Communications on Stochastic Analysis

Volume 5 | Number 3 Article 3

9-1-2011

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Izumi Kubo

Hui-Hsiung Kuo

Communications on Stochastic Analysis
Vol. 5, No. 3 (2011) 481-503

Serials Publications
www.serialspublications.com

TWIN MRM-TRIPLES IN MULTIPLICATIVE RENORMALIZATION METHOD

IZUMI KUBO* AND HUI-HSIUNG KUO

ABSTRACT. MRM-applicable measures for MRM-factors ${}_0F_0(-;-;x) = e^x$, ${}_1F_0(\kappa;-;x) = (1-x)^{-\kappa}$ and ${}_1F_1(1;2;x) = \frac{1}{x}(e^x-1)$ have already been completely determined. Conversely, MRM-factors for which those measures are MRM-applicable have also been completely determined. Some measures have couples of MRM-triples $\big(h(x), \rho(t), B(t)\big)$ and $\big(h_0(x), \rho_0(t), B_0(t)\big)$ such that $h_0(x) = \frac{d}{dx}\big(xh(x)\big)$, $B(t) = \frac{1}{t}\rho(t)$, $\rho_0(t) = \rho(t)$, $B_0(t) = \rho'(t) = \frac{d}{dt}\big(tB(t)\big)$. Such triples are called twin MRM-triples. We determine in this paper all twin MRM-triples for symmetric measures and for measures with the Jacobi-Szegö parameters $\alpha_n = a_2n^2 + a_1n + a_0$.

1. MRM-Applicability of Probability Measures

A probability measure μ on $\mathbb{R}$ with support of infinite set is said to be *applicable to the multiplicative renormalization method for* $h(x)$ (or simply *MRM-applicable*), if there exists a suitable analytic function $\rho(t)$ around $t = 0$ with $\rho(0) = 0$, $r_1 = \rho'(0) \neq 0$ such that

$$\psi(t, x) = \frac{h(\rho(t)x)}{\varphi(t)} \quad \text{with} \quad \varphi(t) = \theta(\rho(t)), \quad \theta(t) = \int h(tx)\, d\mu(x) \tag{1.1}$$

is a generating function of the orthogonal monic polynomials $\{P_n(x)\}$ in $L^2(\mu)$. Here we call a polynomial *monic*, if its leading coefficient is one. Then there exist Jacobi-Szegö parameters $\{\alpha_n, \omega_n\}$ satisfying the recursive relation

$$P_{n+1}(x) = (x - \alpha_n)P_n(x) - \omega_n P_{n-1}(x) \tag{1.2}$$

with $\omega_0 = 1, P_{-1}(x) = 0$. For simplicity, we call them *the Jacobi-Szegö parameters for* μ and $h(x)$ *an MRM-factor for* μ. It is known that

$$\|P_n\|^2 = \lambda_n = \omega_0\omega_1 \cdots \omega_n \quad \text{for} \quad n \geq 0. \tag{1.3}$$

By Favard's theorem (1935), a set of monic polynomials $\{P_n\}$ is orthogonal for a probability measure if and only if they satisfy the recursion relation Eq. (1.2) with $\omega_n > 0$ for all $n \geq 0$ (see [1], [4] and [7]). Then we say that such a measure corresponds to polynomials $\{P_n\}$ or to Jacobi-Szegö parameters $\{\alpha_n, \omega_n\}$.

Received 2010-12-16; Communicated by the editors.
2000 *Mathematics Subject Classification.* Primary 33C45; Secondary 60E05.
Key words and phrases. Orthogonal polynomials, MRM-applicable, Jacobi-Szegö parameters, hypergeometric functions, Meixner class, twin triple, Boas-Buck generating function.
* This research is supported by Grant-in-Aid for Scientific Research (C) no.20540145 from Japan Society for the Promotion of Science.

Let us suppose that

$$h(x) = \sum_{n=0}^{\infty} h_n x^n, \quad h_0 = 1,\ h_n \neq 0, \quad n \geq 1,$$
$$r(t) = \sum_{n=1}^{\infty} r_n t^n, \quad r_1 = 1, \quad B(t) = \sum_{n=0}^{\infty} b_n t^n, \quad b_0 = 1.$$

Then we have the expansion

$$\psi(t,x) = B(t)h(\rho(t)x) = \sum_{n=0}^{\infty} h_n P_n(x) t^n. \tag{1.4}$$

This is called a Boas-Buck generating function of $\{P_n(x)\}$. If $\{P_n(x)\}$ are orthogonal polynomials for a probability measure, then $B(t) = 1/\varphi(t)$ holds. We call $(h(x), \rho(t), B(t))$ an *MRM-triple*. By using Eq. (1.4), we can show that $\rho(t)$ and $B(t)$ are uniquely determined if μ is MRM-applicable for $h(x)$. It is remarkable that a scaling $\tilde{h}_p = h(px)$, $\tilde{\rho}_p(t) = p\rho(pt)$, $\tilde{B}_p(t) = B(pt)$ of an MRM-triple $\big(h(x), \rho(t), B(t)\big)$ for μ is also an MRM-triple for μ. Moreover, if μ is symmetric, $h_c(x) = h_{even}(x) + c\, h_{odd}(x)$ $(c \neq 0)$ is an MRM-factor, where $h_{even}(x) = \frac{1}{2}(h(x) + h(-x))$, $h_{odd}(x) = \frac{1}{2}(h(x) - h(-x))$. We call these *trivial modifications*. Furthermore, $h(x)$ is also an MRM-factor of the scaling $\tau_p\mu$ of μ with $\tilde{\rho}_p(t)$ and $\tilde{B}_p(t)$, where $\tau_p\mu$ is defined by $\int f(px)\, d\mu(x) = \int f(x)\, d\tau_p\mu(x)$. Then Jacobi-Szegö parameters are changed to $\tilde{\alpha}_n = p\alpha$, $\tilde{\omega}_n = p^2\omega_n$.

We have discussed MRM-applicable measures for several MRM-factors and found very special triples $(h_0(x), \rho_0(t), B_0(t))$ and $(h(x), \rho(t), B(t))$ such that

$$h(x) = \int_0^1 h_0(tx)\, dt, \quad h_0(x) = \frac{d}{dx}\big(xh(x)\big), \tag{1.5}$$
$$B_0(t) = \frac{d}{dt}\rho_0(t), \tag{1.6}$$
$$\rho(t) = \rho_0(t), \tag{1.7}$$
$$B(t) = \frac{1}{t}\rho(t). \tag{1.8}$$

Two triples $(h_0(x), \rho_0(t), B_0(t))$ and $(h(x), \rho(t), B(t))$ are called *twin triples* if Equations (1.5)–(1.8) hold. They are called *twin MRM-triples* if in addition they are MRM-triples.

Equations (1.5)–(1.8) are observed in normalizing $(h(x), \rho(t), B(t))$ so that

$$h(0) = 1, \quad \rho'(0) = 1, \quad B(0) = 1. \tag{1.9}$$

Here we show examples satisfying Eqs. (1.5)–(1.8).

Example 1.1. Let

$$h_0(x) = e^x = {}_0F_0(-;-;x) \text{ and } h(x) = \int_0^1 h_0(tx)\, dt = \frac{1}{x}(e^x - 1) = {}_1F_1(1;2;x).$$

MRM-applicable measures for both $h_0(x)$ and $h(x)$ are studied in [11]. They are the shifted negative binomial distribution $\sigma_\beta \mathrm{NegBin}(\kappa, p)$ with $\kappa = 2$, $\beta = 1$, the

Meixner distribution $M_{\kappa,\eta}$ with $\kappa = 2$ and the gamma distribution γ_κ with $\kappa = 2$. In the following cases, Eqs(1.5) – (1.8) are satisfied clearly.

(1) The shifted negative binomial distribution $\sigma_1 \mathrm{NegBin}(2, p)$ $(0 < p < 1)$.
Then $\{\alpha_n, \omega_n\} = \big\{\frac{2-p}{p}(n+1), \frac{1-p}{p^2}n(n+1)\big\}$ and

$$\rho_0(t) = \log\frac{p+t}{p+(1-p)t}, \quad B_0(t) = \frac{p^2}{(p+(1-p)t)(p+t)},$$
$$\rho(t) = \log\frac{p+t}{p+(1-p)t}, \quad B(t) = \frac{1}{t}\log\frac{p+t}{p+(1-p)t}.$$

(2) The Meixner distribution $M_{2,\eta}$.
Then $\{\alpha_n, \omega_n\} = \{2\eta(n+1), (1+\eta^2)n(n+1)\}$ and

$$\rho_0(t) = \tan^{-1}\frac{t}{1+\eta t}, \quad B_0(t) = \frac{1}{1+2\eta t+(1+\eta^2)t^2},$$
$$\rho(t) = \tan^{-1}\frac{t}{1+\eta t}, \quad B(t) = \frac{1}{t}\tan^{-1}\frac{t}{1+\eta t}.$$

In §7 of [11], we did not normalize ρ. By normalizing so as $\rho'(0) = 1$, we get the above formulas.

(3) The gamma distribution γ_κ with $\kappa = 2$.
Then $\{\alpha_n, \omega_n\} = \{2(n+1), n(n+1)\}$ and

$$\rho_0(t) = \frac{t}{1+t},\ B_0(t) = \frac{1}{(1+t)^2},\ \rho(t) = \frac{t}{1+t},\ B(t) = \frac{1}{1+t}.$$

The gamma distribution γ_κ $(\kappa > 0)$ has more MRM-factors ${}_1F_1(c;\kappa;x)$ for any $c \neq 0, -1, -2, \cdots$.

Example 1.2. Let

$$h_0(x) = {}_2F_1(2, \tfrac{3}{2}; p; 4x) \quad \text{and} \quad h(x) = \int_0^1 h_0(tx)\,dt = {}_2F_1(1, \tfrac{3}{2}; p; 4x)$$

for $0 < p < 3$. MRM-applicable measures for both $h_0(x)$ and $h(x)$ are studied in Theorem 2.3 of [9]. The typical one of them is the beta distribution $B(p, 3-p)$ over the interval [0, 1]. The ρ- and B-functions are

$$\rho_0(t) = \frac{t}{(1+t)^2}, \quad B_0(t) = \frac{1-t}{(1+t)^3} = \frac{d}{dt}\rho_0(t),$$
$$\rho(t) = \frac{t}{(1+t)^2}, \quad B(t) = \frac{1}{(1+t)^2} = \frac{1}{t}\rho(t).$$

Thus Eqs. (1.6), (1.7), and (1.8) hold. The Jacobi-Szegö parameters are

$$\{\alpha_n, \omega_n\} = \Big\{\frac{2n^2+4n+p}{(2n+1)(2n+3)}, \frac{(n+2-p)(n-1+p)}{4(2n+1)^2}\Big\}.$$

Example 1.3. Let

$$h_0(x) = {}_1F_0(2; -; 2x) = \frac{1}{(1-2x)^2},$$
$$h(x) = \int_0^1 h_0(tx)\,dt = {}_1F_0(1; -; 2x) = \frac{1}{1-2x}.$$

MRM-applicable measures for both $h(x)$ and $h_0(x)$ are studied in [13] and [15], respectively (see [5] also). From those results, we find the beta distribution $\widetilde{B}(\frac{3}{2}, \frac{3}{2})$ over the interval $[-1,\ 1]$, which is MRM-applicable for the $h_0(x)$ and $h(x)$. The ρ- and B-functions are

$$\rho_0(t) = \frac{t}{1+t^2}, \quad B_0(t) = \frac{(1-t^2)}{(1+t^2)^2} = \frac{d}{dt}\rho_0(t),$$
$$\rho(t) = \frac{t}{1+t^2}, \quad B(t) = \frac{1}{1+t^2} = \frac{1}{t}\rho(t).$$

Thus Eqs. (1.6), (1.7), and (1.8) hold. The Jacobi-Szegö parameters are given by $\{\alpha_n, \omega_n\} = \{0, \frac{1}{4}\}$.

