Journal of Stochastic Analysis

Volume 4 | Number 2 Article 4

June 2023

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Recommended Citation
Accardi, Luigi; Ji, Un Cig; and Saitô, Kimiaki (2023) "Multiplication Operators by White Noise Delta Functions and Associated Differential Equations," *Journal of Stochastic Analysis*: Vol. 4: No. 2, Article 4.
DOI: 10.31390/josa.4.2.04
Available at: https://digitalcommons.lsu.edu/josa/vol4/iss2/4

Journal of Stochastic Analysis
Vol. 4, No. 2 (2023) Article 4 (19 pages)
DOI: 10.31390/josa.4.2.04

MULTIPLICATION OPERATORS BY WHITE NOISE DELTA FUNCTIONS AND ASSOCIATED DIFFERENTIAL EQUATIONS

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Abstract. We establish explicit forms of the multiplication operators induced by white noise delta functions, which are closely related to the Bogoliubov transformation and a quantum analogue of Girsanov transform. Then we study the differential equations for operators associated with the multiplication operators by the white noise delta functions.

1. Introduction

Recently, in [3], motivated by the Donsker's delta function [22, 13, 19, 38, 26, 29, 11], the authors formulated a new white noise delta function of an infinite dimensional Brownian motion as a (two dimensional) white noise distribution, and studied white noise differential equations derived by the white noise delta functions which are associated with the infinite dimensional Laplacians, e.g. Volterra-Gross Laplacian [1, 18], Lévy Laplacian [27] and exotic Laplacian [7]. In fact, the Lévy Laplacian can be considered as a particular type of Gross Laplacian (see [1, 18]), and the exotic Laplacian as a natural generalization of the Lévy Laplacian is defined by higher order Cesàro means (see [5, 6]). In the white noise theory [12, 13, 14, 20, 23, 32] (and references cited therein), the infinite dimensional Laplacians with connections to heat equation and stochastic processes have been studied by many authors from several different points of view (see [2, 7, 28, 30, 34], [9, 10, 23, 24, 25, 35] and references cited therein).

On the other hand, the infinite dimensional Laplacians are closely related to the quadratic quantum white noises $\{a_t, a_t^* : t \in \mathbb{R}\}$, where a_t and a_t^* are the pointwisely defined annihilation and creation operators, respectively. Therefore, roughly speaking, the quantum analogues of the classical results associated with the infinite dimensional Laplacians provide us useful tools to analyze and understand some quantum phenomena.

The main purpose of this paper is to drive a quantum white noise differential equation associated with the multiplication operators induced by the white noise delta functions based on the Kubo-Yokoi delta function. In fact, the multiplication operator is a quantum analogue of the white noise delta function and so it can be considered as a white noise delta operator, and then we demonstrate that the white

Received 2023-5-10; Accepted 2023-6-15; Communicated by the editors.
2020 *Mathematics Subject Classification.* Primary 60H40; Secondary 46G05, 46M40.
Key words and phrases. Multiplication operator, white noise delta function, differential equation, Bogoliubov transformation, quantum Girsanov transform.
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noise delta operator plays an important role in the study of quantum white noise differential equation. For our purpose, we first construct a distribution-valued Brownian motion for which two variable white noise functionals play an important role. Then we establish explicit forms of the multiplication operators in the normal ordered form from which we recognize that the multiplication operators are closely related to the Bogoliubov transformation (see [16]) and a quantum extension of the Girsanov transform (see [17]). The quantum white differential equations studied in this paper provide us a strong motivation to study differential equations for operators in an abstract level.

This paper is organized as follows. In Section 2 we recall the standard setting of white noise functionals and construct a distribution-valued Brownian motion $\{\mathbb{B}(t)\}_{t\geq 0}$. In Section 3 we summarize the white noise operator theory which are needed for our main study. In Section 4 we discuss white noise delta function $\delta_x(\mathbb{B}(t))$ based on the Kubo-Yokoi delta function on the space of test white noise functionals (for more detail, see [3]). For notational convenience and the self contained paper, we also recall the well-known notion of Hermite polynomials. In Section 5 as a simple classical and quantum correspondence, we study the multiplication operators induced by the white noise delta functions and their explicit representation in the normal ordered form. Finally, in Section 6 we drive quantum white noise differential equations associated with the multiplication operators induced by the white noise delta functions which is closely related to a Hamiltonian equation of operators. The equation implies a different expression of the quantum Itô formula.

2. White Noise Functionals

Let H be a complex separable Hilbert space and let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for H. Let $\{\lambda_k\}_{k=1}^{\infty}$ be an increasing sequence of positive real numbers such that

$$1 < \lambda_1 \leq \lambda_2 \leq \dots \quad \text{and} \quad \sum_{j=1}^{\infty} \lambda_j^{-2} < \infty. \tag{2.1}$$

Consider a linear operator $A : \mathrm{Dom}(A) \to H$ defined by

$$A\xi = \sum_{k=1}^{\infty} \lambda_k \langle e_k, \xi\rangle e_k, \quad \xi = \sum_{k=1}^{\infty} \langle e_k, \xi\rangle e_k \in \mathrm{Dom}(A) \subset H,$$

where

$$\mathrm{Dom}(A) = \left\{ \xi \in H : \sum_{k=1}^{\infty} \lambda_k^2 |\langle e_k, \xi\rangle|^2 < \infty \right\}.$$

Then we can easily see that $\mathrm{Dom}(A)$ is dense in H, and A is self-adjoint and positive. Moreover, A is invertible and $\|A^{-1}\| = \lambda_1^{-1} < 1$, and we can also see that A^{-1} is of Hilbert-Schmidt class and $\|A^{-1}\|_{\mathrm{HS}} < \infty$. In fact, we obtain that

$$\|A^{-1}\|_{\mathrm{HS}}^2 = \sum_{k=1}^{\infty} \langle A^{-1} e_k, e_k\rangle^2 = \sum_{k=1}^{\infty} \lambda_k^{-2} < \infty.$$

For each $p \in \mathbb{R}$, define the norms $|\cdot|_p$ by

$$|\,\xi\,|_p^2 = \sum_{k=1}^{\infty} \lambda_k^{2p} |\langle e_k, \xi\rangle|^2, \qquad \xi = \sum_{k=i}^{\infty} \langle e_k, \xi\rangle e_k \in H,$$

and for each $p \geq 0$, we put $E_p = \{\xi \in H \,:\, |\,\xi\,|_p < \infty\}$ and let E_{-p} be the completion of H with respect to $|\cdot|_{-p}$. Here $|\cdot|_0$ is the Hilbertian norm on H. In fact, $|\cdot|_p = |\,A^p\cdot|_0$ and $E_p = \mathrm{Dom}(A^p)$ for $p \geq 0$. Then we construct a Gelfand triple:

$$\operatorname*{proj\,lim}_{p\to\infty} E_p =: E \subset H \subset E^* \cong \operatorname*{ind\,lim}_{p\to\infty} E_{-p}. \tag{2.2}$$

Here the nuclearity of E is from the second condition of (2.1).

Now, we construct a rigging of (Boson) Fock space based on the basic nuclear triple (2.2). For each $p \in \mathbb{R}$, let $\Gamma(E_p)$ be the Fock space over the Hilbert space E_p, i.e.,

$$\Gamma(E_p) = \left\{ \phi = (f_n)_{n=0}^{\infty} \,:\, f_n \in E_p^{\widehat{\otimes} n},\ \|\,\phi\,\|_p^2 = \sum_{n=0}^{\infty} n!\,|\,f_n\,|_p^2 < \infty \right\}.$$

Then by identifying $\Gamma(H)$ with its dual space, we have a chain of Fock spaces:

$$\cdots \subset \Gamma(E_p) \subset \Gamma(E_0) = \Gamma(H) \subset \Gamma(E_{-p}) \subset \cdots$$

and we construct a Gelfand triple:

$$\operatorname*{proj\,lim}_{p\to\infty} \Gamma(E_p) =: (E) \subset \Gamma(H) \subset (E)^* \cong \operatorname*{ind\,lim}_{p\to\infty} \Gamma(E_{-p}). \tag{2.3}$$

In fact, for the second quantization $\Gamma(A^p)$ of A^p defined by

$$\Gamma(A^p)\phi = \left((A^p)^{\otimes n} f_n\right), \quad \phi = (f_n) \in \Gamma(H), \tag{2.4}$$

we have $\|\cdot\|_p = \|\Gamma(A^p)\cdot\|_0$, where $\|\cdot\|_0$ is the Hilbertian norm on $\Gamma(H)$.

An exponential vector (or also called a coherent vector) associated with $x \in E^*$ is defined by

$$\phi_x = \left(1, x, \frac{x^{\otimes 2}}{2!}, \ldots, \frac{x^{\otimes n}}{n!}, \ldots\right). \tag{2.5}$$

Obviously, $\phi_x \in (E)^*$ and $\phi_\xi \in (E)$ for all $\xi \in E$. The *S-transform* of an element $\Phi \in (E)^*$ is defined by

$$S\Phi(\xi) = \langle\!\langle \Phi,\, \phi_\xi \rangle\!\rangle, \qquad \xi \in E,$$

where $\langle\!\langle \cdot,\, \cdot \rangle\!\rangle$ is the canonical $\mathbb{C}$-bilinear form on $(E)^* \times (E)$ which takes the form:

$$\langle\!\langle \Phi,\, \phi \rangle\!\rangle = \sum_{n=0}^{\infty} n!\,\langle F_n,\, f_n\rangle, \qquad \Phi = (F_n) \in (E)^*, \quad \phi = (f_n) \in (E). \tag{2.6}$$

Since $\{\phi_\xi\,;\, \xi \in E\}$ spans a dense subspace of (E), every element $\Phi \in (E)^*$ is uniquely determined by its S-transform $S\Phi$.

