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## Replenishment policies for a tree-type three echelon supply chain system

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**REPLENISHMENT POLICIES FOR A TREE-TYPE  
THREE ECHELON SUPPLY CHAIN SYSTEM**

**A Thesis**

**Submitted to the Graduate Faculty of the  
Louisiana State University and  
Agricultural and Mechanical College  
In partial fulfillment of the  
Requirements for the degree of  
Master of Science in Industrial Engineering**

**in**

**The Department of Construction Management and Industrial Engineering**

**By**

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B.Eng., Thammasat University, 2003  
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August, 2011**

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## ABSTRACT

One of the common goals which most companies have is to maximize profits. There are two way to increase profit: increasing revenue or reducing cost. Lacking of ability to keep the cost down could potentially drive the companies out of the business. In recent years, many researchers have been paying more attention on improving supply chain system due to high potential of creating cost savings. The supply chain network considered in this research is a tree-type, three-echelon single producer, multiple distributors, and multiple retailers system. The goal of this research is to develop a replenishment policy which satisfies customers' demand and minimizes the total production-inventory system cost.

Three inventory models are developed here. First, *tree-type, three-echelon distribution (producer, distributor and retailers) model with end customers' backorders* (TDB) at retailer's level is developed. Second, the *variation of downstream holding cost* (DHV) is studied and a model is developed to investigate the effect downstream holding cost structure. Third, a model is developed to *improve the retailer's service rate (ISR)*. This model combines the features of TDB and DHV models together (allowable backorder and reduced delivery interval at retailer's level). Operational schedules of TDB are constructed and the limitations of DHV model are established. The improvement in the ISR model is confirmed and demonstrated through numerical examples. Significance and conclusions of this research are highlighted along with an indication of future research.



## **CHAPTER 1**

### **INTRODUCTION**

The competition in business nowadays is highly competitive. Customers have more ways to access the information which helps them make the best purchase decision. Businesses do not compete just with competitors in the same region anymore. Globalization and free trade allow companies in the other side of the world to compete with one on another. Ability of the company to use the process of bringing the product and service to the customer at the lowest possible cost is crucial. As the company operates at a low cost, it is able to provide products to the customer at the low prices which make the company more competitive since, in the world of business, every penny counts. One area which has been recently received attention from many researchers is supply chain operation due to many opportunities to create savings. Many experts in both industry and education fields have been trying to find the ways to manage supply chain most efficiently. Supply chain management deals with the processes of transforming and transferring products from raw material to finished goods which are ready to be delivered to the customers.

The complexity of supply chain system varies depending on several factors. It can be simple as there are only a few steps and a few parties involved; or it can be complex as there are too many processes, parts, and several parties involved. Inventory management is actually a part of supply chain management, but it mainly focuses on determining decisions on inventory activities such as how much inventory a company should carry and when orders should be made to satisfy the customers' demands at the possible lowest cost.

Inventory helps companies deal with unexpected and fluctuated demands of the customers, but too low inventory level may cause the companies face a shortage situation which creates customers' dissatisfaction due to not fulfilling the orders in timely fashion and/or

additional expenses to both parties. In many cases, shortage would have a long term impact on the companies' sales reputation which can result in the permanent lost of sales or lost customers. On the other hands, even though having too much inventory could reduce the probability of the shortage, the company could probably end up with the unnecessary high inventory holding cost which would lower the company's competitiveness. Many researchers have developed procedures and solutions to determine the replenishment policies in many different ways. Questions that are normally asked when it comes to inventory situations include how much inventory should be held at each stage and what should be the reorder points.

In the past, as most companies used push-inventory system, where the company keeps producing the products or produces based on the demand forecast, the problem occurred when the actual demands do not match or come close to the forecasted demand. That caused unwanted inventory which generated high inventory costs. At the beginning, researchers looked at the inventory system for an individual company. The Economic Ordering Quantity (EOQ) was introduced by Harris (1913) to determine the ordering quantity which leads to the lowest total cost when finite demand rate, holding cost, and setup cost are given. For many cases, EOQ does not seem to be practical. For example, when an EOQ for a buyer is not economical for a vendor. Therefore, an integrated inventory system was first developed by Goyal (1976) to deal with those mentioned shortcomings of EOQ. The integrated inventory system considers the combined total cost generated throughout the network, not just at an individual. Much research on inventory management has been done in an attempt to find the inventory models and replenishment policies which minimize the total cost of the system.

## **1.1 Research Goals**

The goal of this research is to develop a replenishment policy for a tree-type three-echelon distribution supply chain system to satisfy customer demand at the minimal possible

cost under different situations (backorder, downstream holding cost rate, and shipment interval at retailers). The developed replenishment policies are expected to be implemented in businesses which have similar types of supply chain networks. This will not only benefit the companies that utilize these replenishment policies for cost savings, but also benefits the customers by setting the lower price for products due to cost savings.

## **1.2 General Objectives**

The main purpose of this research is to study and model the inventory of the tree-type supply chain network. In such a system, a producer produces and delivers products to satisfy demand of distributors who hold and deliver products to satisfy retailer demand. The demand rates per year are known. There are costs associated with holding, setup, logistics, and shortage (if applicable) activities. This research also proposes replenishment policies to satisfy the customers demand and to minimize the total system cost of the inventory.

To achieve the research goal, the research objectives are determined. First, related literatures will be reviewed to identify shortcomings which lead to determining research problems. Second, cost models will be developed based on identified research problems. The cost models will cover all costs related to producing and handling products throughout supply chain. Third, optimization techniques will be utilized to obtain the optimal solutions which are in term of basic decisions in inventory control such as ordering schedule and ordering quantity. Forth, to evaluate the models and the solutions, numerical examples will be given. The numerical results will be carefully analyzed to confirm the effect of the model and solutions. Finally, the solutions will be comprehensively explained to help the readers get the idea how to implement the solutions.

### **1.3 Scope and Applications**

There are an infinite possible ways to build a tree-type supply chain network. A network must have at least two stages, but it can technically have an infinite number of stages. At each stage, the number of parties in the succeeding stages must be at least equal to the number of preceding stages. It can consider only one product or multiple products. A network considered in this research is a tree-type three-echelon system with a producer, multiple distributors, and multiple retailer supply chain network for a single product.

This research could be applicable to businesses that provide products to the customers through tree-type networks which consist of a producer, distributors, and retailers. Those businesses include automobiles, electronics, medical equipments, and machinery. In automobile industry, cars are produced at a plant, parked at its off-the-line open warehouse, and they are delivered later to car distributors as orders are received. Similarly, car distributors hold vehicles at their car-lot and ship them to the dealers as they need them. It is also usual that when a customer goes to a car dealer to buy a car he/she wants, but the car is not available at a time. So, the customer book the car at that time, and the date and time that the car will be delivered to the customer is fixed. Similar examples can be framed for many manufacturing systems of variety of products.

This research aims at developing the solution which helps the company to determine replenishment policy to satisfy customers' demands to minimize the total system cost. The inventory model and optimal solution based on the ordering quantity and maximum quantities of shortage are generated. The next chapter will present the current status of literature on this related inventory problem.

## **CHAPTER 2**

### **LITERATURE REVIEW**

This section review the literature related to the integrated inventory supply chain system. Also, the shortcomings of existing models are shown at the end of this chapter. The literature is grouped into categories based on the characteristics of the problems.

#### **2.1 Single-Vendor Single-Buyer Problem**

Many of researchers developed models to deal with a single-vendor, single-buyer inventory problem to simplify the model. Most of early research in integrated inventory problems are on two-echelon supply chains. Goyal (1977) claims to be the first research to contributed to integrated inventory system. The work introduced the integrated inventory model of single-vender single-buyer which is used to determine the economic joint replenishment policy for both vender and buyer to achieve the shared benefits. It was assumed that the demand is finite. Banerjee (1986) developed the model under deterministic condition to deal with the case which a vender produces products to fulfill orders placed by a customer in lot-for-lot basis. A vender deliveries a lot of products which are produce in one batch at one time. This model determines the ordering quantity which helps both a vendor and a buyer. Goyal (1988) extended Banerjee (1986) by allowing the production lot to be shipped in many small shipments. Goyal claimed that the model provide a lower cost to the system. Lu (1995) developed an optimal solution to single-vendor single-buyer problem for the case that all shipment sizes are equal. Lu also allowed shipments to be made before a production batch is completed. With the same assumption as Lu's that the shipment can be made before the lot is completed, Goyal (1995) presented the case that all shipment sizes are not identical. Sarker and Parija (1994 and 1996) developed the ordering policies for the system which consisted of manufacturing and raw material suppliers with fixed intervals. Hill (1999) Wee and Yang

(2002) developed, algebraically, the optimal policy of the integrated vendor–buyer inventory system without using differential calculus. Yang and Pan (2002) introduced the integrated inventory system where the lead time is adjustable, that is that the lead time can be shortened by paying an extra cost. The authors claimed that the model is more profitable than Banerjee (1986) and Goyal (1988). Yang and Pan (2004) presented the integrated inventory model which includes the quality issue and where lead time is controllable. Ouyang *et al.* (2006) studies the inventory model for a single-vendor single-buyer system where the lead times and the ordering cost are controllable. Two models were developed to investigate the effect of the lead time and the ordering cost reduction models. Recently, Sajadieh *et al* (2010) presented a joint economic lot-sizing model for a single-vendor single-buyer problem. The work studied the supply chain network of a supplier and a retailer where the demand has a positive relationship with the quantities at the buyer’s display area. Algorithms were developed to determine the replenishment policy in term of three variables.

## **2.2 Single-Vendor Multi-Buyer Problem**

However, in the real-life business, it is not practical for the supply chain network to have only one vendor and one buyer all the time. Therefore, there is much research in single-vendor multi-buyers for two or more stages supply chain to get the research closer to the cases in the real-world practices. As a very first research in this area, Clark and Scarf (1960) dealt with the problem of identifying the optimal policies on purchasing quantity which will lead to the minimum total cost of the multi-echelon inventory problems. Then, Khouja (2003) studied a supply chain network which has multiple firms at each stage and each firm can serve multiple customers. Khouja identified the optimal cycle times under three coordination mechanisms; the equal cycle time mechanism, integer multipliers for the integer multipliers mechanism, and the integer powers of two multipliers mechanism. Wee and Yang (2004) revised an optimal

solution (Goyal, 1988) and developed a heuristic solution model for a producer-distributors-retailers inventory system by using the principle of strategic partnership. Abdul-Jalbar (2007) addressed an integrated inventory system problem consisting of a single vendor and two buyers in terms of integer-ratio policies and developed a procedure for computing an optimal policy. They also developed a procedure which computes an optimal integer-ratio policy. Haji *et al.* (2009) introduce a new replenishment policy where the authors claimed that it is easy to implement and it also reduces the impacts of the uncertain demand.

### **2.3 Inventory Problem with Allowable Shortage**

Literature reviewed above have one thing in common: shortage is not allowed. Some researchers have investigated inventory problems in the case that the shortage situation is allowed. Grubbstrom and Erdem (1999) extended their previous work regarding to EOQ formula by taking backlogging into consideration. Cardenas-Barron (2001) extended Grubbstrom and Erdem (1999) model to obtain the EPQ with shortage allowed. In this case, it was assumed that one shortage cost per unit and time unit. Wu and Ouyang (2003) extended Grubbstrom and Erdem's method. They developed the optimal replenishment policy which deals with the integrated single-vendor single-buyer inventory system where shortage is allowed. The theoretical result in their research showed that the integrated total cost with shortage is lower than the one without shortage. Zhou and Wang (2007) studied the single-vendor single-buyer problem where shortage is permitted. Their model does not limit the holding cost for the buyer but has to be greater than the one at the vendor, as previous research did.

### **2.4 Lower Downstream Holding Cost Rate**

In general, the holding costs are assumed to increase as the product goes down the chain. It can be explained as the value of product increases as the product goes down the supply

chain. However, the lower holding cost rate downstream is held in some cases. The following research have been done under the assumption that the holding cost at the vendor is greater than the holding cost at the buyer. Braglia and Zavanella (2003) and Valantini and Zavenella (2003) studied on the consignment stock and mentioned that the lower holding cost rate downstream is possible and explained, based on consignment stock concept. Hill and Omar (2006) developed the inventory model of single-vendor single-buyer with the higher holding cost at the vendor and unequal shipment sizes. The authors explained the occurring of the lower holding cost rate downstream based on the situation that the buyer perform as a manufacturer which has a bulk storage that can stores products at the cheaper cost.

## **2.5 Drawbacks of Previous Research**

After reviewing the literature, it is evident that the integrated inventory supply chain system has received attention by researchers and practitioners. However, there are still some flaws in the earlier research. These shortcomings are listed below.

1. **Supply chain network:** The literature mentioned above have studied integrated model mostly limited to two-echelon and single-vendor single-buyer system. Higher order echelons and stages remain to be explored.
2. **Shortage allowance:** Some research allowed the shortages [Grubbstrom and Erdem (1999) and Wu and Ouyang (2003)] but were limited to the above case of a single-vendor single-buyer system. Backorder issues were not included in the higher order cases.
3. **Lower downstream holding cost rate:** Most research assume that the holding cost rate increase as product moves down the supply chain. However, Hill and Omar (2006) presented the cases with the lowered holding cost rate downstream. The networks studied in those cases were only two echelon supply chain.



4. **Improving service rate:** Most research were not concerned about improving service rate which is one of the most important metrics that have significant affect on the satisfaction level of customers, which will lead to repeat customers and future purchases.

Table 2.1 shows the overall related research compared with the proposed models TDB (Tree-type Distribution with Backorder) , DHV (Downstream Holding cost Variation) and ISR (Improving Service Rate) which will be developed in this study.

Table 2.1 Comparison of characteristics between previous research and this research

Properties considered	Braglia & Zavanella (2003)	Valantini & Zavenella (2003)	Wee & Yang (2002)	Wee & Yang (2004)	Zimone & Grubbstrom (2004)	TDB (Model 1)	DHV (Model 2)	ISR (Model 3)
# Echelon	2	2	2	3	2	3	3	3
# Producer	1	1	1	1	1	1	1	1
# Distributor	N/A	N/A	N/A	$M$	N/A	$m$	$m$	$m$
# Retailer	$n$	2	$n$	$n$	1	$n$	$n$	$n$
Production rate	constant	constant	constant	constant	constant	constant	constant	constant
Demand rate	constant	constant	constant	constant	constant	constant	constant	constant
Shortage	No	No	No	No	backorder	backorder	No	backorder
Reduced retailer delivery interval	No	No	No	No	No	No	Yes	Yes
Holding cost	$H_v > H_b$	$H_v > H_b$	$H_v < H_b$	$H_d > H_r$	$H_v > H_b$	$H_d < H_r$	$H_d > H_r$	$H_d < H_r$

## 2.6 Specific Objectives

After reviewing literatures and identifying shortcomings, the specific objectives in this research are listed below:

1. TDB: To develop a supply chain model for a tree-type distribution model with allowable backorder at the retailer levels, that minimizes the total inventory

system cost and finds an optimal policy for producer's run time, replenishment cycle times, and replenishment quantities for both distributors and retailers.

2. DHV: To develop a model for a downstream holding cost variation (with no backorder), that minimizes the total inventory system cost and finds a policy for optimal production run time, distributor's delivery cycle time, and retailer's replenishment quantities.
3. ISR: To develop a model for improving the retailer's service rate (ISR), that minimizes the total inventory system cost and finds an optimal policy for distributor's delivery cycle time, replenishment quantities and the service rate at the retailer's level.

Optimal closed-form solutions will be sought to determine the variables in each of the above three models. Search procedure will be employed either to find the optimal/sub-optimal solutions where closed form solutions are not obtainable or to refine the solutions for some integer variables. Numerical examples will be demonstrated to illustrate the solution procedures.

## **2.7 Methodology**

The methods and techniques used in this research to develop the models and solutions for all cases are basically the same. First, the problems are identified, assumptions are set, and notations are defined. Then, the inventory systems are depicted pictorially at all levels (producer, distributor and retailer) for all cases. Using basic algebraic approach, the total cost models are developed based on the inventory system. The convexity is proved using different approaches for each case as it demands. The closed-form optimal solutions for the TDB model (Chapter 3) and DHV model (Chapter 4) are developed using derivative technique. Since some variables must be integer numbers, integerization techniques are applied to refine them from

*real* numbers to *integer* numbers. In ISR model (Chapter 5), since the total cost model contains some integer function  $\lceil \cdot \rceil$  that leads the total cost function to be piece-wise convex, the branching search procedure was used to find the optimal solutions and the piecewise convex curves were graphically illustrated along with the refinement process. All models and solutions were tested and illustrated through numerical examples.

The next chapter will present the development of TDB model, the optimal solutions, and the numerical example.

## CHAPTER 3

### TREE-TYPE DISTRIBUTION MODEL WITH BACKORDER (TDB)

This chapter proposed a model of integrated inventory system with backorder for a single producer and multiple distributors and retailers. The development of the model will be shown at the beginning which is followed by optimal solutions, numerical example, and sensitivity analysis.

#### 3.1 The Problem

This research deals with a tree-type three-echelon supply chain system consisting of one producer (or interchangeably, manufacturer),  $m$  distributors and  $n$  retailers. The manufacturer produces items to meet the demands of customers through  $m$  distributors and  $n$  retailers. The objective of the problem is to develop the solution procedure that determines the optimal replenishment policy for the variables under backorder at the retailer levels. Let's call this problem as TDB.

The manufacturer produces finished goods and deliveries them to the distributors at the quantities which are determined by the total demand of all retailers served by each distributor. The supply chain network in Figure 3.1 demonstrates which retailers are served by which particular distributor. Each distributor  $j$  serves  $n_j$  retailers. Therefore, the total number of retailers is  $n = \sum_{j=1}^m n_j$ . Those distributors hold products and then deliver them to retailers when the order is received, which is every retailer's replenishment interval.

As shown in the Figure 3.1, the total number of retailers is always greater than the total number of distributors. Here, it is assumed that the producer is allowed to make a shipment during its production which means producer does not need to complete its production cycle before making the first delivery to distributors in each shipment interval.

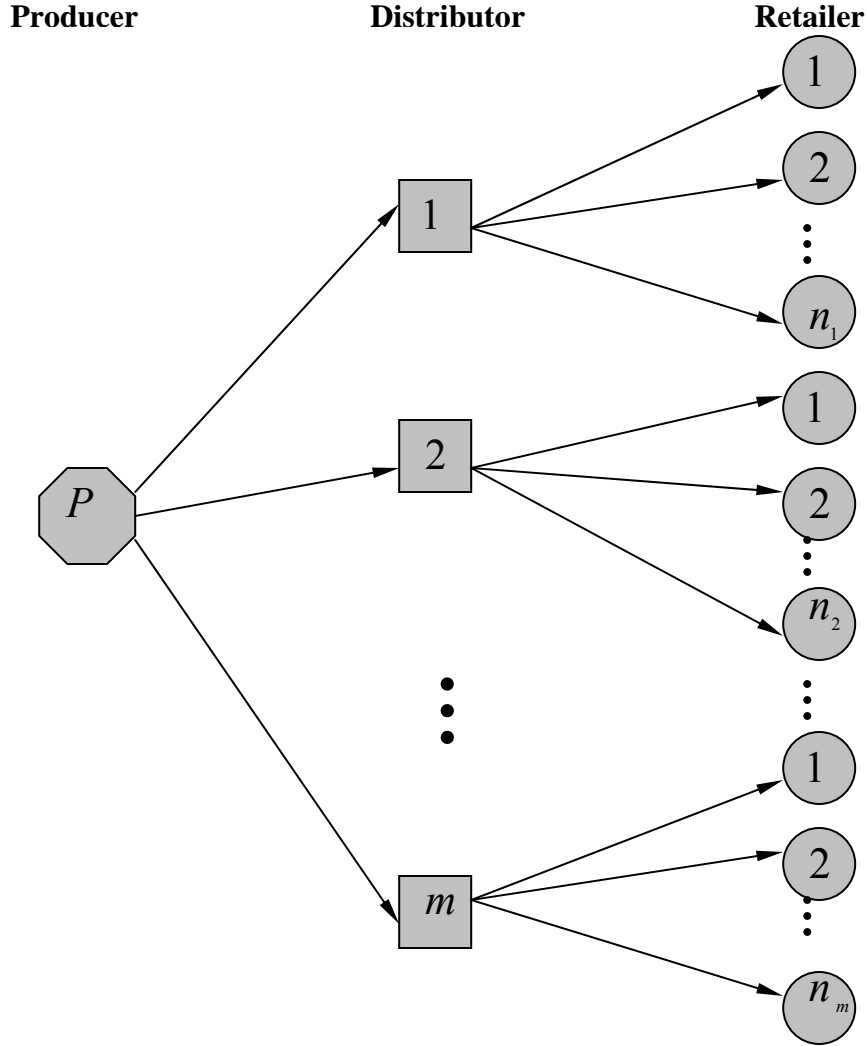


Figure 3.1 A producer-distributor-retailer's supply chain network.

During one cycle of producer's shipment interval,  $T^p$  the producer delivers products to distributor  $j$  for  $N_j^p$  times. Consider distributor  $j$  ( $j = 1, \dots, m$ ) serves a group of  $n_j$  retailers and the retailer  $k$ 's annual demands from distributor  $j$ ,  $d_{jk}$  are known. Assume that during the shipment interval  $T_j^d$ , distributor  $j$  makes  $N_{jk}^d$  deliveries to the retailer  $k$ . So, during producer's one shipment intervals,  $T^p$  the distributor  $j$  makes deliveries to retailer  $k$  for  $N_j^p N_{jk}^d$  times. Thus, the replenishment interval of each retailer  $j$  is  $T^p / N_j^p N_{jk}^d$ . There may be

shortages at the retailer's level depending on the demand,  $d_{jk}$ . So during the retailer  $k$ 's replenishment interval  $T_{jk}^r$ , the retailer  $k$  may stock out for  $t_{jk}^r$  period. Here, we assume instantaneous replenishment at the distributor and retailer levels because of the closer proximity of producer, distributors and retailers. At the time right before the retailer stock is replenished, the maximum backorder  $b_{jk}$  is reached.

### 3.2 Assumptions and Notation

The following assumptions for the multi-distributor, multi-buyer system under consideration are necessary to formulate the problem. This will follow with a set of parametric notation and variables.

#### 3.2.1 Assumptions

1. The system considers a single-producer, multi-distributor and multi-retailer problem with single product.
2. The demand rate of the product is finite and constant, and the production rate is also finite.
3. The distributor's and the retailer's replenishments are instantaneous.
4. The producer, the distributors and the retailers have complete knowledge of each other's information and the total number of retailers is always greater than the total number of distributors.
5. Through information sharing, the sales data is received from the retailers to synchronize the producer's, the distributors' and the retailers' inventory levels or lot sizes.
6. Shortage is allowed to reflect the practicality of the problem and it generates backorder costs only.

### 3.2.2 Notation

All parametric notations and variables used in this research are listed below separately.

#### (a) System parameters

$A_p$	Producer's setup cost (\$/setup)
$A_d$	Distributor's ordering cost including transportation cost (\$/order)
$A_r$	Retailer's ordering cost including transportation cost (\$/order)
$D$	Total annual demand of all retailers (units/year)
$d_{jk}$	Annual demand rate for retailer $k$ ( $k = 1, \dots, n_j$ ) served by distributor $j$ ( $j = 1, \dots, m$ ), (units/year)
$H_p$	Producer's annual holding cost (\$/unit/year)
$H_d$	Distributor's annual holding cost (\$/unit/year)
$H_r$	Retailer's holding cost (\$/unit/year)
$P$	Annual production rate (units/year)
$\pi$	Cost of backorder (\$/unit)

#### (b) Intermediate variables

$\bar{I}^p$	Average inventory level at the producer (unit)
$\bar{I}_j^p$	Average inventory level at producer, proportional to the demand of distributor $j$ (unit)
$\bar{I}_j^d$	Average inventory level at the distributor $j$ (unit)
$\bar{I}_{jk}^r$	Average inventory level at the retailer $jk$ (unit)
$N^p$	Number of production cycle at the producer level
$p_j$	Producer's production rate portioned to the demand of distributor $j$ (units/year)
$TC$	Total cost of the producer, the distributors and the retailers (\$)

$T_j^d$  Distributor  $j$ 's shipment interval  $j$  ( $j = 1, \dots, m$ ) (in year)

$T_{jk}^r$  Retailer  $k$ 's replenishment interval, where retailer  $k$  ( $k = 1, \dots, n_j$ ) is served by distributor  $j$  ( $j = 1, \dots, m$ ) (in year)

$t_{jk}^r$  Stock out time at retailer  $k$  supplied from distributor  $j$  (in year)

### (c) Decision variables

$N_{jk}^d$  Number of deliveries from distributor  $j$  ( $j = 1, \dots, m$ ) to retailer  $k$  in  $T_j^d$

$N_j^p$  Number of deliveries from the producer to the distributor  $j$  in  $T^p$

$Q_{jk}^r$  Lot size (in units) of retailer  $k$  ( $j = 1, \dots, n_j$ ) received from the distributor  $j$  per delivery (unit)

$Q_j^d$  Lot size (in units) of distributor  $j$  ( $j = 1, \dots, m$ ) received from the producer per delivery (unit)

$Q^p$  Total quantity produced during  $T^p$  (unit)

$R_{jk}$  The proportion of retailer  $k$ 's service time to its replenishment interval  $T_{jk}^r$ , that is,  $R_{jk} = (T_{jk}^r - t_{jk}^r) / T_{jk}^r$  for retailer  $k$  ( $k = 1, \dots, n_j$ ) served by distributor  $j$  ( $j = 1, \dots, m$ )

$T^p$  Producer's cycle time during which it makes  $\sum_{j=1}^m N_j^p$  shipments to all distributors (in year)

### 3.3 The TDB Model

In this section, a three-stage total cost function of a tree-type producer-distributor-retailer system is developed. Costs considered in the supply chain are holding cost, ordering and shipping cost, and shortage cost. This total cost function can be described mathematically as

$$TC_{TDB} = TC_{HC} + TC_{SC} + TC_{BC}, \quad (3.1)$$



where  $TC_{HC}$  is the total holding cost,  $TC_{SC}$  is the total setup cost, and  $TC_{BC}$  is the total backorder cost.

