aligned}
$$

By the lower semi-continuity of the norms, we have

$$
\begin{aligned}
&2\liminf_{N\to\infty}E\int_0^T e^{-r(s)}\langle \mathbf{A}\mathbf{u}^{NM_1}(s)+\mathbf{B}^N(\mathbf{u}^{NM_1}(s))-\mathbf{f}^N(\mathbf{u}^{NM_1}(s))+\frac{1}{2}\dot{r}(s)\mathbf{u}^{NM_1}(s),\\
&\qquad\qquad R^{M_1}\mathbf{u}^{NM_1}(s)\rangle ds.
\end{aligned}
$$

$$
\begin{aligned}
&= -E\|\sqrt{R^{M_1}}\phi\|_{\mathbb{L}^2}^2 + \liminf_{N\to\infty} Ee^{-r(0)}\|\sqrt{R^{M_1}}\mathbf{u}^{NM_1}(0)\|_{\mathbb{L}^2}^2 \\
&\quad - gE\|\psi\|_{L^2}^2 + g\liminf_{N\to\infty} Ee^{-r(0)}\|\xi^{NM_1}(0)\|_{L^2}^2 \\
&\quad + g\liminf_{N\to\infty} E\int_0^T e^{-r(s)}\|Z^{NM_1}(s)\|_{L_Q}^2 ds + \liminf_{N\to\infty} E\int_0^T e^{-r(s)}\|\sqrt{R^{M_1}}\mathbf{Z}^{NM_1}(s)\|_{L_{\mathbb{Q}}}^2 ds \\
&\geq -E\|\sqrt{R^{M_1}}\phi\|_{\mathbb{L}^2}^2 + Ee^{-r(0)}\|\sqrt{R^{M_1}}\mathbf{u}(0)\|_{\mathbb{L}^2}^2 - gE\|\psi\|_{L^2}^2 + gEe^{-r(0)}\|\xi(0)\|_{L^2}^2 \\
&\quad + gE\int_0^T e^{-r(s)}\|Z(s)\|_{L_Q}^2 + E\int_0^T e^{-r(s)}\|\sqrt{R^{M_1}}\mathbf{Z}(s)\|_{L_{\mathbb{Q}}}^2 ds.
\end{aligned} \tag{5.11}
$$

Again applying the Itô formula to $e^{-r(s)}\|\sqrt{R^{M_1}}\mathbf{u}(s)\|_{\mathbb{L}^2}^2$ and $\|\xi(s)\|_{L^2}^2$ in (5.6) and (5.7) to get

$$
\begin{aligned}
&E\|\sqrt{R^{M_1}}\phi\|_{\mathbb{L}^2}^2 - Ee^{-r(0)}\|\sqrt{R^{M_1}}\mathbf{u}(0)\|_{\mathbb{L}^2}^2 + gE\|\psi\|_{L^2}^2 - gEe^{-r(0)}\|\xi(0)\|_{L^2}^2 \\
&\quad - gE\int_0^T e^{-r(s)}\|Z(s)\|_{L_Q}^2 ds - E\int_0^T e^{-r(s)}\|\sqrt{R^{M_1}}\mathbf{Z}(s)\|_{L_{\mathbb{Q}}}^2 ds \\
&= -2E\int_0^T e^{-r(s)}\langle \mathbf{F}_1(s) + \mathbf{F}_2(s) + \frac{1}{2}\dot{r}(s)\mathbf{u}(s), R^{M_1}\mathbf{u}(s)\rangle ds.
\end{aligned} \tag{5.12}
$$

Hence (5.11) and (5.12) imply

$$
\begin{aligned}
&2\liminf_{N\to\infty} E\int_0^T e^{-r(s)}\langle \mathbf{A}\mathbf{u}^{NM_1}(s) + \mathbf{B}^N(\mathbf{u}^{NM_1}(s)) - \mathbf{f}^N(\mathbf{u}^{NM_1}(s)) + \frac{1}{2}\dot{r}(s)\mathbf{u}^{NM_1}(s), \\
&\qquad\qquad R^{M_1}\mathbf{u}^{NM_1}(s)\rangle ds. \\
&\geq 2E\int_0^T e^{-r(s)}\langle \mathbf{F}_1(s) + \mathbf{F}_2(s) + \frac{1}{2}\dot{r}(s)\mathbf{u}(s), R^{M_1}\mathbf{u}(s)\rangle ds.
\end{aligned} \tag{5.13}
$$

Together with (5.8), one gets

$$
\begin{aligned}
&E\int_0^T e^{-r(s)}\langle \mathbf{F}_1(s) + \mathbf{F}_2(s) + \frac{1}{2}\dot{r}(s)\mathbf{u}(s), R^{M_1}\mathbf{u}(s) - R^{M_1}\mathbf{v}(s)\rangle ds \\
&\leq \liminf_{N\to\infty} E\int_0^T e^{-r(s)}\langle \mathbf{A}\mathbf{v}(s) + \mathbf{B}^N(\mathbf{v}(s)) - \mathbf{f}^N(\mathbf{v}(s)) + \frac{1}{2}\dot{r}(s)\mathbf{v}(s), R^{M_1}\mathbf{u}^{NM_1}(s) - R^{M_1}\mathbf{v}(s)\rangle ds \\
&= \liminf_{N\to\infty} E\int_0^T e^{-r(s)}\langle \mathbf{A}\mathbf{v}(s) + \mathbf{B}(\mathbf{v}(s)) - \mathbf{f}(\mathbf{v}(s)) + \frac{1}{2}\dot{r}(s)\mathbf{v}(s), \\
&\qquad\qquad P_N\left\{R^{M_1}\mathbf{u}^{NM_1}(s) - R^{M_1}\mathbf{v}(s)\right\}\rangle ds.
\end{aligned} \tag{5.14}
$$