A complex-valued function F on E is called a U-functional if F is Gâteaux entire and there exist constants $C, K \geq 0$ and $p \geq 0$ such that

$$|F(\xi)| \leq C \exp\left(K\,|\,\xi\,|_p^2\right), \qquad \xi \in E.$$

Theorem 2.1 ([33])**.** *A $\mathbb{C}$-valued function F on E is the S-transform of an element in $(E)^*$ if and only if F is a U-functional.*

For each $\Phi, \Psi \in (E)^*$, by applying Theorem 2.1, we can see that there exists a unique element of $(E)^*$, denoted by $\Phi \diamond \Psi$ and called the Wick product of Φ and Ψ, such that

$$S(\Phi \diamond \Psi)(\xi) = (S\Phi)(\xi)(S\Psi)(\xi), \quad \xi \in E. \tag{2.7}$$

Remark 2.2. Let $E_{\mathbb{R}}^*$ be a real subspace of E^* such that $E^* = E_{\mathbb{R}}^* + iE_{\mathbb{R}}^*$. For the convenience, we assume that $\{e_k\}_{k=1}^{\infty} \subset H_{\mathbb{R}}$ (see the beginning of this section). Then by applying the Bochner-Minlos Theorem, we see that there exists a probability measure μ on $E_{\mathbb{R}}^*$ such that

$$\int_{E_{\mathbb{R}}^*} e^{i\langle x,\xi\rangle} d\mu(x) = e^{-\frac{1}{2}\langle \xi,\xi\rangle}, \qquad \xi \in E_{\mathbb{R}}.$$

Then the celebrated Wiener-Itô-Segal isomorphism is a unitary isomorphism between $\Gamma(H)$ and $L^2(E^*, \mu)$ which is uniquely determined by the correspondence:

$$\phi_\xi = \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \dots, \frac{\xi^{\otimes n}}{n!}, \dots\right) \quad \longleftrightarrow \quad e^{\langle x,\xi\rangle - \langle \xi,\xi\rangle/2} = \phi_\xi(x),\ \xi \in E.$$

The Gelfand triple obtained from (2.3) through the Wiener-Itô-Segal isomorphism is denoted also by

$$(E) \subset L^2(E^*, \mu) \subset (E)^*$$

which is referred to as the Hida–Kubo–Takenaka space. An element of (E) (resp. $(E)^*$) is called a test (resp. generalized) white noise function.

We now consider a two variable white noise functional which is convenient for a realization of infinite dimensional Brownian motion (see (2.9)).

Put $\mathcal{E}_{\mathbb{R}} := E_{\mathbb{R}} \otimes E_{\mathbb{R}}$ and $\mathcal{E} := E \otimes E$ which are constructed by using the densely defined, positive, self-adjoint operator $A_2 := A^{\otimes 2}$. Then by using the second quantization $\Gamma(A_2)$ (see (2.4)), we can construct the spaces $(\mathcal{E}), (L^2) \equiv L^2(\mathcal{E}_{\mathbb{R}}^*, \mu_2)$ and $(\mathcal{E})^*$, where μ_2 is a standard Gaussian measure on $\mathcal{E}_{\mathbb{R}}^* \cong E_{\mathbb{R}}^* \otimes E_{\mathbb{R}}^*$ satisfying that

$$\int_{\mathcal{E}_{\mathbb{R}}^*} e^{i\langle w,\xi\rangle} d\mu_2(w) = \exp\left\{-\frac{1}{2}\langle \xi,\xi\rangle\right\},\ \xi \in \mathcal{E}_{\mathbb{R}}.$$

Here we denote the p-norm ($p \in \mathbb{R}$) on $\mathcal{E}$ by the same notation $|\cdot|_p$, i.e., $|\xi|_p := |A_2^p \xi|$ for $\xi \in \mathcal{E}$. We also denote the S-transform on $(\mathcal{E})^*$ by S_2.

From now on, as a concrete construction, we take $L^2_{\mathbb{R}}(\mathbb{R})$ and $\mathcal{S}_{\mathbb{R}}(\mathbb{R})$ as $H_{\mathbb{R}}$ and $E_{\mathbb{R}}$, respectively. Here $\mathcal{S}_{\mathbb{R}}(\mathbb{R})$ is the Schwartz space of real-valued rapidly deceasing C^∞-functions on $\mathbb{R}$, and then $E_{\mathbb{R}}^* = \mathcal{S}_{\mathbb{R}}'(\mathbb{R})$ becomes the space of tempered distributions on $\mathcal{S}_{\mathbb{R}}(\mathbb{R})$.

For each $k \in \mathbb{N}$, put

$$B_k(t)(w) = \begin{cases} \langle w, 1_{[0,t)} \otimes e_k\rangle, & t \geq 0, \\ -\langle w, 1_{[t,0)} \otimes e_k\rangle, & t < 0, \end{cases} \qquad w \in \mathcal{E}_{\mathbb{R}}^* = E_{\mathbb{R}}^* \otimes E_{\mathbb{R}}^*. \tag{2.8}$$

Then we can easily see that for each $k \in \mathbb{N}$, $\{B_k(t)\}_{t\in\mathbb{R}}$ be a Brownian motion, which is called a realization of Brownian motion, and then $\{\{B_k(t)\}_{t\in\mathbb{R}}\}_{k=1}^{\infty}$ becomes a sequence of independent one-dimensional Brownian motions on a probability space $(\mathcal{E}_{\mathbb{R}}^*, \mu_2)$. An infinite dimensional ($E_{\mathbb{R}}^*$-valued) Brownian motion $\{\mathbb{B}(t)\}_{t\in\mathbb{R}}$ is defined by

$$\mathbb{B}(t) := \sum_{k=0}^{\infty} B_k(t) e_k, \tag{2.9}$$

which has values in $E_{\mathbb{R}}^*$ (μ_2-a.e.) for each $t \in \mathbb{R}$. In fact, for any $p \geq 1$, we obtain that

$$\begin{aligned}
\mathbf{E}\left[|\mathbb{B}(t)|_{-p}^2\right] &= \mathbf{E}\left[\left|\sum_{k=0}^{\infty} B_k(t) e_k\right|_{-p}^2\right] = \mathbf{E}\left[\left|\sum_{k=0}^{\infty} \lambda_k^{-p} B_k(t) e_k\right|_0^2\right] \\
&= \sum_{k=0}^{\infty} \lambda_k^{-2p} \mathbf{E}\left[(B_k(t))^2\right] = \sum_{k=0}^{\infty} \lambda_k^{-2p} \left|1_{[0,t)} \otimes e_k\right|_0^2 \\
&= t \sum_{k=0}^{\infty} \lambda_k^{-2p} \\
&< \infty.
\end{aligned}$$

In general, for each given $f \in H_{\mathbb{R}} = L^2_{\mathbb{R}}(\mathbb{R})$, since $E_{\mathbb{R}} = \mathcal{S}_{\mathbb{R}}(\mathbb{R})$ is dense in $H_{\mathbb{R}}$, by using the approximation procedure, we can define

$$Z_f := \sum_{k=0}^{\infty} \langle \cdot, f \otimes e_k \rangle e_k, \tag{2.10}$$

and then, similarly, we can see that Z_f has values in E_{-p} (μ_2-a.e.) for all $p \geq 1$ and Z_f is a E_{-p}-valued Gaussian random variable with mean 0 and the covariance operator $|f|_0^2 I$, where $I : E \to E$ is the identity operator. More precisely, for any $\zeta \in E$, we obtain that

$$\begin{aligned}
\mathbf{E}\left[\langle Z_f, \zeta\rangle^2\right] &= \sum_{k=0}^{\infty} \mathbf{E}\left[\langle \cdot, f \otimes e_k\rangle^2\right] \langle e_k, \zeta\rangle^2 = \sum_{k=0}^{\infty} |f \otimes e_k|_0^2 \langle e_k, \zeta\rangle^2 \\
&= |f|_0^2 \langle \zeta, \zeta\rangle,
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{E}\left[e^{-i\langle Z_f, \zeta\rangle}\right] &= \prod_{k=1}^{\infty} \mathbf{E}\left[e^{-i\langle \cdot, f\otimes e_k\rangle \langle e_k, \zeta\rangle}\right] \\
&= \prod_{k=1}^{\infty} \frac{1}{\sqrt{2\pi}|f|_0} \int_{\mathbb{R}} e^{-ix\langle e_k, \zeta\rangle} e^{-\frac{x^2}{2|f|_0^2}} dx \\
&= \prod_{k=1}^{\infty} e^{-\frac{1}{2}|f|_0^2 \langle e_k, \zeta\rangle^2} \left(\frac{1}{\sqrt{2\pi}|f|_0} \int_{\mathbb{R}} e^{-\frac{1}{2|f|_0^2}(x^2 + i2|f|_0^2 x\langle e_k, \zeta\rangle - |f|_0^4 \langle e_k, \zeta\rangle^2)} dx\right) \\
&= e^{-\frac{1}{2}|f|_0^2 \langle \zeta, \zeta\rangle}.
\end{aligned}$$

3. White Noise Operators

For locally convex nuclear space $\mathfrak{X}$, we take E or $\mathcal{E} = E \otimes E$. Let $\mathcal{L}((\mathfrak{X}), (\mathfrak{X})^*)$ be the space of all continuous linear operators from $(\mathfrak{X})$ into $(\mathfrak{X})^*$. An element of $\mathcal{L}((\mathfrak{X}), (\mathfrak{X})^*)$ is called a white noise operator. Since $\{\phi_\xi : \xi \in \mathfrak{X}\}$ spans a dense subspace of $(\mathfrak{X})$, every white noise operator $\Xi \in \mathcal{L}((\mathfrak{X}), (\mathfrak{X})^*)$ is uniquely determined by

$$\widehat{\Xi}(\xi, \eta) := \langle\!\langle \Xi\phi_\xi, \phi_\eta \rangle\!\rangle, \quad \xi, \eta \in \mathfrak{X},$$

which is called the symbol of Ξ. More precisely, we have an analytic characterization of symbols.