2. Some Lemmas

In this section, we suppose that $h(x)$ is not specified and

$$h(x) = \sum_{n=0}^{\infty} h_n x^n, \quad \rho(t) = \sum_{n=0}^{\infty} r_n t^n \quad \text{and} \quad B(t) = \sum_{n=0}^{\infty} b_n t^n. \tag{2.1}$$

We may normalize so as $h(0) = h_0 = 1$, $B(0) = b_0 = 1$ and $\rho(0) = r_0 = 0, \rho'(0) = r_1 = 1$. For convenience, we put $h_{-1} = b_{-1} = 0$. Suppose that $\psi(t,x) = B(t)h(\rho(t)x)$ is a generating function of polynomials $\{P_n\}$;

$$\psi(t,x) = B(t)h(\rho(t)x) = \sum_{n=0}^{\infty} h_n P_n(x) t^n. \tag{2.2}$$

Then

$$B(t) = \frac{1}{\varphi(t)}, \quad \text{with} \quad \theta(t) = \int_{\mathbb{R}} h(tx)\, d\mu(x) \quad \text{and} \quad \varphi(t) = \theta(\rho(t)) \tag{2.3}$$

holds obviously.

By Favard's theorem [7], a set of monic polynomials $\{P_n\}$ satisfies the recursion relation Eq. (1.2)

$$P_{n+1}(x) = (x - \alpha_n)P_n(x) - \omega_n P_{n-1}(x) \quad \text{for} \quad n \geq 0$$

with Jacobi-Szegö parameters $\{\alpha_n, \omega_n\}$ satisfying $P_{-1}(x) = 0, \alpha_{-1} = 0, \omega_0 = 1$, $\omega_n > 0$ for any $n \geq 0$, if and only if they are orthogonal polynomials with respect to a probability measure μ.

Define

$$W_n(x) = P_n(x) - (x - \alpha_{n-1})P_{n-1}(x) + \omega_{n-1}P_{n-2}(x) \tag{2.4}$$

and let $W_{n,m}$ be the m-th coefficient of $W_n(x)$ for $n \geq m \geq 0$. Since $W_n(x) = 0$ must hold, all $W_{n,m}$ must vanish. Let $B_{m,k}$ be the coefficient of $B(t)\rho^m(t)$, $m \geq 0$. Then $B_{m,k} = 0$ for $m > k$. Hence

$$B_m(t) = B(t)\rho^m(t) = \sum_{k=0}^{\infty} B_{m,k} t^k = \sum_{k=m}^{\infty} B_{m,k} t^k. \tag{2.5}$$

The following Lemmas 2.1, 2.2 and 2.6 are shown in [9]. Lemmas 2.3 and 2.4 are in [10]. Other lemmas are easily seen.

Lemma 2.1. *The following equalities hold:*

$$\begin{aligned}
W_{n+1,0} &= \frac{b_{n+1}}{h_{n+1}} + \alpha_n \frac{b_n}{h_n} + \omega_n \frac{b_{n-1}}{h_{n-1}}, \\
W_{n+1,1} &= \frac{h_1}{h_{n+1}} r_{n+1} + \frac{h_1}{h_{n+1}} \sum_{k=1}^{n} b_{n+1-k} r_k - \frac{b_n}{h_n} \\
&\quad + \alpha_n \frac{h_1}{h_n} \sum_{k=1}^{n} b_{n-k} r_k + \omega_n \frac{h_1}{h_{n-1}} \sum_{k=1}^{n-1} b_{n-1-k} r_k, \\
W_{n+1,m} &= \frac{h_m}{h_{n+1}} B_{m,n+1} - \frac{h_{m-1}}{h_n} B_{m-1,n} + \alpha_n \frac{h_m}{h_n} B_{m,n} + \omega_n \frac{h_m}{h_{n-1}} B_{m,n-1}, \\
W_{n+1,n-1} &= \frac{h_{n-1}}{2h_{n+1}} (2b_2 + 2(n-1)(b_1 r_2 + r_3) + (n-1)(n-2) r_2^2) \\
&\quad - \frac{h_{n-2}}{2h_n} \big(2b_2 + 2(n-2)(b_1 r_2 + r_3) + (n-2)(n-3) r_2^2\big) \\
&\quad + \frac{h_{n-1}}{h_n} (b_1 + (n-1) r_2) \alpha_n + \omega_n, \\
W_{n+1,n} &= \frac{h_n}{h_{n+1}} (b_1 + n r_2) - \frac{h_{n-1}}{h_n} (b_1 + (n-1) r_2) + \alpha_n.
\end{aligned}$$

Moreover, a probability measure μ is MRM-applicable if and only if $W_{n,m} = 0$ for all $n-1 \geq m \geq 0$ and $\omega_n > 0$ for all $n \geq 0$.

Lemma 2.2. *For given $\{\alpha_n, \omega_n\}$ and $\{b_1, r_2, r_3\}$, we have the recursion formulas:*

$$\begin{aligned}
h_{n+1} &= \frac{h_n^2 (b_1 + n r_2)}{h_{n-1}\big(b_1 + (n-1) r_2\big) - h_n \alpha_n}, \\
b_{n+1} &= -\frac{h_{n+1}}{h_n} \alpha_n b_n - \frac{h_{n+1}}{h_{n-1}} \omega_n b_{n-1}, \\
r_{n+1} &= \frac{b_n h_{n+1}}{h_1 h_n} - \sum_{m=1}^{n} b_{n+1-m} r_m - \frac{h_{n+1}}{h_n} \alpha_n \sum_{m=1}^{n} b_{n-m} r_m \\
&\quad - \frac{h_{n+1}}{h_{n-1}} \omega_n \sum_{m=1}^{n-1} b_{n-1-m} r_m,
\end{aligned}$$

for $n \geq 1$ if $h_{n-1}\big(b_1 + (n-1) r_2\big) - h_n \alpha_n \neq 0$. Moreover, $h_1 = -\frac{b_1}{\alpha_0}$ if $\alpha_0 \neq 0$.

Lemma 2.3. (i) *If $A_m = A_{m'} = 0$ for some $m > m' \geq 0$, then $A_n = 0$ holds for any $n \geq 0$.*

(ii) *Suppose that $A_n = \sum_{j=0}^{n} \alpha_j \neq 0$ for any $n \geq 0$. Then for given $\{\alpha_n\ \omega_n\}$ and $\{b_1,\ r_2,\ r_3\}$, we have*

$$\frac{h_{n+1}}{h_n} = -\frac{b_1 + n r_2}{A_n}$$

and

$$\begin{aligned}
h_n &= (-1)^n \prod_{k=0}^{n-1} \frac{b_1 + k r_2}{A_k} \quad \textit{for} \quad n \geq 1, \\
b_{n+1} &= \frac{b_1 + n r_2}{A_n} \alpha_n b_n - \frac{(b_1 + n r_2)(b_1 + (n-1) r_2)}{A_{n-1} A_n} \omega_n b_{n-1} \quad \textit{for} \quad n \geq 1,
\end{aligned}$$

$$\begin{aligned} r_{n+1} = {} & \frac{(b_1+nr_2)A_0}{b_1A_n}b_n - \sum_{m=1}^{n} b_{n+1-m}r_m + \frac{b_1+nr_2}{A_n}\alpha_n \sum_{m=1}^{n} b_{n-m}r_m \\ & - \frac{(b_1+nr_2)(b_1+(n-1)r_2)}{A_{n-1}A_n}\omega_n \sum_{m=1}^{n-1} b_{n-1-m}r_m \quad \text{for } n \geq 2. \end{aligned}$$

(iii) *If* $\alpha_0 = 0$ *and* $A_1 \neq 0$, *then* $b_1 = 0$, $A_n \neq 0$ *for* $n \geq 1$ *and the following formulas hold:*

$$\begin{aligned} h_n &= (-1)^{n-1} r_2^{n-1}(n-1)!h_1 \prod_{k=1}^{n-1} \frac{1}{A_k} \quad \text{for } n \geq 2, \\ b_{n+1} &= \frac{nr_2}{A_n}\alpha_n b_n - \frac{n(n-1)r_2^2}{A_{n-1}A_n}\omega_n b_{n-1} \quad \text{for } n \geq 2, \\ r_{n+1} &= -\frac{r_2 n}{h_1 A_n} b_n - \sum_{m=1}^{n} b_{n+1-m}r_m + \frac{nr_2}{A_n}\alpha_n \sum_{m=1}^{n} b_{n-m}r_m \\ & \quad - \frac{n(n-1)r_2^2}{A_{n-1}A_n}\omega_n \sum_{m=1}^{n-1} b_{n-1-m}r_m \quad \text{for } n \geq 2. \end{aligned}$$

(iv) *If* $\alpha_0 \neq 0$ *and* $A_m = 0$ *for an* $m \geq 1$, *then*

$$h_n = \begin{cases} \displaystyle\prod_{k=0}^{n-1} \frac{(m-k)r_2}{A_k} & \text{if } m \geq n \geq 1, \\ \displaystyle h_{m+1} \prod_{k=m+1}^{n-1} \frac{(m-k)r_2}{A_k} & \text{if } n \geq m+2, \end{cases}$$

where h_{m+1} *is given by*

$$h_{m+1} = \frac{\big((m-1)(m+2)b_1^2 - 2m^2 b_2 - 2(m-1)m^2 r_3\big)h_m h_{m-1}}{\big((m-2)(m+3)b_1^2 - 2m^2 b_2 - 2(m-2)m^2 r_3\big)h_{m-2} + 2mb_1 h_{m-1}\alpha_m + 2m^2 h_m \omega_m}.$$

Lemma 2.4. *If* $\alpha_n = 0$ *for all* $n \geq 0$, *then* $r_{2n} = 0$, $b_{2n+1} = 0$ *for all* $n \geq 0$. *For given* $\{\alpha_n\ \omega_n\}$ *and* $\{h_1,\ b_2,\ r_3\}$, *the following recursion formulas hold,*

$$\begin{aligned} h_{n+1} &= \frac{\big(b_2+(n-1)r_3\big)h_{n-1}h_n}{\big(b_2+(n-2)r_3\big)h_{n-2} - h_n\omega_n}, \quad n \geq 2 \\ b_{n+1} &= -\frac{h_{n+1}}{h_{n-1}}\omega_n b_{n-1}, \quad n \geq 2 \\ r_{n+1} &= \frac{b_n h_{n+1}}{h_1 h_n} - \sum_{m=1}^{n} b_{n+1-m}r_m - \frac{h_{n+1}}{h_{n-1}}\omega_n \sum_{m=1}^{n-1} b_{n-1-m}r_m, \quad n \geq 2. \end{aligned}$$

Furthermore, putting $\Omega_n = \sum_{j=1}^{n} \omega_j$ *for* $n \geq 1$, *we have*

$$\frac{h_{n+1}}{h_{n-1}} = -\frac{b_2+(n-1)r_3}{\Omega_n},$$

$$h_{2n} = (-1)^n \prod_{k=1}^{n} \frac{\big(b_2+2(k-1)r_3\big)}{\Omega_{2k-1}}, \quad h_{2n+1} = (-1)^n h_1 \prod_{k=1}^{n} \frac{\big(b_2+(2k-1)r_3\big)}{\Omega_{2k}},$$

$$b_{2n}=(-1)^n h_{2n}\prod_{k=1}^{n}\omega_{2k-1}=\prod_{k=1}^{n}\frac{\omega_{2k-1}\big(b_2+2(k-1)r_3\big)}{\Omega_{2k-1}},$$

$$\begin{aligned}r_{2n+1}=&\prod_{k=1}^{n}\frac{\omega_{2k-1}\big(b_2+(2k-1)r_3\big)}{\Omega_{2k}}-\sum_{m=1}^{n}b_{2(n+1-m)}r_{2m-1}\\&+\frac{\omega_{2n}\big(b_2+(2n-1)r_3\big)}{\Omega_{2n}}\sum_{m=1}^{n}b_{2(n-m)}r_{2m-1}.\end{aligned}$$