### 3.3.1 Holding Cost

The total holding cost of the system consists of the total holding costs incurred in all three stages, producer, distributors and retailers. Holding costs per unit  $H_p, H_d$ , and  $H_r$  at producer, distributors and retailers, respectively, are known and constant. For the producer's average inventory level,  $\bar{I}^p$  the distributor  $k$ 's average inventory,  $\bar{I}_j^d$  and the retailer  $k$ 's average inventory level,  $\bar{I}_k^r$ , the total cost  $TC_{HC}$  can, therefore, be written as

$$TC_{HC} = TC_{HC}^p + TC_{HC}^d + TC_{HC}^r = H_p \bar{I}^p + \sum_{j=1}^m H_d \bar{I}_j^d + \sum_{j=1}^m \sum_{k=1}^{n_j} H_r \bar{I}_{jk}^r, \quad (3.2)$$

where  $TC_{HC}^p = H_p \bar{I}^p$  is the finished goods inventory cost at the manufacturer level,  $TC_{HC}^d = \sum_{j=1}^m H_d \bar{I}_j^d$  is the inventory cost at the distributor level, and  $TC_{HC}^r = \sum_{j=1}^m \sum_{k=1}^{n_j} H_r \bar{I}_{jk}^r$  is the inventory cost at the retailer's level. While the total individual holding costs are calculated by multiplying holding cost rate with the average inventory, the methods to obtain the average inventory for all cases are different as shown in the following sub-sections.

#### (a) Retailers' inventory level

At time zero, the distributor  $j$  delivers  $d_{jk} T_{jk}^r$  units of products to retailer  $k$ . The retailer  $k$  satisfies  $b_{jk}$  units of products to the customers who had not received their products from the previous cycle  $t_{jk}^r$  yet. Then, the retailer  $k$  starts selling the rest of the products to customers at an annual demand rate of the retailer  $k$ ,  $d_{jk}$ . The retailer  $k$  keeps selling the products until time  $T_k^r - t_k^r$  when the retailer  $k$ 's inventory level falls to zero, and from that point in time, the backorders starts building

Regardless of the zero-inventory level, the retailer  $k$  continues taking the order placed by the customers at the same rate,  $d_{jk}$ , and gives the customers an expected time when the products will become available and delivered to them at the retailer  $k$ 's extra expense of  $\pi$  dollars per item (bookkeeping cost). The cost of backorder is independent of the period of time that the products are being out of the retailer's stock. As time passes and reaches time  $T_{jk}^r$ , the inventory level reaches level  $-b_{jk}$ . It is the same time when distributor  $j$  deliveries  $d_{jk}T_{jk}^r$  units to retailer  $k$  [see Figure 3.2].

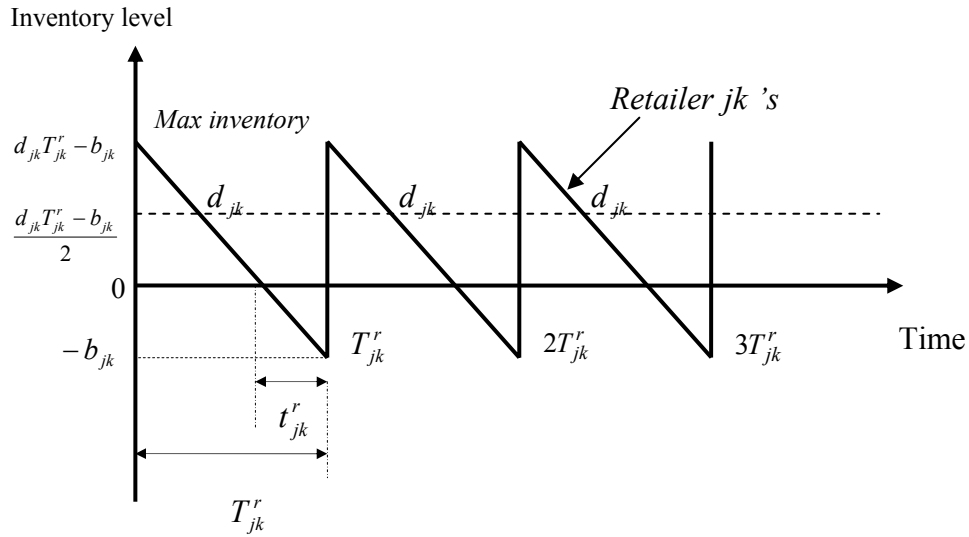


Figure 3.2. Retailer  $k$ 's inventory level (supplied by distributor  $j$ )

As shown in Figure 3.2, during the retailer  $k$ 's replenishment interval  $T_{jk}^r$ , the retailer  $k$ 's maximum inventory level,  $I_{jk}^r$ , is  $d_{jk}T_{jk}^r - b_{jk}$ . The average inventory level at retailer  $k$  is  $\bar{I}_{jk}^r = (d_{jk}T_{jk}^r - b_{jk})(T_{jk}^r - t_{jk}^r)/2T_{jk}^r$ . In order to ease the computation, the retailer  $k$ 's average inventory can be written as  $\bar{I}_{jk}^r = (d_{jk}T_{jk}^r - b_{jk})R_{jk}/2$ , where  $R_{jk} = (T_{jk}^r - t_{jk}^r)/T_{jk}^r$  for  $(0 \leq R_{jk} \leq 1)$ . In the extreme case,  $R_{jk}$  is equal to 0 when the retailer's inventory is out of stock at all times. In other words, the retailer has zero inventory. Another extreme case is when  $R_{jk}$

is equal to 1; it means there is no shortage occurring at all during each retailer replenishment interval. As  $T_{jk}^r = T^p / N_j^p N_{jk}^d$ ,  $\bar{I}_{jk}^r$  can be rewritten as  $(d_{jk} T^p / N_j^p N_{jk}^d - b_{jk}) R_{jk} / 2$ . Moreover, as  $R_{jk} = (T_{jk}^r - t_{jk}^r) / T_{jk}^r$ ,  $b_{jk}$  can be written as  $b_{jk} = d_{jk} T_{jk}^r (1 - R_{jk})$  or  $b_{jk} = d_{jk} T^p (1 - R_{jk}) / N_j^p N_{jk}^d$ . So, the average inventory  $\bar{I}_{jk}^r$  can be written as  $\bar{I}_{jk}^r = d_{jk} T^p R_{jk}^2 / 2 N_j^p N_{jk}^d$  and the retailer's total holding cost,  $TC_{HC}^r$ , is given by

$$TC_{HC}^r(T^p, R_{jk}, N_j^p, N_{jk}^d) = \sum_{j=1}^m \sum_{k=1}^{n_j} H_r (d_{jk} T^p R_{jk}^2 / 2 N_j^p N_{jk}^d). \quad (3.3)$$

### (b) Distributor's inventory level

The producer instantaneously delivers products to distributor  $j$  every  $T_j^d$  period. During producer's cycle time,  $T^p$ ,  $N_j^d$  shipments are made to distributor  $j$  (i.e.,  $N_j^d T_j^d \leq T^p$ ). At the time, the producer is replenishing distributor  $j$ 's stock with  $d_{jk} T_j^d$  each time, and at the same time, the distributor  $j$  supplies  $d_{jk} T_{jk}^r$  to retailer  $k$  such that  $d_{jk} T_j^d = N_{jk}^r d_{jk} T_{jk}^r$ . Therefore, immediately after the first shipment from distributor  $j$  to retailer  $k$ , the distributor's inventory level goes down to  $d_{jk} T_j^d - d_{jk} T_{jk}^r$ . During the retailer's first interval  $T_{jk}^r$ , distributor  $j$ 's inventory level stays constant at  $d_{jk} T_j^d - d_{jk} T_{jk}^r$  until the second shipment of  $d_{jk} T_{jk}^r$  units to retailer  $k$  needs to be made after time  $T_{jk}^r$ . As a result, the distributor  $j$ 's inventory level falls to  $d_{jk} T_j^d - 2d_{jk} T_{jk}^r$ . This cycle repeats  $N_{jk}^r$  times during one distributor  $j$ 's shipment interval. After the  $N_{jk}^r$ th shipment is made to the retailer, the inventory level become zero and stays zero for  $T_{jk}^r$ . Then the producer instantaneously ships  $d_{jk} T_j^d$  to distributor  $j$  as shown in Figure 3.3.

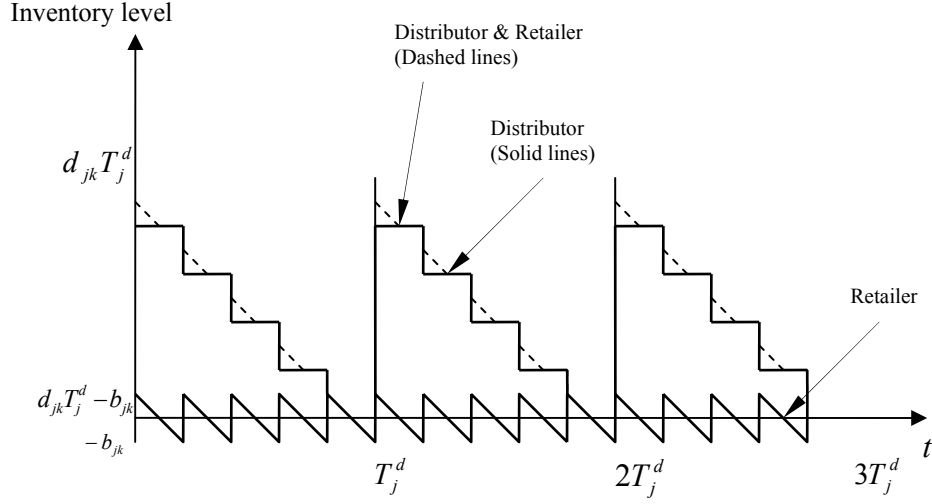


Figure 3.3. Distributor's and retailer's inventory

Thus, the distributor  $j$ 's average inventory level,  $\bar{I}_j^d$ , is  $\sum_{k=1}^{n_j} d_{jk} (T_j^d - T_{jk}^r) / 2$ ,  $\forall j$ . As

$T_{jk}^r = T^p / N_j^p N_{jk}^d$  and  $T_j^d = T^p / N_j^p$ , the average inventory level  $\bar{I}_j^d$  can be rewritten as

$\bar{I}_j^d = \sum_{k=1}^{n_j} d_{jk} (T^p / N_j^p - T^p / N_j^p N_{jk}^d) / 2$ , which, upon simplification and substituting into the

distributor's inventory cost,  $TC_{HC}^d$ , yields

$$TC_{HC}^d(T^p, N_j^p, N_{jk}^d) = H_d \sum_{j=1}^m \sum_{k=1}^{n_j} \frac{d_{jk} T^p}{2 N_j^p} \left( 1 - \frac{1}{N_{jk}^d} \right). \quad (3.4)$$

### (c) Producer's Inventory Level

The manufacturer produces the items at the production rate  $P$ . To ease the calculation, the production rate  $P$  is partitioned to  $p_j$  ( $j = 1, \dots, m$ ) according to the weight manifested by the relative demand at each distributor such that

$$p_j = (\sum_{k=1}^{n_j} d_{jk} / \sum_{j=1}^m \sum_{k=1}^{n_j} d_{jk}) P, \quad (3.5)$$

And  $P = \sum_{j=1}^m p_j$ . Figure 3.4 shows, the producer's inventory considered just for the distributor  $j$ 's portion. At the beginning, the inventory level at the producer level for distributor

$j$ ,  $I_j^p$ , is zero. So after  $Q_j^d / p_j$  time units, the first shipment is made to distributor  $j$ . This first shipment is assumed to happen at time zero while the production actually starts  $Q_j^d / p_j$  time units earlier. Producer begins its production at the rate of  $P$ , of which,  $p_j$  contributes to fulfill distributor  $j$ 's demand. The inventory is being built from  $Q_j^d / p_j$  time units prior to the first shipment of  $Q_j^d$  units, resulting to zero inventory at the producer level. The second shipment of  $Q_j^d$  units is made by the producer to the distributor  $j$  after  $T^p / N_j^p$  time units. The production run continues until time  $(N_j^p - 1)Q_j^d / p_j$  when the inventory level reaches its peak and stays constant until the next shipment is made after the production stops. So during this downtime period, the producer keeps delivering  $Q_j^d$  units/shipment to distributor  $j$  at every  $T^p / N_j^p$  until the time  $(N_j^p - 1)T^p / N_j^p$  when the inventory level falls to zero. At the time  $T^p - Q_j^d / p_j$ , the producer starts its new production cycle again. Since  $p_j \geq \sum_{k=1}^{n_j} d_{jk}$ , the inventory at the producer level builds up to be shipped  $N_j^p$  times to distributor  $j$  during one cycle time.

Computing the producer's average inventory level is involved. The time-weighted inventory ( $TWI$ ) is calculated for each shipment from the producer to a distributor in the producer's cycle time. All  $TWIs$  are summed up for all inventories generated at the producer for entire period  $T^p$ . The average inventory at the producer level is  $\bar{I}^p = \sum_{j=1}^{n_j} \bar{I}_j^p$ . Each  $\bar{I}_j^p$  is  $\bar{I}_j^p = \sum TWI / T^p$  where  $T^p$  is the production cycle time.

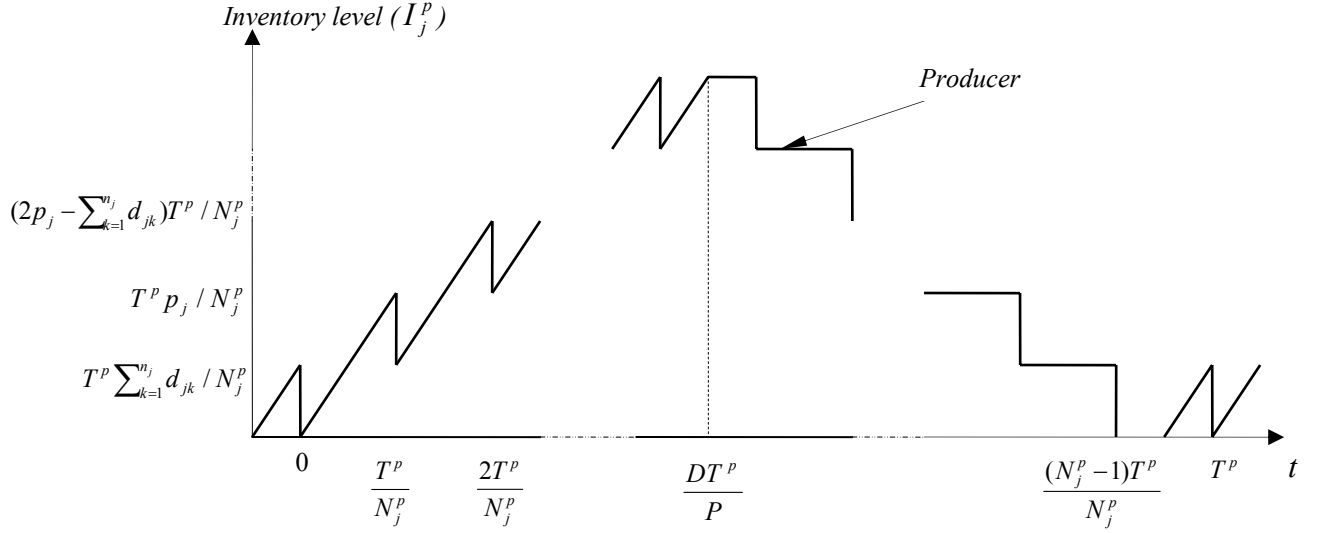


Figure 3.4 The producer's inventory level portioned to the distributor  $j$

According to Figure 3.5,  $TWI$  for each distributor's inventory is the summation of the areas  $\alpha$ ,  $\beta$ ,  $\gamma$ , ...,  $\delta$ ; that is, for example,  $TWI_\alpha$ ,  $TWI_\beta$  and  $TWI_\delta$  are the  $TWI$  for the first, second and the last shipment to retailers, respectively. To calculate  $TWI$ , each left-angled trapezium is divided into two parts; triangle and rectangle except the first time-weighted inventory  $TWI_\alpha$  which is only triangle shaped. Figure 3.5 shows that there are  $N_j^P$  triangles while there are  $N_j^P - 1$  rectangles for each production cycle time. All  $N_j^P$  triangles are identical with the size of  $(Q_j^d)^2 / 2p_j$ . Therefore, the total area of all  $N_j^P$  triangles is  $N_j^P (Q_j^d)^2 / 2p_j$ . As shown in Figure 3.5, the rectangular area  $\beta$  defined by the width of  $T^P / N_j^P - Q_j^d / p_j$  and the height of  $Q_j^d$ . Therefore, the area of rectangular section of  $\beta$  is  $(T^P / N_j^P - Q_j^d / p_j)Q_j^d$ . Next, the rectangular area  $\gamma$  is consisted of the width of  $T^P / N_j^P - Q_j^d / p_j$  and the height of  $Q_j^d$ . Therefore, the rectangular area of  $\gamma$  is  $2Q_j^d(T^P / N_j^P - Q_j^d / p_j)$ . Similarly, the rectangular the area  $\delta$  is equal to  $Q_j^d(N_j^P - 1)(T^P / N_j^P - Q_j^d / p_j)$ .

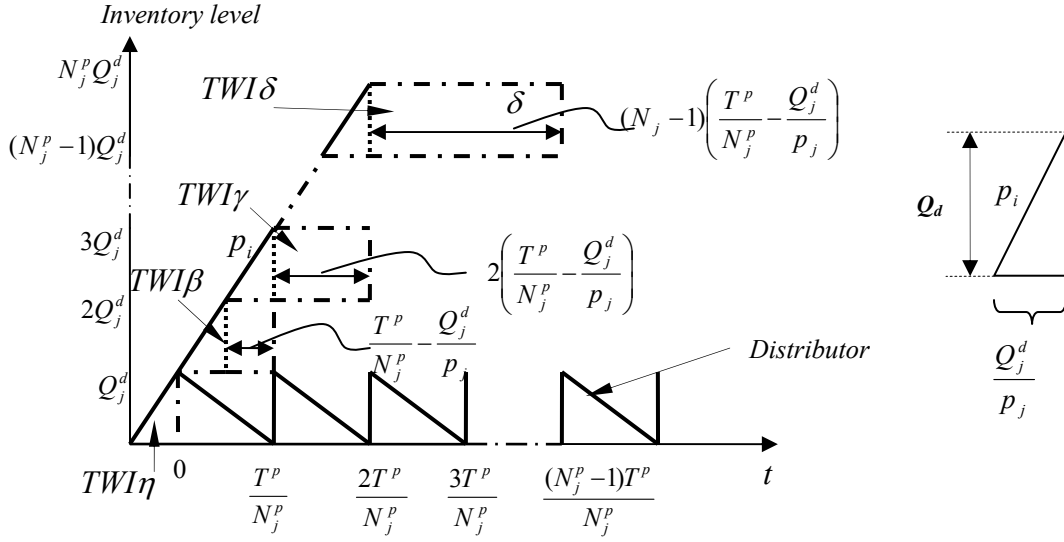


Figure 3.5 Producer's  $TWI$  for the distributor  $j$ 's and the distributor  $j$ 's inventory

Therefore, the total area of all rectangles is

$$\Sigma_{Rectangles} = (T^p / N_j^p - Q_j^d / p_j) Q_j^d + 2Q_j^d (T^p / N_j^p - Q_j^d / p_j) + \dots + Q_j^d (N_j^p - 1) (T^p / N_j^p - Q_j^d / p_j),$$

which reduces to

$$\Sigma_{Rectangles} = Q_j^d (T^p / N_j^p - Q_j^d / p_j) [1 + 2 + \dots + (N_j^p - 1)].$$

As  $T^p / N_j^p$  is  $T_j^d$  and  $T_j^d$  is  $Q_j^d / \sum_{k=1}^{n_j} d_{jk}$ , the previous term becomes

$$\Sigma_{Rectangles} = \frac{(Q_j^d)^2}{\sum_{k=1}^{n_j} d_{jk}} \left( 1 - \sum_{k=1}^{n_j} d_{jk} / p_j \right) [1 + 2 + \dots + (N_j^p - 1)].$$

Since  $1 + 2 + \dots + (N_j^p - 1)$  is replaced by  $(N_j^p - 1)N_j^p / 2$ ,  $\sum_{k=1}^{n_j} d_{jk} / p_j$  is replaced by  $D / P$ , and

$Q_j^d / \sum_{k=1}^{n_j} d_{jk}$  is replaced by  $T^p / N_j^p$ , the total of rectangular areas for distributor  $j$  is

$$\Sigma_{Rectangles} = \frac{T^p Q_j^d}{2} \left( 1 - \frac{D}{P} \right) (N_j^p - 1). \quad (3.6)$$

After adding all triangular and rectangular areas in Figure 3.5 for all distributors together and replacing  $Q_j^d$  by  $T^p \sum_{k=1}^{n_j} d_{jk} / N_j^d$ , the average producer inventory is finally obtained, and the total finished goods inventory cost,  $TC_{HC}^P$ , can be written as

$$TC_{HC}^P(T^p, N_j^p) = H_p \sum_{j=1}^m \left[ \frac{DT^p \sum_{k=1}^{n_j} d_{jk}}{2PN_j^p} + \frac{T^p \sum_{k=1}^{n_j} d_{jk}}{2N_j^p} \left(1 - \frac{D}{P}\right) (N_j^p - 1) \right]. \quad (3.7)$$

### 3.3.2 Setup Cost

The setup cost,  $A_p$ , refers to the production run setup cost at the producer level, and  $A_d$  and  $A_r$  refer to the ordering costs at distributor and retailer levels, respectively. These ordering costs include all associated costs including transportation, and  $A_d$  are same for each of the distributors, and  $A_r$  is same for each of the retailers. The distributor  $j$  places orders  $N_j^p$  times to producer during production cycle time  $T^p$ . The retailer  $k$  orders  $N_{jk}^d$  times to the distributor  $j$  during distributor's one shipment interval  $T_j^d$ ; hence, there are  $N_j^p N_{jk}^d$  orders from the retailers during one production cycle time  $T^p$ . Thus, the total yearly setup cost is written as

$$TC_{SC}(T^p, N_j^p, N_{jk}^d) = \frac{A_p + \sum_{j=1}^m A_d N_j^p + \sum_{j=1}^m \sum_{k=1}^{n_j} A_r N_j^p N_{jk}^d}{T^p}. \quad (3.8)$$

### 3.3.3 Backorder Cost

As we mentioned earlier, the nature of products to which we refer are high-cost products. So, shortage is assumed to have no effect to their buying decision since not many inventory systems maintain high inventory of such costly items. In other words, customers who have already made up their minds will not go and buy product somewhere else but wait for the products to become available at the stores. Therefore, cost of lost sale and penalty cost which depend on the time period which the products are being out of stock are not applied here. The only cost is an extra cost incurred by activities to complete backorders which we determine



here as  $\pi$ . For each retailer, during each retailer replenishment interval, at the time when retailer receives instantaneous shipment from the distributor, the inventory level immediately goes up and reaches level  $d_{jk}T_{jk}^r - b_{jk}$ . Then, the inventory level constantly goes down at the retailer's constant demand rate  $d_{jk}$ . The inventory level reaches zero level at the time  $T_{jk}^r - t_{jk}^r$ . At the time, there is no product left in retailer inventory, and the customers who place the orders after that are informed about the situation and the expected time that products will be available to be delivered to them. There are no actual products delivered to customer during that time but the inventory level shown on graph keep going down to show the demand which is not met and accounted to be satisfied when the products become available. Since, the stock out time  $t_{jk}^r$  equals to  $(1 - R_{jk})T_{jk}^r$ , the backorder during  $T_{jk}^r$  is  $(1 - R_{jk})d_{jk}T_{jk}^r$ . Therefore, the total amount of backorder ( $B_{jk}$ ) per year is

$$B_{jk} = (1 - R_{jk})d_{jk}T_{jk}^r. \quad (3.9)$$

At the time products are replenished by the distributor, the inventory level reaches negative  $b_{jk}$ . The total number of products which are replenished is  $d_{jk}T_{jk}^r$ . We assume  $\pi$  as time-independent cost for number of units short. The backorder cost which retailer  $k$  needs to pay extra for, those activities which are required to complete backorder during one replenishment interval, is  $\pi b_{jk}$ . So, for each retailer cycle, retailer pays  $\pi b_{jk}$  regardless of the length of time that products are being out of stock. The total cost of backorder of the whole system is  $TC_{BC} = \sum_{j=1}^m \sum_{k=1}^{n_j} \pi B_{jk}$ . Since  $B_{jk} = (1 - R_{jk})d_{jk}T_{jk}^r$ , we can rewrite the equation for  $TC_{BC}$  as

$$TC_{BC}(R_{jk}) = \sum_j^m \sum_{k=1}^{n_j} d_{jk} (1 - R_{jk}) \pi. \quad (3.10)$$

### 3.3.4 Total Cost Function, $TC_{TDB}$

All cost functions described above are collected and combined into one equation. As mentioned earlier, the total cost function consists of four costs: holding cost, shipping and ordering cost, and backorder cost while the holding cost is composed of three other costs due to finished product inventory, distributors' inventory and retailers' inventory. The total cost of this system is a function of four variables  $T^p, N_j^p, N_{jk}^d$ , and  $R_{jk}$  while all other parameters are known. Thus, the total cost function,  $TC_{TDB}(T^p, N_j^p, N_{jk}^d, R_{jk})$ , can be written as

$$\begin{aligned}
 TC_{TDB}(T^p, N_j^p, N_{jk}^d, R_{jk}; j=1, \dots, m; k=1, \dots, n_j) \\
 &= TC_{HC}^p(T^p, N_j^p) + TC_{HC}^d(T^p, N_j^p, N_{jk}^d) + TC_{HC}^r(T^p, R_{jk}, N_j^p, N_{jk}^d) \\
 &\quad + TC_{SC}(T^p, N_j^p, N_{jk}^d) + TC_{BC}(R_{jk}) \\
 &= H_p \sum_{j=1}^m \left[ \frac{DT^p \sum_{k=1}^{n_j} d_{jk}}{2PN_j^p} + \frac{T^p \sum_{k=1}^{n_j} d_{jk}}{2N_j^p} \left(1 - \frac{D}{P}\right) (N_j^p - 1) \right] \\
 &\quad + H_d \sum_{j=1}^m \sum_{k=1}^{n_j} \frac{d_{jk} T^p}{2N_j^p} \left(1 - \frac{1}{N_{jk}^d}\right) + H_r \sum_{j=1}^m \sum_{k=1}^{n_j} \frac{d_{jk} T^p R_{jk}^2}{2N_j^p N_{jk}^d} \\
 &\quad + \frac{A_p + \sum_{j=1}^m A_d N_j^p + \sum_{j=1}^m \sum_{k=1}^{n_j} A_r N_j^p N_{jk}^d}{T^p} + \sum_{j=1}^m \sum_{k=1}^{n_j} d_{jk} (1 - R_{jk}) \pi . \quad (3.11)
 \end{aligned}$$

### 3.4 Optimization Solution Procedure

The integrated total cost function in Equation (3.11) is a convex function in  $R_{jk}$ ,  $N_{jk}^d$ ,  $N_j^p$ , and  $T^p$  [see Appendix A3.1]. Hence,  $\partial TC_{TDB} / \partial R_{jk} = 0$ ,  $\partial TC_{TDB} / \partial N_{jk}^d = 0$ ,  $\partial TC_{TDB} / \partial N_j^p = 0$ , and  $\partial TC_{TDB} / \partial T^p = 0$  leads to the solutions as [see Appendix A3.1 for details]:

$$T^{p*} = \sqrt{\frac{2PA_p}{H_p D(P-D)}} \quad , \quad (3.12)$$

$$N_j^{p*} = \sqrt{\frac{\sum_{k=1}^{n_j} d_{jk} A_p (PH_d + 2DH_p - PH_p)}{A_d H_p D(P-D)}} \quad , \quad (3.13)$$

$$N_{jk}^{d*} = \sqrt{\frac{2PA_d H_r H_d d_{jk}}{\sum_{k=1}^{n_j} d_{jk} (PH_d + 2DH_p - PH_p) (d_{jk} \pi^2 - 2H_r A_r)}} \quad , \quad (3.14)$$

and

$$R_{jk}^* = \frac{\pi}{H_r} \sqrt{\frac{H_r H_d d_{jk}}{(d_{jk} \pi^2 - 2H_r A_r)}} \quad . \quad (3.15)$$

### 3.5 Computational Results

This section provides a numerical example to illustrate the solution procedure for a three-stage system with one manufacturer, two distributors, and four retailers (two retailers for each distributor).