Theorem 3.1 ([31, 8]). *A $\mathbb{C}$-valued function Θ on $\mathfrak{X} \times \mathfrak{X}$ is the symbol of an operator $\Xi \in \mathcal{L}((\mathfrak{X}), (\mathfrak{X})^*)$ if and only if*

(i) *Θ is Gâteaux entire,*

(ii) *there exist $C \geq 0$, $K \geq 0$ and $p \geq 0$ such that*

$$|\Theta(\xi, \eta)| \leq C \exp K\big(|\,\xi\,|_p^2 + |\,\eta\,|_p^2\big), \qquad \xi, \eta \in \mathfrak{X}.$$

Moreover, Θ is the symbol of an operator $\Xi \in \mathcal{L}((\mathfrak{X}), (\mathfrak{X}))$ if and only if Θ satisfies (i) and

(ii′) *for any $p \geq 0$ and $\epsilon > 0$, there exist constants $C \geq 0$ and $q \geq 0$ such that*

$$|\Theta(\xi, \eta)| \leq C \exp \epsilon \left(|\xi|_{p+q}^2 + |\eta|_{-p}^2 \right), \qquad \xi, \eta \in \mathfrak{X}.$$

For each $\Xi_1, \Xi_2 \in \mathcal{L}((\mathfrak{X}), (\mathfrak{X})^*)$, by applying Theorem 3.1, we can see that there exists a unique element of $\mathcal{L}((\mathfrak{X}), (\mathfrak{X})^*)$, denoted by $\Xi_1 \diamond \Xi_2$ and called the Wick product of Ξ_1 and Ξ_2, such that

$$\widehat{\Xi_1 \diamond \Xi_2}(\xi, \eta) = \widehat{\Xi_1}(\xi, \eta)\widehat{\Xi_2}(\xi, \eta) e^{-\langle \xi,\, \eta \rangle}, \quad \xi, \eta \in \mathfrak{X}.$$

Theorem 3.2 ([31, 8]). *Let $\Xi_n \in \mathcal{L}((\mathfrak{X}), (\mathfrak{X})^*)$ and $\Theta_n = \widehat{\Xi}_n$ for each $n \in \mathbb{N}$. Then $\{\Xi_n\}_{n=1}^\infty$ converges in $\mathcal{L}((\mathfrak{X}), (\mathfrak{X})^*)$ if and only if the following conditions are satisfied:*

(i) *$\lim_{n\to\infty} \Theta_n(\xi, \eta)$ exists for each $\xi, \eta \in \mathfrak{X}$,*

(ii) *there exist nonnegative constants C, K and p (independent of n) such that*

$$|\Theta_n(\xi, \eta)| \leq C \exp K \left(|\xi|_p^2 + |\eta|_p^2 \right)$$

for all $n \in \mathbb{N}$ and $\xi, \eta \in \mathfrak{X}$.

Moreover, $\{\Xi_n\}_{n=1}^\infty$ converges in $\mathcal{L}((\mathfrak{X}), (\mathfrak{X}))$ if and only if the condition (i) holds and Θ_n satisfies the condition (ii′):

(ii′) *for any $p \geq 0$ and $\epsilon > 0$, there exist constants $C \geq 0$ and $q \geq 0$ such that*

$$|\Theta_n(\xi, \eta)| \leq C \exp \epsilon \left(|\xi|_{p+q}^2 + |\eta|_{-p}^2 \right), \qquad \xi, \eta \in \mathfrak{X}.$$

For each $\Xi \in \mathcal{L}((\mathfrak{X}), (\mathfrak{X})^*)$, the Wick exponential $\mathrm{wexp}\,\{\Xi\}$ of Ξ is defined by

$$\mathrm{wexp}\,\{\Xi\} = \sum_{k=0}^{\infty} \frac{1}{k!} \Xi^{\diamond k}$$

if the series converges to a white noise operator in $\mathcal{L}((\mathfrak{X}),(\mathfrak{X})^*)$ (see [15, 17]). For each $\Xi \in \mathcal{L}((\mathfrak{X}),(\mathfrak{X})^*)$ with $\mathrm{wexp}\,\{\Xi\} \in \mathcal{L}((\mathfrak{X}),(\mathfrak{X})^*)$, from the definition, for any $\xi,\eta \in \mathfrak{X}$, we obtain that

$$\begin{aligned}\Theta(\xi,\eta) := \widehat{\mathrm{wexp}\,\{\Xi\}}(\xi,\eta) &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\widehat{\Xi}(\xi,\eta)e^{-\langle\xi,\eta\rangle}\right)^k e^{\langle\xi,\eta\rangle} \\ &= e^{\langle\xi,\eta\rangle + \widehat{\Xi}(\xi,\eta)e^{-\langle\xi,\eta\rangle}}. \end{aligned} \tag{3.1}$$

For each $n \in \mathbb{N}$, put

$$\Theta_n(\xi,\eta) := \sum_{k=0}^{n} \frac{1}{k!} \left(\widehat{\Xi}(\xi,\eta)e^{-\langle\xi,\eta\rangle}\right)^k e^{\langle\xi,\eta\rangle}, \quad \xi,\eta \in \mathfrak{X}.$$

Then, by Theorem 3.2, it is obvious that for each $\Xi \in \mathcal{L}((\mathfrak{X}),(\mathfrak{X})^*)$, $\mathrm{wexp}\,\{\Xi\} \in \mathcal{L}((\mathfrak{X}),(\mathfrak{X})^*)$ if and only if Θ_n satisfies the condition (ii) in Theorem 3.2. Moreover, for each $\Xi \in \mathcal{L}((\mathfrak{X}),(\mathfrak{X}))$, $\mathrm{wexp}\,\{\Xi\} \in \mathcal{L}((\mathfrak{X}),(\mathfrak{X}))$ if and only if Θ_n satisfies the condition (ii′) in Theorem 3.2.

Examples 3.3. (1) For each $x \in \mathfrak{X}$, the annihilation operator $a(x)$ on $(\mathfrak{X})$ is defined by

$$a(x)\phi_\xi = \langle x,\xi\rangle\phi_\xi, \quad \xi \in \mathfrak{X}.$$

Then by applying Theorem 3.1, we can see that $a(x) \in \mathcal{L}((\mathfrak{X}),(\mathfrak{X}))$ (see [15, 32]). In this case, $\widehat{a(x)}(\xi,\eta) = \langle x,\,\xi\rangle\, e^{\langle\xi,\,\eta\rangle}$ for all $\xi,\eta \in \mathfrak{X}$. Also, for any $\xi,\eta \in \mathfrak{X}$, we have

$$\widehat{e^{a(x)}}(\xi,\eta) = e^{\langle x,\xi\rangle + \langle\xi,\eta\rangle},$$

from which by applying Theorem 3.1, we see that $e^{a(x)} \in \mathcal{L}((\mathfrak{X}),(\mathfrak{X}))$ and $e^{a(x)} = \mathrm{wexp}\,\{a(x)\}$. For each $x \in \mathfrak{X}$, the creation operator $a^*(x)$ is the adjoint operator with respect to the canonical $\mathbb{C}$-bilinear form $\langle\langle\cdot,\cdot\rangle\rangle$, and then we have $a^*(x) \in \mathcal{L}((\mathfrak{X})^*,(\mathfrak{X})^*)$ and $e^{a^*(x)} = \mathrm{wexp}\,\{a^*(x)\}$ in $\mathcal{L}((\mathfrak{X})^*,(\mathfrak{X})^*)$.