Lemma 2.5. *Suppose that $\alpha_n=0$ for any $n\geq 0$. Then*

$$\begin{aligned}W_{2n,0}&=\frac{b_{2n}}{h_{2n}}+\omega_{2n-1}\frac{b_{2n-2}}{h_{2n-2}},\\W_{2n+1,1}&=\frac{h_1}{h_{2n+1}}\sum_{k=1}^{n}b_{2n-2k}r_{2k+1}-\frac{b_{2n}}{h_{2n}}+\omega_{2n}\frac{h_1}{h_{2n-1}}\sum_{k=1}^{n-1}b_{2n-2-2k}r_{2k+1},\\W_{2n,2m}&=\frac{h_{2m}}{h_{2n}}B_{2m,2n}-\frac{h_{2m-1}}{h_{2n-1}}B_{2m-1,2n-1}+\omega_{2n-1}\frac{h_{2m}}{h_{2n-2}}B_{2m,2n-2},\\W_{2n+1,2m+1}&=\frac{h_{2m+1}}{h_{2n+1}}B_{2m+1,2n+1}-\frac{h_{2m}}{h_{2n}}B_{2m,2n}+\omega_{2n}\frac{h_{2m+1}}{h_{2n-1}}B_{2m+1,2n-1},\\W_{2n,2n-2}&=\frac{h_{2n-2}}{h_{2n}}(b_2+(2n-2)r_3)-\frac{h_{2n-3}}{h_{2n-1}}\big(b_2+(2n-3)r_3\big)+\omega_{2n-1},\\W_{2n+1,2n-1}&=\frac{h_{2n-1}}{h_{2n+1}}(b_2+(2n-1)r_3)-\frac{h_{2n-2}}{h_{2n}}\big(b_2+(2n-2)r_3\big)+\omega_{2n},\\W_{2n,2m+1}&=W_{2n+1,2m}=0.\end{aligned}$$

Moreover, a probability measure μ is MRM-applicable if and only if $W_{n,m}=0$ for all $n-1\geq m\geq 0$ and $\omega_n>0$ for all $n\geq 0$.

Lemma 2.6. *For given $h(x)$ and fixed $\{b_1,b_2,r_2,r_3\}$, the Jacobi-Szegö parameters are uniquely determined by*

$$\begin{aligned}\alpha_n&=-\frac{h_n}{h_{n+1}}(r_2n+b_1)+\frac{h_{n-1}}{h_n}\big(r_2n+b_1-r_2\big)\quad \text{for}\quad n\geq 1,\\\omega_n&=\frac{h_{n-1}}{2h_{n+1}}\big(r_2^2n^2+(2b_1r_2+r_2^2-2r_3)n+2(b_1^2-b2-r_2^2+r_3)\big)\\&\quad+\frac{h_{n-2}}{2h_n}\big(r_2^2n^2-(3r_2^3-2r_3)n+2(r_2^2-r_3)\big)\\&\quad-\left(\frac{h_{n-1}}{h_n}\big(r_2n+b_1-r_2\big)\right)^2\quad \text{for}\ \ n\geq 2,\end{aligned}$$

and $\alpha_0=-\frac{b_1}{h_1}$, $\omega_0=1$, $\omega_1=\frac{b_1^2-b_2+b_1r_2}{h_2}-\frac{b_1^2}{h_1^2}$. Furthermore, for $n\geq 2$, the recursion formulas for $\{b_n,r_n\}$ are given by

$$\begin{aligned}b_{n+1}&=-\frac{h_{n+1}}{h_n}\alpha_nb_n-\frac{h_{n+1}}{h_{n-1}}\omega_nb_{n-1},\\r_{n+1}&=\frac{b_nh_{n+1}}{h_1h_n}-\sum_{m=1}^{n}b_{n+1-m}r_m-\frac{h_{n+1}}{h_n}\alpha_n\sum_{m=1}^{n}b_{n-m}r_m-\frac{h_{n+1}}{h_{n-1}}\omega_n\sum_{m=1}^{n-1}b_{n-1-m}r_m.\end{aligned}$$

3. Twin Triples

First we give a general theorem for twin triples:

Theorem 3.1. *Suppose that the triples* $(h_0(x), \rho_0(x), B_0(t))$ *and* $(h(x), \rho(x), B(t))$ *satisfy Eqs. (1.5) and (1.6). Let*

$$\psi_0(t,x) = B_0(t)h_0(\rho_0(t)x), \quad \psi(t,x) = B(t)h(\rho(t)x)$$

be the corresponding Boas-Buck generating functions. Then $\psi_0(t,x)$ *and* $\psi(t,x)$ *are generating functions of the common polynomials if and only if Eqs. (1.7) and (1.8) are also satisfied.*

Proof. Let $h_0(x) = \sum_{n=0}^{\infty} h_n^0 x^n$, $\rho_0(t) = \sum_{n=0}^{\infty} r_n^0 t^n$, $B_0(t) = \sum_{n=0}^{\infty} b_n^0 t^n$, $h(x) = \sum_{n=0}^{\infty} h_n x^n$, $\rho(t) = \sum_{n=0}^{\infty} r_n t^n$, and $B(t) = \sum_{n=0}^{\infty} b_n t^n$. From Eqs. (1.5) and (1.6), we obtain the relations

$$h_n^0 = (n+1)h_n \quad \text{and} \quad b_n^0 = (n+1)r_{n+1}^0. \tag{3.1}$$

Suppose that monic polynomials $\{P_n(x)\}$ satisfy

$$\psi_0(t,x) = \sum_{n=0}^{\infty} h_n^0 P_n(x)t^n \quad \text{and} \quad \psi(t,x) = \sum_{n=0}^{\infty} h_n P_n(x)t^n.$$

From Eq. (3.1), $\frac{\partial}{\partial t}\big(t\psi(t,x)\big) = \psi_0(t,x)$ must hold. Hence

$$\begin{aligned} &B(t)h(\rho(t)x) + tB'(t)h(\rho(t)x) + tx\rho'(t)B(t)h'(\rho(t)x) \\ &= B_0(t)h_0(\rho_0(t)x). \end{aligned}$$

Letting $x \to 0$, we have $B(t) + tB'(t) = B_0(t)$, which implies $\big(tB(t)\big)' = B_0(t)$ and $\rho_0(t) = tB(t)$. By letting $x \to 0$, the equality $\frac{\partial^2}{\partial t \partial x}\psi_0(t,x) = \frac{\partial}{\partial x}\psi(t,x)$ becomes

$$h_1\rho(t)\big(B(t) + tB'(t)\big) + h_1 t\rho'(t)B(t) = h_1^0 \rho_0(t)B_0(t).$$

Since $h_1^0 = 2h_1$, $B_0(t) = \rho_0'(t)$, $\big(tB(t)\big)' = B(t) + tB'(t) = B_0(t)$, we have

$$\rho(t)\rho_0'(t) + \rho'(t)\rho_0(t) = 2\rho_0(t)\rho_0'(t).$$

This implies $\big(\rho(t)\rho_0(t)\big)' = \big(\rho_0(t)^2\big)'$ and hence $\rho(t)\rho_0(t) = \rho_0(t)^2$ because $\rho(0) = \rho_0(0) = 0$. Hence we see that $\rho_0(t) = \rho(t) = tB(t)$, or equivalently, Eqs. (1.7) and (1.8) hold.

Conversely, suppose that Eqs. (1.7) and (1.8) are satisfied. Since $B_0(t) = \big(tB(t)\big)' = \rho_0'(t) = \rho'(t)$, we have

$$\begin{aligned} &\frac{\partial}{\partial t}\Big(t\psi(t,x)\Big) \\ &= \frac{\partial}{\partial t}\Big(tB(t)h(\rho(t)x)\Big) = \big(B(t) + tB'(t)\big)h(\rho(t)x) + tB(t)\rho'(t)xh'(\rho(t)x) \\ &= B_0(t)h(\rho_0(t)x) + B_0(t)\rho_0(t)xh'(\rho_0(t)x) = B_0(t)h_0(\rho(t)x) = \psi_0(t,x) \end{aligned}$$

since $h_0(x) = h(x) + xh'(x)$. It is easily seen that $\psi(t,x)$ and $\frac{\partial}{\partial t}\big(t\psi(t,x)\big)$ generate the same polynomials. □

Theorem 3.2. *Suppose that* $\big(h_0(x), \rho_0(t), B_0(t)\big)$ *and* $\big(h(x), \rho(t), B(t)\big)$ *satisfy Eqs. (1.5)–(1.8). Then the triple* $\big(h_0(x), \rho_0(t), B_0(t)\big)$ *is an MRM-triple for some* μ *if and only if* $\big(h(x), \rho(t), B(t)\big)$ *is an MRM-triple for the* μ.

Proof. According to Theorem 3.1, the assertion is clear. □

Proposition 3.3. *Suppose that* $\big(h(x), \rho(t), B(t) = \frac{1}{t}\rho(t)\big)$ *is an MRM-triple. If* $A_m = 0$ *for an* m*, then* $\alpha_n = 0$ *for all* $n \geq 0$*, that is, the corresponding measure is symmetric.*

Proof. By using $W_{n+1,n} = 0$ in Lemma 2.1 and $r_{n+1} = b_n$ implied from $B(t) = \frac{1}{t}\rho(t)$, we have

$$A_n = -\frac{h_n}{h_{n+1}}(b_1 + r_2 n) = -(n+1)b_1\frac{h_n}{h_{n+1}}$$

for any n. Hence $b_1 = 0$ holds, if $A_m = 0$. Therefore we have that $A_n = 0$ for any n, equivalently that $\alpha_n = 0$ for any n. □

4. Twin Triples for Symmetric Measures

It is difficult to study conditions under which triples $(h_0(x), \rho_0(t), B_0(t))$ and $(h(x), \rho(t), B(t))$ satisfying Eqs. (1.5)–(1.8) are MRM-triples for a measure. Hence we discuss symmetric cases. From Theorem 3.2, we may discuss conditions only for one of them. Suppose that $(h(x), \rho(t), B(t))$ satisfies Eq. (1.8) and that μ is symmetric, that is,

$$r_{n+1} = b_n \quad \text{and} \quad \alpha_n = 0 \quad \text{for any} \quad n \geq 0. \tag{4.1}$$

From Lemma 2.4, we have

Lemma 4.1. *Suppose that* $b_n = r_{n+1}$ *and* $\alpha_n = 0$ *for all* $n \geq 0$*. Then the following equalities hold:*

$$\frac{h_{n+1}}{h_{n-1}} = -\frac{nb_2}{\Omega_n}, \quad h_{2n} = (-b_2)^n \prod_{k=1}^{n} \frac{2k-1}{\Omega_{2k-1}}, \quad h_{2n+1} = (-b_2)^n h_1 \prod_{k=1}^{n} \frac{2k}{\Omega_{2k}},$$

$$b_{2n} = r_{2n+1} = (-1)^n h_{2n} \prod_{k=1}^{n} w_{2k-1} = b_2^n \prod_{k=1}^{n} \frac{(2k-1)\omega_{2k-1}}{\Omega_{2k-1}},$$

where $\Omega_n = \sum_{j=1}^{n} \omega_j$.