#### Example 3.1: TDB Model

Assume a system with parameters obtained from the example presented in Wee (2003) as  $A_p = \$400/\text{set up}$ ,  $A_d = \$25/\text{order}$ ,  $A_r = \$2/\text{order}$ ,  $H_p = \$4/\text{unit/year}$ ,  $H_d = \$5/\text{unit/year}$ ,  $H_r = \$15/\text{unit/year}$ , and  $P = 5,000 \text{ units/year}$ . The other input data are  $d_{11} = 100 \text{ units/year}$ ,  $d_{12} = 150 \text{ units/year}$ ,  $d_{21} = 225 \text{ units/year}$ ,  $d_{22} = 337 \text{ units/year}$ , such that  $D = 812 \text{ units/year}$ . The unit shortage cost is assumed as  $\pi = \$1/\text{unit}$  (which is not in Wee 2003).

First, closed-form solutions of all variables,  $N_j^p (j = 1, 2)$ ,  $N_{jk}^d (j, k = 1, 2)$ ,  $R_{jk} (j, k = 1, 2)$ , and  $T^p$  were computed by using the Equations (9), (10), (11), and (12). The calculated results are shown in Table 3.1 and the corresponding total cost,  $TC_{TDB}(T^p, N_j^p, N_{jk}^d, R_{jk})$ , is obtained as \$2,320.

Table 3.1 Closed-form solutions

$N_1^p$	$N_2^p$	$N_{11}^d$	$N_{12}^d$	$N_{21}^d$	$N_{22}^d$	$R_{11}$	$R_{12}$	$R_{21}$	$R_{22}$	$T^p$
1.8385	2.7577	4.0379	3.2969	1.9881	1.8776	0.9129	0.7454	0.6742	0.6367	0.5421

Since the computed values of  $N_j^p (j=1,2)$  and  $N_{jk}^d (j,k=1,2)$  are integer, the real values obtained for them need to be refined to some integer values. A refinement scheme is pursued by changing *one particular variable* to integers on both sides of the real value of that variable, holding *all other variables* fixed to previously obtained values. As sample computations, the corresponding total costs are recorded in Tables 3.2 and 3.3 for ten different variations of those particular variables. The effects of variations are observed and the convexities of the total cost function are characterized through these variables.

Table 3.2. Effects of variation of  $N_j^p$  and  $N_{jk}^d$  on  $TC_{TDB}$ 

$N_1^p$	$TC(N_1^p)$	$N_2^p$	$TC(N_2^p)$	$N_{11}^d$	$TC(N_{11}^d)$	$N_{12}^d$	$TC(N_{12}^d)$	$N_{21}^d$	$TC(N_{21}^d)$	$N_{22}^d$	$TC(N_{22}^d)$
1.00	2,371	1.75	2,355	1	2,383	1	2,356	1	2,330	1	2,328
1.25	2,340	2.00	2,337	2	2,334	2	2,326	2	2,320	2	2,320
1.50	2,326	2.25	2,327	3	2,322	3	2,320	3	2,324	3	2,324
1.75	2,320	2.50	2,322	4	2,320	4	2,321	4	2,331	4	2,331
2.00	2,321	2.75	2,320	5	2,321	5	2,324	5	2,339	5	2,340
2.25	2,326	3.00	2,321	6	2,324	6	2,328	6	2,348	6	2,349
2.50	2,333	3.25	2,325	7	2,329	7	2,333	7	2,357	7	2,358
2.75	2,342	3.50	2,330	8	2,333	8	2,339	8	2,366	8	2,368
3.00	2,353	3.75	2,336	9	2,339	9	2,345	9	2,376	9	2,377
3.25	2,365	4.00	2,343	10	2,344	10	2,350	10	2,386	10	2,387

Tables 3.2 and 3.3 present the impacts of changing the values of  $N_j^p$ ,  $N_{jk}^d$ ,  $T^p$ , and  $R_{11}$  on the total costs. The results show that as the values of each of variables is changing, one at a time and all parameters and other variables remain constant, the total cost reduces at the beginning and reaches the lowest point of \$2,320 at the optimum point presented in Table 3.1.

Then, the total costs start to increase as the variables increase from the optimal value of each variable.

Figure 3.6 presents the graphs plotted for  $N_j^p$  and  $N_{jk}^d$  to  $TC_{TDB}$ . The range of the input values for  $N_1^p$  is from 0.25 to 3.75 and from 1.75 to 4 for  $N_2^d$ . The values of  $TC_{TDB}(N_1^p)$  quickly decreases at the beginning, slows down as  $N_1^p$  approaches the optimal point, and reaches the minimum point of \$2,320 when  $N_1^p$  reaches 1.84. Then,  $TC_{TDB}$  slowly increases as  $N_1^p$  increases.

Table 3.3. Effects of variation of  $T^p$  and  $R_{jk}$  on  $TC_{TDB}$

$T^p$	$TC(T^p)$	$R_{11}$	$TC(R_{11})$	$R_{12}$	$TC(R_{12})$	$R_{21}$	$TC(R_{21})$	$R_{22}$	$TC(R_{22})$
0.1	6,066	0.1	2,356	0.1	2,362	0.1	2,375	0.1	2,396
0.2	3,442	0.2	2,348	0.2	2,350	0.2	2,357	0.2	2,370
0.3	2,694	0.3	2,340	0.3	2,340	0.3	2,343	0.3	2,350
0.4	2,417	0.4	2,334	0.4	2,332	0.4	2,332	0.4	2,335
0.5	2,327	0.5	2,329	0.5	2,326	0.5	2,325	0.5	2,325
0.6	2,331	0.6	2,325	0.6	2,322	0.6	2,321	0.6	2,320
0.7	2,388	0.7	2,322	0.7	2,320	0.7	2,320	0.7	2,321
0.8	2,479	0.8	2,320	0.8	2,320	0.8	2,322	0.8	2,327
0.9	2,593	0.9	2,320	0.9	2,322	0.9	2,328	0.9	2,338
1.0	2,722	1.0	2,320	1.0	2,326	1.0	2,337	1.0	2,355

Similarly,  $TC_{TDB}(N_2^p)$  quickly decreases at the beginning, slows down as  $N_{11}^d$  approaches the optimal point, and reaches the minimum point of \$2,320 when  $N_1^p$  reaches 2.76. Then,  $TC$  slowly increases as  $N_1^p$  increases. The range of the input values for all  $N_{jk}^d$  is from 1 to 10. Similarly, The graphs  $TC_{TDB}(N_{11}^d)$ ,  $TC_{TDB}(N_{12}^d)$ ,  $TC_{TDB}(N_{21}^d)$ , and  $TC_{TDB}^*(N_{22}^d)$  are in convex forms where the minimum's of the graphs reach the total cost at \$2,320.

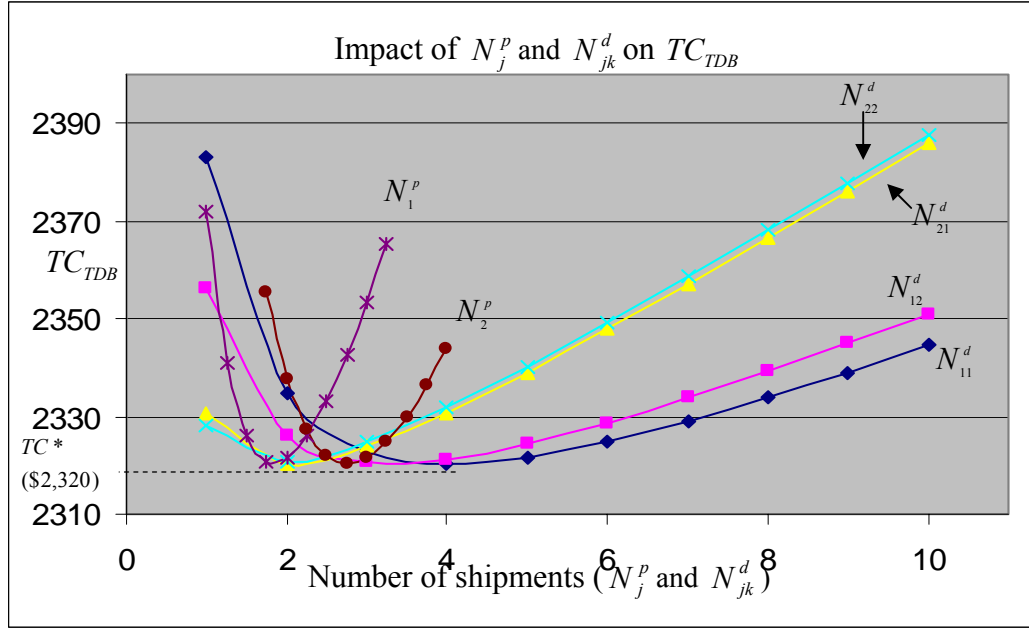


Figure 3.6. Impacts of variation of  $N_j^p$  and  $N_{jk}^d$  on  $TC_{TDB}$

Figure 3.7 shows the trend of  $TC_{TDB}$  when  $T^p$  is changed from 0.1 to 1 at an interval of 0.1. The total cost,  $TC_{TDB}$ , quickly decreases at the beginning, slows down as  $T^p$  approaches the optimal point, and reaches the minimum point of \$2,320 when  $T^p$  reaches 0.5421 year when  $TC_{TDB}$  slowly starts increasing as  $T^p$  increases.

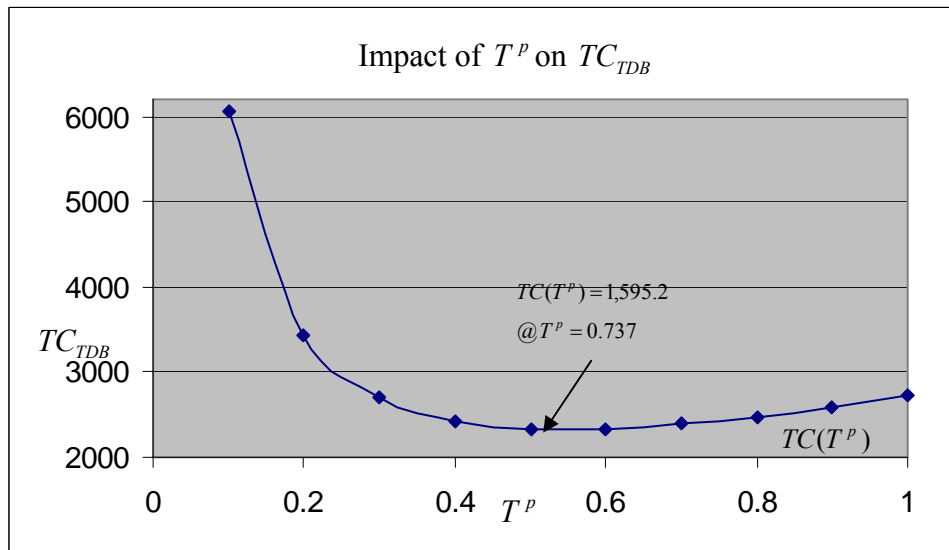


Figure 3.7 Effect of producer cycle time on total cost

Again, Figure 3.8 shows the trend of  $TC_{TDB}$  when  $R_{11}$ ,  $R_{12}$ ,  $R_{21}$ , and  $R_{22}$  are changed from 0.1 to 1.0 at an interval of 0.1. It is evident that  $TC_{TDB}$  behaves as a convex function in  $R_{jk}$ , and the minimum cost is \$2,320 at  $R_{11} = 0.9129$ ,  $R_{12} = 0.7454$ ,  $R_{21} = 0.6742$ , and  $R_{22} = 0.6367$ . Observing Figure 2.8, after passing the optimal point of each  $R_{jk}$ ,  $TC_{TDB}$  starts to increase as it approaches to higher values.

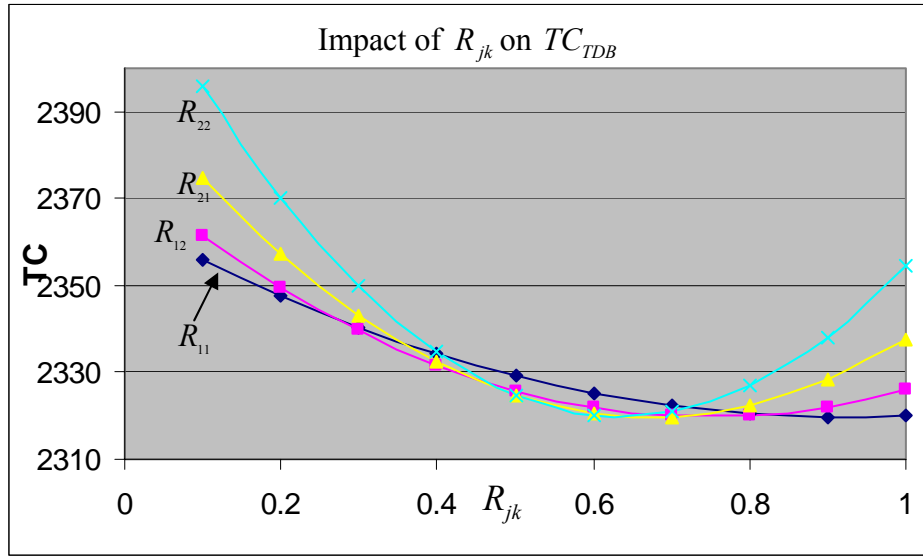


Figure 3.8 Effect of service rate on total cost

While the number of shipments are integer values, the optimal number of shipments computed for the producer to distributors ( $N_1^p$  and  $N_2^p$ ) and for distributors to retailers ( $N_{11}^d$ ,  $N_{12}^d$ ,  $N_{21}^d$ , and  $N_{22}^d$ ) are not integers. A sequential approach (decision tree) is used to find the combinations of the integer numbers of shipments that yield the lowest total cost as presented in Figure 3.9.

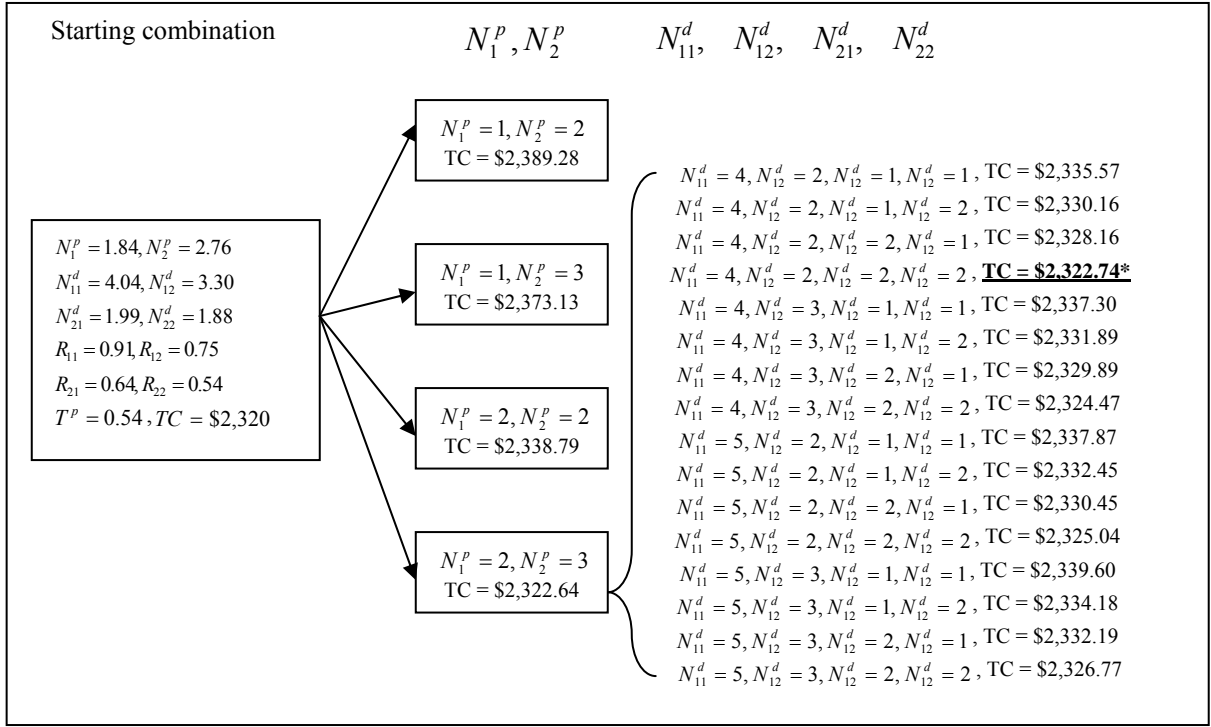


Figure 3.9 Decision tree to adjust  $N_j^p$  and  $N_{jk}^d$  to integer number of shipments

As shown in Figure 3.9, the starting condition is that  $N_1^p = 1.84$ ,  $N_2^p = 2.76$  and  $N_{11}^d = 4.04$ ,  $N_{12}^d = 3.30$ ,  $N_{21}^d = 1.99$ , and  $N_{22}^d = 1.88$  which gives the total cost equal to \$2,320. First, for  $N_1^p$  and  $N_2^p$ , the calculated optimums are 1.84 and 2.76 for which the possible integers could be either 1 or 2 and 2 or 3, respectively. There are four ( $2^2$ ) possible combinations of  $N_1^p$  and  $N_2^p$ . The results show that the combination of  $N_1^p = 2$  and  $N_2^p = 3$  gives the lowest total cost of  $TC_{TDB} = \$2,322.64$ . Next, for  $N_{11}^d$ ,  $N_{12}^d$ ,  $N_{21}^d$  and  $N_{22}^d$ , there are 16 possible combinations. The combination of  $N_{11}^d = 4$ ,  $N_{12}^d = 2$ ,  $N_{21}^d = 2$  and  $N_{22}^d = 2$  gives the lowest total cost of  $TC_{TDB} = \$2,322.74$ . Table 3.4 shows the final version of the decision variables that the  $N_j^p$  and  $N_{jk}^d$  have been adjusted to integer number of shipments.



Table 3.4 Adjusted sub-optimal variables

$N_1^p$	$N_2^p$	$N_{11}^d$	$N_{12}^d$	$N_{21}^d$	$N_{22}^d$	$T^p$	$TC_{TDB}$
2	3	4	2	2	2	0.2941	\$2,322.74

Comparing the total costs computed using calculated variables in Table 2.1 and the one using integerized variables in Table 3.4 shows the slightly difference. The total cost computed by using non integerized variables is \$2,320, while the one computed by using integerized variables is \$2,322.74 which is \$2.24 higher than the non integerized total cost. The percentage of error is 0.09% which is minimum. The next chapter will show the proposition of the research which includes what have been done and the plan to complete the research.

### 3.6 Operational Schedule and Implementation

This chapter deals with creating operational schedules to illustrate the replenishment policy developed here. These schedules show specific values for all variables and parameters to assist in understanding how the solutions are implemented.

This replenishment policy considers the case of three stage supply chain network with one producer, multiple distributors, and multiple retailers, and, the case where backorders are allowed for certain period of time. The producer makes products to satisfy the demand at the distributors that also are ultimately determined by the demands at the retailers. The parameters used in Example 3.1 are also used to show the operational schedule.

Table 3.5 Integer solutions for number of shipments

$N_1^p$	$N_2^p$	$N_{11}^d$	$N_{12}^d$	$N_{21}^d$	$N_{22}^d$	$R_{11}$	$R_{12}$	$R_{21}$	$R_{22}$	$T^p$	$TC_{TDB}$
2	3	4	2	2	2	0.9129	0.7454	0.6742	0.6367	0.5421	\$2,322

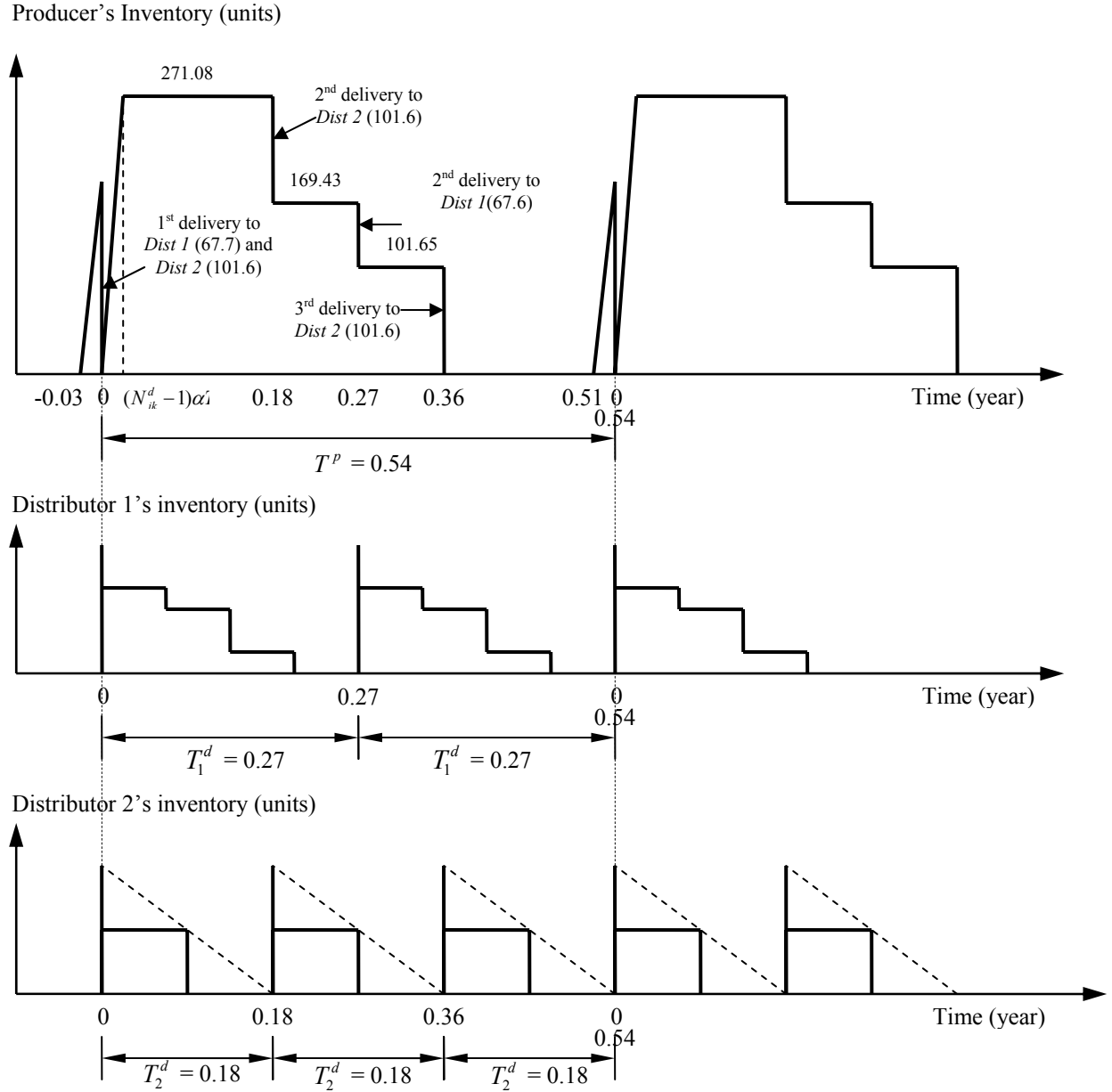


Figure 3.10 Operation schedule between the producer and distributors 1 and 2

By utilizing the sub-optimal solutions demonstrated in Section 3.4, the results are as shown in Table 3.5. Figure 3.10 shows the operational schedules of the activities and transactions between the producer and distributor 1 and 2. The producer starts its production at the production rate of 5,000 units/year 0.08 year before the first shipments to distributors 1 and 2 are made. At time zero, the production, which has been running for 0.8 year, is stopped and

the inventory reaches the peak to 440.49 units which will exactly meet all combined downstream demands for entire production cycle time of 0.54 year.