Since $\mathbf{u}^{NM_1} \xrightarrow{w} \mathbf{u}$ in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{H}^1_0(G)))$, it is easy to show that

$$
P_N\left\{R^{M_1}\mathbf{u}^{NM_1}\right\} \xrightarrow{w} R^{M_1}\mathbf{u}
$$

in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{H}^1_0(G)))$ as well. Thus (5.14) becomes

$$
\begin{aligned}
&E\int_0^T e^{-r(s)}\langle \mathbf{F}_1(s) + \mathbf{F}_2(s) + \frac{1}{2}\dot{r}(s)\mathbf{u}(s), R^{M_1}\mathbf{u}(s) - R^{M_1}\mathbf{v}(s)\rangle ds \\
&\leq E\int_0^T e^{-r(s)}\langle \mathbf{A}\mathbf{v}(s) + \mathbf{B}(\mathbf{v}(s)) - \mathbf{f}(\mathbf{v}(s)) + \frac{1}{2}\dot{r}(s)\mathbf{v}(s), R^{M_1}\mathbf{u}(s) - R^{M_1}\mathbf{v}(s)\rangle ds.
\end{aligned}
$$

Since the above inequality holds for all $\mathbf{v} \in L^\infty_{\mathcal{F}}([0,T]\times\Omega;\mathbb{H}^1_{0M_2}(G))$ and all $M_2 \in \mathbb{N}$, we know that it holds true for all $\mathbf{v} \in L^\infty_{\mathcal{F}}([0,T]\times\Omega;\mathbb{H}^1_0(G))$. Let us choose $\mathbf{v}=\mathbf{u}+\lambda\mathbf{w}$ where $\mathbf{w} \in L^\infty_{\mathcal{F}}([0,T]\times\Omega;\mathbb{H}^1_0(G))$ and $\lambda > 0$, one gets

$$
\begin{aligned}
&E\int_0^T e^{-r(s)}\langle \mathbf{F}_1(s)+\mathbf{F}_2(s), R^{M_1}\mathbf{w}\rangle ds\\
\geq &E\int_0^T e^{-r(s)}\langle \mathbf{A}\mathbf{v}(s)+\mathbf{B}(\mathbf{v}(s))-\mathbf{f}(\mathbf{v}(s))+\lambda\frac{1}{2}\dot{r}(s)\mathbf{w}(s), R^{M_1}\mathbf{w}(s)\rangle ds.
\end{aligned}
$$

Letting λ vanish to 0, and by the arbitrariness of $\mathbf{w}$ and the continuity of the coefficients, we know that

$$
\mathbf{F}_1(s)+\mathbf{F}_2(s) = \mathbf{A}\mathbf{u}(s)+\mathbf{B}(\mathbf{u}(s))-\mathbf{f}(\mathbf{u}(s)) \quad \text{P-a.s.}
$$

for all $M_1 \in \mathbb{N}$. The regularity of the solution is guaranteed by Proposition 5.2. The proof can then be completed by letting M_1 go to infinity. □

6. Uniqueness and Continuity

In this section we deal with the uniqueness and continuity of the solution. Again we assume the uniform bound of the terminal conditions under $\mathbb{H}^1_0$-norm. Such circumstances arise in certain other nonlinear stochastic partial differential equations such as stochastic Euler equations.

Theorem 6.1. *Suppose that the terminal conditions satisfy* $\phi \in L^\infty_{\mathcal{F}_T}(\Omega;\mathbb{H}^1_0(G))$ *and* $\psi \in L^\infty_{\mathcal{F}_T}(\Omega;H^1_0(G))$. *Then system* (5.1) *admits a unique adapted solution* $(\mathbf{u},\mathbf{Z},\xi,Z)$ *in*

$$
\begin{aligned}
&L^\infty_{\mathcal{F}}([0,T]\times\Omega;\mathbb{H}^1_0(G))\times L^2_{\mathcal{F}}(\Omega;L^2(0,T;L_{\mathbb{Q}}))\\
&\times L^\infty_{\mathcal{F}}([0,T]\times\Omega;H^1_0(G))\times L^2_{\mathcal{F}}(\Omega;L^2(0,T;L_Q)).
\end{aligned}
$$