(2) For each operator $K \in \mathcal{L}(\mathfrak{X},\mathfrak{X}^*)$, the generalized Gross Laplacian $\Delta_{\mathrm{G}}(K)$ is defined by

$$\Delta_{\mathrm{G}}(K)\phi_\xi = \langle K\xi,\,\xi\rangle\,\phi_\xi, \quad \xi \in \mathfrak{X}.$$

Again, by applying Theorem 3.1, we can see that $\Delta_{\mathrm{G}}(K)$ is extended to $(\mathfrak{X})$ as a continuous linear operators in $\mathcal{L}((\mathfrak{X}),(\mathfrak{X}))$, and also we see that $e^{\Delta_{\mathrm{G}}(K)} = \mathrm{wexp}\,\{\Delta_{\mathrm{G}}(K)\} \in \mathcal{L}((\mathfrak{X}),(\mathfrak{X}))$. In fact, we obtain that

$$\widehat{\Delta_{\mathrm{G}}(K)}(\xi,\eta) = \langle K\xi,\,\xi\rangle\, e^{\langle\xi,\,\eta\rangle}, \quad \widehat{e^{\Delta_{\mathrm{G}}(K)}}(\xi,\eta) = e^{\langle K\xi,\,\xi\rangle + \langle\xi,\,\eta\rangle}.$$

In particular, $\Delta_{\mathrm{G}} := \Delta_{\mathrm{G}}(I) \in \mathcal{L}((\mathfrak{X}),(\mathfrak{X}))$ is called the Gross Laplacian. The adjoint operator of $\Delta_{\mathrm{G}}(K)$ with respect to the canonical $\mathbb{C}$-bilinear form $\langle\langle\cdot,\cdot\rangle\rangle$ is denoted by $\Delta_{\mathrm{G}}^*(K)$. Then we see that $\Delta_{\mathrm{G}}^*(K) \in \mathcal{L}((\mathfrak{X})^*,(\mathfrak{X})^*)$ and $e^{\Delta_{\mathrm{G}}^*(K)} = \mathrm{wexp}\,\{\Delta_{\mathrm{G}}^*(K)\} \in \mathcal{L}((\mathfrak{X})^*,(\mathfrak{X})^*)$. In this case, we obtain that

$$\widehat{\Delta_{\mathrm{G}}^*(K)}(\xi,\eta) = \langle K\eta,\,\eta\rangle\, e^{\langle\xi,\,\eta\rangle}, \quad \widehat{e^{\Delta_{\mathrm{G}}^*(K)}}(\xi,\eta) = e^{\langle K\eta,\,\eta\rangle + \langle\xi,\,\eta\rangle}.$$

(3) For each operator $K \in \mathcal{L}(\mathfrak{X}, \mathfrak{X}^*)$, consider the function $\Theta : \mathfrak{X} \times \mathfrak{X} \to \mathbb{C}$ defined by $\Theta(\xi, \eta) = \langle K\xi, \eta\rangle e^{\langle \xi, \eta\rangle}$ for all $\xi, \eta \in \mathfrak{X}$. Then by applying Theorem 3.1, we see that there exists a unique operator $\Lambda(K) \in \mathcal{L}((\mathfrak{X}), (\mathfrak{X})^*)$ such that

$$\widehat{\Lambda(K)}(\xi, \eta) = \Theta(\xi, \eta) = \langle K\xi, \eta\rangle e^{\langle \xi, \eta\rangle}, \quad \xi, \eta \in \mathfrak{X}.$$

The operator $\Lambda(K)$ is called the conservation operator. Moreover, by applying Theorem 3.1, we can see that if $K \in \mathcal{L}(\mathfrak{X}, \mathfrak{X})$, then $\Lambda(K) \in \mathcal{L}((\mathfrak{X}), (\mathfrak{X}))$. In particular, $N := \Lambda(I) \in \mathcal{L}((\mathfrak{X}), (\mathfrak{X}))$ is called the number operator. Furthermore, for each operator $K \in \mathcal{L}(\mathfrak{X}, \mathfrak{X}^*)$, we see that $\operatorname{wexp}\{\Lambda(K)\} \in \mathcal{L}((\mathfrak{X}), (\mathfrak{X})^*)$ and from (3.1), we have

$$\widehat{\operatorname{wexp}\{\Lambda(K)\}}(\xi, \eta) = e^{\langle K\xi, \eta\rangle + \langle \xi, \eta\rangle}, \quad \xi, \eta \in \mathfrak{X}. \tag{3.2}$$

If $K \in \mathcal{L}(\mathfrak{X}, \mathfrak{X})$, then $\operatorname{wexp}\{\Lambda(K)\} \in \mathcal{L}((\mathfrak{X}), (\mathfrak{X}))$.

(4) Let $K \in \mathcal{L}(\mathfrak{X}, \mathfrak{X}^*)$. Then the second quantization $\Gamma(K)$ of K is defined by

$$\Gamma(K)\phi = \left(K^{\otimes n} f_n\right), \quad \phi = (f_n) \in (\mathfrak{X}) \tag{3.3}$$

(see (2.4)), and so we have

$$\widehat{\Gamma(K)}(\xi, \eta) = \langle\langle \phi_{K\xi}, \phi_\eta \rangle\rangle = e^{\langle K\xi, \eta\rangle}, \quad \xi, \eta \in \mathfrak{X}.$$

Therefore, from (3.2), we see that

$$\Gamma(I + K) = \operatorname{wexp}\{\Lambda(K)\}. \tag{3.4}$$

Also, from (3.2), we see that

$$\widehat{\operatorname{wexp}\{-N\}}(\xi, \eta) = 1, \tag{3.5}$$

and hence $\operatorname{wexp}\{-N\} = \operatorname{wexp}\{-\Lambda(I)\} = \Gamma(I - I) = \Gamma(0)$ is the vacuum projection operator.

4. White Noise Delta Functions

4.1. Kubo-Yokoi Delta Function. In [21], Kubo and Yokoi proved that every $\varphi \in (E)$ is continuous as a functional on $E_{\mathbb{R}}^*$ (see also [23] and [32]), from which it has been shown that for each $x \in E_{\mathbb{R}}^*$, the evaluation map

$$\delta_x : (E) \ni \varphi \mapsto \langle\langle \delta_x, \varphi\rangle\rangle := \varphi(x) \in \mathbb{C}$$

is a continuous linear functional and so $\delta_x \in (E)^*$. The functional δ_x is called the Kubo-Yokoi delta function. Then from the definition, for each $x \in E_{\mathbb{R}}^*$, we have

$$S\delta_x(\xi) = \langle\langle \delta_x, \phi_\xi\rangle\rangle = \phi_\xi(x) = e^{\langle x, \xi\rangle - \frac{1}{2}\langle \xi, \xi\rangle}, \quad \xi \in E,$$

from which we see that

$$\delta_x = \phi_x \diamond g_{-\frac{1}{2}}, \tag{4.1}$$

where $\diamond$ is the Wick product (see (2.7) and also [23]) and $g_{-\frac{1}{2}}$ is a generalized white noise functional corresponding to a Gaussian measure such that $Sg_{-\frac{1}{2}}(\xi) = e^{-\frac{1}{2}\langle \xi, \xi\rangle}$ for $\xi \in E$.

The Hermite polynomials and their basic properties are well-known, however, for the notational convenience, we now recall the definition of the Hermite polynomials. The Hermite polynomial of degree n with parameter $\sigma^2 > 0$ is defined by

$$:x^n:_{\sigma^2} := n! H_n(\sigma^2; x) := H_n^\sigma(x) := (-\sigma^2)^n e^{\frac{x^2}{2\sigma^2}} D_x^n \left(e^{-\frac{x^2}{2\sigma^2}} \right),$$

which can also be defined by the generating function:

$$e^{tx - \frac{1}{2}\sigma^2 t^2} = \sum_{n=0}^\infty \frac{t^n}{n!} :x^n:_{\sigma^2} .$$

By applying the generating function, we can easily see that for any $\sigma^2 > 0$,

$$:x^n:_{\sigma^2} = \left(\frac{\sigma}{\sqrt{2}} \right)^n \left[2^n : \left(\frac{x}{\sqrt{2}\sigma} \right)^n :_{1/2} \right], \quad n = 0, 1, 2, \cdots .$$

and by taking $H_n(x) := 2^n :x^n:_{1/2}$, we have $:x^n:_{\sigma^2} = \left(\frac{\sigma}{\sqrt{2}} \right)^n H_n\left(\frac{x}{\sqrt{2}\sigma} \right)$. Then it is well-known that the set $\{\zeta_n(\sigma^2; x)\}_{n=0}^\infty$ becomes a complete orthonormal basis for the Hilbert space $L^2\left(\mathbb{R}, \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx\right)$, where

$$\zeta_n(\sigma^2; x) = \frac{1}{\sqrt{n!}\sigma^n} :x^n:_{\sigma^2} = \frac{1}{\sqrt{2^n n!}} H_n\left(\frac{x}{\sqrt{2}\sigma} \right).$$

For each multi-index $\mathbf{n} = (n_1, n_2, n_3, \cdots) \in \mathbb{N}_0^\infty$ and $n \in \mathbb{N}_0$, we put

$$|\mathbf{n}| := \sum_{k=1}^\infty n_k, \quad \mathcal{I} := \{\mathbf{n} \in \mathbb{N}_0^\infty \, : \, |\mathbf{n}| < \infty\}, \quad \mathcal{I}_n := \{\mathbf{n} \in \mathcal{I} \, : \, |\mathbf{n}| = n\},$$

and for each $\mathbf{n} \in \mathcal{I}$, we put

$$\mathbb{H}_{\mathbf{n}}(x) := \prod_{k=1}^\infty \left[\frac{1}{\sqrt{2^{n_k} n_k!}} H_{n_k}\left(\frac{\langle x, e_k \rangle}{\sqrt{2}} \right) \right].$$

Then $\{\mathbb{H}_{\mathbf{n}}; \ \mathbf{n} \in \mathcal{I}\} \subset (E)$ and it is a complete orthonormal basis for $(L^2) \equiv L^2(E_{\mathbb{R}}^*, \mu)$, and we have the following series expansion of δ_x.

Proposition 4.1 ([3])**.** *For any $x \in E_{\mathbb{R}}^*$, the equality*

$$\delta_x = \sum_{\mathbf{n} \in \mathcal{I}} \mathbb{H}_{\mathbf{n}}(x) \mathbb{H}_{\mathbf{n}}$$

holds.