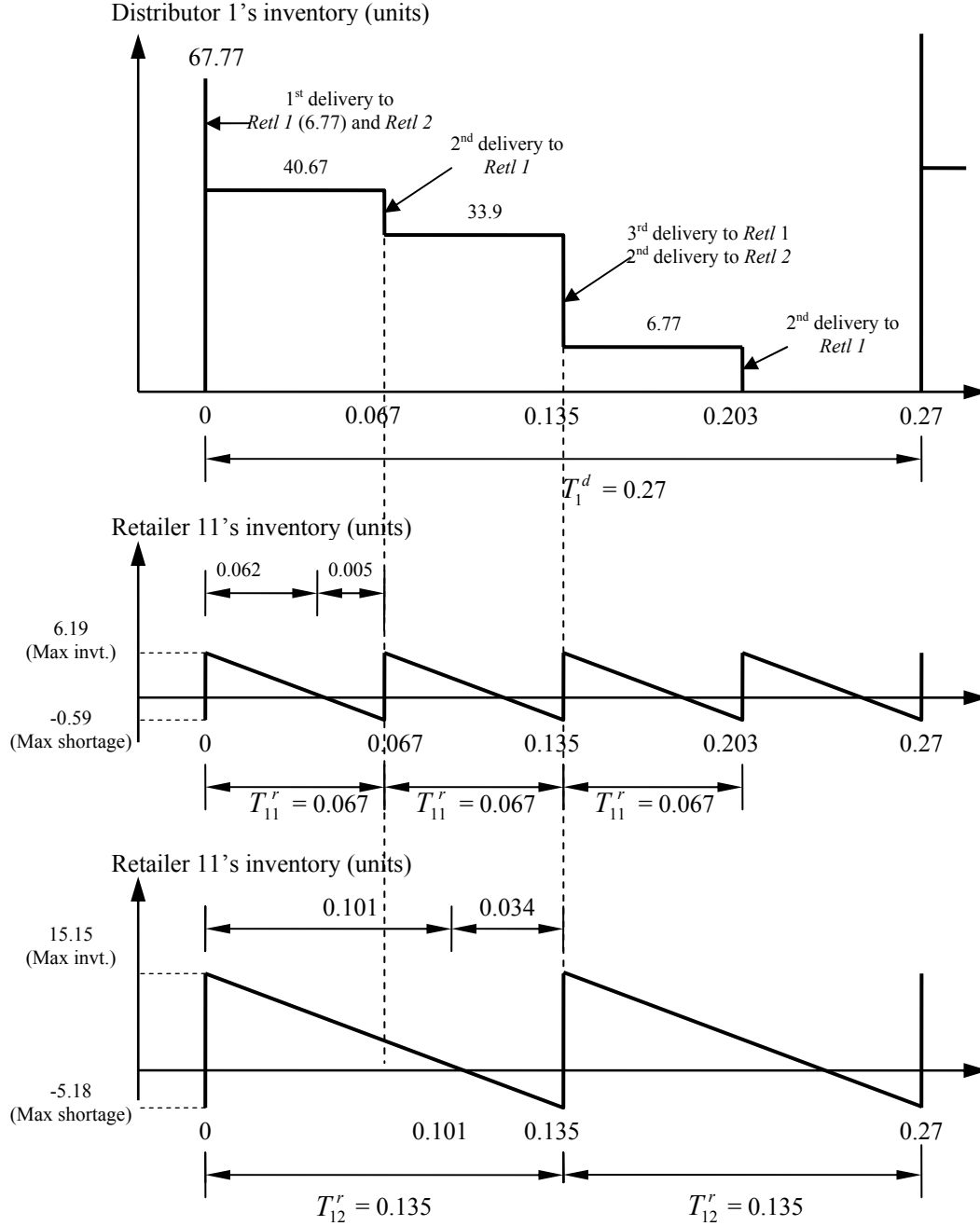


Figure 3.11 Operational schedules between the distributor 1 and retailer 11 and 12.

Using formula  $T_j^d = T^p / N_j^p$ , the distributors 1 and 2's shipment intervals ( $T_1^d$  and  $T_2^d$ ) are calculated as  $0.54/2 = 0.27$  and  $0.54/3 = 0.18$  year, respectively. After the shipment of 67.7

and 101.6 units have been shipped to distributor 1 and 2, the inventory is falls to 271.08 units and stay constant until the second shipment to distributor 2 is made at time 0.18 year, which results in the inventory declining to 169.43 units. The second delivery to distributor 1 and third delivery to distributor 2 are made at time 0.27 and 0.36 year, respectively, which bring the inventory down to zero until time 0.45 at which time the production of next cycle is started.

Figure 3.11 shows the operational schedule of activities and transactions between the distributor 1 and retailer 11 and 12. At time zero, distributor 1's inventory is replenished by producer resulting in the inventory rising to 67.77, and, at the same time, distributor 1 delivers the first shipments to retailer 11 and retailer 12 which brings distributor 1's inventory down to 40.67 units. The inventory at the distributor stays constant until the second delivery is made to retailer 11, and the distributor 1's inventory has dropped to 33.9 and stays constant until time 0.135 at which time the third delivery to retailer 11 and second delivery to retailer 12 is made. Therefore, the inventory falls to 6.77 and the remainder is shipped to retailer 11 which leads distributor 1's inventory drop to zero before being replenished at time 0.27 which is a starting point for the next cycle. For retailer 11, at time zero, its inventory is replenished by distributor 1 with the shipment size of 6.7 units of which 0.59 units will be immediately utilized to fulfilled the backorders from the previous cycle, and the remainder of products will go to retailer 11's warehouse which will cause retailer 11's inventory to rise to 6.19. As time elapse, the inventory shrinks at retailer 11's demand rate which is 100 units/year. When time reaches 0.061, retailer 11's inventory falls to zero. The orders which are placed after this point are processed as backorder. At time 0.067, the inventory of retailer 11 is replenished and backorders are fulfilled.

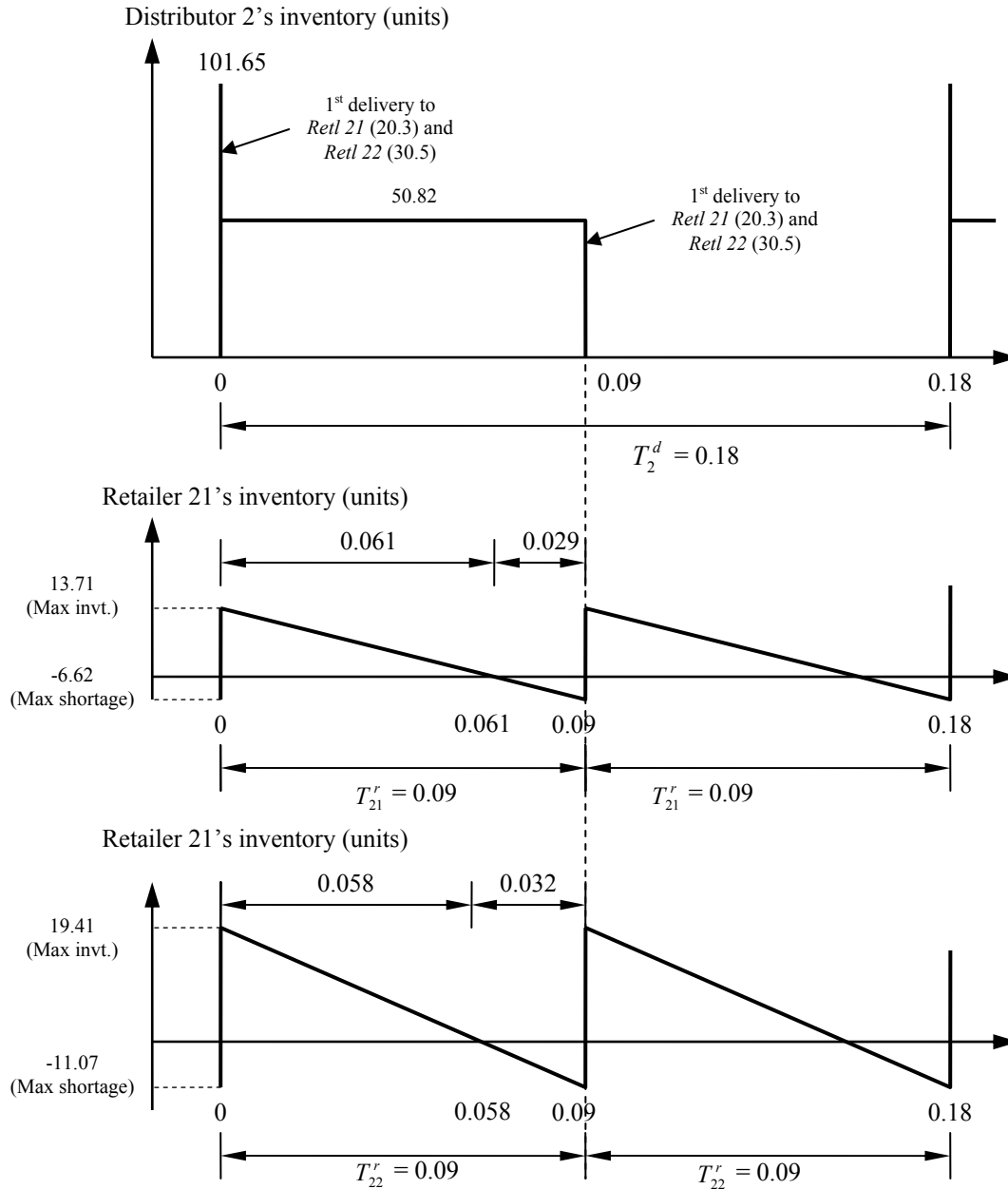


Figure 3.12 Operational schedules between the distributor 2 and retailer 21 and 22.

Figure 3.12 presents operational schedules between the distributor 2 and retailers 21 and 22. At time zero, the distributor 2's inventory is replenishment and inventory climbs to 101.65, but immediately goes down to 50.82 units since the shipments to retailer 21 and 22 are made. Distributor 2's inventory has stayed constant at 50.82 before dropping to zero at time 0.09 when the second deliveries to retailers 21 and 22 were made. Next, at time zero, retailer 21's

inventory is replenished. The backorder of 11.07 units is fulfilled and the inventory goes up to 19.41 units. Then, the inventory is declining at the retailer 21's demand rate of 337 units/year. At time 0.058, retailer 21's inventory approaches zero, but the order from customers are still accepted as backorder. At time 0.09, the inventory at retailer's 21 is replenished and backorders are completed.

## CHAPTER 4

### DOWNSTREAM HOLDING COST VARIATION (DHV)

This chapter considers an inventory supply chain problem comprised of a single producer, multiple distributors and multiple retailers. Unlike the first model, this model assumes a nonincreasing holding cost rate in downstream supply stages; that is, it assumes that the holding cost rate at the distributor(s) is at least equal to or higher than the holding cost rate at the retailer levels downstream, in which situation results in inventories build-up at the retailers' levels. This assumption has been explained in Zanomi and Grubbstrom (2004), and Hill and Omar (2006). This model extends the combined effect of Hill's model (2006) and the TDB cost model, the first model in which the holding costs are in nondecreasing order downstream the supply line. While Hill's model considers a two-echelon problem, this model deals with a three-echelon problem. Due to the higher holding cost at the distributor, the products tend to be transferred from the distributor more rapidly to keep the average inventory at the distributor low, generating cost saving. On the other hand, the inventories at the retailers build up resulting in increasing of the average inventory. The proposed three-echelon model will be more practical since the two-echelon supply chain system is rarely found in the real business world. Moreover, the model for three-echelon supply chain system is relatively easier to be expanded to the  $n$ -echelon model in the future.

The TDB model is extended into two models: Model 2 (DHV) and Model 3 (IRS). First, in Model 2, we assume the lower downstream holding cost rate. The distributor is assumed to delivers products to the retailer more rapidly to benefit from the lower holding cost at the retailer. Next, since we experienced backorder in Model 1, Model 3 is developed to improve the service rate which will increase the company's goodwill. The delivery interval at retailer is

reduced which is expected to reduce shortage at the retailer. The diagram, which illustrates the relationship among TDB, DHV, and IRS models, is shown in Figure 4.1.

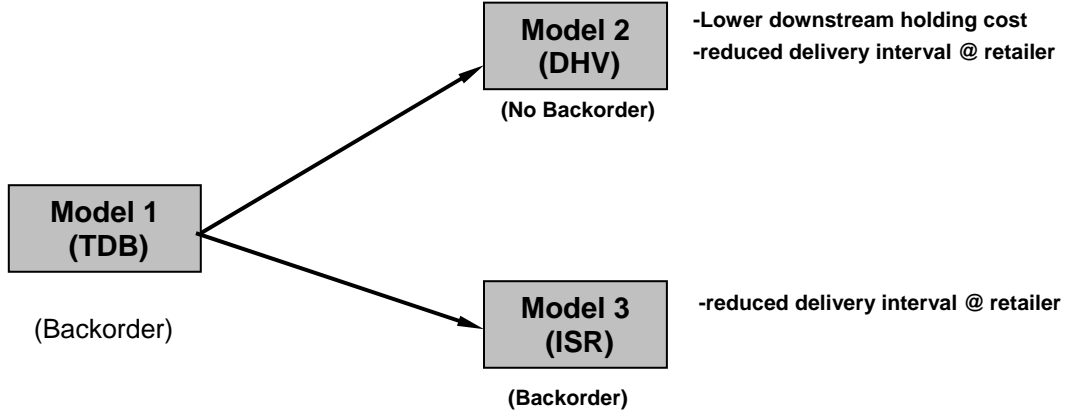


Figure 4.1 Extension of Model 1 (TDB)

The supply chain network in this study is basically the same as the network of the first model (see Figure 4.2); that is, a tree-type three-echelon supply chain network in which a producer produces and delivers the items to  $m$  distributors and each distributor serves subsequently  $n_i$  retailers ( $i = 1, 2, \dots, m$ ). All demand rates at retailers are known and constant. We assume that the producer, distributors, and retailers in the system have complete knowledge of each other's data. As the holding cost rate is higher at the distributors' (upstream) level than the holding cost rate at the retailers' (downstream) level, the retailers shorten their shipment intervals in a proportion of  $\alpha$  ( $0 < \alpha < 1$ ). As a result, the average inventories at the distributors will decrease while those at the retailers will increase. Since the higher holding cost rate at distributor level is greater, decreasing in the inventory at the distributor will reduce the total cost. Thus, the objective of this problem is to find the optimal solutions for the replenishment policy to minimize the total cost of three-echelon problem under the *nonincreasing holding cost* (DHV) model down the supply chain stream. The next section will introduce assumptions and notation which will be used here.



## 4.1 Assumptions and Notation

The following assumptions for the multi-distributor, multi-buyer system under consideration are necessary to formulate the problem. This will follow with a set of parametric notation and variables.

### 4.1.1 Assumptions

- 1.The system considers a single-producer, multi-distributor and multi-retailer problem with a single product.
- 2.The demand rate of the product is finite and constant, and the production rate is also finite and greater than the demand rate.
- 3.The distributor's and the retailer's replenishments are instantaneous.
- 4.The producer, the distributors and the retailers have complete knowledge of each other's information, and the total number of retailers is always at least equal to the total number of distributors, to ensure the flow materials at the distributors and delivery to customers from a distributor.
- 5.Through information sharing, the sales data is received from the retailers to synchronize the producer's, the distributors' and the retailers' inventory levels or lot sizes.
- 6.Shortage is not allowed at any level in the supply tree.
- 7.The holding cost at the distributor is assumed to be higher than the holding cost at the retailers, resulting in the inventory build-up at the retailers.

### 4.1.2 Notation

Most parameters and variables used here have been introduced in Chapter 3. Two new notations used are listed below.

- $\alpha$       Ratio of the shortened replenishment interval of any retailer to its original replenishment interval (assumed it is the same for all).

$\tau_{jk}^r$  Replenishment interval of retailer  $k$  ( $k = 1, \dots, n_j$ ) served by distributor  $j$ , where  $\tau_{jk}^r =$

$$\alpha_{jk} T_{jk}^r \text{ and } T_{jk}^r = T^P / (N_{jk}^d N_j^P).$$

## 4.2 DHV Cost Model

The total inventory cost function consists of two sub-cost functions which are holding costs and setup costs. It can be mathematically written as

$$TC_{DHV} = TC_{HC} + TC_{SC}, \quad (4.1)$$

where  $TC_{HC}$  is the total holding cost and  $TC_{SC}$  is the total setup cost. Each of these costs is described in detail in the following sections.

### 4.2.1 Holding Cost

Holding costs incur at all three stages, producer, distributors, and retailers due to storing products in their respective warehouses. Thus, the total holding cost at three stages is given by

$$TC_{HC} = TC_{HC}^P + TC_{HC}^d + TC_{HC}^r = H_p \bar{I}^P + \sum_{j=1}^m H_d \bar{I}_j^d + \sum_{j=1}^m \sum_{k=1}^{n_j} H_r \bar{I}_{jk}^r. \quad (4.2)$$

### 4.2.2 Retailers' Inventory Level

As the holding cost rate at the distributor is higher than the one at the retailers, a distributor tends to deliver products to the retailer(s) rapidly, resulting in faster inventory built-up at the retailers, which also prevents shortages. Unlike the TDB cost model, the time interval of the deliveries from the distributors to the retailers is less due to the higher holding cost rate at the distributor. In DHV model,  $T_j^d / N_{jk}^d$  is greater than  $\tau_{jk}^r$  while  $T_j^d / N_{jk}^d$  is equal to  $T_{jk}^r$  in the TDB model (i.e., nondecreasing holding cost rate downstream). Here, it is assumed that  $T_j^d / N_{jk}^d = \alpha T_{jk}^r$  where  $\alpha$  is determined as the proportion of  $T_j^d$  to  $T_{jk}^r N_{jk}^d$ . The time when the distributor delivers the first shipment to the retailer, after receiving the delivery from the producer, is set to time zero. At the same time (i.e., at time zero), the distributor delivers the first  $Q_{jk}^r$  units of

products to the retailer resulting in reducing inventory at the distributor from  $d_{jk}T_j^d$  to  $(d_{jk}T_j^d - Q_{jk}^r)$  and increasing inventory at the retailer from zero to  $Q_{jk}^r$ . At time  $\alpha T_{jk}^r$ , the distributor delivers the second shipment of  $Q_{jk}^r$  units of the products to the retailer resulting in a drop in the distributor's inventory to  $d_{jk}T_j^d - 2Q_{jk}^r$  and the retailer adds  $2Q_{jk}^r - \alpha d_{jk}T_{jk}^r$ , because the demand at the retailers during the shipment interval  $\tau_{jk}^r = \alpha T_{jk}^r$  is  $d_{jk}(\alpha T_{jk}^r)$ , which is the reduced amount from total cumulative shipment of  $2Q_{jk}^r$ . During one distributor replenishment cycle ( $T_j^d$ ), there are total  $N_{jk}^d$  shipments from the distributor to the retailer. Since the first shipment occurs at time zero and each of consecutive shipments occur  $\alpha T_j^d / N_{jk}^d$  time units after the previous one, the  $N_{jk}^{d\text{th}}$  shipment is delivered at time  $(N_{jk}^d - 1)\alpha T_j^d / N_{jk}^d$  which is also the time that inventory reaches its peak. At that time (inventory peak time), the cumulative shipment amount is  $N_{jk}^d Q_{jk}^r$  and the demand at retailer during that time  $(N_{jk}^d - 1)\alpha T_j^d / N_{jk}^d$  is  $(N_{jk}^d - 1)\alpha d_{jk} T_{jk}^r$  units which is deducted from the cumulative shipment amount giving the peak inventory of  $N_{jk}^d Q_{jk}^r - (N_{jk}^d - 1)\alpha d_{jk} T_{jk}^r$  units.

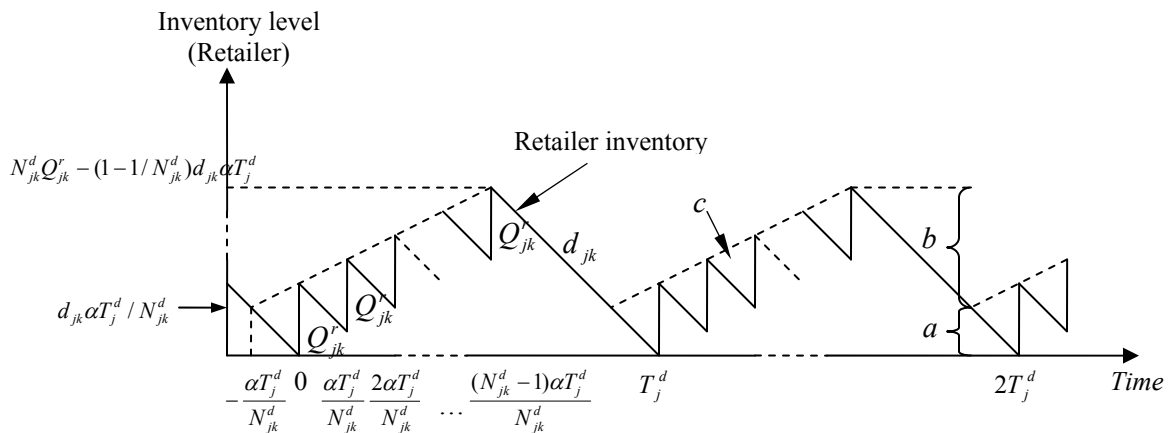


Figure 4.2 Retailer's inventory

The notation  $c$  represents the total area of all triangles in a cycle as indicated in Figure 4.2. At first, by ignoring area  $c$ , the average inventory at the retailer is the summation of  $a$  and a half of  $b$  where  $a$  refers to the retailer's inventory at the time  $\alpha T_j^d / N_{jk}^d$  prior to time zero and  $b$  refers to the difference between the peak inventory and  $a$  ( $b = \text{peak inventory} - a$ ). The size of  $a$  is equal to the amount of the inventory reduced during time zero to  $\alpha T_j^d / N_{jk}^d$  at the demand rate  $d_{jk}$ , which, thus, equals to  $d_{jk} \alpha T_j^d / N_{jk}^d$ . As  $b$  is the result of a subtraction of  $a$  from the peak inventory,  $b$  can be written as  $N_{jk}^d Q_{jk}^r - (1 - 1/N_{jk}^d) d_{jk} \alpha T_j^d - \alpha T_j^d / N_{jk}^d$ . Then the area  $c$  is taken into consideration by subtracting it from the summation of  $a$  and half of  $b$ . The total area  $c$  consists of  $N_{jk}^d$  identical triangles with the height of  $Q_{jk}^r$ , which causes the average inventory, incurred by area  $c$ , to become  $\alpha d_{jk} T_j^d / 2 N_{jk}^d$ . After substituting  $d_{jk} T_j^d / N_{jk}^d$  into  $Q_{jk}^r$ , the average inventory at one retailer,  $\bar{I}_{avg}$ , is

$$\begin{aligned} \bar{I}_{avg} &= \frac{d_{jk} \alpha T_j^d}{N_{jk}^d} + \left( d_{jk} T_j^d - \left(1 - \frac{1}{N_{jk}^d}\right) d_{jk} \alpha T_j^d - \frac{d_{jk} \alpha T_j^d}{N_{jk}^d} \right) / 2 - \frac{\alpha d_{jk} T_j^d}{2 N_{jk}^d} \\ &= \frac{d_{jk} T_j^d}{2 N_{jk}^d} [\alpha_{jk} + N_{jk}^d (1 - \alpha)]. \end{aligned} \quad (4.3)$$

[see Appendix B4.1 for details]. As  $T_j^d = T^p / N_j^p$ , the total holding cost at all retailers for the system can be written as

$$TC_{HC}^r = H_r \sum_{j=1}^m \sum_{k=1}^{n_m} \frac{d_{jk} T^p}{2 N_j^p N_{jk}^d} [\alpha_{jk} + N_{jk}^d (1 - \alpha)]. \quad (4.4)$$

### 4.2.3 Distributor's Inventory Level

It is assumed that the distributor received delivery of  $N_{jk}^d Q_{jk}^r$  units of products from the producer at every  $T_j^d$  time unit. Unlike TDB model, the distributor in this model delivers products to the retailers more rapidly due to the higher holding cost at the distributor.