Lemma 4.2. *Put* $C(n; i_1, i_2, \ldots, i_k) = \frac{n!}{i_1! i_2! \cdots i_k! (n - i_1 - i_2 - \cdots - i_k)!}$ *if* $i_1 + i_2 + \cdots + i_k \leq n$ *and* $= 0$ *otherwise. Then*

$$B_{2m,2m+2k} = \sum_{2j_2+4j_4+\cdots+2kj_{2k}=2k} C(2m+1; j_2, j_4, \ldots, j_{2k}) b_2^{j_2} b_4^{j_4} \cdots b_{2k}^{j_{2k}},$$

$$B_{2m+1,2m+2k+1} = \sum_{2j_2+4j_4+\cdots+2kj_{2k}=2k} C(2m+2; j_2, j_4, \ldots, j_{2k}) b_2^{j_2} b_4^{j_4} \cdots b_{2k}^{j_{2k}}.$$

In particular,

$$B_{2m,2m+2} = (2m+1)b_2, \quad B_{2m,2m+4} = (2m+1)(mb_2^2 + b_4),$$

$$B_{2m+1,2m+3} = (2m+2)b_2, \quad B_{2m+1,2m+5} = (m+1)\big((2m+1)b_2^2 + 2b_4\big).$$

From Lemma 4.1, we see that

$$\begin{aligned}&h_2 = -\frac{b_1}{\Omega_1},\ h_3 = -\frac{2b_2h_1}{\Omega_2},\ h_4 = \frac{3b_2^2}{\Omega_1\Omega_3},\ h_5 = \frac{8b_2^2h_1}{\Omega_2\Omega_4},\ h_6 = -\frac{15b_2^3}{\Omega_1\Omega_3\Omega_5},\\ &h_7 = -\frac{48b_2^3h_1}{\Omega_2\Omega_4\Omega_6},\ b_4 = \frac{3b_2^2\omega_3}{\Omega_3},\ b_6 = \frac{15b_2^3\omega_3\omega_5}{\Omega_3\Omega_5},\ r_1 = b_0 = 1,\ r_3 = b_2,\\ &r_5 = b_4,\ r_7 = b_6,\ b_1 = b_3 = b_5 = b_7 = r_0 = r_2 = r_4 = r_6 = 0\end{aligned}$$

and

$$\frac{h_{2m}}{h_{2m+2}} = -\frac{\Omega_{2m+1}}{(2m+1)b_2},\qquad \frac{h_{2m}}{h_{2m+4}} = \frac{\Omega_{2m+1}\Omega_{2m+3}}{(2m+1)(2m+3)b_2^2},$$
$$\frac{h_{2m-1}}{h_{2m+1}} = -\frac{\Omega_{2m}}{(2m)b_2},\qquad \frac{h_{2m-1}}{h_{2m+3}} = \frac{\Omega_{2m}\Omega_{2m+2}}{(2m)(2m+2)b_2^2}.$$

From Lemma 2.5 and Lemma 4.2, we see that

$$\begin{aligned}W_{2m+4,2m} &= \frac{h_{2m}}{h_{2m+4}}B_{2m,2m+4} - \frac{h_{2m-1}}{h_{2m+3}}B_{2m-1,2m+3} + \omega_{2m+3}\frac{h_{2m}}{h_{2m+2}}B_{2m,2m+2}\\ &= \frac{\Omega_{2m+1}\Omega_{2m+2}}{(2m+3)b_2^2}(mb_2^2+b_4) - \frac{\Omega_{2m}\Omega_{2m+2}}{2(2m+2)b_2^2}((2m-1)b_2^2+2b_4)\\ &\quad -\frac{\Omega_{2m+1}}{(2m+3)b_2^2}\big((m+3)b_2^2-b_4\big)\omega_{2m+3}\end{aligned}$$

and

$$\begin{aligned}&W_{2m+5,2m+1}\\ &= \frac{h_{2m+1}}{h_{2m+5}}B_{2m+1,2m+5} - \frac{h_{2m}}{h_{2m+4}}B_{2m,2m+4} + \omega_{2m+4}\frac{h_{2m+1}}{h_{2m+3}}B_{2m+1,2m+3}\\ &= \frac{\Omega_{2m+2}\Omega_{2m+4}}{2(2m+4)b_2^2}((2m+1)b_2^2+2b_4) - \frac{\Omega_{2m+1}\Omega_{2m+3}}{(2m+3)b_2^2}(mb_2^2+b_4) - \omega_{2m+4}\Omega_{2m+2}\\ &\quad -\frac{\Omega_{2m+2}}{2(2m+4)b_2^2}((2m+7)b_2^2-2b_4)\omega_{2m+4}.\end{aligned}$$

From the equations $W_{2m+4,2m} = W_{2m+5,2m+1} = 0$, we have the following recursion formulas:

$$\begin{aligned}&\omega_{2m+3}\\ &= \frac{((2m+2)(2mb_2^2+2b_4)\Omega_{2m+1} - (2m+3)((2m-1)b_2^2+2b_4)\Omega_{2m})\Omega_{2m+2}}{(2m+2)((2m+6)b_2^2-2b_4)\Omega_{2m+1}},\\ &\omega_{2m+4}\\ &= \frac{(2m+3)((2m+1)b_2^2+2b_4)\Omega_{2m+2} - (2m+4)(2mb_2^2+2b_4)\Omega_{2m+1})\Omega_{2m+3}}{(2m+3)((2m+7)b_2^2-2b_4)\Omega_{2m+2}}.\end{aligned}$$

In both cases, we can use the equality $b_4 = \frac{3b_2^2\omega_3}{\omega_1+\omega_2+\omega_3}$ to derive

$$\begin{aligned}\omega_{n+1} = \frac{1}{n((n+4)\Omega_2+(n-2)\omega_3)\Omega_{n-1}}\Big(&(n((n-2)\Omega_2+(n+4)\omega_3)\Omega_{n-1}\\ &-(n+1)((n-3)\Omega_2+(n+3)\omega_3)\Omega_{n-2})\Omega_n\Big).\end{aligned} \tag{4.2}$$

From Eq. (4.2), we have

$$\begin{aligned}
\omega_4 &= \frac{(\omega_1+\omega_2+\omega_3)\big((\omega_1+\omega_2)^2-\omega_3(\omega_1-7\omega_2)\big)}{(\omega_1+\omega_2)(7\omega_1+7\omega_2+\omega_3)},\\
\omega_5 &= \frac{\big((\omega_1+\omega_2)^2+\omega_2\omega_3\big)\big(3(\omega_1+\omega_2)^2+5(\omega_1+\omega_2)\omega_3+32\omega_3^2\big)}{(\omega_1+\omega_2)\big(4(\omega_1+\omega_2)+\omega_3\big)\big(7(\omega_1+\omega_2)+\omega_3\big)},\\
\omega_6 &= \big(7(\omega_1+\omega_2)^2+9(\omega_1+\omega_2)\omega_3+8\omega_3^2\big)\\
&\quad\times\frac{\big(3(\omega_1+\omega_2)^3+(\omega_1+\omega_2)(\omega_1+11\omega_2)\omega_3-(4\omega_1-26\omega_2)\omega_3^2\big)}{2(\omega_1+\omega_2)\big(3(\omega_1+\omega_2)+\omega_3\big)\big(4(\omega_1+\omega_2)+\omega_3\big)\big(7(\omega_1+\omega_2)+\omega_3\big)}.
\end{aligned}$$

We get

$$\begin{aligned}
W_{7,1} &= (\omega_1+\omega_2-\omega_3)(\omega_1+\omega_2-2\omega_3)\Big(21(\omega_1+\omega_2)^3\\
&\quad+(\omega_1+\omega_2)(19\omega_1-37\omega_2)\omega_3+8\omega_2\omega_3^2\Big)\\
&\quad\times\frac{(\omega_1+\omega_2+\omega_3)\big((\omega_1+\omega_2)^2+\omega_2\omega_3\big)}{2(\omega_1+\omega_2)(3\omega_1+3\omega_2+\omega_3)(4\omega_1+4\omega_2+\omega_3)(7\omega_1+7\omega_2+\omega_3)^2}.
\end{aligned}$$

Solving the equation $W_{7,1}=0$ in w_3, we obtain solutions

$$\omega_3=\omega_1+\omega_2,\quad \omega_3=\frac{1}{2}(\omega_1+\omega_2)$$

and an equation

$$21(\omega_1+\omega_2)^3+(\omega_1+\omega_2)(19\omega_1-37\omega_2)\omega_3-8\omega_2\omega_3^2=0. \tag{4.3}$$

Case I. $\omega_3=\omega_1+\omega_2$:

From Eq. (4.1),

$$\omega_{n+1}=\frac{2n(n+1)\Omega_2(\Omega_{n-1}-\Omega_{n-2})\Omega_n}{2n(n+1)\Omega_2\Omega_{n-1}}=\frac{\omega_{n-1}\Omega_n}{\Omega_{n-1}}. \tag{4.4}$$

Therefore,

$$\omega_3=\omega_1+\omega_2,\quad \omega_4=2\omega_2,\quad \omega_5=\omega_1+2\omega_2,\quad \omega_6=3\omega_2.$$

Suppose that

$$\omega_{2k}=k\omega_2\quad\text{and}\quad \omega_{2k+1}=\omega_1+k\omega_2$$

hold for $k\le m$. Then

$$\Omega_{2k}=k(\omega_1+k\omega_2)\quad\text{and}\quad \Omega_{2k+1}=(k+1)(\omega_1+k\omega_2)$$

for $k\le m$. From Eq. (4.4),

$$\begin{aligned}
\omega_{2m+2} &= \frac{m\omega_2(m+1)(\omega_1+m\omega_2)}{m(\omega_1+m\omega_2)}=(m+1)\omega_2,\\
\Omega_{2m+2} &= (m+1)(\omega_1+m\omega_2)+(m+1)\omega_2=(m+1)\big(\omega_1+(m+1)\omega_2\big),\\
\omega_{2m+3} &= \frac{(\omega_1+m\omega_2)(m+1)(\omega_1+(m+1)\omega_2)}{(m+1)(\omega_1+m\omega_2)}=\omega_1+(m+1)\omega_2.
\end{aligned}$$

Thus we have shown that $\omega_{2m} = m\omega_2$ for $m \geq 1$ and $\omega_{2m+1} = \omega_1 + m\omega_2$ for any $m \geq 0$. From Lemma 4.1,

$$\begin{aligned} h_{2n} &= (-b_2)^n \prod_{k=1}^{n} \frac{2k-1}{k(\omega_1 + (k-1)\omega_2)} = \Big(-\frac{2b_2}{\omega_2}\Big)^n \frac{\big(\frac{1}{2}\big)_n}{n!\big(\frac{\omega_1}{\omega_2}\big)_n}, \\ h_{2n+1} &= (-b_2)^n h_1 \prod_{k=1}^{n} \frac{2k}{k(\omega_1 + k\omega_2)} = \Big(-\frac{2b_2}{\omega_2}\Big)^n \frac{1}{\big(\frac{\omega_1}{\omega_2}+1\big)_n}, \\ b_{2n} &= b_2^n \prod_{k=1}^{n} \frac{(2k-1)(\omega_1 + (k-1)\omega_2)}{k(\omega_1 + (k-1)\omega_2)} = (2b_2)^n \frac{\big(\frac{1}{2}\big)_n}{n!}. \end{aligned}$$

Hence

$$\begin{aligned} h(x) &= {}_1F_1\big(\tfrac{1}{2}; \tfrac{\omega_1}{\omega_2}; -\tfrac{2b_2}{\omega_2}x^2\big) + h_1\, {}_1F_1\big(1; \tfrac{\omega_1}{\omega_2}+1; -\tfrac{2b_2}{\omega_2}x^2\big)x, \\ b(t) &= \frac{1}{\sqrt{1-2b_2t^2}}, \quad \rho(t) = \frac{t}{\sqrt{1-2b_2t^2}}. \end{aligned}$$