4.2. White Noise Delta Functions of Gaussian Random Variables. For each $f \in L^2_{\mathbb{R}}(\mathbb{R})$, as given as in (2.10), we define a function $Z_f : \mathcal{E}_{\mathbb{R}}^* \to E^*$ by

$$Z_f(w) := \sum_{k=0}^\infty \langle w, f \otimes e_k \rangle e_k, \quad w \in \mathcal{E}_{\mathbb{R}}^*, \tag{4.2}$$

where $Z_f(w)$ is well-defined μ_2-a.e. $w \in \mathcal{E}_{\mathbb{R}}^*$. Also, for notational convenience, for each $f \in L^2_{\mathbb{R}}(\mathbb{R})$, let $K_f : \mathcal{E} \to \mathcal{E}$ be a continuous linear operator defined by

$$K_f(\xi) = f \otimes \langle \xi, f \rangle, \quad \xi \in \mathcal{E}, \tag{4.3}$$

where $\langle y, \langle \xi, f\rangle\rangle = \langle \xi, f \otimes y\rangle$ for any $y \in E^*$. Then we can easily see that $K_f^* = K_f$ for all $f \in L^2_{\mathbb{R}}(\mathbb{R})$, and we have the following proposition.

Proposition 4.2 ([3]). *For any $x \in E^*_{\mathbb{R}}$ and $f \in L^2_{\mathbb{R}}(\mathbb{R})$ with $|f|_0 = 1$,*

$$\delta_{x,f} := \sum_{\mathbf{n}\in\mathcal{I}} \mathbb{H}_{\mathbf{n}}(x)\mathbb{H}_{\mathbf{n}}(Z_f)$$

is an element of $(\mathcal{E})^*$. Moreover, the S_2-transform $S_2\delta_{x,f}$ of $\delta_{x,f}$ is given by

$$S_2\delta_{x,f}(\xi) = \exp\left(\langle x, \langle \xi, f\rangle\rangle - \frac{1}{2}\langle K_f\xi, \xi\rangle\right), \quad \xi \in \mathcal{E}. \tag{4.4}$$

By Propositions 4.1 and 4.4, the generalized white noise functional $\delta_{x,f}$ is considered as $\delta_x(Z_f)$, i.e., $\delta_{x,f} =: \delta_x(Z_f)$.

As discussed in [3], for each $t > 0$, we put $h_t := \frac{1}{\sqrt{t}}1_{[0,t)}$, and then we have $|h_t|_0 = 1$ for all $t > 0$ and

$$Z_{h_t} = \sum_{k=0}^{\infty}\langle \cdot, h_t \otimes e_k\rangle e_k = \frac{1}{\sqrt{t}}\mathbb{B}(t) = \frac{1}{\sqrt{t}}\left(\sum_{k=0}^{\infty} B_k(t)e_k\right)$$

for all $t > 0$ (see (2.9)). Therefore, from Proposition 4.2, for any $x \in E^*_{\mathbb{R}}$ and $t > 0$, we consider a white noise delta functions $\delta_x\left(\frac{\mathbb{B}(t)}{\sqrt{t}}\right)$ and $\delta_x(\mathbb{B}(t))$ as distributions in $(\mathcal{E})^*$ defined by

$$\delta_x\left(\frac{\mathbb{B}(t)}{\sqrt{t}}\right) = \delta_x(Z_{h_t}) = \delta_{x,h_t} = \sum_{\mathbf{n}\in\mathcal{I}} \mathbb{H}_{\mathbf{n}}(x)\mathbb{H}_{\mathbf{n}}(Z_{h_t}) = \sum_{\mathbf{n}\in\mathcal{I}} \mathbb{H}_{\mathbf{n}}(x)\mathbb{H}_{\mathbf{n}}\left(\frac{\mathbb{B}(t)}{\sqrt{t}}\right).$$

On the other hand, since $x \in E^*_{\mathbb{R}}$ can be chosen arbitrarily for $\delta_{x,f}$, we can also consider $\delta_{\frac{x}{\sqrt{t}},h_t}$ and then for our convenience, we define

$$\delta_x(\mathbb{B}(t)) := \delta_{\frac{x}{\sqrt{t}}}\left(\frac{\mathbb{B}(t)}{\sqrt{t}}\right).$$

Then from (4.4), for any $t > 0$, we obtain that

$$S_2\left[\delta_x\left(\frac{\mathbb{B}(t)}{\sqrt{t}}\right)\right](\xi) = \exp\left(\langle x, \langle \xi, h_t\rangle\rangle - \frac{1}{2}\langle K_{h_t}\xi, \xi\rangle\right),$$
$$S_2[\delta_x(\mathbb{B}(t))](\xi) = \exp\left(\frac{1}{t}\langle x, \langle \xi, 1_{[0,t]}\rangle\rangle - \frac{1}{2t}\langle K_{1_{[0,t]}}\xi, \xi\rangle\right)$$

for all $\xi \in \mathcal{E}$.

For the notational convenience, for each $\xi \in \mathcal{E}$ and $t \geq 0$, put

$$\boldsymbol{I}_\xi(t) = \langle \xi, 1_{[0,t]}\rangle. \tag{4.5}$$

Then $\boldsymbol{I}_\xi(t)$ is an element of E and we have $\langle K_{1_{[0,t]}}\xi, \xi\rangle = \langle \boldsymbol{I}_\xi(t), \boldsymbol{I}_\xi(t)\rangle$ (see (4.3)). Also, for each $t > 0$ and $x \in E^*_{\mathbb{R}}$, by introducing a functional $\boldsymbol{e}_{t,x}$ defined on E by

$$\boldsymbol{e}_{t,x}(\eta) = \exp\left(\frac{1}{t}\langle x, \eta\rangle - \frac{1}{2t}\langle \eta, \eta\rangle\right), \quad \eta \in E, \tag{4.6}$$

we have $S_2[\delta_x(\mathbb{B}(t))](\xi) = \boldsymbol{e}_{t,x}(\boldsymbol{I}_\xi(t))$, i.e.,

$$S_2[\delta_x(\mathbb{B}(t))](\xi) = \exp\left(\frac{1}{t}\langle x, \boldsymbol{I}_\xi(t)\rangle - \frac{1}{2t}\langle \boldsymbol{I}_\xi(t), \boldsymbol{I}_\xi(t)\rangle\right) = \phi_{\boldsymbol{I}_\xi(t)/\sqrt{t}}\left(\frac{x}{\sqrt{t}}\right) \tag{4.7}$$

for all $\xi \in \mathcal{E}$.

5. Multiplication Operators by White Noise Delta Functions

5.1. Classical and Quantum Correspondence. We first note that (E) is closed under pointwise product (Wiener product). Therefore, each $\Phi \in (E)^*$ can be considered as a multiplication operator $M_\Phi \in \mathcal{L}((E),(E)^*)$ defined by

$$\langle\langle M_\Phi \phi, \varphi\rangle\rangle = \langle\langle \Phi, \phi\varphi\rangle\rangle, \quad \phi, \varphi \in (E),$$

see [15, 32]. Then for the vacuum vector $\phi_0 = (1, 0, 0, \cdots)$, it is obvious that

$$M_\Phi \phi_0 = \Phi \tag{5.1}$$

for any $\Phi \in (E)^*$. For each given $\Phi, \Psi \in (E)^*$ and any $\xi, \eta \in E$, we obtain that

$$\begin{aligned}\widehat{M_{\Phi\diamond\Psi}}(\xi,\eta) &= \langle\langle (\Phi \diamond \Psi)\,\phi_\xi, \phi_\eta\rangle\rangle = \langle\langle \Phi\diamond\Psi, \phi_\xi\phi_\eta\rangle\rangle = e^{\langle \xi, \eta\rangle}\,\langle\langle \Phi\diamond\Psi, \phi_{\xi+\eta}\rangle\rangle \\ &= e^{\langle \xi, \eta\rangle}\,\langle\langle \Phi, \phi_{\xi+\eta}\rangle\rangle\,\langle\langle \Psi, \phi_{\xi+\eta}\rangle\rangle \\ &= \widehat{M_\Phi}(\xi,\eta)\widehat{M_\Psi}(\xi,\eta)e^{-\langle \xi, \eta\rangle},\end{aligned}$$

which implies that

$$M_{\Phi\diamond\Psi} = M_\Phi \diamond M_\Psi.$$

5.2. Multiplication Operators.

Proposition 5.1. *For each $x \in E_{\mathbb{R}}^*$, we have*

$$\begin{aligned} M_{\delta_x} &= e^{a^*(x)} e^{-\frac{1}{2}\Delta_{\mathrm{G}}^*} \mathrm{wexp}\,\{-N\} e^{-\frac{1}{2}\Delta_{\mathrm{G}}} e^{a(x)} &(5.2)\\ &= \mathrm{wexp}\left\{a^*(x) - \frac{1}{2}\Delta_{\mathrm{G}}^* - N - \frac{1}{2}\Delta_{\mathrm{G}} + a(x)\right\}. &(5.3)\end{aligned}$$

Proof. For any $\xi, \eta \in E$, we obtain that

$$\begin{aligned}\langle\langle M_{\delta_x}\phi_\xi, \phi_\eta\rangle\rangle &= \langle\langle \delta_x\phi_\xi, \phi_\eta\rangle\rangle = \langle\langle \delta_x, \phi_\xi\phi_\eta\rangle\rangle = \phi_\xi(x)\phi_\eta(x) \\ &= e^{\langle x, \xi\rangle + \langle x, \eta\rangle - \frac{1}{2}\langle \xi, \xi\rangle - \frac{1}{2}\langle \eta, \eta\rangle}.\end{aligned}$$