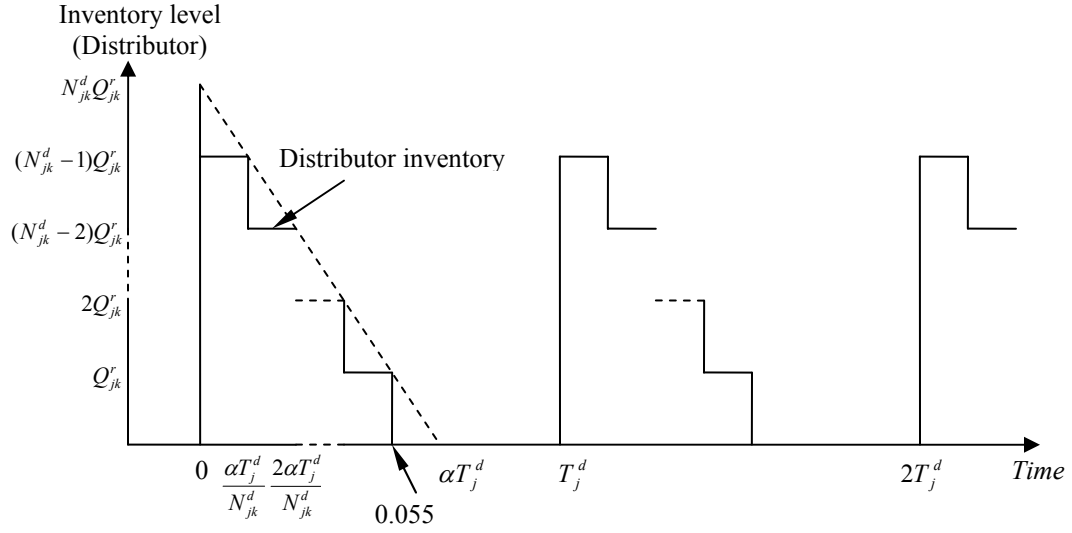


Figure 4.3 Distributor's inventory

As shown in Figure 4.3, at time zero, the producer delivers  $N_{jk}^d Q_{jk}^r$  units of products to the distributor  $j$ , and at the same time the distributor  $j$  ships  $q$  units of the products to the retailer  $k$ , resulting in the remaining inventory of  $(N_{jk}^d - 1)Q_{jk}^r$  units at the distributor  $j$ . This inventory level remains at  $(N_{jk}^d - 1)Q_{jk}^r$  until the distributor  $j$  ships the second shipment to the retailer  $k$  at time  $\alpha T_j^d / N_{jk}^d$  when the inventory drops to  $(N_{jk}^d - 2)Q_{jk}^r$ . The distributor  $j$  delivers the last shipment of  $Q_{jk}^r$  units in its cycle time  $T_j^d$ , to the retailer at time  $(N_{jk}^d - 1)\alpha T_j^d / N_{jk}^d$  when the inventory at the distributor  $j$  becomes zero.

In Figure 4.3, consider the dashed-line triangle. The average inventory during the time zero to  $\alpha T_j^d$  is  $N_{jk}^d Q_{jk}^r / 2$ . However, the distributor  $j$ 's inventory is represented by solid-line graph. The different gap between the dashed line and the solid line graph can be seen as  $N_{jk}^d$  small triangles. So, the average inventory from time zero to  $\alpha T_j^d$  is  $N_{jk}^d Q_{jk}^r / 2 - Q_{jk}^r / 2$  or

$Q_{jk}^r(N_{jk}^d - 1)/2$ . As  $Q_{jk}^r$  is equal to  $d_{jk}T^p / N_j^p, N_{jk}^d$ , the function can be simplified to  $(d_{jk}T^p / 2N_j^p) \times (1 - 1/N_{jk}^d)$ . In this case, since the average inventory is considered for whole  $T_j^d$ , the function is reduced by a factor of  $\alpha$ , resulting in

$$TC_{HC}^d = (T^p, N_j^p, N_{jk}^d) = H_d \sum_{j=1}^m \sum_{k=1}^{n_j} \frac{\alpha d_{jk} T^p}{2N_j^p} \left( 1 - \frac{1}{N_{jk}^d} \right). \quad (4.5)$$

An alternative approach to find the total holding cost at the distributor is shown in Appendix B4.2.

#### 4.2.4 Producer's Inventory Level

The production inventory in this case is similar to that in TDB Model. The producer serves  $m$  distributors and the production rate,  $P$ , is greater than the integrated demand rate (i.e.,  $P \geq \sum_{j=1}^m \sum_{k=1}^{n_j} d_{jk}$ ) combined for all distributors, resulting in inventory being built-up during the time the production is running. Figure 3.4 shows the producer's inventory which is partitioned for the distributor  $j$ 's portion. During one producer cycle time, the production running time is  $DT^p / P$ . To simpler calculation, we assume  $p_j$ , by partitioning the production rate  $P$ , as  $p_j = (\sum_{k=1}^{n_j} d_{jk} / \sum_{j=1}^m \sum_{k=1}^{n_j} d_{jk})P$  that also equates to  $\sum_{k=1}^{n_j} d_{jk} / p_j$  equal to  $D / P$ . The producer starts its production at time  $T^p \sum_{k=1}^{n_j} d_{jk} / p_j N_j^p$  prior to time zero at the production rate  $p_j$ . At time zero, the inventory rises to  $T^p \sum_{k=1}^{n_j} d_{jk} / N_j^p$  (equal to  $Q_j^d$ ) and that entire inventory is delivered to the distributor  $j$  which lowers the inventory to zero. As production is still running, the inventory is built up until the time  $T^p / N_j^p$  when the inventory reaches  $p_j T^p / N_j^p$  and the second delivery of  $T^p \sum_{k=1}^{n_j} d_{jk} / N_j^p$  is made to the distributor  $j$  which

brings the inventory down to  $(p_j - \sum_{k=1}^{n_j} d_{jk})T^p / N_j^p$ . This process repeats until time  $DT^p / P$  which is the time that the exact amount of product for distributor  $j$  has been produced, and production ceases. After that time, due to no production, inventory stays constant and falls every time shipment is made to distributor  $j$ . The last shipment of the cycle is made at time  $(N_j^p - 1)T^p / N_j^p$ , which brings inventory to zero. The production begins to serve the distributor  $j$ 's demand for the next cycle at time  $T^p - T^p \sum_{k=1}^{n_j} d_{jk} / p_j N_j^p$ .

To find the producer's average inventory, the *time-weighted inventory* ( $TWI$ ) is computed. The production starts at  $Q_j^d / p_j$  time units prior to time zero at the production rate  $p_j$  dedicated to distributor  $j$ , and the first shipment is made to the distributor  $j$  at time zero.

In Figure 3.5, the area of  $TWI\eta$  shows the *time-weighted inventory* ( $TWI$ ) for the first shipment. Similarly, the area of  $TWI\beta$  is being produced from time zero to  $Q_j^d / p_j$  and is delivered to the distributor at the time  $T^p / N_j^p$ . Each area from  $TWI\beta$  to  $TWI\delta$  (which is the last one) is composed of a triangle and a rectangular. All triangle parts are identical in shape and size which is  $(Q_j^d)^2 / 2p_j$ . During one producer's production cycle time, there are  $N_j^p$  shipments to the distributor  $j$  and there are also identical  $N_j^p$  triangles. Those triangles have the height of  $Q_j^d$  and the width of the bases of  $Q_j^d / p_j$ . As  $T^p \sum_{k=1}^{n_j} d_{jk} / N_j^p = Q_j^d$  and  $\sum_{k=1}^{n_j} d_{jk} / p_j = D / P$ , the area of one triangle becomes  $D(T^p)^2 \sum_{k=1}^{n_j} d_{jk} / 2P(N_j^p)^2$ . The sum of all  $N_j^p$  areas of those triangles are

$$\sum_{\Delta}^{DHV} = \frac{D(T^p)^2 \sum_{k=1}^{n_j} d_{jk}}{2PN_j^p}. \quad (4.6)$$

In Figure 3.5, the rectangle area of  $TWI\beta$  is calculated by multiplying the length of the base,  $T^p / N_j^p - Q_j^d / p_j$ , and the height,  $Q_j^d$ , which brings the result to  $(T^p / N_j^p - Q_j^d / p_j)Q_j^d$ . Similarly, the rectangle part of  $\gamma$  area is  $2Q_j^d(T^p / N_j^p - Q_j^d / p_j)$ , and the rectangle area of  $TWI\delta$  which is the last one in this cycle is  $Q_j^d(N_j^p - 1)(T^p / N_j^p - Q_j^d / p_j)$ . Adding all rectangular areas, we get

$$\sum_{\text{Rectangles}} = Q_j^d \left( \frac{T^p}{N_j^p} - \frac{Q_j^d}{p_j} \right) + 2Q_j^d \left( \frac{T^p}{N_j^p} - \frac{Q_j^d}{p_j} \right) + \dots + (N_j^p - 1)Q_j^d \left( \frac{T^p}{N_j^p} - \frac{Q_j^d}{p_j} \right). \quad (4.7)$$

As  $T^p / N_j^p = T_j^d$  and  $T_j^d = Q_j^d / \sum_{k=1}^{n_j} d_{jk}$ , the function in Equation (4.7) can be rewritten as

$$\sum_{\text{Rectangles}} = \frac{(Q_j^d)^2}{\sum_{k=1}^{n_j} d_{jk}} \left( 1 - \frac{\sum_{k=1}^{n_j} d_{jk}}{p_j} \right) [1 + 2 + \dots + (N_j^p - 1)]. \quad (4.8)$$

Since  $[1 + 2 + \dots + (N_j^p - 1)] = (N_j^p - 1)N_j^p / 2$  and  $Q_j^d = T^p \sum_{k=1}^{n_j} d_{jk} / N_j^p$ , the function in

Equation (4.8) becomes

$$\sum_{\text{Rectangles}} = \frac{(T^p)^2 \sum_{k=1}^{n_j} d_{jk}}{2 N_j^p} \left( 1 - \frac{D}{P} \right) (N_j^p - 1). \quad (4.9)$$

After adding the rectangular and triangular areas,  $\sum_{\text{Rectangles}}$  is divided by  $T^p$  and multiplied by

$H_p$ , the total holding cost at producer level is obtained as

$$TC_{HC}^P(T^p, N_j^p) = H_p \sum_{j=1}^m \left[ \frac{DT^p \sum_{k=1}^{n_j} d_{jk}}{2PN_j^p} + \frac{T^p \sum_{k=1}^{n_j} d_{jk}}{2N_j^p} \left( 1 - \frac{D}{P} \right) (N_j^p - 1) \right]. \quad (4.10)$$

#### 4.2.5 Setup Cost

Each year, the production runs  $1/T^p$  times and spends  $A_p$  dollars for each setup. During one production cycle time,  $T^p$ , the distributor  $j$  places orders to the producer and the products are delivered  $N_j^p$  times to the distributor  $j$ , and costing the distributor  $A_d$  each cycle. Similarly,



during one distributor  $j$ 's shipment interval period,  $T_j^d$ , the retailer  $k$  places orders to the distributor  $j$  and the products are delivered to the retailer  $k$   $N_{jk}^d$  times, and the retailer  $k$  incurs  $A_r$  cost per shipment. Thus, the total setup cost of the system per year is

$$TC_{SC}(T^p, N_j^p, N_{jk}^d) = \frac{A_p + \sum_{j=1}^m A_d N_j^p + \sum_{j=1}^m \sum_{k=1}^{n_j} A_r N_j^p N_{jk}^d}{T^p}. \quad (4.11)$$

The first numerator term,  $A_p$  in Equation (4.11), represents the unit setup cost for the producer. The second and third numerator terms,  $\sum_{j=1}^m A_d N_j^p$  and  $\sum_{j=1}^m \sum_{k=1}^{n_j} A_r N_j^p N_{jk}^d$ , represent the total setup costs for all distributors and retailers, respectively, during one producer's production cycle.

#### 4.2.6 Total Cost Function

Thus, the total cost function of the whole inventory system is the sum of the total holding cost and the total setup cost, as given in the following expression:

$$\begin{aligned} TC_{DHV}(T^p, N_j^p, N_{jk}^d; j=1, \dots, m; k=1, \dots, n_j) \\ &= TC_{HC}^p(T^p, N_j^p) + TC_{HC}^d(T^p, N_j^p, N_{jk}^d) + TC_{HC}^r(T^p, N_j^p, N_{jk}^d) + TC_{SC}(T^p, N_j^p, N_{jk}^d) \\ &= H_p \sum_{j=1}^m \left[ \frac{DT^p \sum_{k=1}^{n_j} d_{jk}}{2PN_j^p} + \frac{T^p \sum_{k=1}^{n_j} d_{jk}}{2N_j^p} \left( 1 - \frac{D}{P} \right) (N_j^p - 1) \right] \\ &\quad + H_d \sum_{j=1}^m \sum_{k=1}^{n_j} \frac{\alpha d_{jk} T^p}{2N_j^p} \left( 1 - \frac{1}{N_{jk}^d} \right) + H_r \sum_{j=1}^m \sum_{k=1}^{n_j} \frac{d_{jk} T^p (\alpha_{jk} + N_{jk}^d (1 - \alpha))}{2N_j^p N_{jk}^d} \\ &\quad + \frac{A_p + \sum_{j=1}^m A_d N_j^p + \sum_{j=1}^m \sum_{k=1}^{n_j} A_r N_j^p N_{jk}^d}{T^p}. \end{aligned} \quad (4.12)$$

#### 4.3 Optimization Solution Procedure

The total cost function  $TC_{DHV}(T^p, N_j^p, N_{jk}^d; j=1, \dots, m; k=1, \dots, n_j)$  in Equation (4.12) can easily be proved to be a convex function, and the closed-form solutions are obtained by solving

$\partial TC_{DHV} / \partial N_{jk}^d = 0$ ,  $\partial TC_{DHV} / \partial N_j^p = 0$ , and  $\partial TC_{DHV} / \partial T^p = 0$  simultaneously, which leads to the optimal production cycle time as

$$T^{p*} = \sqrt{\frac{2A_p P}{H_p D (P - D)}} , \quad (4.13)$$

and the number of deliveries from the producer to the distributor  $j$  in  $T^{p*}$  time periods as

$$N_j^{p*} = \sqrt{\frac{\sum d_{jk} A_p [H_p (2D - P) + H_d P \alpha_{jk} + H_r P (1 - \alpha)]}{H_p D (P - D) A_d}} . \quad (4.14)$$

These optimal solutions in equations (4.13) and (4.14) along with  $\partial TC_{DHV} / \partial N_{jk}^d = 0$  lead to the optimal number of deliveries  $N_{jk}^{d*}$  from distributor  $j$  ( $j=1, \dots, m$ ) to retailer  $k$  in distributor  $j$ 's shipment interval  $T_j^d$  [see Appendix B4.3 for details]:

$$N_{jk}^{d*} = \sqrt{\frac{A_d A_p P d_{jk} \alpha_{jk} (H_r - H_d)}{A_r \sum d_{jk} A_p [H_p (2D - P) + H_d P \alpha_{jk} + H_r P (1 - \alpha)]}} , \quad H_d \leq H_r . \quad (4.15)$$

Equation (4.15) becomes immediately invalid when  $H_d > H_r$  as we initially assumed. Typically  $H_d \leq H_r$  in most industrial systems. If the inventory holding cost  $H_r$  at a downstream stage can be reduced significantly as compared to the inventory holding cost  $H_d$  at an upstream stage in a supply chain system, then there is a possibility of quickly reducing (or shipping) the upstream higher inventory cost holding, and subsequently increasing downstream lower inventory holding cost. This situation forms an equilibrium condition that leads an optimal strategy of shifting inventories between two consecutive stages.

The closed-form solutions, thus, indicate that we can get such a feasible improved situation as long as the reduction in  $H_r$  holds in the condition,  $H_d \leq H_r$ . In other words, this 3-stage system will always benefit from relocating or quickly shipping the materials from the

upstream stages (in this case, distributors) to the downstream stages (retailers), if the downstream stage find such a storage system with lower holding cost rate than previous rate.

#### Example 4.1: Integer solutions

The following parameters are used for this example:  $A_p = \$400/\text{set up}$ ,  $A_d = \$25/\text{order}$ ,  $A_r = \$2/\text{order}$ ,  $H_p = \$4/\text{unit/year}$ ,  $H_d = \$5/\text{unit/year}$ ,  $H_r = \$15/\text{unit/year}$ , and  $P = 5,000$  units/year,  $d_{11} = 100$  units/year,  $d_{12} = 150$  units/year,  $d_{21} = 225$  units/year,  $d_{22} = 337$  units/year such that  $D = 812$  units/year.

##### (a) Closed-form solutions

Using the given parameters for  $j=1,2$  and  $k=1,2$ , and Equations (4.13), (4.14), and (4.15), the closed-form solutions to the problem and the corresponding minimal total cost for the system are shown in Table 4.1 under the assumption of continuous variables. Since all but the production cycle time  $T^p$  are integers (i.e.,  $N_j^p$  and  $N_{jk}^d$ ), a further refinement is warranted for integerization of the other variables, which, as expected, will generate a higher total cost.

Table 4.1 Closed-form solutions

$T^p$	$N_1^p$	$N_2^p$	$N_{11}^d$	$N_{12}^d$	$N_{21}^d$	$N_{22}^d$	$TC_{DHV}$
0.54	2.71	4.06	2.51	3.07	2.51	3.07	\$2,379

##### (b) Integerization procedure (Branching Process)

A set of starting initial values for the variables is chosen arbitrarily, but logically, and these values are shown in Table 4.2.

Table 4.2 Initial starting solutions for the branching process

$T^p$	$N_1^p$	$N_2^p$	$N_{11}^d$	$N_{12}^d$	$N_{21}^d$	$N_{22}^d$	$TC_{DHV}$
0.7	2	2	4	4	4	4	\$2,980

Table 4.3 Branching/sequential process of finding integer solutions

Table 4.3(a)  $T^P$  vs  $TC$

$T^P$	$TC$
0.1	5,951
0.2	3,441
0.3	2,812
<b>0.4*</b>	<b>2,652</b>
0.5	2,681
0.6	2,804
0.7	2,980
0.8	3,190
0.9	3,422
1	3,670

Table 4.3(b)  $N_1^P$  vs  $TC$  ( $T^P = 0.4$ )

$N_1^P$	$TC$
1	2,765
<b>2*</b>	<b>2,652</b>
3	2,683
4	2,750
5	2,831
6	2,919
7	3,011
8	3,106
9	3,203
10	3,300

Table 4.3(c)  $N_2^P$  vs  $TC$  ( $N_1^P = 2$ )

$N_2^P$	$TC$
1	3,033
2	2,652
<b>3*</b>	<b>2,594</b>
4	2,616
5	2,670
6	2,740
7	2,820
8	2,905
9	2,994
10	3,086

Table 4.3(d)  $N_{1k}^d$  vs  $TC$  ( $N_2^P = 3$ )

$N_{1k}^d$	$TC$
1	2,591
<b>2*</b>	<b>2,583</b>
3	2,587
4	2,594
5	2,602
6	2,611
7	2,620
8	2,629
9	2,639
10	2,648

Table 4.3(e)  $N_{2k}^d$  vs  $TC$  ( $N_{1k}^d = 2$ )

$N_{2k}^d$	$TC$
1	2,593
<b>2*</b>	<b>2,576</b>
3	2,577
4	2,583
5	2,590
6	2,598
7	2,607
8	2,616
9	2,625
10	2,635

Table 4.3(f)  $\alpha$  vs  $TC$  ( $N_{2k}^d = 2$ )

$\alpha$	$TC$
1	2,572
<b>2*</b>	<b>2,560</b>
3	2,566
4	2,576
5	2,589
6	2,602
7	2,615
8	2,630
9	2,644
10	2,658

Table 4.3(g)  $X$  vs  $TC$  ( $\alpha = 2$ )

$X$	$TC$
1	2,575
<b>2*</b>	<b>2,550</b>
3	2,552
4	2,560
5	2,571
6	2,583
7	2,596
8	2,610
9	2,624
10	2,638

The initial total cost is computed as \$2,980 using Equation (4.12) with the initial variables in Table 1. First, fixing all other variables to the initial values, the total costs are computed by varying the production cycle time  $T^P$  over the range from 0.1 to 1 at an interval of 0.1. Table 4.3(a) shows range of values of  $T^P$  and the corresponding total costs in columns 1 and 2, respectively. These results indicate that the lowest total cost occurs at \$2,652 for the production cycle time  $T^P = 0.4$ . Therefore,  $T^P$  is set to be constant at 0.4 for the rest of the calculation, and the value of  $N_1^P$  is then changed from 1 to 10 at an interval of 1 when the total cost is re-computed. The total cost is lowest at  $N_1^P = 2$ , and the current lowest total cost is still \$2,652. Then, by setting  $N_1^P$  at 2, the total cost is calculated with  $N_2^P$  changing from 1 to 10 at

an interval of 1. Similarly, the lowest total cost for  $N_2^p = 2$  occurred at \$2,594. The same process is repeated for the remainder of the variables ( $N_{11}^d$ ,  $N_{12}^d$ ,  $N_{21}^d$ , and  $N_{22}^d$ ).

The repetitive computational results of the branching procedure for finding the solutions at the minimal total cost are shown in Table 4.3 and the final set of variables is shown in Table 4.4. The minimal total cost with the solutions by sequential process is \$2,550 which is a 14.4 % reduction from the initial total cost of \$2,980.

Table 4.4 Integerized solutions by branching process

$T^p$	$N_1^p$	$N_2^p$	$N_{11}^d$	$N_{12}^d$	$N_{21}^d$	$N_{22}^d$	$TC_{DHV}$
0.4	2	3	2	2	2	2	\$2.550

Figure 4.3 depicts the piece-wise convex function of the total cost in Equation (4.12), which strongly proves the convexity within the range of each variable. This piece-wise convex functional characteristic is exploited to find the sub-optimal solution through the branching search process.

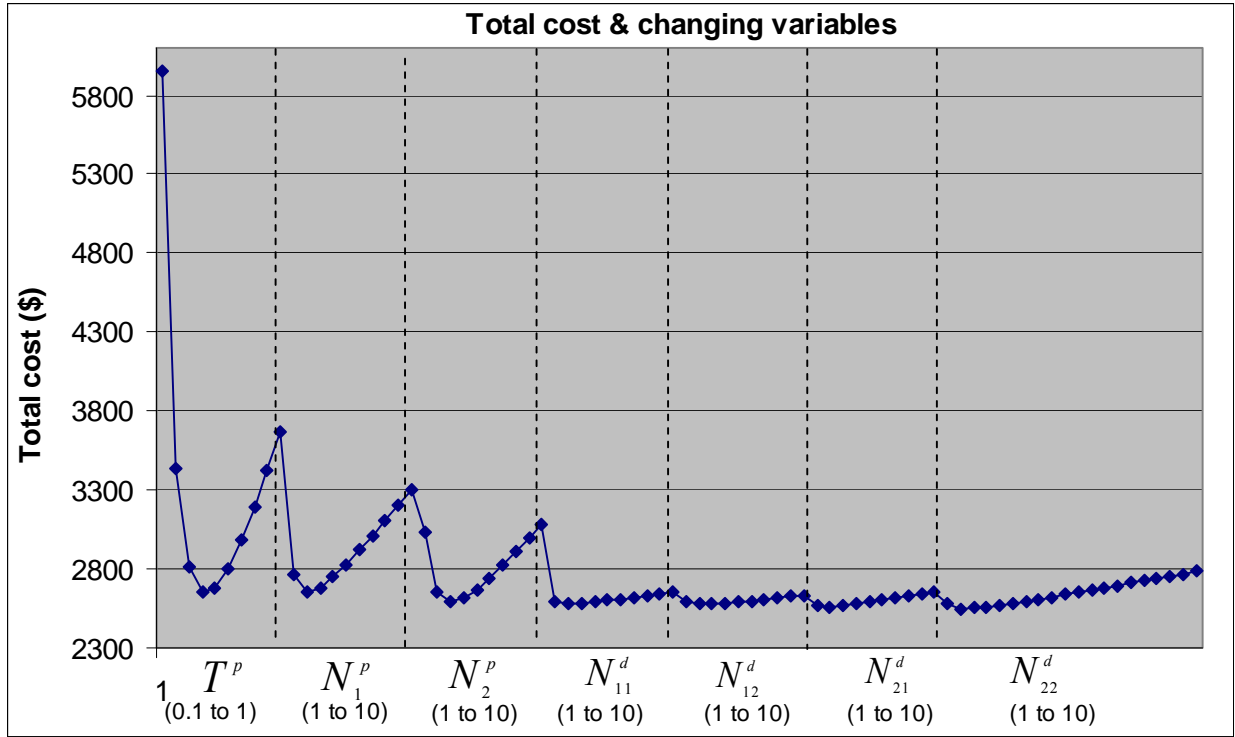


Figure 4.4 Total cost and changing values of the variables (Piecewise convex curves)

Figure 4.4 shows that y-axis is the total cost (in dollars) and x-axis is the changing values of different variables during the search process. The x-axis is divided into 7 sections by vertical dashed lines separating 7 variables. The lowest point of each piece-wise convex function is decreasing from left to right, conforming the minimal total cost at the zero gradient of the curve. This property helps immediately isolate the optimal values.