Since $B_m(t) = t^m B(t)^{m+1} = t^m(1-2b_2t^2)^{-(m+1)/2}$,

$$\begin{aligned} B_{2m,2n} &= \begin{cases} (2b_2)^{n-m} \dfrac{\big(m+\frac{1}{2}\big)_{n-m}}{(n-m)!} & \text{for } \ n \geq m, \\ 0 & \text{for } \ n < m, \end{cases} \\ B_{2m+1,2n+1} &= \begin{cases} (2b_2)^{n-m} \dfrac{n!}{m!(n-m)!} & \text{for } \ n \geq m, \\ 0 & \text{for } \ n < m. \end{cases} \end{aligned}$$

Obviously,

$$\begin{aligned} \frac{h_{2m}}{h_{2n}} &= \Big(-\frac{2b_2}{\omega_2}\Big)^{m-n} \frac{n!\big(\frac{1}{2}\big)_m \big(\frac{\omega_1}{\omega_2}\big)_n}{m!\big(\frac{1}{2}\big)_n \big(\frac{\omega_1}{\omega_2}\big)_m}, \\ \frac{h_{2m+1}}{h_{2n+1}} &= \Big(-\frac{2b_2}{\omega_2}\Big)^{m-n} \frac{\big(\frac{\omega_1}{\omega_2}+1\big)_n}{\big(\frac{\omega_1}{\omega_2}+1\big)_m}. \end{aligned}$$

Then apply Lemma 2.5, $(a)_m(a+m)_{n-m} = (a)_n$, and $\frac{(a+1)_n}{(a+1)_m} = \frac{(a)_{n+1}}{(a)_{m+1}}$ to get

$$\begin{aligned} W_{2n,2m} &= \Big(-\frac{2b_2}{\omega_2}\Big)^{m-n} \frac{n!\big(\frac{1}{2}\big)_m \big(\frac{\omega_1}{\omega_2}\big)_n}{m!\big(\frac{1}{2}\big)_n \big(\frac{\omega_1}{\omega_2}\big)_m} (2b_2)^{n-m} \frac{\big(m+\frac{1}{2}\big)_{n-m}}{(n-m)!} \\ &\quad - \Big(-\frac{2b_2}{\omega_2}\Big)^{m-n} \frac{\big(\frac{\omega_1}{\omega_2}+1\big)_{n-1}}{\big(\frac{\omega_1}{\omega_2}+1\big)_{m-1}} (2b_2)^{n-m} \frac{(n-1)!}{(m-1)!(n-m)!} \\ &\quad + \Big(\omega_1 + (n-1)\omega_2\Big) \Big(-\frac{2b_2}{\omega_2}\Big)^{m-n+1} \frac{(n-1)!\big(\frac{1}{2}\big)_m \big(\frac{\omega_1}{\omega_2}\big)_{n-1}}{m!\big(\frac{1}{2}\big)_{n-1} \big(\frac{\omega_1}{\omega_2}\big)_m} \\ &\qquad \times (2b_2)^{(n-m-1)} \frac{\big(m+\frac{1}{2}\big)_{n-m-1}}{(n-m-1)!} = 0, \\ W_{2n+1,2m+1} &= \Big(-\frac{2b_2}{\omega_2}\Big)^{m-n} \frac{\big(\frac{\omega_1}{\omega_2}+1\big)_n}{\big(\frac{\omega_1}{\omega_2}+1\big)_m} (2b_2)^{n-m} \frac{n!}{m!(n-m)!} \end{aligned}$$

$$
\begin{aligned}
&- \left(-\frac{2b_2}{\omega_2}\right)^{m-n} \frac{n!\left(\frac{1}{2}\right)_m \left(\frac{\omega_1}{\omega_2}\right)_n}{m!\left(\frac{1}{2}\right)_n \left(\frac{\omega_1}{\omega_2}\right)_m} (2b_2)^{n-m} \frac{\left(m+\frac{1}{2}\right)_{n-m}}{(n-m)!} \\
&+ n\omega_2 \left(-\frac{2b_2}{\omega_2}\right)^{m+1-n} \frac{\left(\frac{\omega_1}{\omega_2}+1\right)_{n-1}}{\left(\frac{\omega_1}{\omega_2}+1\right)_m} (2b_2)^{n-m-1} \frac{(n-1)!}{m!(n-m-1)!} \\
= \ & 0
\end{aligned}
$$

By Lemma 2.5, we see that the triple $(h(x), \rho(t), B(t))$ is an MRM-triple. From Theorem 3.2, $(h(x), \rho(t), B(t))$ and $(h_0(x), \rho_0(t), B_0(t))$ are twin triples with

$$
\begin{aligned}
h_0(x) = \ & {}_1F_1\left(\tfrac{1}{2}; \tfrac{\omega_1}{\omega_2}; -\tfrac{2b_2}{\omega_2}x^2\right) - \frac{\omega_1}{2b_2} {}_1F_1\left(\tfrac{3}{2}; \tfrac{\omega_1}{\omega_2}+1; -\tfrac{2b_2}{\omega_2}x^2\right)x^2 \\
& + 2h_1 \left\{ {}_1F_1\left(1; \tfrac{\omega_1}{\omega_2}; -\tfrac{2b_2}{\omega_2}x^2\right)x - \frac{\omega_1}{2b_2} {}_1F_1\left(2; \tfrac{\omega_1}{\omega_2}+1; -\tfrac{2b_2}{\omega_2}x^2\right)x^3 \right\}, \\
\rho_0(t) = \ & \rho(t) = \frac{t}{\sqrt{1-2b_2t^2}}, \quad B_0(t) = \frac{d}{dt}\frac{t}{\sqrt{1-2b_2t^2}} = \left(1-2b_2t^2\right)^{-3/2}.
\end{aligned}
$$

Case II. $\omega_3 = \frac{1}{2}(\omega_1 + \omega_2)$:

From (4.2),

$$
\omega_{n+1} = \frac{\left(n^2\omega_{n-1} + \Omega_{n-2}\right)\Omega_n}{n(n+2)\Omega_{n-1}}. \tag{4.5}
$$

Therefore,

$$
\omega_3 = \frac{1}{2}(\omega_1+\omega_2),\ \omega_4 = \frac{1}{10}(\omega_1+9\omega_2),\ \omega_5 = \frac{1}{5}(2\omega_1+3\omega_2),\ \omega_6 = \frac{1}{7}(\omega_1+6\omega_2).
$$

Suppose that

$$
\omega_{2k} = \frac{(k-1)\omega_1 + 3(k+1)\omega_2}{2(2k+1)} \quad \text{and} \quad \omega_{2k+1} = \frac{(k+2)\omega_1 + 3k\omega_2}{2(2k+1)} \tag{4.6}
$$

hold for $k \le m$. Then

$$
\Omega_{2k} = \frac{k\left((k+2)\omega_1 + 3k\omega_2\right)}{2k+1} \quad \text{and} \quad \Omega_{2k+1} = \frac{1}{2}\left((k+2)\omega_1 + 3k\omega_2\right) \tag{4.7}
$$

for $k \le m$. From Eqs. (4.6) and (4.7), we see that

$$
\begin{aligned}
\omega_{2m+2} &= \frac{(2m+1)^2\omega_{2m} + \Omega_{2m-1}}{(2m+1)(2m+3)\Omega_{2m}}\Omega_{2m+1} = \frac{m\omega_1 + 3(m+2)\omega_2}{2(2m+3)} \\
&= \frac{(m+1-1)\omega_1 + 3((m+1)+1)\omega_2}{2\left(2(m+1)+1\right)}, \\
\Omega_{2m+2} &= \frac{1}{2}\left((m+2)\omega_1 + 3m\omega_2\right) + \frac{m\omega_1 + 3(m+2)\omega_2}{2(2m+3)} \\
&= \frac{(m+1)\left((m+3)\omega_1 + 3(m+1)\omega_2\right)}{2m+3}, \\
\omega_{2m+3} &= \frac{(2m+2)^2\omega_{2m+1} + \Omega_{2m}}{(2m+2)(2m+4)\Omega_{2m+1}}\Omega_{2m+2} = \frac{(m+3)\omega_1 + 3(m+1)\omega_2}{2(2m+3)} \\
&= \frac{\left((m+1)+2\right)\omega_1 + 3(m+1)\omega_2}{2\left(2(m+1)+1\right)}.
\end{aligned}
$$

Thus the parameters $\{\omega_n\}$ are given by Eq. (4.6). From Lemma 4.1,

$$\begin{aligned} h_{2n} &= \left(-\frac{4b_2}{\omega_1+3\omega_2}\right)^n \frac{\left(\frac{1}{2}\right)_n}{\left(\frac{2\omega_1}{\omega_1+3\omega_2}\right)_n}, \\ h_{2n+1} &= h_1\left(-\frac{4b_2}{\omega_1+3\omega_2}\right)^n \frac{\left(\frac{3}{2}\right)_n}{\left(\frac{2\omega_1}{\omega_1+3\omega_2}+1\right)_n}, \\ b_{2n} &= b_2^n \prod_{k=1}^{n} \frac{(2k-1)\omega_{2k-1}}{\Omega_{2k-1}} = b_2^n. \end{aligned}$$

Hence

$$\begin{aligned} h(x) &= {}_2F_1\left(1, \tfrac{1}{2}; \tfrac{2\omega_1}{\omega_1+3\omega_2}; -\tfrac{4b_2}{\omega_1+3\omega_2}x^2\right) \\ &\qquad + h_1\ {}_2F_1\left(1, \tfrac{3}{2}; \tfrac{2\omega_1}{\omega_1+3\omega_2}+1; -\tfrac{4b_2}{\omega_1+3\omega_2}x^2\right)x, \\ b(t) &= \frac{1}{1-b_2t^2}, \quad \rho(t) = \frac{t}{1-b_2t^2}. \end{aligned}$$

Since $B_m(t) = t^m B(t)^{m+1} = t^m(1-b_2t^2)^{-(m+1)}$,

$$\begin{aligned} B_{2m,2n} &= \begin{cases} b_2^{n-m}\dfrac{(n+m)!}{(2m)!(n-m)!} & \text{for} \quad n \geq m, \\ 0 & \text{for} \quad n < m, \end{cases} \\ B_{2m+1,2n+1} &= \begin{cases} b_2^{n-m}\dfrac{(n+m+1)!}{(2m+1)!(n-m)!} & \text{for} \quad n \geq m, \\ 0 & \text{for} \quad n < m. \end{cases} \end{aligned}$$

Obviously,

$$\begin{aligned} \frac{h_{2m}}{h_{2n}} &= \left(-\frac{4b_2}{\omega_1+3\omega_2}\right)^{m-n} \frac{\left(\frac{1}{2}\right)_m\left(\frac{2\omega_1}{\omega_1+3\omega_2}\right)_n}{\left(\frac{1}{2}\right)_n\left(\frac{2\omega_1}{\omega_1+3\omega_2}\right)_m}, \\ \frac{h_{2m+1}}{h_{2n+1}} &= \left(-\frac{4b_2}{\omega_1+3\omega_2}\right)^{m-n} \frac{\left(\frac{3}{2}\right)_m\left(\frac{2\omega_1}{\omega_1+3\omega_2}+1\right)_n}{\left(\frac{3}{2}\right)_n\left(\frac{2\omega_1}{\omega_1+3\omega_2}+1\right)_m}. \end{aligned}$$