Therefore, by taking into account (3.5), we have

$$\langle\langle M_{\delta_x}\phi_\xi, \phi_\eta\rangle\rangle = \left\langle\!\!\left\langle e^{a^*(x)} e^{-\frac{1}{2}\Delta_{\mathrm{G}}^*} \mathrm{wexp}\,\{-N\} e^{-\frac{1}{2}\Delta_{\mathrm{G}}} e^{a(x)}\phi_\xi, \phi_\eta\right\rangle\!\!\right\rangle,$$

which implies (5.2). Since the Wick product is commutative, the identity given as in (5.3) is immediate from Examples 3.3. □

We now consider the multiplication operator $M_{\delta_{x,f}}$ induced by the white noise delta function $\delta_{x,f} \in (\mathcal{E})^*$ for $x \in E_{\mathbb{R}}^*$ and $f \in L^2_{\mathbb{R}}(\mathbb{R})$ with $|f|_0 = 1$. Then by Proposition 4.2, $M_{\delta_{x,f}} \in \mathcal{L}((\mathcal{E}), (\mathcal{E})^*)$ and for any $\xi, \eta \in \mathcal{E}$, we obtain that

$$\begin{aligned}
\widehat{M_{\delta_{x,f}}}(\xi,\eta) &= \langle\langle M_{\delta_{x,f}}\phi_\xi, \phi_\eta\rangle\rangle = \langle\langle \delta_{x,f}, \phi_{\xi+\eta}\rangle\rangle e^{\langle \xi,\,\eta\rangle} \\
&= \exp\left(\langle x, \langle \xi+\eta, f\rangle\rangle - \frac{1}{2}\langle\langle \xi+\eta, f\rangle, \langle \xi+\eta, f\rangle\rangle\right) e^{\langle \xi,\,\eta\rangle} \\
&= e^{\langle \xi,\,\eta\rangle} \exp\left(\langle x, \langle \xi, f\rangle\rangle - \frac{1}{2}\langle\langle \xi, f\rangle, \langle \xi, f\rangle\rangle\right) \\
&\quad \times \exp\left(-\langle\langle \xi, f\rangle, \langle \eta, f\rangle\rangle\right) \\
&\quad \times \exp\left(\langle x, \langle \eta, f\rangle\rangle - \frac{1}{2}\langle\langle \eta, f\rangle, \langle \eta, f\rangle\rangle\right) \\
&= e^{\langle x,\langle \eta, f\rangle\rangle - \frac{1}{2}\langle K_f\eta,\eta\rangle + \langle (I-K_f)\xi,\eta\rangle - \frac{1}{2}\langle K_f\xi,\xi\rangle + \langle x,\langle \xi,f\rangle\rangle}. \qquad (5.4)
\end{aligned}$$

Theorem 5.2. *For any $x \in E_{\mathbb{R}}^*$ and $f \in L^2_{\mathbb{R}}(\mathbb{R})$ with $|f|_0 = 1$, we have*

$$\begin{aligned}
M_{\delta_{x,f}} &= e^{a^*(f\otimes x)} e^{-\frac{1}{2}\Delta_{\mathrm{G}}^*(K_f)} \Gamma(I-K_f) e^{-\frac{1}{2}\Delta_{\mathrm{G}}(K_f)} e^{a(f\otimes x)} \\
&= e^{a^*(f\otimes x)} e^{-\frac{1}{2}\Delta_{\mathrm{G}}^*(K_f)} \mathrm{wexp}\,\{-\Lambda(K_f)\} e^{-\frac{1}{2}\Delta_{\mathrm{G}}(K_f)} e^{a(f\otimes x)}. \qquad (5.5)
\end{aligned}$$

Proof. The proof of the first equality is immediate from (5.4) and then the second equality is from Examples 3.3 by taking into account (3.4). □

Remark 5.3. From the explicit form, given as in (5.5), of the multiplication operator induced by a white noise delta function, we know that those are closely related to the Bogoliubov transformation [16] and a quantum analogue of Girsanov transform [17].

By similar arguments used in (5.4) and by taking into account (4.7), we obtain that

$$\begin{aligned}
&\widehat{M_{\delta_x(\mathbb{B}(t))}}(\xi,\eta) = e^{\langle \xi,\,\eta\rangle} \boldsymbol{e}_{t,x}(\boldsymbol{I}_\xi(t)) e^{-\langle \boldsymbol{I}_\xi(t), \boldsymbol{I}_\eta(t)\rangle} \boldsymbol{e}_{t,x}(\boldsymbol{I}_\eta(t)) \\
&\quad = e^{\frac{1}{t}\langle 1_{[0,t]}\otimes x,\eta\rangle - \frac{1}{2t}\langle K_{1_{[0,t]}}\eta,\eta\rangle + \left\langle \left(I - \frac{1}{t}K_{1_{[0,t]}}\right)\xi,\eta\right\rangle - \frac{1}{2t}\langle K_{1_{[0,t]}}\xi,\xi\rangle + \frac{1}{t}\langle 1_{[0,t]}\otimes x,\xi\rangle}, \qquad (5.6)
\end{aligned}$$

where $\boldsymbol{e}_{t,x}$ is given as in (4.6). Therefore, we have the following theorem.

Theorem 5.4. *For any $x \in E_{\mathbb{R}}^*$ and $t > 0$, we have*

$$\begin{aligned}
M_{\delta_x(\mathbb{B}(t))} &= e^{\frac{1}{t}a^*(1_{[0,t]}\otimes x)} e^{-\frac{1}{2t}\Delta_{\mathrm{G}}^*(K_{1_{[0,t]}})} \Gamma\left(I - \frac{1}{t}K_{1_{[0,t]}}\right) e^{-\frac{1}{2t}\Delta_{\mathrm{G}}(K_{1_{[0,t]}})} e^{\frac{1}{t}a(1_{[0,t]}\otimes x)} \\
&= e^{\frac{1}{t}a^*(1_{[0,t]}\otimes x)} e^{-\frac{1}{2t}\Delta_{\mathrm{G}}^*(K_{1_{[0,t]}})} \mathrm{wexp}\left\{-\frac{1}{t}K_{1_{[0,t]}}\right\} e^{-\frac{1}{2t}\Delta_{\mathrm{G}}(K_{1_{[0,t]}})} e^{\frac{1}{t}a(1_{[0,t]}\otimes x)}.
\end{aligned}$$

Proof. For the proof of the second equality, (3.4) is applied. □

6. Differential Equations for Operators

From the notation given as in (4.3), we used the notation $\langle \xi, x\rangle$ for $\xi \in \mathcal{E} = E\otimes E$ and $x \in E^*$, which means that $\langle \xi, x\rangle \in E$ and for any $y \in E^*$, it holds that

$\langle y, \langle \xi, x\rangle\rangle = \langle x \otimes y, \xi\rangle$. Therefore, from now on, for more clear meaning, we use the following notations:

$$\langle y, \langle \xi, x\rangle_l\rangle = \langle x \otimes y, \xi\rangle, \quad \langle y, \langle \xi, x\rangle_r\rangle = \langle y \otimes x, \xi\rangle$$

for all $x, y \in E$ and $\xi \in \mathcal{E}$, where the subindex l and r mean the left (contraction) and right (contraction), respectively (see [32]). Also, for notational convenience, for each $x, y \in \mathcal{E}^*$, we define a continuous linear operator $K_{x,y} : \mathcal{E} \to \mathcal{E}^*$ by

$$K_{x,y}\xi = x \otimes \langle y, \xi\rangle_l\,, \quad \xi \in \mathcal{E}. \tag{6.1}$$

Then we can easily see that for each $x, y \in E^*$, the adjoint $K^*_{x,y}$ of $K_{x,y}$ coincides with $K_{y,x}$, i.e. $K^*_{x,y} = K_{y,x}$. Also, we have $K_f = K_{f,f}$, where K_f is defined as in (4.3).

Examples 6.1. (1) For any $x \in E^*$ and $t > 0$, from the definition of $\boldsymbol{I}_\xi(t)$ given as in (4.5), we obtain that

$$\frac{\partial}{\partial t}\langle x, \boldsymbol{I}_\xi(t)\rangle = \frac{\partial}{\partial t}\langle x, \langle \xi, 1_{[0,t]}\rangle_l\rangle = \langle x, \langle \xi, \delta_t\rangle_l\rangle = \langle \delta_t \otimes x, \xi\rangle \tag{6.2}$$

for all $\xi \in \mathcal{E}$.

(2) For any $\xi, \eta \in \mathcal{E}$ and $t > 0$, we obtain that

$$\begin{aligned}\frac{\partial}{\partial t}\langle \boldsymbol{I}_\xi(t), \boldsymbol{I}_\eta(t)\rangle &= \left\langle \frac{\partial}{\partial t}\boldsymbol{I}_\xi(t), \boldsymbol{I}_\eta(t)\right\rangle + \left\langle \boldsymbol{I}_\xi(t), \frac{\partial}{\partial t}\boldsymbol{I}_\eta(t)\right\rangle \\ &= \big\langle \langle \xi, \delta_t\rangle_l, \langle \eta, 1_{[0,t]}\rangle_l\big\rangle + \big\langle \langle \xi, 1_{[0,t]}\rangle_l, \langle \eta, \delta_t\rangle_l\big\rangle \\ &= \big\langle K_{1_{[0,t]},\delta_t}\xi, \eta\big\rangle + \big\langle K_{\delta_t,1_{[0,t]}}\xi, \eta\big\rangle \\ &= \left\langle \left(K_{1_{[0,t]},\delta_t} + K^*_{1_{[0,t]},\delta_t}\right)\xi, \eta\right\rangle.\end{aligned} \tag{6.3}$$

Lemma 6.2. *For any $t > 0$ and $x \in E^*_{\mathbb{R}}$, we have*

$$\begin{aligned}\frac{\partial}{\partial t}\widehat{M_{\delta_x(\mathbb{B}(t))}}(\xi, \eta) = \Big[&\boldsymbol{Q}(x, \xi; t) - \left\langle \left(K_{1_{[0,t]},\delta_t} + K^*_{1_{[0,t]},\delta_t}\right)\xi, \eta\right\rangle + \boldsymbol{Q}(x, \eta; t)\Big] \\ &\times \widehat{M_{\delta_x(\mathbb{B}(t))}}(\xi, \eta),\end{aligned}$$

where $\boldsymbol{Q}(x, \xi; t)$ is given as in (6.4).