The values by both type of solutions vary with wide ranges. It appears that the total cost in the integerization process is  $TC_{DHV}^{Int} = \$2,550$  as compared to  $TC_{DHV}^{closed} = \$2,379$  by closed-form solution. The increase in total cost due to integerization is about 7.19%.

## CHAPTER 5

### IMPROVING THE RETAILER'S SERVICE RATE (ISR)

This chapter deals with a three-echelon tree-type inventory system with allowable backorder and decreased carrying cost at retailer. As all parties, the producer, distributors, and retailers, are assumed to be allies, this model aims at minimizing the total cost of the entire inventory system. Allowing shortage will reduce average inventories at the retailer despite some backorder occurring. Shrinking the shipment interval at the distributor level will reduce the average inventory at distributor and increase the average inventory level at the retailers so as to meet the customers' demand with higher service rate. In order to achieve this goal, the following models are developed.

#### 5.1 Backorder Cost

During a distributor shipment interval ( $T_j^d$ ), there are  $N_{jk}^d$  shipments from distributor  $j$  ( $j = 1, \dots, m$ ) to retailer  $k$  ( $k = 1, \dots, n$ ). The shipment quantity from a distributor per delivery is  $d_{jk} T_{jk}^r$ . Three scenarios in backorder phenomena can occur at any shipment: (a) *complete backorder* (i.e., the amount required to meet the remaining backorder is equal to or more than a complete shipment size), (b) *partial backorder* (i.e., the amount required to meet the remaining backorder is no more than a complete shipment size), and/or (c) *no backorder*.

We call it '*complete backorders*' if the actual shortage quantity is equal to or greater than a shipment quantity following which there is a partial backorder. In other words, the number of complete backorders,  $n^c$  is  $\lfloor b_{jk} / (d_{jk} T_{jk}^r) \rfloor - 1 < n^c \leq \lfloor b_{jk} / (d_{jk} T_{jk}^r) \rfloor$ . The amount of shipment for a 'complete backorder' equals to the quantity  $d_{jk} T_{jk}^r$ , and, thus, the number of 'complete backorders' is equal to the largest integer number of shipments that amount to no more than the total shortage quantity.

In Figure 5.1, the first, second, and third shipments are considered a shipment with a complete backorder. The shipments are made to the retailers and the retailers ship them to the customers immediately to meet the backordered demand. As a result, no inventory remains at the retailer's warehouse.

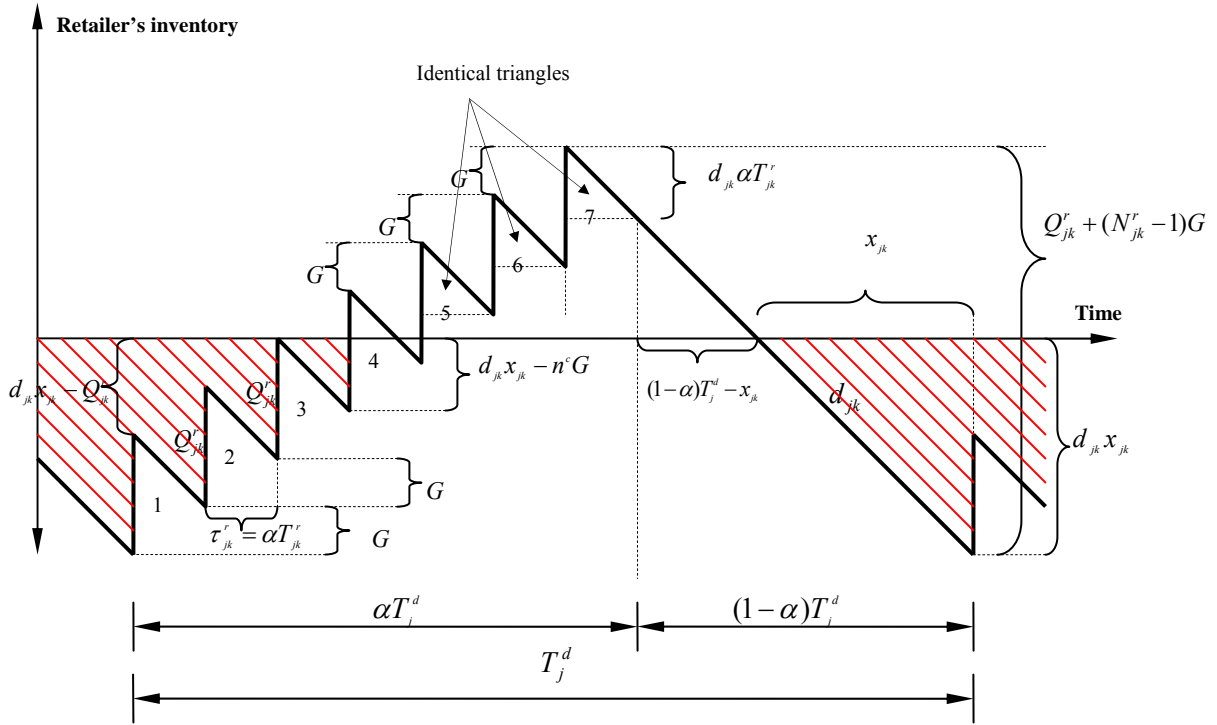


Figure 5.1 Retailer's inventory with shortage

The forth and fifth shipments are considered shipments with *partial backorder* (shortages) at the retailers since each of these shipments from the distributor (after the receipt by the retailer) is divided into two parts; one part (backorder) fulfills the customer's backorder amount immediately while the other part remains at the retailer's warehouse creating a positive inventory. The sixth and seventh shipments are considered shipments to meet the regular demand since the entire quantity received remains in the retailer's warehouse to meet no immediate shortage.

The distance between the timeline at zero inventory level and the lowest (negative) inventory (shortage) before the retailer receives the first shipment from the distributor is  $d_{jk} x_{jk}$ .



The reduction,  $G$ , in backorder quantity after each shipment is given by  $G = (1 - \alpha)d_{jk}T_{jk}^r$ , where  $\alpha$  is the *cycle shrinking factor* as defined earlier. The shortage amount for the  $i$ th partial backorder is  $d_{jk}x_{jk} - (i - 1)(1 - \alpha)d_{jk}T_{jk}^r$ . For the specific cases of fourth and fifth shipments (which are partial backorders), these amount to  $d_{jk}x_{jk} - 3(1 - \alpha)d_{jk}T_{jk}^r$  and  $d_{jk}x_{jk} - 4(1 - \alpha)d_{jk}T_{jk}^r$ , respectively.

In order to find the amount of shortage at the retailer's, we have to know how many shipments in each cycle fall into each type of shortage motioned above [complete backorder(s), partial backorder(s), or no backorder]. Let  $n_{jk}^c$  be the number of shipments with complete backorders,  $n_{jk}^p$  be the number of shipments with partial backorder, and  $n_{jk}^0$  be the number of shipments with no backorder, then,  $N_{jk}^d = n_{jk}^c + n_{jk}^p + n_{jk}^0$ .

#### (a) Complete backorders

In Figure 5.1, the first shipment of  $(Q_{jk}^r)$  is delivered to retailer and the entire shipment is sent to the waiting customers as backorders. The difference between the zero inventory line and the highest inventory (least shortage) immediately after the first shipment is made is  $d_{jk}x_{jk} - Q_{jk}^r$ . To find how many more shipments can fit in this space, the amount  $d_{jk}x_{jk} - Q_{jk}^r$  is, then, divided by the reduction amount  $(1 - \alpha)d_{jk}T_{jk}^r$  and rounded up to  $n_{jk}^c$ . The number of shipments with complete backorders,  $n_{jk}^c$ , is given by

$$n_{jk}^c = \left\lceil \frac{x_{jk} - T_{jk}^r}{(1 - \alpha)T_{jk}^r} \right\rceil, 0 \leq n_{jk}^c, x_{jk} > T_{jk}^r. \quad (5.1)$$

#### (b) Partial Backorders

In Figure 5.1, there is a space between the zero inventory line and the inventory immediately before the first partial shipment (i.e., the fourth shipment) is made by the

distributor to the retailer, which is  $d_{jk}x_{jk} - n_{jk}^c(1-\alpha)d_{jk}T_{jk}^r$ . This leads to the number of partial backorders,  $n_{jk}^p$ , to be found as

$$n_{jk}^p = \left\lceil \frac{x_{jk} - n_{jk}^c(1-\alpha)T_{jk}^r}{(1-\alpha)T_{jk}^r} \right\rceil$$

$$= \left\lceil \frac{x_{jk} - \left\lceil \frac{x_{jk} - T_{jk}^r}{(1-\alpha)T_{jk}^r} \right\rceil (1-\alpha)T_{jk}^r}{(1-\alpha)T_{jk}^r} \right\rceil, \quad 0 \leq n_{jk}^p, x_{jk} > T_{jk}^r \quad (5.2)$$

since  $N_{jk}^d = n_{jk}^c + n_{jk}^p + n_{jk}^0$ , the number of shipments with *no* backorder can be found as

$$n_{jk}^0 = N_{jk}^d - n_{jk}^c - n_{jk}^p. \quad (5.3)$$

The total amount of shortage due to  $n^c$  complete backorders is  $n_{jk}^c d_{jk} T_{jk}^r$ . The amount of shortage for  $i$ th shipment with partial backorder is  $d_{jk}x_{jk} - (i-1)(1-\alpha)d_{jk}T_{jk}^r$ , where, in this case,  $i = (n_{jk}^c + 1), (n_{jk}^c + 2), \dots, (n_{jk}^c + n_{jk}^p)$ . The total shortage in partial backorder,  $TS^p$ , is given by

$$TS^p = \sum_{i=n_{jk}^c+1}^{n_{jk}^c+n_{jk}^p} [d_{jk}x_{jk} - (i-1)(1-\alpha)d_{jk}T_{jk}^r], \quad (5.4)$$

which can be simplified as [see Appendix C5.1]:

$$TS^p = n_{jk}^p d_{jk} x_{jk} - \left( \frac{(n_{jk}^p)^2 + 2n_{jk}^c n_{jk}^p - n_{jk}^p}{2} \right) (1-\alpha) d_{jk} T_{jk}^r. \quad (5.5)$$

The shortage of shipments with no shortage is obviously zero. The total shortage at a retailer for one year,  $B_{jk}$ , is the accumulation of three types of shortages (the last one ‘no shortage’ being zero):

$$B_{jk} = \frac{d_{jk} N_j^p}{T^p} \left[ n_{jk}^c T_{jk}^r + n_{jk}^p x_{jk} - \left( \frac{(n_{jk}^p)^2 + 2n_{jk}^c n_{jk}^p - n_{jk}^p}{2} \right) (1-\alpha) T_{jk}^r \right]. \quad (5.6)$$

Since the value of  $x_{jk}$  is between 0 and  $(1-\alpha)T_j^d$ , to facilitate the calculation, let  $X$  be the ratio of  $x_{jk}$  to  $(1-\alpha)T_j^d$ , when  $0 < X < 1$ . Substituting  $X(1-\alpha)T_j^d$  for  $x_{jk}$ , Equation (5.6) becomes

$$B_{jk} = \frac{d_{jk} N_j^p}{T^p} \left( n_{jk}^c T_{jk}^r + n_{jk}^p X(1-\alpha)T_j^d - \left[ \frac{(n_{jk}^p)^2 + 2n_{jk}^c n_{jk}^p - n_{jk}^p}{2} \right] (1-\alpha)T_{jk}^r \right). \quad (5.7)$$

## 5.2 Retailer's Inventory

Unlike the average inventory functions for the producer and distributors, the function to find the retailer's average inventory is more complicated because parts of deliveries received at retailers are devoted to backorder.

In Figure 5.1, since the received inventories corresponding to the complete backorders (say, first, second and third shipments for example) creates no positive inventory at the retailers side; these shipments are excluded in computing the average inventory at the retailer. We are basically to find the total areas which are highlighted. Observing Figure 5.1, there are  $N_{jk}^d$  deliveries of materials from the distributors to the retailer. The peak of inventory level at the  $i^{th}$  receipt of shipment at the retailer is  $[Q_{jk}^r + (i-1)(1-\alpha)d_{jk}T_{jk}^r] - d_{jk}x_{jk}$ . There are  $n_{jk}^0 + 1$  identical triangles corresponding to  $n_{jk}^0 + 1$  shipments above the zero inventory line, the height of each triangle is  $d_{jk}\alpha T_{jk}^r$  with a base length of  $\alpha T_{jk}^r$ . This results in the area of all  $n_{jk}^0 + 1$  triangles as

$$\sum_{\Delta}^{IST} = \frac{(n_{jk}^0 + 1)d_{jk}(\alpha T_{jk}^r)^2}{2}. \quad (5.8)$$

There are  $n_{jk}^0 + 1$  rectangles below each of the identical triangles, and  $n_{jk}^0$  rectangles if the height of peak inventory of the last partial backorder is exact  $d_{jk}\alpha T_{jk}^r$ . The rectangle generated from the last *partial backorder* has the height of

$$[Q_{jk}^r + (n_{jk}^c + n_{jk}^p - 1)(1 - \alpha)d_{jk}T_{jk}^r] - d_{jk}x_{jk} - d_{jk}\alpha T_{jk}^r,$$

and the base length of  $\alpha T_{jk}^r$  resulting in the area

$$\{[Q_{jk}^r + (n_{jk}^c + n_{jk}^p - 1)(1 - \alpha)d_{jk}T_{jk}^r] - d_{jk}x_{jk} - d_{jk}\alpha T_{jk}^r\}\alpha T_{jk}^r.$$

For all rectangular areas,

$$\Sigma_{Rectangles} = \sum_{i=n_{jk}^c + n_{jk}^p}^{N_{jk}^d} \{[Q_{jk}^r + (i - 1)(1 - \alpha)d_{jk}T_{jk}^r] - d_{jk}x_{jk} - d_{jk}\alpha T_{jk}^r\}\alpha T_{jk}^r \quad (5.9)$$

which can also written in non-sigma form as:

$$\begin{aligned} \Sigma_{Rectangles} &= (n_{jk}^p + 1) \{d_{jk}\alpha T_{jk}^r (T_{jk}^r - x_{jk} - d_{jk}\alpha T_{jk}^r)\} \\ &\times \left( \frac{(n_{jk}^0)^2 + n_{jk}^0(2n_{jk}^c + 2n_{jk}^p - 1) + 2n_{jk}^c + 2n_{jk}^p - 2}{2} \right) (1 - \alpha)d_{jk}\alpha (T_{jk}^r)^2. \end{aligned} \quad (5.10)$$

(See Appendix C5.2 for the detailed derivation).

Each pair of consecutive partial backorders create a triangle which is smaller than the previously noted identical triangle; thus, there are total  $(n_{jk}^p - 1)$  non-identical triangles. The first triangle which is between the first and the second partial backorders has the height of  $[Q_{jk}^r + (n_{jk}^c)(1 - \alpha)d_{jk}T_{jk}^r] - d_{jk}x_{jk}$  and a base with length  $[(Q_{jk}^r + (n_{jk}^c)(1 - \alpha)d_{jk}T_{jk}^r) - d_{jk}x_{jk}] / d_{jk}$ , resulting in an area of the first triangle as  $([Q_{jk}^r + (n_{jk}^c)(1 - \alpha)d_{jk}T_{jk}^r] - d_{jk}x_{jk})^2 / 2d_{jk}$ . Thus, the general form of the total area of non-identical triangles (NIT) is given by

$$\Sigma_{NIT} = \sum_{i=n_{jk}^c + 1}^{n_{jk}^c + n_{jk}^p - 1} \frac{([Q_{jk}^r + (i - 1)(1 - \alpha)d_{jk}T_{jk}^r] - d_{jk}x_{jk})^2}{2d_{jk}} \quad (5.11)$$

The last triangle immediately before the in the positive inventory ends has a height of  $[Q_{jk}^r + (N_{jk}^d - 1)(1 - \alpha)d_{jk}T_{jk}^r] - d_{jk}x_{jk}$  and a base with length  $(1 - \alpha)T_{jk}^d - x_{jk}$ . Therefore, the area of this triangle is given by

$$\Delta_{last} = \frac{[(Q_{jk}^r + (N_{jk}^d - 1)(1 - \alpha)d_{jk}T_{jk}^r) - d_{jk}x_{jk}][(1 - \alpha)T_j^d - x_{jk}]}{2} \quad (5.12)$$

Thus, the total inventory  $I_{Total}$  at the retailer during one cycle period  $T_j^d$  is the accumulated value of Equations (5.9), (5.10), (5.11), and (5.12):

$$\begin{aligned} I_{Total} = & \frac{(n_{jk}^0 + 1)d_{jk}(\alpha T_{jk}^r)^2}{2} + \sum_{i=n_{jk}^c + n_{jk}^p}^{N_{jk}^d} \{[Q_{jk}^r + (i - 1)(1 - \alpha)d_{jk}T_{jk}^r] - d_{jk}x_{jk} - d_{jk}\alpha T_{jk}^r\} \alpha T_{jk}^r \\ & + \sum_{i=n_{jk}^c + 1}^{n_{jk}^c + n_{jk}^p - 1} \frac{([Q_{jk}^r + (i - 1)(1 - \alpha)d_{jk}T_{jk}^r] - d_{jk}x_{jk})^2}{2d_{jk}} \\ & + \frac{[(Q_{jk}^r + (N_{jk}^d - 1)(1 - \alpha)d_{jk}T_{jk}^r) - d_{jk}x_{jk}][(1 - \alpha)T_j^d - x_{jk}]}{2}. \end{aligned} \quad (5.13)$$

The average inventory  $\bar{I}_{Total}$  over the entire horizon is found by dividing the Equation (5.10) by  $T_j^d$ :

$$\begin{aligned} \bar{I}_{Total} = & \frac{(n_{jk}^0 + 1)d_{jk}(\alpha T_{jk}^r)^2}{2T_j^d} + \sum_{i=n_{jk}^c + n_{jk}^p}^{N_{jk}^d} \frac{\{[Q_{jk}^r + (i - 1)(1 - \alpha)d_{jk}T_{jk}^r] - d_{jk}x_{jk} - d_{jk}\alpha T_{jk}^r\} \alpha T_{jk}^r}{T_j^d} \\ & + \sum_{i=n_{jk}^c + 1}^{n_{jk}^c + n_{jk}^p - 1} \frac{([Q_{jk}^r + (i - 1)(1 - \alpha)d_{jk}T_{jk}^r] - d_{jk}x_{jk})^2}{2d_{jk}T_j^d} \\ & + \frac{[(Q_{jk}^r + (N_{jk}^d - 1)(1 - \alpha)d_{jk}T_{jk}^r) - d_{jk}x_{jk}][(1 - \alpha)T_j^d - x_{jk}]}{2T_j^d}. \end{aligned} \quad (5.14)$$

Substituting  $X(1 - \alpha)T_j^d$  for  $x_{jk}$ ,  $\frac{d_{jk}T^p}{N_j^p N_{jk}^d}$  for  $Q_{jk}^r$ ,  $\frac{T^p}{N_j^p N_{jk}^d}$  for  $T_{jk}^r$ , and  $\frac{T^p}{N_j^p}$  for  $T_j^d$ , Equation

(5.14) becomes transforms in original variables to

$$\bar{I}_{Total} = \frac{(n^0 + 1)d_{jk}\alpha T_{jk}^r}{2N_j^p (N_{jk}^d)^2} + \sum_{i=n_{jk}^c + n_{jk}^p}^{N_{jk}^d} \frac{\left\{ \frac{T^p d_{jk}}{N_j^p N_{jk}^d} [1 + (i - 1)(1 - \alpha)] - d_{jk}X(1 - \alpha)T_j^d - d_{jk}\alpha \frac{T^p}{N_j^p N_{jk}^d} \right\} \alpha}{N_{jk}^d}$$

$$\begin{aligned}
& + \sum_{i=n_{jk}^c+1}^{n_{jk}^c+p_{jk}-1} \frac{\left[ \left( \frac{T^p}{N_j^p N_{jk}^d} + (i-1)(1-\alpha) \frac{T^p}{N_j^p N_{jk}^d} \right) - X(1-\alpha)T_j^d \right]^2 N_j^p}{2T^p} \\
& + \frac{\left[ \left( \frac{d_{jk} T^p}{N_j^p N_{jk}^d} + (N_{jk}^d - 1)(1-\alpha) d_{jk} \frac{T^p}{N_j^p N_{jk}^d} \right) - d_{jk} X(1-\alpha)T_j^d \right] \left[ (1-\alpha) \frac{T^p}{N_j^p} - X(1-\alpha)T_j^d \right] N_j^p}{T^p}. \quad (5.15)
\end{aligned}$$

### 5.3 Total Cost Function, $TC_{ISR}$

The total cost function,  $TC_{ISR}$ , is the sum of the total holding costs, the total backorder costs, and the total setup costs. The producer holding cost, distributor holding cost, and total setup cost are identical to those in DHV model, which are Equations (4.10), (4.5), and (4.11), respectively. The total cost function is given by

$$\begin{aligned}
& TC_{ISR}(T^p, N_j^p, N_{jk}^d, \alpha, X; j = 1, \dots, m; k = 1, \dots, n_j) \\
& = TC_{HC}^p(T^p, N_j^p) + TC_{HC}^d(T^p, N_j^p, N_{jk}^d, \alpha) + TC_{HC}^r(T^p, N_j^p, N_{jk}^d, \alpha, X) \\
& + TC_{SC}(T^p, N_j^p, N_{jk}^d) + TC_{BC}(T^p, N_j^p, \alpha, X) \\
& = H_p \sum_{j=1}^m \left[ \frac{DT^p \sum_{k=1}^{n_j} d_{jk}}{2PN_j^p} + \frac{T^p \sum_{k=1}^{n_j} d_{jk}}{2N_j^p} \left( 1 - \frac{D}{P} \right) (N_j^p - 1) \right] \\
& + H_d \sum_{j=1}^m \sum_{k=1}^{n_j} \frac{\alpha d_{jk} T^p}{2N_j^p} \left( 1 - \frac{1}{N_{jk}^d} \right) + \sum_{j=1}^m \sum_{k=1}^{n_j} \frac{H_r (n_{jk}^0 + 1) d_{jk} \alpha T_{jk}^r}{2N_j^p (N_{jk}^d)^2} \\
& + \sum_{j=1}^m \sum_{k=1}^{n_j} \sum_{i=n_{jk}^c+1}^{n_{jk}^c+p_{jk}-1} \frac{H_r \left\{ \frac{T^p d_{jk}}{N_j^p N_{jk}^d} [1 + (i-1)(1-\alpha)] - d_{jk} X(1-\alpha)T_j^d - d_{jk} \alpha \frac{T^p}{N_j^p N_{jk}^d} \right\} \alpha}{N_{jk}^d} \\
& + \sum_{j=1}^m \sum_{k=1}^{n_j} \sum_{i=n_{jk}^c+1}^{n_{jk}^c+p_{jk}-1} \frac{H_r \left[ \left( \frac{T^p}{N_j^p N_{jk}^d} + (i-1)(1-\alpha) \frac{T^p}{N_j^p N_{jk}^d} \right) - X(1-\alpha)T_j^d \right]^2 N_j^p}{2T^p}
\end{aligned}$$

$$\begin{aligned}
& + \frac{\sum_{j=1}^m \sum_{k=1}^{n_j} H_r \left[ \left( \frac{d_{jk} T^p}{N_j^p N_{jk}^d} + (N_{jk}^d - 1)(1 - \alpha) d_{jk} \frac{T^p}{N_j^p N_{jk}^d} \right) - d_{jk} X(1 - \alpha) T_j^d \right] \left[ (1 - \alpha) \frac{T^p}{N_j^p} - X(1 - \alpha) T_j^d \right] N_j^p}{T^p} \\
& + \frac{A_p + \sum_{j=1}^m A_d N_j^p + \sum_{j=1}^m \sum_{k=1}^{n_j} A_r N_j^p N_{jk}^d}{T^p} \\
& + \sum_{j=1}^m \sum_{k=1}^{n_j} \frac{d_{jk} N_j^p \pi}{T^p} \left( n_{jk}^c T_{jk}^r + n_{jk}^p X(1 - \alpha) T_j^d - \left[ \frac{(n_{jk}^p)^2 + 2n_{jk}^c n_{jk}^p - n_{jk}^p}{2} \right] (1 - \alpha) T_{jk}^r \right). \quad (5.16)
\end{aligned}$$

This is a complicated cost function which is a piece-wise convex function in multiple variables. The closed-form solution to it is unknown, and finding the optimal solution to it is computationally intensive.

#### 5.4 Solution Optimality

The optimal solution for this total cost model cannot be found by using derivative technique as it was used in the model DHV and TDB since the total cost function Equation (5.16) contains integerized functions (piece-wise convex function). The optimal solution of it is explained through the following instance.