Therefore,

$$\begin{aligned} &W_{2n,2m} \\ &= \left(-\frac{4b_2}{\omega_1+3\omega_2}\right)^{m-n} \frac{\left(\frac{1}{2}\right)_m\left(\frac{2\omega_1}{\omega_1+3\omega_2}\right)_n}{\left(\frac{1}{2}\right)_n\left(\frac{2\omega_1}{\omega_1+3\omega_2}\right)_m} b_2^{n-m}\frac{(n+m)!}{(2m)!(n-m)!} \\ &\quad - \left(-\frac{4b_2}{\omega_1+3\omega_2}\right)^{m-n} \frac{\left(\frac{3}{2}\right)_{m-1}\left(\frac{2\omega_1}{\omega_1+3\omega_2}+1\right)_{n-1}}{\left(\frac{3}{2}\right)_{n-1}\left(\frac{2\omega_1}{\omega_1+3\omega_2}+1\right)_{m-1}} b_2^{n-m}\frac{(n+m-1)!}{(2m-1)!(n-m)!} \\ &\quad + \frac{(n+1)\omega_1+3(n-1)\omega_2}{2(2n-1)}\left(-\frac{4b_2}{\omega_1+3\omega_2}\right)^{m-n+1} \frac{\left(\frac{1}{2}\right)_m\left(\frac{2\omega_1}{\omega_1+3\omega_2}\right)_{n-1}}{\left(\frac{1}{2}\right)_{n-1}\left(\frac{2\omega_1}{\omega_1+3\omega_2}\right)_m} \\ &\qquad \times b_2^{n-m-1}\frac{(n+m-1)!}{(2m)!(n-m-1)!} = 0. \end{aligned}$$

From Lemma 2.5, we see that the triple $(h(x), \rho(t), B(t))$ is an MRM-triple. From Theorem 3.2, we conclude that $(h(x), \rho(t), B(t))$ and $(h_0(x), \rho_0(t), B_0(t))$ are twin triples with

$$\begin{aligned} h_0(x) &= {}_2F_1\big(1, \tfrac{1}{2}; \tfrac{2\omega_1}{\omega_1+3\omega_2}; -\tfrac{4b_2}{\omega_1+3\omega_2}x^2\big) \\ &\quad - \frac{2b_2}{\omega_1}x^2\, {}_2F_1\big(2, \tfrac{3}{2}; \tfrac{2\omega_1}{\omega_1+3\omega_2}+1; -\tfrac{4b_2}{\omega_1+3\omega_2}x^2\big) \\ &\quad + 2h_1 x\Big\{{}_2F_1\big(1, \tfrac{3}{2}; \tfrac{2\omega_1}{\omega_1+3\omega_2}+1; -\tfrac{4b_2}{\omega_1+3\omega_2}x^2\big) \\ &\qquad - \frac{2b_2}{\omega_1+\omega_2}x^2\, {}_2F_1\big(2, \tfrac{5}{2}; \tfrac{2\omega_1}{\omega_1+3\omega_2}+2; -\tfrac{4b_2}{\omega_1+3\omega_2}x^2\big)\Big\}, \\ \rho_0(t) &= \rho(t) = \frac{t}{1-b_2t^2}, \quad B_0(t) = \frac{d}{dt}\frac{t}{1-b_2t^2} = \frac{1+b_2t^2}{(1-b_2t^2)^2}. \end{aligned}$$

Case III. $21(\omega_1+\omega_2)^3 + (\omega_1+\omega_2)(19\omega_1 - 37\omega_2)\omega_3 - 8\omega_2\omega_3^2 = 0$:

Again by the recursion formulas, we have

$$\begin{aligned} \omega_7 &= \frac{\Omega_6\big(-7\Omega_4(\Omega_2+3\omega_3) + 4\Omega_5(2\omega_2+5\omega_3)\big)}{4\Omega_5(5\Omega_2+2\omega_3)}, \\ h_8 &= -\frac{105b_2^4}{\Omega_1\Omega_3\Omega_5\Omega_7}, \quad b_8 = \frac{105b_2^4\omega_3\omega_5\omega_7}{\Omega_3\Omega_5\Omega_7}, \end{aligned}$$

which can be expressed only in terms of $\{\omega_1, \omega_2, \omega_3, b_2, h_1\}$. Solving the equations $W_{8,2} = 0$ and $W_{8,4} = 0$ in ω_3, we get solutions

$$\omega_3 = \omega_1+\omega_2, \quad \omega_3 = \frac{1}{2}(\omega_1+\omega_2), \quad \omega_3 = \frac{3}{2}(\omega_1+\omega_2), \quad \omega_3 = \frac{7}{2}(\omega_1+\omega_2)$$

and an equation

$$33(\omega_1+\omega_2)^3 + (\omega_1+\omega_2)(11\omega_1+51\omega_2)\omega_3 - 4(\omega_1 - 9\omega_2)\omega_3^2 = 0. \tag{4.8}$$

The former two cases have been discussed in Case I and Case II already.

Subcase III-1. $\omega_3 = \frac{3}{2}(\omega_1+\omega_2)$:

In this subcase, Eq. (4.3) becomes $-\frac{33}{2}(\omega_2 - 3\omega_1)(\omega_2 + \omega_1^2) = 0$. Therefore, $\omega_2 = 3\omega_1$ must hold. Then

$$\omega_3 = 6\omega_1, \quad \omega_4 = 10\omega_1, \quad \omega_5 = 15\omega_1, \quad \omega_6 = 28\omega_1, \ \omega_7 = 36\omega_1.$$

Suppose that $\omega_k = \frac{1}{2}k(k+1)\omega_1$ for $k \le m$. Then $\Omega_k = \frac{1}{6}n(n+1)(2n+1)\omega_1$ for $k \le m$. From $\omega_2 = 3\omega_1$, $\omega_3 = 6\omega_1$, Eq. (4.2) becomes

$$\omega_{n+1} = \frac{\Omega_n\big(n(5n+8)\Omega_{n-1} - (n+1)(5n+3)\Omega_{n-2}\big)}{n(5n+2)\Omega_{n-1}}.$$

Therefore,

$$\begin{aligned} \omega_{m+1} &= \frac{m(m+1)(m+2)\omega_1}{6m^2(m+1)(m-1)(5m+2)}\Big(m^2(m+1)(m-1)(5m+8) \\ &\qquad - m(m+1)(m-1)(m-2)(5m+3)\Big) \\ &= \frac{1}{2}(m+1)(m+2)\omega_1. \end{aligned}$$

By induction, we can show that $\omega_n = \frac{1}{2}n(n+1)\omega_1$. This is the case of Example 1.1 (2) with $\eta = 0$ up to a scaling.

Subcase III-2. $\omega_3 = \frac{7}{2}(\omega_1+\omega_2)$:

In this subcase, Eq. (4.3) becomes $-\frac{7}{2}(3\omega_2 - 25\omega_1)(\omega_2+\omega_1)^2 = 0$. Therefore, $\omega_2 = \frac{25}{3}\omega_1$ must hold. Then

$$\omega_3 = \frac{98}{3}\omega_1,\ \omega_4 = 90\omega_1,\ \omega_5 = \frac{605}{3}\omega_1,\ \omega_6 = \frac{1183}{3}\omega_1,\ \omega_7 = 700\omega_1.$$

Suppose that $\omega_k = \frac{1}{18}k(k+1)(2k+1)^2\omega_1$ for $k \le m$. Then $\Omega_k = \frac{1}{6}n(n+1)(2n+1)\omega_1$ for $k \le m$. By virtue of $\omega_2 = \frac{25}{3}\omega_1,\ \omega_3 = \frac{98}{3}\omega_1$, Eq. (4.2) becomes

$$\omega_{n+1} = \frac{\Omega_n\big(n(3n+8)\Omega_{n-1} - (n+1)(3n+5)\Omega_{n-2}\big)}{n(3n-2)\Omega_{n-1}}.$$

Therefore,

$$\omega_{m+1} = \frac{1}{18}(m+1)(m+2)(2m+3)^2\omega_1$$

holds. By induction, we see that $\omega_n = \frac{1}{18}n(n+1)(2n+1)^2\omega_1$ for any $n \ge 1$. From Lemma 4.1, we have

$$h_{2n} = \frac{(-3b_2)^n}{(4\omega_1)^n n!\big(\frac{3}{4}\big)_n}, \quad h_{2n+1} = h_1\frac{(-3b_2)^n}{(8\omega_1)^n\big(\frac{1}{2}\big)_n\big(\frac{5}{4}\big)_n}.$$

Taking $\omega_1 = 18,\ b_2 = -\frac{4}{5},\ h_1 = \frac{1}{3}$, we have

$$h_n = \frac{1}{(n+1)!\big(\frac{3}{2}\big)_n}, \quad h(x) = {}_1F_2\big(1;2,\tfrac{3}{2};x\big) = \frac{1}{x}\Big(\sinh\sqrt{x}\Big)^2.$$

From $W_{n,m} = 0$ in Lemma 2.1, we have differential equations

$$\begin{aligned}&2t(1+4t^2)B_m''(t) + t(5+28t^2)B_m'(t)\\ &\quad + (1+12t^2)B_m(t) - (m+1)(2m+1)tB_{m-1}(t) = 0.\end{aligned}$$

Since $B_m(t) = t^m B(t)^{m+1}$, we have differential equations for $B(t)$;

$$\begin{aligned}&2t^2(1+4t^2)B(t)B''(t) + t\big(4m+5+4(4m+7)t^2\big)B(t)B'(t)\\ &+ 2mt^2(1+4t^2)B'(t)^2 + \big(2m+1+4(2m+3)t^2\big)B(t)^2\\ &-(2m+1)B(t) = 0\end{aligned}$$

for any m. Hence we must solve

$$4t^2(1+4t^2)B'(t)^2 + 8t(1+4t^2)B(t)B'(t) + 4(1+4t^2)B(t)^2 - 4B(t) = 0$$

with $B(0) = 1$. Putting $B(t) = \frac{1}{1+4t^2}X(t)^2$ with $X(0) = 1$, we have a differential equation for $X(t)$;

$$2t(1+4t^2)X'(t) + (1+12t^2)X(t) = 1, \quad X(0) = 1.$$

The solution is $X(t) = \frac{\sqrt{1+4t^2}}{2\sqrt{t}}E(4,0;t)$, where $E(a,b;t)$ is an elliptic integral of the first kind

$$E(a,b;t) = \int_0^t \frac{du}{\sqrt{u(1+bu+au^2)}}. \tag{4.9}$$

Thus we have $B(t) = \frac{1}{4t}E(4,0;t)^2$ and $\rho(t) = \frac{1}{4}E(4,0;t)^2$. It is easily seen that $B(t)$ satisfies $W_{m,n} = 0$ for any $n \ge 0$.

Subcase III-3. $33(\omega_1+\omega_2)^3+\big((\omega_1+\omega_2)(51\omega_1+11\omega_2)+4(9\omega_2-\omega_1)\omega_3\big)\omega_3=0$:

Solutions of Eqs. (4.3) and (4.8) in (ω_2,ω_3) are

$$(\omega_2,\omega_3)=\Big(\frac{1}{6}\omega_1,-\frac{49}{6}\omega_1\Big),\ \ \Big(\frac{3}{17}\omega_1,-\frac{60}{17}\omega_1\Big),\ \ \Big(\frac{5}{27}\omega_1,-\frac{128}{27}\omega_1\Big).$$

The results contradict to the positivity of $\{\omega_n\}$.