Proof. From (5.6), we have

$$\widehat{M_{\delta_x(\mathbb{B}(t))}}(\xi, \eta) = e^{\langle \xi, \eta\rangle}\boldsymbol{e}_{t,x}(\boldsymbol{I}_\xi(t))e^{-\langle \boldsymbol{I}_\xi(t), \boldsymbol{I}_\eta(t)\rangle}\boldsymbol{e}_{t,x}(\boldsymbol{I}_\eta(t)),$$

where $\boldsymbol{e}_{t,x}(\eta)$ is given as in (4.6). Therefore, we have

$$\boldsymbol{e}_{t,x}(\boldsymbol{I}_\xi(t)) = \exp\left(\frac{1}{t}\langle x, \boldsymbol{I}_\xi(t)\rangle - \frac{1}{2t}\langle \boldsymbol{I}_\xi(t), \boldsymbol{I}_\xi(t)\rangle\right),$$

and so by applying Examples 6.1, we have

$$\frac{\partial}{\partial t}\boldsymbol{e}_{t,x}(\boldsymbol{I}_\xi(t)) = \boldsymbol{Q}(x, \xi; t)\boldsymbol{e}_{t,x}(\boldsymbol{I}_\xi(t)),$$

where

$$\begin{aligned}\boldsymbol{Q}(x,\xi;t) &:= \frac{\partial}{\partial t}\left[\frac{1}{t}\langle x,\boldsymbol{I}_\xi(t)\rangle - \frac{1}{2t}\langle \boldsymbol{I}_\xi(t),\boldsymbol{I}_\xi(t)\rangle\right] \\ &= \left\{-\frac{1}{t^2}\langle x,\boldsymbol{I}_\xi(t)\rangle + \frac{1}{2t^2}\langle \boldsymbol{I}_\xi(t),\boldsymbol{I}_\xi(t)\rangle + \frac{1}{t}\langle \delta_t\otimes x,\xi\rangle - \frac{1}{t}\left\langle K_{1_{[0,t]},\delta_t}\xi,\xi\right\rangle\right\} \\ &= \left\{-\frac{1}{t^2}\langle 1_{[0,t]}\otimes x,\xi\rangle + \frac{1}{2t^2}\left\langle K_{1_{[0,t]},1_{[0,t]}}\xi,\xi\right\rangle\right. \\ &\qquad \left. +\frac{1}{t}\langle \delta_t\otimes x,\xi\rangle - \frac{1}{t}\left\langle K_{1_{[0,t]},\delta_t}\xi,\xi\right\rangle\right\}. \end{aligned} \tag{6.4}$$

Therefore, by direct computation, we obtain that

$$\begin{aligned}\frac{\partial}{\partial t}\widehat{M_{\delta_x(\mathbb{B}(t))}}(\xi,\eta) &= e^{\langle \xi,\eta\rangle}\left[\frac{\partial}{\partial t}\boldsymbol{e}_{t,x}(\boldsymbol{I}_\xi(t))\right]e^{-\langle \boldsymbol{I}_\xi(t),\boldsymbol{I}_\eta(t)\rangle}\boldsymbol{e}_{t,x}(\boldsymbol{I}_\eta(t)) \\ &\quad + \boldsymbol{e}_{t,x}(\boldsymbol{I}_\xi(t))\left[\frac{\partial}{\partial t}e^{-\langle \boldsymbol{I}_\xi(t),\boldsymbol{I}_\eta(t)\rangle}\right]\boldsymbol{e}_{t,x}(\boldsymbol{I}_\eta(t)) \\ &\quad + \boldsymbol{e}_{t,x}(\boldsymbol{I}_\xi(t))e^{-\langle \boldsymbol{I}_\xi(t),\boldsymbol{I}_\eta(t)\rangle}\left[\frac{\partial}{\partial t}\boldsymbol{e}_{t,x}(\boldsymbol{I}_\eta(t))\right] \\ &= \left[\boldsymbol{Q}(x,\xi;t) - \left\langle\left(K_{1_{[0,t]},\delta_t} + K^*_{1_{[0,t]},\delta_t}\right)\xi,\eta\right\rangle + \boldsymbol{Q}(x,\eta;t)\right] \\ &\quad \times \widehat{M_{\delta_x(\mathbb{B}(t))}}(\xi,\eta),\end{aligned}$$

which gives the proof. □

The characterization of the convergence of sequence of white noise operators in terms of operator symbol given as in Theorem 3.2 can be extended to the convergence of continuous variable functions valued in white noise operators. Therefore, by applying Lemma 6.2, we can prove the following theorem.

Theorem 6.3. *For any $t > 0$ and $x \in E^*_{\mathbb{R}}$, we have*

$$\frac{\partial}{\partial t}M_{\delta_x(\mathbb{B}(t))} = \left[\boldsymbol{A}(x,t) - \left(\Lambda\left(K_{1_{[0,t]},\delta_t}\right) + \Lambda\left(K^*_{1_{[0,t]},\delta_t}\right)\right) + \boldsymbol{A}^*(x,t)\right]\diamond M_{\delta_x(\mathbb{B}(t))}, \tag{6.5}$$

where $\boldsymbol{A}(x,t)$ and $\boldsymbol{A}^*(x,t)$ are given as in (6.6) *and* (6.7)*, respectively.*

Proof. From (6.4) and Examples 3.3, we can easily see that

$$\begin{aligned}\boldsymbol{Q}(x,\xi;t)e^{\langle \xi,\eta\rangle} &= \widehat{\boldsymbol{A}(x,t)}(\xi,\eta), \\ \boldsymbol{Q}(x,\eta;t)e^{\langle \xi,\eta\rangle} &= \widehat{\boldsymbol{A}^*(x,t)}(\xi,\eta)\end{aligned}$$

where $\boldsymbol{A}(x,t)$ is given by

$$\begin{aligned}\boldsymbol{A}(x,t) &= -\frac{1}{t^2}a(1_{[0,t]}\otimes x) + \frac{1}{2t^2}\Delta_{\mathrm{G}}\left(K_{1_{[0,t]},1_{[0,t]}}\right) \\ &\quad + \frac{1}{t}a(\delta_t\otimes x) - \frac{1}{t}\Delta_{\mathrm{G}}\left(K_{1_{[0,t]},\delta_t}\right),\end{aligned} \tag{6.6}$$

and $\boldsymbol{A}^*(x,t) = (\boldsymbol{A}(x,t))^*$, i.e.,

$$\begin{aligned}\boldsymbol{A}^*(x,t) &= -\frac{1}{t^2}a^*(1_{[0,t]}\otimes x) + \frac{1}{2t^2}\Delta_{\mathrm{G}}^*\left(K_{1_{[0,t]},1_{[0,t]}}\right)\\ &\qquad + \frac{1}{t}a^*(\delta_t\otimes x) - \frac{1}{t}\Delta_{\mathrm{G}}^*\left(K_{1_{[0,t]},\delta_t}\right). \qquad (6.7)\end{aligned}$$

Also, by applying Examples 3.3, we can easily see that

$$\left\langle\left(K_{1_{[0,t]},\delta_t} + K^*_{1_{[0,t]},\delta_t}\right)\xi, \eta\right\rangle e^{\langle\xi,\,\eta\rangle} = \left\langle\!\!\left\langle\left(\Lambda\left(K_{1_{[0,t]},\delta_t}\right) + \Lambda\left(K^*_{1_{[0,t]},\delta_t}\right)\right)\phi_\xi,\, \phi_\eta\right\rangle\!\!\right\rangle$$

for all $\xi,\eta\in\mathcal{E}$. Therefore, by applying Lemma 6.2, we prove the assertion. □

We now consider a classical and quantum correspondence. For any given white noise operators $\Xi_1,\Xi_2\in\mathcal{L}((\mathcal{E}),(\mathcal{E})^*)$ and $\xi\in\mathcal{E}$, we obtain that

$$S\left((\Xi_1\diamond\Xi_2)\,\phi_0\right)(\xi) = \langle\!\langle(\Xi_1\diamond\Xi_2)\,\phi_0,\,\phi_\xi\rangle\!\rangle = \langle\!\langle\Xi_1\phi_0,\,\phi_\xi\rangle\!\rangle\,\langle\!\langle\Xi_2\phi_0,\,\phi_\xi\rangle\!\rangle\,,$$

from which we see that

$$(\Xi_1\diamond\Xi_2)\,\phi_0 = (\Xi_1\phi_0)\diamond(\Xi_2\phi_0)\,. \qquad (6.8)$$

For each $L\in\mathcal{L}(\mathcal{E},\mathcal{E}^*)$, by the kernel theorem, there exists $\tau_L\in(\mathcal{E}\otimes\mathcal{E})^*$, called the L-trace, such that

$$\langle\tau_L,\eta\otimes\xi\rangle = \langle L\xi,\eta\rangle,\quad \xi,\eta\in\mathcal{E}. \qquad (6.9)$$

Corollary 6.4. *For any $t>0$ and $x\in E^*_{\mathbb{R}}$, we have*

$$\frac{\partial}{\partial t}\delta_x\left(\mathbb{B}(t)\right) = \left[\Phi_{1;t,x} + \Phi_{2;t,K}\right]\diamond\delta_x\left(\mathbb{B}(t)\right), \qquad (6.10)$$

where $\Phi_{1;t,x}$ and $\Phi_{2;t,K}$ are given by

$$\begin{aligned}\Phi_{1;t,x} &= \left(0, \frac{1}{t}\left(\delta_t - \frac{1}{t}1_{[0,t]}\right)\otimes x, 0, 0, \cdots\right),\\ \Phi_{2;t,K} &= (0,0,\tau_{K_t},0,0,\cdots),\quad K_t = \frac{1}{2t^2}K_{1_{[0,t]},1_{[0,t]}} - \frac{1}{t}K_{1_{[0,t]},\delta_t}, \qquad (6.11)\end{aligned}$$

with K_t-trace τ_{K_t} defined as in (6.9).