##### Example 5.1:

The Equation (5.16) was tested by substituting in numerical parameters:  $A_p = \$400/\text{set up}$ ,  $A_d = \$25/\text{order}$ ,  $A_r = \$2/\text{order}$ ,  $H_p = \$4/\text{unit/year}$ ,  $H_d = \$5/\text{unit/year}$ ,  $H_r = \$15/\text{unit/year}$ , and  $P = 5,000 \text{ units/year}$ ,  $d_{11} = 100 \text{ units/year}$ ,  $d_{12} = 150 \text{ units/year}$ ,  $d_{21} = 225 \text{ units/year}$ ,  $d_{22} = 337 \text{ units/year}$  such that  $D = 812 \text{ units/year}$ . The unit shortage cost is assumed as  $\pi = \$1/\text{unit}$ . The numerical solutions are shown through the following computational procedure:

Table 5.1 Initial starting solutions

$T^p$	$N_1^p$	$N_2^p$	$N_{11}^d$	$N_{21}^d$	$\alpha$	$X$	$TC_{ISR}$
0.8	2	2	4	4	0.4	0.4	\$2,471

## Branching and integerization procedure

Initial variables are arbitrarily considered as shown in Table 5.1 to be substituting into the total cost function in Equation 5.16. The initial total cost is calculated using the initial variables is \$2,471. First, with all other variables fixed, the total costs are calculated with the value of  $T^p$  ranging from 0.1 to 1 at an interval of 0.1. These results shown in Table 5.2(a) indicate that the lowest total cost occurs with \$2,303 at the production cycle time  $T^p = 0.5$ . Then,  $T^p$  is fixed at 0.5 and the value of  $N_1^p$  is changed from 1 to 10 when the total cost is being calculated. The total cost is lowest at  $N_1^p = 2$  (which also is the initial value of  $N_1^p$  incidentally), so the current lowest total cost is still is \$2,303. Then, by fixing both  $T^p$  and  $N_1^p$  at 0.5 and 2, respectively, the total cost is calculated with  $N_2^p$  changing from 1 to 10 at an interval of 1. Similarly, the lowest total cost for  $N_2^p$  occurred at \$2,303 when  $N_2^p = 2$ . The same process is repeated for the remaining variables ( $N_{1k}^d$ ,  $N_{2k}^d$ ,  $\alpha$ , and  $X$ ).

The repetitive computational results from the branching procedure of finding the mixed-integer solutions at the minimal total cost are shown in Table 5.2 and the final set of optimal variables is shown in Table 5.3. The minimal total cost with the mixed-integer solutions is  $TC_{ISR}$  of \$2,098 which is a 15.4 % reduction from the initial total cost of \$2,471.

Table 5.2 Branching process of finding integer solutions

$T^p$	$TC$
0.1	6,081
0.2	3,439
0.3	2,681
0.4	2,396
<b>0.5*</b>	<b>2,303</b>

$N_1^p$	$TC$
1	2,323
<b>2*</b>	<b>2,303</b>
3	2,358
4	2,427

$N_2^p$	$TC$
1	2,443
<b>2*</b>	<b>2,303</b>
3	2,326

$N_{1k}^d$	$TC$
1	2,203
<b>2*</b>	<b>2,179</b>

$N_{2k}^d$	$TC$
1	2,735

$\alpha$	$TC$
<b>0.5</b>	<b>2,098</b>



0.6	2,308	5	2,502	4	2,381	3	2,260	2	2,259	0.1	2,325	<i>X</i>	<i>TC</i>
0.7	2,371	6	2,580	5	2,448	4	2,303	3	2,190	0.2	2,291	0.1	2,278
0.8	2,471	7	2,658	6	2,520	5	2,347	4	2,179	0.3	2,167	0.2	2,233
0.9	2,597	8	2,738	7	2,596	6	2,393	5	2,161	0.4	2,152	0.3	2,196
1	2,743	9	2,818	8	2,673	7	2,460	<b>6*</b>	<b>2,152</b>	<b>0.5*</b>	<b>2,148</b>	0.4	2,148
		10	2,899	9	2,751	8	2,499	7	2,237	0.6	2,152	<b>0.5*</b>	<b>2,098</b>
				10	2,830	9	2,585	8	2,230	0.7	2,147	0.6	2,184
						10	2,675	9	2,228	0.8	2,153	0.7	2,253
								10	2,289	0.9	2,170	0.8	2,234
										1	n/a	0.9	2,322
												1	2,322

(a) (b) (c) (d) (e) (f) (g)

Table 5.3 Mixed-integer solutions

$T^p$	$N_1^p$	$N_2^p$	$N_{11}^d$	$N_{21}^d$	$\alpha$	$X$	$TC_{ISR}^{Int}$
0.5	2	2	2	6	0.5	0.5	\$2,098

Figure 5.2 graphically shows the piece-wise convex functional behavior of the total cost function in Equation (5.16) and it indicates the convexity within the range of each variable, forming a piece-wise convex function. This piece-wise convex functional characteristic is utilized in the branching searching process. Figure 5.2 shows that y-axis is the total cost (in dollars) and x-axis is the changing values of different variables during the search process. The x-axis is divided into 7 parts by dashed lines separating 7 variables. From left to right, the lowest point of each curve is declining, assuring the minimal valued solution.

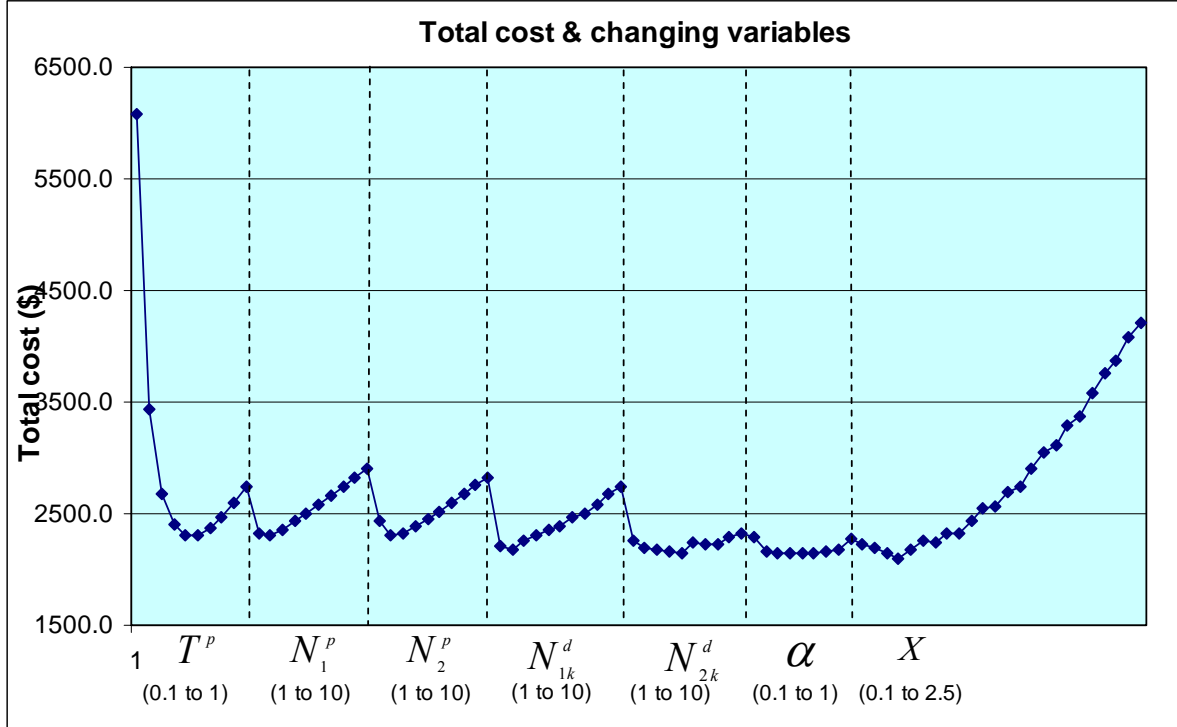


Figure 5.2 Total cost and changing values of the variables (piecewise convex curves)

### 5.5 Service Rate Evaluation

Referring to Example 3.1 in Chapter 3, the total backorders per year for TDB model are computed using the expression:  $B_{jk} = (1 - R_{jk})d_{jk}$ , and the computed backorders at retailers  $(j, k) = (1, 1), (1, 2), (2, 1),$  and  $(2, 2)$  are 8.7, 38.2, 73.3, and 122.4, respectively [see Table 5.4]. The total backorder of all retailers amounts to 242.6 units per year while the total yearly demand  $D$  was 812 units. Therefore, the service rate is computed as  $R_{TDB} = (D - \sum_{j=1}^m \sum_{k=1}^{n_j} B_{jk}) / D \equiv [(812 - 242.6) / 812] \times 100 = 70.1\%$ .

Table 5.4 Computation of backorders in TDB and ISR Models

Distribution Route, ( $i, j$ )	$d_{jk}$	TDB Model		ISR Model
		Service rate $R_{jk}$	Backorders size $B_{jk} = (1 - R_{jk})d_{jk}$	Backorders size $B_{jk}$ [using Eq. (5.7)]
(1,1)	100	0.9129	8.7	25.0
(1,2)	150	0.7454	38.2	37.5
(2,1)	225	0.6742	73.3	37.5
(2,2)	337	0.6367	122.4	56.1
Total:	812	-	242.6	156.1

Next, using Equation (5.7) and the mixed-integer solutions shown in Table 5.3, the yearly backorders at retailers  $(j, k) = (1, 1), (1, 2), (2, 1),$  and  $(2, 2)$  are obtained as 25, 37.5, 37.5, and 56.1, respectively, totaling 156.1 units per year (see Table 5.4). So the improved service rate,  $R_{ISR}$ , for this case is  $R_{ISR} = [(812-156.1)/812] \times 100 = 80.70\%$  which is improved from the service rate  $R_{TDB} = 70.10\%$  computed for Example 3.1.

## **CHAPTER 6**

### **DISCUSSION AND CONCLUSION**

The TDB model (Chapter 3) has to do with a tree-type distribution model of a single producer, multiple distributors and retailers where backorders are allowed at the retailers' levels. The total cost in this case is dependent on the producer's cycle time, numbers of shipment from producer to distributors and from distributor to retailers, and retailer's service rates. Even though there will be an extra cost to operate the backorder, the system will benefit from the savings on the lower inventory at retailers. As the demands at each retailer are known and constant, the producer cycle time, the numbers of shipment from producer to distributors, and the number of shipments from distributor to retailers determine the quantities of shipment which directly impact the inventory at producer, distributors, and retailers. During one producer cycle time, as the number of shipments from the producer to the distributor increases, the overall inventory falls, resulting in an increase of the total holding cost. In contrast, the setup cost, which has positive correlation with the number of shipment, will decrease. The total cost function and the optimal solutions are developed and tested with numerical examples.

The DHV model (Chapter 4) has to do with the TDB models with no shortage allowed, but with a primary assumption of greater holding cost at the distributor level than at the retailer level. It is mathematically proved that this assumption of higher holding at preceding stages (distributors) is not possible when searching for a feasible solution. In other words, the holding cost at the downstream stages (retailers) must be at least equal to the holding cost of the preceding stages (distributors).

The third ISR model (Chapter 5) was developed based on combining the features TDB (allowable shortage) and DHV model (by shrinking the retailer's delivery interval) in order to improve the service rate. In this case, the distributors have lower average inventory (due to the

reduced shipment interval) and the retailers have lower average inventory due to allowable shortage that minimize the inventory cost; but, the backorder at the retailers add to the total cost of inventory system. An optimal solution by balancing both situations to improve the overall performance in service rate (meaning increasing it) was sought. The derivative method could not be used to find the optimal solution since the model itself contains some integerization function. Instead, the branching procedure was employed to obtain the sub-optimal solutions.

## 6.1 Summary and Conclusion

This research studied and developed the replenishment policies for a tree-type, three-echelon supply chain system. After reviewing the related literature, shortcomings were identified and categorized into 4 groups: allowable shortage, complication of the supply chain network, holding cost, and service rate. The objectives were to build supply chain models for three situations: (1) allowable backorder, (2) downstream holding cost variation, and (3) Improving the retailers' service rates by reducing the delivery interval [which is the combination of situations (1) and (2)].

The total cost model and closed-form solutions were developed for the first model (TDB). The solutions were numerically tested by both closed-form formulas and branching procedure. Since the outcome of the closed-form solutions provide real number for some integer variables, these integer variables must be integerized to have practical solutions. We learned from the TDB model that allowing backorders at the optimal level could create cost saving by reducing total holding cost at the retailers.

We found from DHV model that the assumption of downstream lower holding cost than upstream holding cost is mathematically impossible for a feasible solution. In other words,  $H_r$  at a downstream stage can be reduced only to  $H_d$ , the upstream holding cost rate to maintain the feasibility of the solutions.

Finally, the ISR model, which combines allowable shortage and reduced retailer's delivery intervals, was developed to improve the service rate at the retailer level. The optimal solutions were found by the branching search procedure. The improved service rate at retailers is proved by numerical examples.

## **6.2 Significance of Research**

Many researches in replenishment policies in the past have some limitations and shortcomings. Those limitations included no shortage allowed, the complexity of the supply chain network, and the attempt to improve the service rate. This research has, to some extent, fulfilled those limitations and shortcomings.

This research is applicable to many kinds of businesses which have tree-type supply chain network. By implementing the optimal replenishment policies developed in this research, companies could optimally replenish their inventories. Based on the policies, they are able to determine when each party in the supply chain network has to order the product and how many units of products are needed for each order at a particular time. Most importantly, the end customer demands are guaranteed to be satisfied. The examples which could benefit from this research include automobile, electronic goods, retail store, medical supply distribution, and machinery businesses.

## **6.3 Future Research**

The studies done here relaxed some practical limitations of tree-type multi-echelon supply chain systems. There are still several other aspects which can be explored along these research objectives. The following issues can be studied for further refinement of the problems addressed here:

- (a) Uncertain demand: It is assumed that the retailer yearly demand is known and constant through time. In real life, demand is changing, affected by several factors.

Future research can be directed to investigate the case with retailers' varying demands.

- (b) Controllable Production rate: The production rate here is assumed to be given and constant. In manufacturing world, production rate can be adjusted to account for fluctuating demand. It could be done in several ways including increasing number of machines or replacing with the machine with higher capacity. In future research, the production rate could be treating as decision variable.
- (c) Multiple products: This research considers a single type of product in the system. In general, each party handles more than one product at a time. Considering multiple types of products in the system could bring the future research closer to the real world practices.
- (d) N-echelon supply chain network: This research consider tree-type three-echelon distribution network. In real world, the supply chain network can be more than three-echelon. From the producer to the end customer, products can go through multiple levels of distributors. The future research could contribute to develop models which considers any number of echelons, but computational tractability has to be assessed ahead of time.

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## APPENDIX A3.1

### PROOF OF CONVEXITY OF THE COST FUNCTION IN MODEL TDB

In this section, the derivatives of the integrated total cost functions with respect to four variables are shown. The purpose is to determine the optimal solutions which will lead to the minimal integrated total cost. Here, the minimal total cost is obtained by differentiating total cost functions with respect to all four variables  $R_{jk}$ ,  $N_{jk}^d$ ,  $N_j^p$  and  $T^p$ .

To obtain the optimal  $R_{jk}$ , first set  $dTC_{TDB}/dR_{jk} = 0$  which yields the optimal  $R_{jk}$  as

$$R_{rk} = \frac{\pi N_j^p N_{jk}^d}{H_r T^p}, \quad (\text{A3.1.1})$$

where the  $TC_{TDB}$  is defined in equation (3.11). The optimal  $R_{jk}$ , (A3.1.1) is then substituted in Eq. (3.11). Next, the integrated total cost function  $TC_{TDB}|_{R_{jk}}$  which has been adjusted with the

optimal  $R_{jk}$  is differentiated respect to  $N_{jk}^d$ . Then, set  $\frac{\partial TC_{TDB}|_{R_{jk}}}{\partial N_{jk}^d} = 0$  that leads to

$$N_{jk}^d = \frac{T^p}{N_j^p} \sqrt{\frac{H_r H_d d_{jk}}{(d_{jk} \pi^2 - 2H_r A_r)}}. \quad (\text{A3.1.2})$$

Let  $B = \sqrt{H_r H_d d_{jk} / (d_{jk} \pi^2 - 2H_r A_r)}$ , then  $N_{jk}^d = T^p B / N_j^p$  and is substituted into the total cost function. Next, to obtain the optimal  $N_j^p$ , the total cost function, which has been

adjusted optimal  $R_{jk}$  and  $N_{jk}^d$  as  $\partial TC_{TDB}|_{R_{jk}, N_{jk}^d} / \partial N_j^p = 0$ , yields

$$N_j^p = T^p \sqrt{\frac{\sum_{k=1}^{n_j} d_{jk} (PH_d + 2DH_p - PH_p)}{2PA_d}}. \quad (\text{A3.1.3})$$

Let  $C = \sqrt{\sum_{k=1}^{n_j} d_{jk} (P + 2DH_p - PH_p) / 2PA_d}$ , then  $N_j^p = T^p C$  and is substituted into the total

cost function. Again, the total cost function which has been adjusted optimal  $R_{jk}$ ,  $N_{jk}^d$  and  $N_j^p$

as  $TC_{TDB}|_{R_{jk}, N_{jk}^d, N_j^p}$  and  $\frac{\partial TC_{TDB}|_{R_{jk}, N_{jk}^d, N_j^p}}{\partial T^p} = 0$  yields

$$T^{p*} = \sqrt{\frac{2PA_p}{H_p D(P-D)}}. \quad (A3.1.4)$$

Observing Equation (A3.1.4),  $T^{p*}$  is in term of only parameters. Equation (A3.1.4) is substituted into (A3.1.3). The optimal  $N_j^p$  is obtained as

$$N_j^{p*} = \sqrt{\frac{\sum_{k=1}^{n_j} d_{jk} A_p (PH_d + 2DH_p - PH_p)}{A_d H_p D(P-D)}}. \quad (A3.1.5)$$

Similarly, (A3.1.4) and (A3.1.5) are substituted into (A3.1.2). The optimal  $N_{jk}^d$  is derived as

$$N_{jk}^{d*} = \sqrt{\frac{2PA_d H_r H_d d_{jk}}{\sum_{k=1}^{n_j} d_{jk} (PH_d + 2DH_p - PH_p) (d_{jk} \pi^2 - 2H_r A_r)}}. \quad (A3.1.6)$$

Then, substituting (A3.1.4), (A3.1.5), and (A3.1.6) into (A3.1.1), the optimal  $R_{jk}$  is earned as

$$R_{jk}^* = \frac{\pi}{H_r} \sqrt{\frac{H_r H_d d_{jk}}{(d_{jk} \pi^2 - 2H_r A_r)}}. \quad (A3.1.7)$$

### Proof of Convexity

Now, substituting (A3.1.1), (A3.1.2), and (A3.1.3) into the total cost function (3.11), the total cost becomes

$$TC_{TDB}|_{R_{jk}, N_{jk}^d, N_j^p} = \sum_{j=1}^m \sum_{k=1}^{n_j} \frac{d_{jk} \pi^2 B}{2H_r} + \sum_{j=1}^m \sum_{k=1}^{n_j} \frac{H_d d_{jk}}{2C} \left(1 - \frac{C}{B}\right)$$

$$\begin{aligned}
& + H_p \sum_{j=1}^m \left[ \frac{d \sum_{k=1}^{n_j} d_{jk}}{2PC} + \frac{\sum_{k=1}^{n_j} d_{jk}}{2C} \left( 1 - \frac{D}{P} \right) (T^p C - 1) \right] \\
& + \frac{A_p}{T^p} + \sum_{j=1}^m A_d C + \sum_{j=1}^m \sum_{k=1}^{n_j} A_r B + \sum_{j=1}^m \sum_{k=1}^{n_j} \left( d_{jk} \pi - \frac{\pi^2 d_{jk} B}{H_r} \right)
\end{aligned}$$

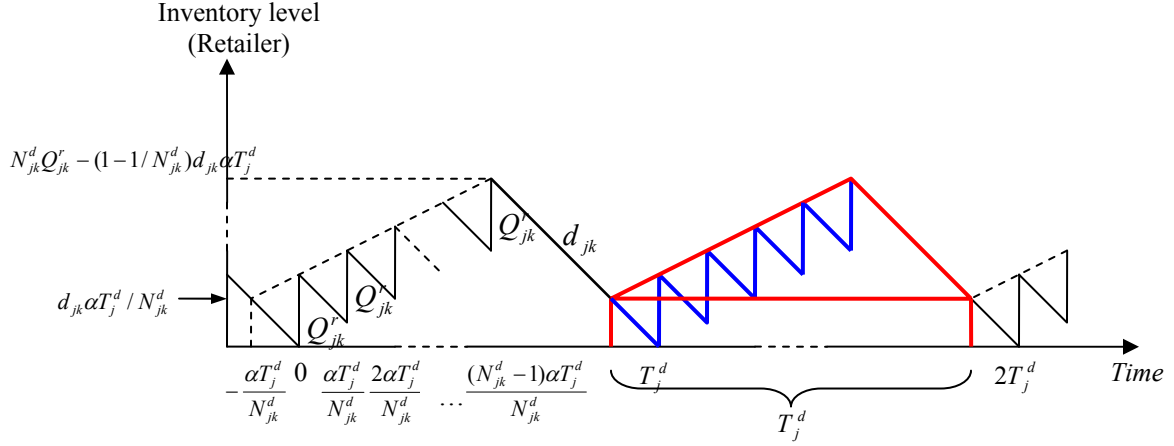
where,  $B = \sqrt{\frac{H_r H_d d_{jk}}{(d_{jk} \pi^2 - 2H_r A_r)}}$  and  $C = \sqrt{\frac{\sum_{k=1}^{n_j} d_{jk} (PH_d + 2DH_p - PH_p)}{2PA_d}}$ . It can be shown that

$$\frac{\partial^2 TC_{TDB} \big|_{R_{jk}, N_{jk}^d, N_{jk}^d}}{\partial T_p^2} = \frac{2A_p}{T_p^3}$$

which is always greater than 0. So, it is concluded that the total cost function in Equation (3.11) is convex.

## APPENDIX B4.1

### AN ALTERNATIVE WAY TO FIND RETAILERS' AVERAGE INVENTORY



#### Total area

$$\text{Total area} = (\text{Rectangle area}) + (\text{Triangle area}) - (N_{jk}^d \text{ small triangles' area}) \quad (\text{B4.1.1})$$

#### Area of rectangle

The area of rectangle = (base) x (height)

Where,  $\text{base} = T_j^d$ , and  $\text{height} = (\text{the amount of inventory consumed at demand rate } d_{jk} \text{ during time } \alpha T_j^d / N_{jk}^d) = d_{jk} \alpha T_j^d / N_{jk}^d$ . Therefore,

$$\text{Rectangle area} = d_{jk} \alpha (T_j^d)^2 / N_{jk}^d. \quad (\text{B4.1.2})$$

#### Area of triangle

The area of triangle =  $(1/2) \times (\text{base}) \times (\text{height})$

Base =  $T_j^d$  and the height =  $\text{peak inventory} - \text{the height (of the rectangle in the previous section)}$ ,

where  $\text{inventory peak} = (\text{the total of quantity of the shipments from time zero to time } (N_{jk}^d - 1) \alpha T_{jk}^r) - (\text{total amount inventory consumed during time zero to } (N_{jk}^d - 1) \alpha T_{jk}^r)$ .

$$\text{Inventory peak} = N_{jk}^d Q_{jk}^r - (N_{jk}^d - 1) \frac{d_{jk} \alpha T_j^d}{N_{jk}^d} - \frac{d_{jk} \alpha T_j^d}{N_{jk}^d}.$$

$$\text{Therefore, the area of triangle} = \frac{T_j^d}{2} \left( N_{jk}^d Q_{jk}^r - (N_{jk}^d - 1) \frac{d_{jk} \alpha T_j^d}{N_{jk}^d} - \frac{d_{jk} \alpha T_j^d}{N_{jk}^d} \right) \quad (\text{B4.1.3})$$

**Area of  $N_{jk}^d$  small triangles**

The area of  $N_{jk}^d$  small rectangles =  $N_{jk}^d \times \frac{1}{2}$  (base x height), where

Base = shipment quantity ( $Q_{jk}^r$ ), and height = time  $\alpha T_j^d / N_{jk}^d$  (which is also equal to  $\tau_{jk}$ )

Therefore, the area =  $N_{jk}^d \times \frac{1}{2} \times Q_{jk}^r \times \alpha T_j^d / N_{jk}^d$ ,  $Q_{jk}^r = d_{jk} T_j^d / N_{jk}^d$

$$\sum_{\text{small-triangles}} = \frac{d_{jk} \alpha (T_j^d)^2}{2 N_{jk}^d}. \quad (\text{B4.1.4})$$

Substitute (B4.1.2), (B4.1.3), and (B4.1.4) into (B4.1.1)

$$\text{The total area, } \sum_{\text{area}}^d = \frac{d_{jk} \alpha (T_j^d)^2}{N_{jk}^d} + \frac{T_j^d}{2} \times \left( N_{jk}^d Q_{jk}^r - (N_{jk}^d - 1) \frac{d_{jk} \alpha T_j^d}{N_{jk}^d} - \frac{d_{jk} \alpha T_j^d}{N_{jk}^d} \right) - \frac{d_{jk} \alpha (T_j^d)^2}{2 N_{jk}^d}.$$