Theorem 4.3. *The following cases are all twin MRM-triples $\big(h(x),\rho(t),B(t)\big)$ and $\big(h_0(x),\rho_0(t),B_0(t)\big)$ for symmetric probability measures up to trivial modification defined in* [11]*:*

(1) $h(x)={}_1F_1\big(\frac{1}{2};\kappa;\pm x^2\big)+x\,{}_1F_1\big(1;\kappa+1;\pm x^2\big),\ \rho(t)=\dfrac{t}{\sqrt{1\pm 2t^2}},\ B(t)=\dfrac{1}{\sqrt{1\pm 2t^2}},$

$$h_0(x)={}_1F_1\big(\tfrac{1}{2};\kappa;\pm x^2\big)\pm\tfrac{x^2}{\kappa}\,{}_1F_1\big(\tfrac{3}{2};\kappa+1;\pm x^2\big)+2x\,{}_1F_1\big(1;\kappa+1;\pm x^2\big)\pm\frac{2x^3}{1+\kappa}\,{}_1F_1\big(2;\kappa+2;\pm x^2\big),$$

$$\rho_0(t)=\frac{t}{\sqrt{1\pm 2t^2}},\quad B_0(t)=\frac{1}{\big(\sqrt{1\pm 2t^2}\big)^3}.$$

They generate the orthogonal polynomials with Jacobi-Szegö parameters $\omega_{2n}=2n,\ n\ge 1$ and $\omega_{2n+1}=2n+2\kappa,\ n\ge 0$.

(2) $h(x)={}_2F_1\big(1,\frac{1}{2};\kappa;\pm 2x^2\big)+x\,{}_2F_1\big(1,\frac{3}{2};\kappa+1;\pm 2x^2\big),\ \rho(t)=\dfrac{t}{1\pm t^2},\ B(t)=\dfrac{1}{1\pm t^2},$

$$h_0(x)={}_2F_1\big(1,\tfrac{1}{2};\kappa;\pm 2x^2\big)\pm\frac{x^2}{\kappa}\,{}_2F_1\big(2,\tfrac{3}{2};\kappa+1;\pm 2x^2\big)+2x\,{}_2F_1\big(1,\tfrac{3}{2};\kappa;\pm 2x^2\big)\pm\frac{3x^3}{\kappa+1}\,{}_2F_1\big(2,\tfrac{3}{2};\kappa+1;\pm 2x^2\big)$$

$$\rho_0(t)=\frac{t}{1\pm t^2},\quad B_0(t)=\frac{1\mp t^2}{(1\pm t^2)^2}$$

for $0<\kappa<2$. They generate the orthogonal polynomials with Jacobi-Szegö parameters $\omega_{2n}=\frac{n+1-\kappa}{2n+1},\ n\ge 1$, and $\omega_{2n+1}=\frac{n+\kappa}{2n+1},\ n\ge 0$.

(3) $h(x)={}_1F_1(1;2;x)=\dfrac{1}{x}(e^x-1),\ \rho(t)=\tan^{-1}t,\ B(t)=\dfrac{1}{t}\tan^{-1}t,$

$$h_0(x)={}_0F_0(-;-;x)=e^x,\ \rho_0(t)=\tan^{-1}t,\ B_0(t)=\frac{1}{1+t^2}.$$

They generate the orthogonal polynomials with Jacobi-Szegö parameters $\omega_n=n(n+1),\ n\ge 1$, whose corresponding measure is a Meixner distribution $M_{2,0}$.

(4) $h(x)={}_1F_2\big(1;2,\frac{3}{2};x\big),\ \rho(t)=\dfrac{1}{4}E(4,0;t)^2,\ B(t)=\dfrac{1}{4t}E(4,0;t)^2,$

$$h_0(x)={}_0F_1\big(-;\tfrac{3}{2};x\big),\ \rho_0(t)=\frac{1}{4}E(4,0;t)^2,\ B_0(t)=\frac{1}{2\sqrt{1+4t^2}}E(4,0;t).$$

They generate the orthogonal polynomials with Jacobi-Szegö parameters $\omega_n=n(n+1)(2n+1)^2,\ n\ge 1$.

Remark 4.4. We make some remarks on Theorem 4.3: (i) The case (2) with $\kappa=\frac{1}{2}$ is Example 1.3 up to a scaling; (ii) The case (3) is the special case of Example 1.1 (2) with $\eta=0$ up to a scaling.

5. Twin Triples With α_n Quadratic in n

It is very complicated to determine all possible twin MRM-triples. In this section, we assume that an MRM-applicable measure μ is not symmetric and that α_n is quadratic in n, say $\alpha_n = a_2n^2 + a_1n + a_0$. From Proposition 3.3, we see that $A_n \neq 0$ for any n and

$$A_n = \sum_{n=0}^{\infty} \alpha_n = \frac{1}{6}(n+1)(2a_2n^2 + (a_2 + 3a_1)n + 6a_0).$$

It is clear that a_0 is the mean of μ and $a_0 \neq 0$. Then we have

$$\begin{aligned} b_2 &= \frac{2b_1^2\big(a_0(a_2 + a_1 + a_0) - \omega_1\big)}{a_0(a_2 + a_1 + 2a_0)}, \\ h_n &= \frac{(-6b_1)^n}{\prod_{j=0}^{n-1}(2a_2j^2 + 3a_1j + 6a_0)}. \end{aligned}$$

Putting $q_n = 2a_2n^2 + 3a_1n + 6a_0$, we obtain that

$$\frac{h_{n-2}}{h_n} = \frac{1}{36b_1^2}q_{n-1}q_{n-2}, \qquad \frac{h_{n-2}}{h_{n-1}} = -\frac{1}{6b_1}q_{n-2}, \qquad \frac{h_{n-1}}{h_n} = -\frac{1}{6b_1}q_{n-1},$$

and by Lemmas 2.3 (ii) and 2.6,

$$\begin{aligned} \omega_n = \frac{q_{n-1}}{18a_0(a_2 + a_1 + 2a_0)} &\Big(a_0\big(- 2a_2(a_2 + a_1 - 3a_0)n^2 \\ &+(5a_2^2 + 2a_2a_1 - 3a_2a_0 - 3a_1^2 + 3a_1a_0)n - 3(a_2 - a_1 + a_0)(a_2 + a_1)\big) \\ &+\big(10a_2n^2 - (13a_2 - 9a_1)n + 6(a_2 - a_1 + a_0)\big)\omega_1\Big). \end{aligned}$$

From the equation $W_{n,m} = 0$ in Lemma 2.1, we have

$$\frac{h_{n-2}}{h_n}B_{m,n} - \frac{h_{n-2}h_{m-1}}{h_{n-1}h_m}B_{m-1,n-1} + \alpha_{n-1}\frac{h_{n-2}}{h_{n-1}}B_{m,n-1} + \omega_{n-1}B_{m,n-2} = 0,$$

which implies a system of differential equations

$$\begin{aligned} &\Big\{p_{0,1}(t)B(t)^2 + p_{0,2}(t)B(t) + p_{0,3}(t)B(t)B'(t) + p_{0,4}(t)B(t)B''(t)\Big\} \\ &\quad + m\Big\{p_{1,1}(t)B(t)^2 + p_{1,2}(t)B(t) + p_{1,3}(t)B(t)B'(t) + p_{1,4}(t)B'(t)^2 \\ &\qquad\qquad +p_{1,5}(t)B(t)B''(t)\Big\} \\ &\quad + m^2\Big\{p_{2,1}(t)B(t)^2 + p_{2,2}(t)B(t) + p_{2,3}(t)B(t)B'(t) + p_{2,4}(t)B'(t)^2\Big\} = 0 \end{aligned}$$

for

$$B_m(t) = \sum_{n=0}^{\infty} B_{m,n}t^n = B(t)\rho(t)^m = t^mB(t)^{m+1}.$$

Since the explicit forms of polynomials $\{p_{*,*}(t)\}$ are very complicated, we do not give their formulas. The system coincides with the following system:

$$\begin{aligned} D_0 &= p_{0,1}(t)B(t)^2 + p_{0,2}(t)B(t) + p_{0,3}(t)B(t)B'(t) + p_{0,4}(t)B(t)B''(t) = 0, \\ D_1 &= p_{1,1}(t)B(t)^2 + p_{1,2}(t)B(t) + p_{1,3}(t)B(t)B'(t) + p_{1,4}(t)B'(t)^2 \\ &\quad +p_{1,5}(t)B(t)B''(t) = 0, \end{aligned}$$

$$D_2 = p_{2,1}(t)B(t)^2 + p_{2,2}(t)B(t) + p_{2,3}(t)B(t)B'(t) + p_{2,4}(t)B'(t)^2 = 0.$$

Since

$$\begin{aligned} p_{0,4}(t) &= p_{2,4}(t) \\ &= 2a_2t^2\Big(a_0(a_2+a_1+2a_0)(1-3b_1t) + 2b_1^2\big(-a_0(a_2+a_1-3a_0)+5\omega_1\big)t^2\Big), \end{aligned}$$

we have $a_2 = 0$ or $(a_2 + a_1 + 2a_0) = -a_0(a_2 + a_1 - 3a_0) + 5\omega_1) = 0$ if $p_{0,4}(t) = p_{2,4}(t) = 0$ for any t. If $a_2 + a_1 + 2a_0 = 0$, then $-a_0(a_2 + a_1 - 3a_0) + 5\omega_1) = -a_0(-2a_0 - 3a_0) + 5\omega_1 = 5(a_0^2 + \omega_1) > 0$. Therefore, $a_2 = 0$ must hold. Then

$$\begin{aligned} D_0 = 3B(t)\Big(&a_0(a_1+2a_0)(a_1-2a_0) - (a_1+2a_0)\big(a_0(a_1-2a_0) \\ &+2a_0^2b_1t - 2b_1^2\omega_1t^2\big)B(t) - a_1t\big(a_0(a_1+2a_0) - 2a_0(a_1+2a_0)b_1t \\ &+2b_1^2(-a_0(a_1-a_0)+3\omega_1)t^2\big)B'(t)\Big) = 0. \end{aligned}$$

It is easily seen from $\omega_1 > 0$ that the coefficient of $B'(t)$ does not vanish. Hence $B'(t)$ is given by

$$B'(t) = -\frac{(a_1+2a_0)\Big(a_0(a_1-2a_0) - \big(a_0(a_1-2a_0) + 2a_0^2b_1t - 2b_1^2\omega_1t^2\big)B(t)\Big)}{a_1\big(a_0(a_1+2a_0) - 2a_0(a_1+2a_0)b_1t + 2b_1^2(-a_0(a_1-a_0)+3\omega_1)t^2\big)t}.$$

Apply this equality to D_1 to get

$$\begin{aligned} D_1 = -6(a_1-a_0)\Big(&a_0(a_1+2a_0) + \big(-a_0(a_1+2a_0)(1-b_1t) \\ &+b_1^2(a_1a_0-2\omega_1)t^2\big)B(t)\Big)B(t). \end{aligned}$$

Therefore,

(1) $\quad a_1 = a_0,\ a_2 = 0$

or

$$B(t) = -\frac{a_0(a_1+2a_0)}{-a_0(a_1+2a_0)(1-b_1t) + b_1^2(a_1a_0-2\omega_1)t^2}$$

must hold. If $B(t)$ is given by this, $D_0 = D_1$ holds and the condition $D_0 = 0$ implies $3a_1(a_1 + 2a_0)(a_1a_0 - 2\omega_1) = 0$. If $a_1 = -2a_0$, then $A_1 = 0$. From Proposition 3.3, we have $a_0 = \alpha_0 = 0$, which contradicts to $a_0 \neq 0$. Hence we have the cases:

(2) $\quad a_1 = 0,\ a_2 = 0,$

(3) $\quad a_1 = \frac{2\omega_1}{a_0},\ a_2 = 0.$

If $p_{0,4}(t) = p_{2,4}(t) \neq 0$, we can have expressions of $B''(t)$ and $B'(t)^2$ from $D_0 = 0$ and $D_2 = 0$, respectively, as

$$B''(t) = F_2(B, B', t) \quad \text{and} \quad B'(t)^2 = F_1(B, B', t).$$

Applying them to the equation $D_1 = 0$, we get an equation of the form

$$\begin{aligned} (a_2-a_1+a_0)\Big(&a_0(a_2+a_1+2a_0) + \big(-a_0(a_2+a_1+2a_0)(1-b_1t) \\ &+ (a_2a_0+a_1a_0-2\omega_1)b_1^2t^2\big)B(t)\Big)B(t) = 0. \end{aligned}$$

Similarly, we can show that $a_0(a_2+a_1+2a_0)+\{-a_0(a_2+a_1+2a_0)(1-b_1t)+(a_2a_0+a_1a_0-2\omega_1)b_1^2t^2\}B(t)\neq 0$. Hence we have the condition $a_2=a_1-a_0$. Then the compatible condition

$$\frac{d}{dt}F_1(B(t),B'(t),t)\Big(=\frac{d}{dt}B'(t)^2=2B'(t)B''(t)\Big)=2B'(t)F_2(B(t),B'(t),t)$$

must be satisfied. We see that one of the two conditions $a_1=2a_0$ and $a_1=-\frac{1}{2}a_0$ must satisfied. If the later condition is satisfied, then we see that $A_1=0$ from the formula $A_n=-\frac{1}{2}(n-1)(n+1)(n+2)$. Therefore, this case is removed. Thus our last case is

(4) $a_1=2a_0,\ a_2=a_1-a_0\quad(a_1=2a_0,\ a_2=a_0)$.