Moreover, the solution is continuous with respect to the terminal conditions in

$$
\begin{aligned}
&L^\infty([0,T];L^2_{\mathcal{F}}(\Omega;\mathbb{L}^2(G)))\times L^2_{\mathcal{F}}(\Omega;L^2(0,T;L_{\mathbb{Q}}))\\
&\times L^\infty([0,T];L^2_{\mathcal{F}}(\Omega;L^2(G)))\times L^2_{\mathcal{F}}(\Omega;L^2(0,T;L_Q)).
\end{aligned}
$$

Proof. Suppose that $(\mathbf{u}_1,\mathbf{Z}_1,\xi_1,Z_1)$ and $(\mathbf{u}_2,\mathbf{Z}_2,\xi_2,Z_2)$ are solutions of system (5.1) according to terminal conditions (ϕ_1,ψ_1) and (ϕ_2,ψ_2), respectively. Denote

$$
\begin{aligned}
&\hat{\mathbf{u}} = \mathbf{u}_1-\mathbf{u}_2, \quad \hat{\mathbf{Z}} = \mathbf{Z}_1-\mathbf{Z}_2, \quad \hat{Z} = Z_1-Z_2,\\
&\hat{\xi} = \xi_1-\xi_2, \quad \hat{\phi} = \phi_1-\phi_2, \quad \hat{\psi} = \psi_1-\psi_2.
\end{aligned}
$$

Then the differences satisfy

$$
\begin{cases}
\frac{\partial\hat{\mathbf{u}}(t)}{\partial t} = -\mathbf{A}\hat{\mathbf{u}}(t)-\mathbf{B}(\mathbf{u}_1(t))+\mathbf{B}(\mathbf{u}_2(t))-g\nabla\hat{\xi}(t)+\mathbf{f}(\mathbf{u}_1(t))-\mathbf{f}(\mathbf{u}_2(t))+\hat{\mathbf{Z}}(t)\frac{d\mathbb{W}(t)}{dt};\\
\frac{\partial\hat{\xi}(t)}{\partial t}+\nabla\cdot(R\hat{\mathbf{u}}(t)) = \hat{Z}(t)\frac{dW(t)}{dt};\\
\hat{\mathbf{u}}(T) = \hat{\phi} \text{ and } \hat{\xi}(T) = \hat{\psi}.
\end{cases}
\tag{6.1}
$$

Define

$$
r(t) \triangleq \int_t^T \Big\{2\alpha+\frac{3}{2^{\frac{5}{3}}}\Big(\frac{C_1^4C_5^4}{(\nu-1)\kappa_h C_0}\Big)^{\frac{1}{3}}K^{\frac{4}{3}}\Big\}ds,
$$

where

$$K = \sup_{(t,\omega)\in[0,T]\times\Omega} \|\mathbf{u}_1\|_{\mathbb{L}^4} + \sup_{(t,\omega)\in[0,T]\times\Omega} \|\mathbf{u}_2\|_{\mathbb{L}^4}.$$

Then an application of the Itô formula to $e^{-r(s)}\|\sqrt{R}\hat{\mathbf{u}}(s)\|_{\mathbb{L}^2}^2$ and $\|\hat{\xi}(s)\|_{L^2}^2$ yields

$$\begin{aligned} &Ee^{-r(t)}\|\sqrt{R}\hat{\mathbf{u}}(t)\|_{\mathbb{L}^2}^2 + gE\|\hat{\xi}(t)\|_{L^2}^2 + gE\int_t^T \|\hat{Z}(s)\|_{L_Q}^2 ds + E\int_t^T e^{-r(s)}\|\sqrt{R}\hat{\mathbf{Z}}(s)\|_{L_{\mathbb{Q}}}^2 ds \\ =&E\|\sqrt{R}\hat{\phi}\|_{\mathbb{L}^2}^2 + gE\|\hat{\psi}\|_{L^2}^2 + 2E\int_t^T e^{-r(s)}\langle \mathbf{A}\hat{\mathbf{u}}(s) + \mathbf{B}(\mathbf{u}_1(s)) - \mathbf{B}(\mathbf{u}_2(s)) - \mathbf{f}(\mathbf{u}_1(s)) + \mathbf{f}(\mathbf{u}_2(s)) \\ &\qquad + \frac{1}{2}\dot{r}(s)\hat{\mathbf{u}}(s), R\hat{\mathbf{u}}(s)\rangle ds \\ \le&E\|\sqrt{R}\hat{\phi}\|_{\mathbb{L}^2}^2 + gE\|\hat{\psi}\|_{L^2}^2. \end{aligned}$$

Thus we have shown the uniqueness and continuity of solutions. □

Now we have established the well-posedness of the backward stochastic tidal dynamics equation. Such well-posedness holds when the terminal conditions are uniformly bounded under $\mathbb{H}_0^1$-norm. One may want to relax the conditions on terminal values to a weaker sense, such as uniformly boundedness in $\mathbb{L}^2$ sense. However, such problems are still open. The difficulty lies in the nonadaptiveness nature of the backward stochastic differential equations. For instance, the function r defined in Lemma 5.1 is not adaptive to the forward filtration. So in the proof of Theorem 5.3 and Theorem 6.1, we redefined r so that it is adaptive to the system. For this approach, we have to improve the regularity of the solution **u** appeared in the definition of r. Such obstacles do not arise in the forward stochastic systems.

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