The average inventory,  $I_{\text{avg}} = \sum_{\text{area}}^d / T_j^d$ , where  $Q_{jk}^r = \frac{d_{jk} T_j^d}{N_{jk}^d}$ ,

$$I_{\text{avg}} = \frac{d_{jk} \alpha T_j^d}{N_{jk}^d} + (\frac{1}{2}) \left( \frac{N_{jk}^d d_{jk} T_j^d}{N_{jk}^d} - (N_{jk}^d - 1) \frac{d_{jk} \alpha T_j^d}{N_{jk}^d} - \frac{d_{jk} \alpha T_j^d}{N_{jk}^d} \right) - \frac{d_{jk} \alpha T_j^d}{2 N_{jk}^d},$$

$$I_{\text{avg}} = \frac{d_{jk} \alpha T_j^d}{N_{jk}^d} + \left( d_{jk} T_j^d - (1 - \frac{1}{N_{jk}^d}) d_{jk} \alpha T_j^d - \frac{d_{jk} \alpha T_j^d}{N_{jk}^d} \right) / 2 - \frac{d_{jk} \alpha T_j^d}{2 N_{jk}^d},$$

$$I_{\text{avg}} = \frac{d_{jk} \alpha T_j^d}{N_{jk}^d} + \frac{d_{jk} T_j^d}{2} - \frac{d_{jk} \alpha T_j^d}{2} + \frac{d_{jk} \alpha T_j^d}{2 N_{jk}^d} - \frac{d_{jk} \alpha T_j^d}{2 N_{jk}^d} - \frac{d_{jk} \alpha T_j^d}{2 N_{jk}^d},$$

$$I_{\text{avg}} = \frac{d_{jk} \alpha T_j^d}{2 N_{jk}^d} + \frac{d_{jk} T_j^d}{2} - \frac{d_{jk} \alpha T_j^d}{2},$$



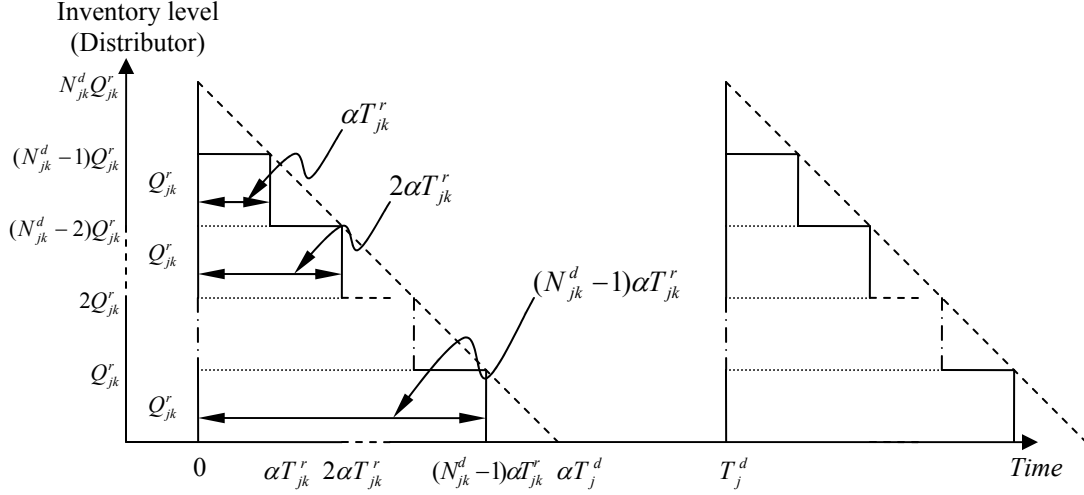
$$I_{avg} = \frac{d_{jk}\alpha T_j^d + N_{jk}^d d_{jk} T_j^d - N_{jk}^d d_{jk} \alpha T_j^d}{2N_{jk}^d}, \text{ and,}$$

$$I_{avg} = \frac{d_{jk} T_j^d (\alpha + N_{jk}^d - N_{jk}^d \alpha)}{2N_{jk}^d} = \frac{d_{jk} T_j^d (\alpha + N_{jk}^d (1 - \alpha))}{2N_{jk}^d}$$

$$\text{Average inventory, } \bar{I}_{avg} = \frac{d_{jk} T^p [\alpha + N_{jk}^d (1 - \alpha)]}{2N_j^p N_{jk}^d}. \quad (\text{B4.1.5})$$

## APPENDIX B4.2

### AN ALTERNATIVE WAY TO FIND THE DISTRIBUTOR'S HOLDING COST



During one distributor shipment interval, there are a total  $N_{jk}^d - 1$  rectangles as shown in above figure. The first rectangle (the top one) has the width of  $\alpha T_{jk}^r$  and the height of  $Q_{jk}^r$ . which leads to the area of the first rectangle to become  $\alpha T_{jk}^r Q_{jk}^r$ . Similarly, the area of the second, third, and the last rectangles are  $2\alpha T_{jk}^r Q_{jk}^r$ ,  $3\alpha T_{jk}^r Q_{jk}^r$ , and  $(N_{jk}^d - 1)\alpha T_{jk}^r Q_{jk}^r$ . Thus, the summation of all rectangles becomes  $\alpha T_{jk}^r Q_{jk}^r + 2\alpha T_{jk}^r Q_{jk}^r + \dots + (N_{jk}^d - 1)\alpha T_{jk}^r Q_{jk}^r$  which can also be written as  $\alpha T_{jk}^r Q_{jk}^r (1 + 2 + \dots + (N_{jk}^d - 1))$ . Since  $(1 + 2 + \dots + (N_{jk}^d - 1)) = (N_{jk}^d - 1)N_{jk}^d / 2$ , the function becomes  $\alpha T_{jk}^r Q_{jk}^r (N_{jk}^d - 1)N_{jk}^d / 2$ . As  $Q_{jk}^r = d_{jk} T_j^d / N_{jk}^d$ , the previous function becomes  $\alpha T_{jk}^r d_{jk} T_j^d (N_{jk}^d - 1) / 2$ . To find the average inventory, the summation of all areas has to be divided by the distributor's shipment cycle time  $T_j^d$  and the result is  $\alpha T_{jk}^r d_{jk} (N_{jk}^d - 1) / 2$ .

Substituting  $T^p / N_j^p N_{jk}^d$  into  $T_{jk}^r$ , the average inventory at distributor becomes

$$I_{avg}^d = \frac{\alpha d_{jk} T^p}{2N_j^p N_{jk}^d} (N_{jk}^d - 1), \quad (\text{B4.2.1})$$

which leads to the total holding cost at distributor for the entire system to be

$$TC_{HC}^d = (T^p, N_j^p, N_{jk}^d) = H_d \sum_{j=1}^m \sum_{k=1}^{n_j} \frac{\alpha d_{jk} T^p}{2N_j^p} \left( 1 - \frac{1}{N_{jk}^d} \right) . \quad (\text{B4.2.2})$$

### APPENDIX B4.3

#### DEVELOPMENT OF CLOSED-FORM SOLUTIONS TO PROBLEM DJV AND A PROOF OF CONVEXITY

From Equation (4.12) the total cost function is expressed as:

$$TC_{DHV} = H_p \sum_{j=1}^m \left[ \frac{DT^p \sum_{k=1}^{n_j} d_{jk}}{2PN_j^p} + \frac{T^p \sum_{k=1}^{n_j} d_{jk}}{2N_j^p} \left(1 - \frac{D}{P}\right) (N_j^p - 1) \right] + H_d \sum_{j=1}^m \sum_{k=1}^{n_j} \frac{\alpha d_{jk} T^p}{2N_j^p} \left(1 - \frac{1}{N_{jk}^d}\right) \\ + H_r \sum_{j=1}^m \sum_{k=1}^{n_m} \frac{d_{jk} T^p [\alpha + N_{jk}^d (1 - \alpha)]}{2N_j^p N_{jk}^d} + \frac{A_p + \sum_{j=1}^m A_d N_j^p + \sum_{j=1}^m \sum_{k=1}^{n_j} A_r N_j^p N_{jk}^d}{T^p},$$

which can be written in terms of original variables and parameters as

$$TC_{DHV} = H_p \sum_{j=1}^m \left[ \frac{DT^p \sum_{k=1}^{n_j} d_{jk}}{2PN_j^p} + \frac{T^p \sum_{k=1}^{n_j} d_{jk}}{2} - \frac{T^p D \sum_{k=1}^{n_j} d_{jk}}{2P} - \frac{T^p \sum_{k=1}^{n_j} d_{jk}}{2N_j^p} + \frac{T^p D \sum_{k=1}^{n_j} d_{jk}}{2PN_j^p} \right] \\ + H_d \sum_{j=1}^m \sum_{k=1}^{n_j} \frac{\alpha d_{jk} T^p}{2N_j^p} - H_d \sum_{j=1}^m \sum_{k=1}^{n_j} \frac{\alpha d_{jk} T^p}{2N_j^p N_{jk}^d} + H_r \sum_{j=1}^m \sum_{k=1}^{n_m} \frac{d_{jk} T^p \alpha}{2N_j^p N_{jk}^d} + \sum_{j=1}^m \sum_{k=1}^{n_m} \frac{H_r d_{jk} T^p}{2N_j^p} \\ - \sum_{j=1}^m \sum_{k=1}^{n_m} \frac{H_r d_{jk} T^p \alpha}{2N_j^p} + \frac{A_p + \sum_{j=1}^m A_d N_j^p + \sum_{j=1}^m \sum_{k=1}^{n_j} A_r N_j^p N_{jk}^d}{T^p}. \quad (B4.3.1)$$

Hence, taking the derivative

$$TC_{DHV} / dN_{jk}^d = H_d \frac{\alpha d_{jk} T^p}{2N_j^p (N_{jk}^d)^2} - \frac{H_r d_{jk} T^p \alpha}{2N_j^p (N_{jk}^d)^2} + \frac{A_r N_j^p}{T^p} = 0$$

$$\text{or, } \frac{A_r N_j^p}{T^p} = \frac{H_r \sum_{j=1}^m \sum_{k=1}^{n_m} d_{jk} T^p \alpha}{2N_j^p (N_{jk}^d)^2} - H_d \sum_{j=1}^m \sum_{k=1}^{n_j} \frac{\alpha d_{jk} T^p}{2N_j^p (N_{jk}^d)^2}$$

$$\text{or, } N_{jk}^d = \frac{T^p}{N_j^p} \sqrt{\frac{d_{jk} \alpha (H_r - H_d)}{2A_r}}$$

$$\text{or } N_{jk}^d = \frac{T^p}{N_j^p} B, \text{ where } B = \sqrt{d_{jk} \alpha (H_r - H_d) / 2 A_r}. \quad (\text{B4.3.2})$$

Substitute (B4.3.2) back into (B4.3.1), we get

$$\begin{aligned} TC_{DHV} \Big|_{N_{jk}^d} &= H_p \sum_{j=1}^m \left[ \frac{DT^p \sum_{k=1}^{n_j} d_{jk}}{2PN_j^p} + \frac{T^p \sum_{k=1}^{n_j} d_{jk}}{2} - \frac{T^p D \sum_{k=1}^{n_j} d_{jk}}{2P} - \frac{T^p \sum_{k=1}^{n_j} d_{jk}}{2N_j^p} + \frac{T^p D \sum_{k=1}^{n_j} d_{jk}}{2PN_j^p} \right] \\ &+ H_d \sum_{j=1}^m \sum_{k=1}^{n_j} \frac{\alpha d_{jk} T^p}{2N_j^p} - H_d \sum_{j=1}^m \sum_{k=1}^{n_j} \frac{\alpha d_{jk}}{2B} + \sum_{j=1}^m \sum_{k=1}^{n_m} \frac{H_r d_{jk} \alpha}{2B} + \sum_{j=1}^m \sum_{k=1}^{n_m} \frac{H_r d_{jk} T^p}{2N_j^p} \\ &- \sum_{j=1}^m \sum_{k=1}^{n_m} \frac{H_r d_{jk} T^p \alpha}{2N_j^p} + \frac{A_p + \sum_{j=1}^m A_d N_j^p}{T^p} + \sum_{j=1}^m \sum_{k=1}^{n_j} A_r B. \end{aligned} \quad (\text{B4.3.3})$$

Now,

$$\begin{aligned} \frac{\partial TC_{DHV} \Big|_{N_{jk}^d}}{\partial N_j^p} &= \sum_{j=1}^m \left[ -\frac{DT^p H_p \sum_{k=1}^{n_j} d_{jk}}{2P(N_j^p)^2} + \frac{T^p H_p \sum_{k=1}^{n_j} d_{jk}}{2(N_j^p)^2} - \frac{T^p D H_p \sum_{k=1}^{n_j} d_{jk}}{2P(N_j^p)^2} \right] - H_d \sum_{j=1}^m \sum_{k=1}^{n_j} \frac{\alpha d_{jk} T^p}{2(N_j^p)^2} \\ &- \frac{H_r \sum_{j=1}^m \sum_{k=1}^{n_m} d_{jk} T^p}{2(N_j^p)^2} + \frac{H_r \sum_{j=1}^m \sum_{k=1}^{n_m} d_{jk} T^p \alpha}{2(N_j^p)^2} + \frac{\sum_{j=1}^m A_d}{T^p} = 0, \end{aligned}$$

or

$$\begin{aligned} \frac{DT^p H_p \sum_{k=1}^{n_j} d_{jk}}{2P(N_j^p)^2} - \frac{T^p H_p \sum_{k=1}^{n_j} d_{jk}}{2(N_j^p)^2} + \frac{T^p D H_p \sum_{k=1}^{n_j} d_{jk}}{2P(N_j^p)^2} + H_d \sum_{j=1}^m \sum_{k=1}^{n_j} \frac{\alpha d_{jk} T^p}{2(N_j^p)^2} + \frac{H_r \sum_{j=1}^m \sum_{k=1}^{n_m} d_{jk} T^p}{2(N_j^p)^2} \\ - \frac{H_r \sum_{j=1}^m \sum_{k=1}^{n_m} d_{jk} T^p \alpha}{2(N_j^p)^2} = \frac{\sum_{j=1}^m A_d}{T^p} \end{aligned}$$

from which, we get

$$N_j^p = T^p \sqrt{\frac{\sum d_{jk} (H_p (2D - P) + H_d P \alpha + H_r P (1 - \alpha))}{2 A_d P}},$$

$$N_j^p = T^p C, \quad (\text{B4.3.4})$$

where C is the quantity under square root.

Substituting (B4.3.4) into (B4.3.3) we get

$$\begin{aligned} TC_{DHV} \Big|_{N_{jk}^d, N_j^p} = & H_p \sum_{j=1}^m \left[ \frac{D \sum_{k=1}^{n_j} d_{jk}}{2PC} + \frac{T^p \sum_{k=1}^{n_j} d_{jk}}{2} - \frac{T^p D \sum_{k=1}^{n_j} d_{jk}}{2P} - \frac{\sum_{k=1}^{n_j} d_{jk}}{2C} + \frac{D \sum_{k=1}^{n_j} d_{jk}}{2PC} \right] \\ & + H_d \sum_{j=1}^m \sum_{k=1}^{n_j} \frac{\alpha d_{jk}}{2C} - H_d \sum_{j=1}^m \sum_{k=1}^{n_j} \frac{\alpha d_{jk}}{2B} + \frac{H_r \sum_{j=1}^m \sum_{k=1}^{n_m} d_{jk} \alpha_{jk}}{2B} + \frac{H_r \sum_{j=1}^m \sum_{k=1}^{n_m} d_{jk}}{2C} - \frac{H_r \sum_{j=1}^m \sum_{k=1}^{n_m} d_{jk} \alpha}{2C} + \\ & \frac{A_p}{T^p} + \sum_{j=1}^m A_d C + \sum_{j=1}^m \sum_{k=1}^{n_j} A_r B. \end{aligned} \quad (\text{B4.3.5})$$

We know  $\sum_{j=1}^m \sum_{k=1}^{n_m} d_{jk} = D$  and

$$\frac{\partial TC_{DHV} \Big|_{N_{jk}^d, N_j^p}}{\partial T^p} = \sum_{j=1}^m \left[ \frac{H_p \sum_{k=1}^{n_j} d_{jk}}{2} - \frac{DH_p \sum_{k=1}^{n_j} d_{jk}}{2P} \right] - \frac{A_p}{(T^p)^2} = 0 \quad (\text{B4.3.6})$$

$$\frac{H_p D}{2} - \frac{H_p D^2}{2P} = \frac{A_p}{(T^p)^2},$$

which leads to

$$T^{p*} = \sqrt{\frac{2A_p P}{H_p D (P - D)}}. \quad (\text{B4.3.7})$$

Substituting (B4.3.7) into (B4.3.4) we get

$$\begin{aligned} N_j^{p*} &= \sqrt{\frac{2A_p P}{H_p D (P - D)}} \sqrt{\frac{\sum d_{jk} (H_p (2D - P) + H_d P \alpha + H_r P (1 - \alpha))}{2A_d P}} \\ &= \sqrt{\frac{\sum d_{jk} A_p (H_p (2D - P) + H_d P \alpha + H_r P (1 - \alpha))}{H_p D (P - D) A_d}}. \end{aligned} \quad (\text{B4.3.8})$$

Substituting (B4.3.7) and (B4.3.8) into (B4.3.2) we get

$$\begin{aligned}
N_{jk}^d &= \frac{T^p}{N_j^p} \sqrt{\frac{d_{jk} \alpha (H_r - H_d)}{2A_r}} \\
N_{jk}^{d*} &= \sqrt{\frac{H_p D (P-D) A_d}{\sum d_{jk} A_p (H_p (2D-P) + H_d P \alpha + H_r P (1-\alpha))}} \sqrt{\frac{2A_p P}{H_p D (P-D)}} \sqrt{\frac{d_{jk} \alpha (H_r - H_d)}{2A_r}} \\
&= \sqrt{\frac{A_d A_p P d_{jk} \alpha (H_r - H_d)}{A_r \sum d_{jk} A_p (H_p (2D-P) + H_d P \alpha_{jk} + H_r P (1-\alpha))}} \quad (B4.3.9)
\end{aligned}$$

Proof of Convexity

The integrated total cost function  $\partial TC_{DHV} \big|_{N_{jk}^d, N_j^p}$  which has been adjusted with the optimal  $N_{jk}^d$  and

$N_j^p$  is double differentiated respect to  $T^p$ . The result is

$$\frac{\partial^2 TC_{TDB} \big|_{R_{jk}, N_{jk}^d, N_{jk}^d}}{\partial T_p^2} = \frac{2A_p}{T_p^3},$$

which is always positive value, since  $A_p$  and  $T_p$  always have positive values. Therefore, the convexity of this total cost function (Equation 4.12) is proved.

## APPENDIX C5.1

### SIMPLIFYING THE TOTAL SHORTAGE FUNCTION OF PARTIAL

From Equation (5.4),

$$TS^P = \sum_{i=n_{jk}^c+1}^{n_{jk}^c+n_{jk}^p} [d_{jk} x_{jk} - (i-1)(1-\alpha) d_{jk} T_{jk}^r], \quad (C5.1.1)$$

which can be rewritten as

$$TS^P = n^p d_{jk} x - \sum_{i=n^c+1}^{n^c+n^p} [(i-1)(1-\alpha_{jk}) d_{jk} T_{jk}^r], \quad (C5.1.2)$$

and

$$\begin{aligned} \sum_{i=n^c+1}^{n^c+n^p} [(i-1)(1-\alpha_{jk}) d_{jk} T_{jk}^r] &= \sum_{i=1}^{n^c+n^p} [(i-1)(1-\alpha_{jk}) d_{jk} T_{jk}^r] - \sum_{i=1}^{n^c} [(i-1)(1-\alpha_{jk}) d_{jk} T_{jk}^r] \\ &= \left( \sum_{i=1}^{n^c+n^p} (i-1) - \sum_{i=1}^{n^c} (i-1) \right) (1-\alpha_{jk}) d_{jk} T_{jk}^r. \end{aligned} \quad (C5.1.3)$$

Since

$$\begin{aligned} \sum_{i=1}^{n^c+n^p} (i-1) &= \sum_{i=1}^{n^c+n^p} (i) - (n^c + n^p) = \frac{(n^c + n^p)(n^c + n^p + 1)}{2} - (n^c + n^p) \\ &= \frac{(n^c)^2 + (n^p)^2 + 2n^c n^p + n^c + n^p}{2} - (n^c + n^p) \\ &= \frac{(n^c)^2 + (n^p)^2 + 2n^c n^p - n^c - n^p}{2}, \end{aligned} \quad (C5.1.4)$$

$$\text{and } \sum_{i=1}^{n^c} (i-1) = \sum_{i=1}^{n^c} (i) - (n^c) = \frac{(n^c)(n^c + 1)}{2} - (n^c) = \frac{(n^c)^2 - n^c}{2}, \quad (C5.1.5)$$

$$\begin{aligned} \text{Then, } \left( \sum_{i=1}^{n^c+n^p} (i-1) - \sum_{i=1}^{n^c} (i-1) \right) &= \frac{(n^c)^2 + (n^p)^2 + 2n^c n^p - n^c - n^p}{2} - \frac{(n^c)^2 - n^c}{2} \\ &= \frac{(n^p)^2 + 2n^c n^p - n^p}{2}. \end{aligned} \quad (C5.1.6)$$

Therefore, the total shortage of partial backorder is

$$TS^P = n^p d_{jk} x - \left( \frac{(n^p)^2 + 2n^c n^p - n^p}{2} \right) (1-\alpha_{jk}) d_{jk} T_{jk}^r. \quad (C5.1.7)$$



## APPENDIX C5.2

### TOTAL RECTANGULAR AREAS

From Equation (5.9),

$$\sum_{Rectangles} = \sum_{i=n^c+n^p}^{N_{jk}^d} \{ [Q_{jk}^r + (i-1)(1-\alpha_{jk})d_{jk}T_{jk}^r] - d_{jk}x - d_{jk}\alpha T_{jk}^r \} \alpha T_{jk}^r \quad (C5.2.1)$$

$$\begin{aligned} &= \sum_{i=n^c+n^p}^{N_{jk}^d} \{ d_{jk}\alpha(T_{jk}^r)^2 + (i-1)(1-\alpha_{jk})d_{jk}\alpha(T_{jk}^r)^2 - d_{jk}x\alpha T_{jk}^r - d_{jk}(\alpha T_{jk}^r)^2 \} \\ &= (n^0 + 1) \{ d_{jk}\alpha(T_{jk}^r)^2 - d_{jk}x\alpha T_{jk}^r - d_{jk}(\alpha T_{jk}^r)^2 \} + \sum_{i=n^c+n^p}^{N_{jk}^d} (i-1)(1-\alpha)d_{jk}\alpha(T_{jk}^r)^2 \end{aligned} \quad (C5.2.2)$$

$$\begin{aligned} \sum_{i=n^c+n^p}^{N_{jk}^d} (i-1)(1-\alpha)d_{jk}\alpha(T_{jk}^r)^2 &= \sum_{i=1}^{N_{jk}^d} (i-1)(1-\alpha)d_{jk}\alpha(T_{jk}^r)^2 - \sum_{i=1}^{n^c+n^p-1} (i-1)(1-\alpha)d_{jk}\alpha(T_{jk}^r)^2 \\ &= \left( \sum_{i=1}^{N_{jk}^d} (i-1) - \sum_{i=1}^{n^c+n^p-1} (i-1) \right) (1-\alpha_{jk})d_{jk}\alpha(T_{jk}^r)^2 \end{aligned} \quad (C5.2.3)$$

$$\begin{aligned} \sum_{i=1}^{N_{jk}^d} (i-1) &= \sum_{i=1}^{N_{jk}^d} (i) - N_{jk}^d = \frac{N_{jk}^d(N_{jk}^d + 1)}{2} - N_{jk}^d \\ &= \frac{(N_{jk}^d)^2 + N_{jk}^d}{2} - N_{jk}^d = \frac{(n^c + n^p + n^0)^2 - (n^c + n^p + n^0)}{2} \\ &= \frac{(n^c)^2 + (n^p)^2 + (n^0)^2 + 2n^cn^p + 2n^cn^0 + 2n^pn^0 - n^c - n^p - n^0}{2} \end{aligned} \quad (C5.2.4)$$

$$\begin{aligned} \sum_{i=1}^{n^c+n^p-1} (i-1) &= \sum_{i=1}^{n^c+n^p-1} (i) - (n^c + n^p - 1) = \frac{(n^c + n^p - 1)(n^c + n^p)}{2} - (n^c + n^p - 1) \\ &= \frac{(n^c)^2 + (n^p)^2 + 2n^cn^p - n^c - n^p}{2} - (n^c + n^p - 1) \\ &= \frac{(n^c)^2 + (n^p)^2 + 2n^cn^p - 3n^c - 3n^p + 2}{2} \end{aligned} \quad (C5.2.5)$$

$$\begin{aligned}
\left( \sum_{i=1}^{N_{jk}^d} (i-1) - \sum_{i=1}^{n^c+n^p-1} (i-1) \right) &= \frac{(n^0)^2 + 2n^c n^0 + 2n^p n^0 - n^c - n^p - n^0 - (-3n^c - 3n^p + 2)}{2} \\
&= \frac{(n^0)^2 + 2n^c n^0 + 2n^p n^0 + 2n^c + 2n^p - n^0 - 2}{2}.
\end{aligned} \tag{C5.2.6}$$

Therefore, the function of total rectangular areas is

$$\begin{aligned}
\Sigma_{Rectangles} &= (n^0 + 1) \{ d_{jk} \alpha T_{jk}^r (T_{jk}^r - x - \alpha T_{jk}^r) \} \\
&+ \left( \frac{(n^0)^2 + n^0 (2n^c + 2n^p - 1) + 2n^c + 2n^p - 2}{2} \right) (1 - \alpha) d_{jk} \alpha (T_{jk}^r)^2.
\end{aligned} \tag{C5.2.7}$$

## **VITA**

Ratkrit Rochanaluk (Nai) was born to Boonlert Rochanaluk (father) and Pathra Rochanaluk (mother) in Bangkok, Thailand, in July, 1981. He joined Thammasat University in 1999 and graduated with a Bachelor of Engineering in Industrial Engineering degree in 2003. He worked at Navasri Manufacturing Company, Pathumthani, Thailand, as a Production Supervisor during 2003-2004 and at Nidec-Copal as a Process Engineer during 2004-2005.

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Ratkrit's research interest is in supply chain management, inventory management, manufacturing system, and distribution logistics.

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