Theorem 5.1. *A triplet $\big(h(x),\rho(t),B(t)=\frac{1}{t}\rho(t)\big)$ is an MRM-triplet for probability measures with quadratic $\{\alpha_n=a_2n^2+a_1n+a_0\}$ $(a_0=\beta\neq 0)$ if and only if it is in the following four cases. In those cases, we have twin triples $\big(h(x),\rho(t),B(t)\big)$ and $\big(h_0(x),\rho_0(t),B_0(t)\big)$ up to a scaling.*

(1) *The case $a_1=a_0,\ a_2=0$:*

$$h(x)={}_1F_1\big(1;2;x\big)=\frac{1}{x}\big(e^x-1\big),\quad \rho(t)=\int_0^t\frac{du}{1+\beta u+\gamma u^2},\ B(t)=\frac{1}{t}\rho(t),$$
$$h_0(x)={}_0F_0\big(-;-;x\big)=e^x,\quad \rho_0(t)=\rho(t),\quad B_0(t)=\frac{1}{1+\beta t+\gamma t^2},$$
$$\alpha_n=\beta(n+1),\quad \omega_n=\gamma n(n+1).$$

(2) *The case $a_1=0,\ a_2=0$:*

$$h(x)={}_1F_0\big(1;-;x\big)=\frac{1}{1-x},\quad \rho(t)=\frac{t}{1+\beta t+\gamma t^2},\quad B(t)=\frac{1}{1+\beta t+\gamma t^2},$$
$$h_0(x)={}_1F_0\big(2;-;x\big)=\frac{1}{\big(1-x\big)^2},\quad \rho_0(t)=\frac{t}{1+\beta t+\gamma t^2},$$
$$B_0(t)=\frac{1-\gamma t^2}{(1+\beta t+\gamma t^2)^2},\quad \alpha_n=\beta,\quad \omega_n=\gamma.$$

(3) *The case $a_1=\frac{2\omega_1}{a_0},\ a_2=0$:*

$$h(x)={}_1F_1\big(1;\kappa;x\big),\quad \rho(t)=\frac{t}{1+\gamma t},\quad B(t)=\frac{1}{1+\gamma t},$$
$$h_0(x)={}_1F_1\big(2;\kappa;x\big),\quad \rho_0(t)=\frac{t}{1+\gamma t},\quad B_0(t)=\frac{1}{(1+\gamma t)^2},$$
$$\alpha_n=\gamma(2n+\kappa),\quad \omega_n=\gamma^2n\big(n+\kappa-1\big).$$

(4) *The case $a_1=2a_0,\ a_2=a_0$:*

$$h(x)={}_1F_2\big(1;2,\tfrac{3}{2};x\big)=\frac{1}{2x}\big(\cosh 2\sqrt{x}-1\big),$$
$$\rho(t)=\frac{1}{4}E\big(4\gamma,\beta;t\big)^2,\quad B(t)=\frac{1}{t}\rho(t),$$
$$h_0(x)={}_0F_1\big(-;\tfrac{3}{2};x\big)=\frac{1}{2\sqrt{x}}\sinh 2\sqrt{x},$$

$$\rho_0(t)=\rho(t),\quad B_0(t)=\frac{1}{2\sqrt{t\big(1+\beta t+4\gamma t^2\big)}}E\big(4\gamma,\beta;t\big),$$
$$\alpha_n=\beta(n+1)^2,\quad \omega_n=\gamma n(n+1)(2n+1)^2.$$

Proof. The necessity of those conditions on parameters has already been illustrated above. We omit the detail since it is too complicated. The sufficiency part of the theorem is proved as follows.

(1) $a_1=a_0,\ a_2=0$:

From the above formula for ω_n, we have

$$\alpha_n=a_0(n+1),\quad \omega_n=\frac{\omega_1}{2}n(n+1).$$

From the formula of h_n, we have

$$h_n=\left(-\frac{2b_1}{a_0}\right)^n\frac{1}{(n+1)!},$$
$$h(x)={}_1F_0\big(1;-;-\tfrac{2b_1}{a_0}x\big)=\frac{a_0}{2b_1x}\big(1-e^{-\frac{2b_1}{a_0}}\big).$$

But the equation $D_0=0$ implies that

$$\big(a_0^2-2a_0^2b_1t+2b_1^2\omega_1t^2\big)(B(t)+tB'(t))=a_0^2.$$

Solving it, we obtain

$$\rho(t)=tB(t)=\int_0^t\frac{du}{1-2b_1u+\frac{2b_1^2\omega_1}{a_0^2}u^2}.$$

We can show that $D_0=D_1=D_2=0$ for the solution. Then $h_0(x)=\frac{d}{dx}(xh(x))$ and $B_0(t)=\frac{d}{dt}\rho(t)$ are easily calculated. Putting $b_1=-\frac{a_0}{2},\ a_0=\beta,\ \omega_1=2\gamma$, we have the assertion.

(2) $a_1=0,\ a_2=0$:

Similarly to (1), we can obtain

$$h_n=(-1)^n\frac{b_1^n}{a_0^n},\quad h(x)={}_1F_0\big(1;-;-\tfrac{b_1}{a_0}x\big)=\frac{1}{1+\frac{b_1}{a_0}x},\quad \alpha_n=a_0,\quad \omega_n=\omega_1.$$

In this case, we see that D_0 does not depend on $\big(B'(t),B''(t)\big)$. Hence solving the algebraic equation $D_0=0$ in $B(t)$, we have

$$B(t)=\frac{1}{1-b_1t+\frac{b_1^2\omega_1}{a_0^2}}.$$

Then $h_0(x)$ and $B_0(t)$ are easily obtained. Putting $b_1=-a_0,\ \omega_1=\gamma,\ a_0=\beta$, we have the assertion.

(3) $a_1=\frac{\omega_1}{a_0^2},\ a_2=0$:

We can obtain

$$h(x)={}_1F_1\big(1;\tfrac{a_0^2}{\omega_1};-\tfrac{a_0b_1}{\omega_1}x\big),\ \ \alpha_n=\frac{1}{a_0}(2\omega_1n+a_o^2),\ \ \omega_n=\frac{\omega_1}{a_0^2}n(\omega_1(n-1)+a_0^2).$$

From the equation $D_1 = 0$, we have

$$B'(t) = \frac{1 - (1 - 2b_1 t + b_1^2 \omega_1 t^2) B(t)}{t(1 - b_1 t)^2}.$$

Substituting $B'(t)$ in D_0 with the expression, we have the equation

$$(a_0^2 - 2\omega_1)(1 - (1 - b_1 t) B(t)) = 0.$$

If $a_0^2 - 2\omega_1 \neq 0$, then $B(t) = \frac{1}{1 - b_1 t}$. This $B(t)$ satisfies $D_0 = D_1 = D_2 = 0$. If $a_0^2 - 2\omega_1 = 0$, then $D_0 = 9a_0^2 B(t)\big((1 - b_1 t)^2 (B(t) + tB'(t)) - 1\big) = 0$. Hence

$$\rho(t) = tB(t) = \int_0^t \frac{du}{(1 - b_1 u)^2} = \frac{t}{1 - b_1 t}.$$

Thus we have the $B(t)$. Furthermore, $\rho(t)$ and $B_0(t)$ are easily obtained and the assertion is seen by putting $a_0 = \beta = \kappa\gamma,\ \omega_1 = \frac{a_0^2}{\kappa} = \gamma^2 \kappa,\ b_1 = -\frac{\omega_1}{a_0} = -\gamma$.

(4) $a_1 = 2a_0,\ a_2 = a_0$:

We obtain

$$h(x) = {}_1F_2\big(1; 2, \tfrac{3}{2}; -\tfrac{3b_1}{a_0} x\big) = -\frac{a_0}{6b_1 x} \Big(\cosh \sqrt{-\frac{12 b_1 x}{a_0}} - 1 \Big),$$

$$\alpha_n = a_0 (n+1)^2, \quad \omega_n = \frac{\omega_1}{18} n(n+1)(2n+1)^2.$$

Put

$$B(t) = \frac{X(t)^2}{a_0^2 - 3a_0^2 b_1 t + 2b_1^2 \omega_1 t^2}, \quad X(0) = a_0,$$

and solve $B'(t)^2 = F_1(B(t), B'(t), t)$ for $X'(t)$. Then we have an equation

$$X'(t) = \frac{a_0}{2t} - \frac{a_0^2 - 2b_1^2 \omega_1 t^2}{2t(a_0^2 - 3a_0^2 b_1 t + 2b_1^2 \omega_1 t^2)} X(t).$$

We can solve this differential equation as

$$\begin{aligned} X(t) &= \frac{\sqrt{1 - 3b_1 t + \frac{2b_1^2 \omega_1}{a_0^2} t^2}}{2\sqrt{t}} \int_0^t \frac{du}{\sqrt{u\big(1 - 3b_1 u + \frac{2b_1^2 \omega_1}{a_0^2} u^2\big)}} \\ &= \frac{\sqrt{1 - 3b_1 t + \frac{2b_1^2 \omega_1}{a_0^2} t^2}}{2\sqrt{t}} E\big(2b_1^2 \omega_1, -3b_1; t\big), \end{aligned}$$

where $E(a, b; t)$ is defined by Eq. (4.9). Thus we get $B(t)$. Others are proved easily. Putting $b_1 = -\frac{a_0}{3},\ \omega_1 = 18\gamma$, and $a_0 = \beta$ into the relevant equations, we ottain the assertion. □

Remark 5.2. We make some remarks on Theorem 5.1:

(i) The case (1) is just Example 1.1 (possibly $a_0 = 0$). The measures are special negative binomial distributions, Meixner distributions, and gamma distributions stated in the example.

(ii) The case (2) is a class of affine transformations of the semi-circle distribution (possibly $a_0 = 0$).

(iii) The case (3) with $\kappa = 2$ is Example 1.1 (3).

Acknowledgment. The authors would like to give their deep appreciation to a referee for many suggestions which lead to the improvement of the paper.

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Izumi Kubo: Department of Mathematics, Graduate School of Science, Hiroshima University, Higashi-Hiroshima, 739-8526, Japan
E-mail address: izumi_kubo@ccv.ne.jp

Hui-Hsiung Kuo: Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA
E-mail address: kuo@math.lsu.edu