Proof. By acting the vacuum vector ϕ_0 on the both sides of (6.5) and using (5.1) and (6.8), we obtain that

$$\frac{\partial}{\partial t}\delta_x\left(\mathbb{B}(t)\right) = \left(\left[\boldsymbol{A}(x,t) - \left(\Lambda\left(K_{1_{[0,t]},\delta_t} + K^*_{1_{[0,t]},\delta_t}\right)\right) + \boldsymbol{A}^*(x,t)\right]\phi_0\right)\diamond\delta_x\left(\mathbb{B}(t)\right). \qquad (6.12)$$

On the other hand, from (6.6) and (6.7), we obtain that

$$\left[\boldsymbol{A}(x,t) - \left(\Lambda\left(K_{1_{[0,t]},\delta_t} + K^*_{1_{[0,t]},\delta_t}\right)\right) + \boldsymbol{A}^*(x,t)\right]\phi_0 = \boldsymbol{A}^*(x,t)\phi_0 \qquad (6.13)$$

and

$$\begin{aligned}\boldsymbol{A}^*(x,t)\phi_0 &= \left[a^*\left(\frac{1}{t}\left(\delta_t - \frac{1}{t}1_{[0,t]}\right)\otimes x\right) + \Delta_{\mathrm{G}}^*\left(\frac{1}{2t^2}K_{1_{[0,t]},1_{[0,t]}} - \frac{1}{t}K_{1_{[0,t]},\delta_t}\right)\right]\phi_0\\ &= \Phi_{1;t,x} + \Phi_{2;t,K}. \qquad (6.14)\end{aligned}$$

Therefore, by (6.12), (6.13) and (6.14), we have (6.10). □

Remark 6.5. The white noise delta function $\delta_x(\mathbb{B}(t))$ also satisfies the equation

$$\frac{\partial}{\partial t}\delta_x(\mathbb{B}(t)) = \left[D_{\dot{\mathbb{B}}(t)} + \frac{1}{2}\Delta_V\right]\delta_x(\mathbb{B}(t)), \tag{6.15}$$

where $D_{\dot{\mathbb{B}}(t)}$ is the differential operator directed for $\dot{\mathbb{B}}(t)$ and Δ_V is the generalized Volterra Laplacian given as in [3]. In fact, the S_2-transforms of $\Phi_{1:t,x} \diamond \delta_x(\mathbb{B}(t))$ and $\Phi_{2:t,K} \diamond \delta_x(\mathbb{B}(t))$ are given by

$$\begin{aligned}S_2[\Phi_{1:t,x} \diamond \delta_x(\mathbb{B}(t))](\xi) &= S_2[\Phi_{1:t,x}](\xi)S_2[\delta_x(\mathbb{B}(t))](\xi)\\ &= \left[\frac{1}{t}\langle x, \mathbf{I}'_\xi(t)\rangle - \frac{1}{t^2}\langle x, \mathbf{I}_\xi(t)\rangle\right]\phi_{\mathbf{I}_\xi(t)/\sqrt{t}}\left(\frac{x}{\sqrt{t}}\right),\end{aligned}$$

and

$$\begin{aligned}S_2[\Phi_{2:t,K} \diamond \delta_x(\mathbb{B}(t))](\xi) &= S_2[\Phi_{2:t,K}](\xi)S_2[\delta_x(\mathbb{B}(t))](\xi)\\ &= K_t\xi\phi_{\mathbf{I}_\xi(t)/\sqrt{t}}\left(\frac{x}{\sqrt{t}}\right)\\ &= \left[\frac{1}{2t^2}\langle \mathbf{I}_\xi(t), \mathbf{I}_\xi(t)\rangle - \frac{1}{t}\langle \mathbf{I}'_\xi(t), \mathbf{I}_\xi(t)\rangle\right]\phi_{\mathbf{I}_\xi(t)/\sqrt{t}}\left(\frac{x}{\sqrt{t}}\right),\end{aligned}$$

respectively. On the other hand, the S_2-transforms of $D_{\dot{\mathbb{B}}(t)}\delta_x(\mathbb{B}(t))$ and $\Delta_V\delta_x(\mathbb{B}(t))$ are given by

$$S_2[D_{\dot{\mathbb{B}}(t)}\delta_x(\mathbb{B}(t))](\xi) = \frac{1}{t}\langle x - \mathbf{I}_\xi(t), \mathbf{I}'_\xi(t)\rangle\phi_{\mathbf{I}_\xi(t)/\sqrt{t}}\left(\frac{x}{\sqrt{t}}\right),$$

and

$$S_2[\Delta_V\delta_x(\mathbb{B}(t))](\xi) = \left[-\frac{2}{t^2}\langle x, \mathbf{I}_\xi(t)\rangle + \frac{1}{t^2}\langle \mathbf{I}_\xi(t), \mathbf{I}_\xi(t)\rangle\right]\phi_{\mathbf{I}_\xi(t)/\sqrt{t}}\left(\frac{x}{\sqrt{t}}\right),$$

respectively. Consequently by the comparison of the above S_2-transforms we obtain the equation (6.15). This means that the equation (6.10) is a different expression of the Itô formula (6.15).

7. Concluding Remark

In [3], we introduced a white noise distribution $\check{\Phi}(\mathbb{B}(t))$ for each $\Phi \in (E)^*$ and $t > 0$ by

$$S_2[\check{\Phi}(\mathbb{B}(t))](\xi) = \langle\!\langle \Phi, \check{\phi}_\xi \rangle\!\rangle, \quad \xi \in \mathcal{E},$$

where $\check{\phi}_\xi$ is given by

$$\check{\phi}_\xi := \phi_{\mathbf{I}_\xi(t)/\sqrt{t}}\left(\frac{\cdot}{\sqrt{t}}\right) = e^{\frac{1}{2t^2}(1-t)\langle \mathbf{I}_\xi(t), \mathbf{I}_\xi(t)\rangle}\phi_{\mathbf{I}_\xi(t)/t},$$

and obtained the following theorem as an extension of the Itô formula.

Theorem 7.1. *For any $t > 0$ and $\Phi \in (E)^*$, the distribution $\check{\Phi}(\mathbb{B}(t))$ satisfies the equation:*

$$\frac{\partial}{\partial t}\check{\Phi}(\mathbb{B}(t)) = D_{\dot{\mathbb{B}}(t)}\check{\Phi}(\mathbb{B}(t)) + \frac{1}{2}\Delta_V\check{\Phi}(\mathbb{B}(t)).$$

This theorem means that the equation (6.5) in Theorem 6.3 gives an expression of the quantum Itô formula for $M_{\delta_x(\mathbb{B}(t)}$ and also implies an expression of the formula for $M_{\check{\Phi}(\mathbb{B}(t))}$. In fact, we have the following formula.

$$\frac{\partial}{\partial t} M_{\check{\varphi}(\mathbb{B}(t))} = \left[\boldsymbol{A}(t) - \left(\Lambda\left(K_{1_{[0,t]},\delta_t}\right) + \Lambda\left(K^*_{1_{[0,t]},\delta_t}\right)\right) + \boldsymbol{A}^*(t)\right] \diamond M_{\check{\varphi}(\mathbb{B}(t))}, \quad (7.1)$$

for any $\varphi \in (E)$ and $t > 0$, where $\boldsymbol{A}(t)$ and $\boldsymbol{A}^*(t)$ are given by

$$\boldsymbol{A}(t) \diamond M_{\check{\varphi}(\mathbb{B}(t))} := \int_{E^*_{\mathbb{R}}} \varphi(x) \boldsymbol{A}(x,t) \diamond M_{\delta_x(\mathbb{B}(t))} d\mu(x)$$

and

$$\boldsymbol{A}^*(t) \diamond M_{\check{\varphi}(\mathbb{B}(t))} := \int_{E^*_{\mathbb{R}}} \varphi(x) \boldsymbol{A}^*(x,t) \diamond M_{\delta_x(\mathbb{B}(t))} d\mu(x),$$

respectively. The formula (7.1) can be extended to that for $M_{\check{\Phi}(\mathbb{B}(t))}$ by using the limiting method.

Acknowledgments. UCJ is supported by Basic Science Research Program through the NRF funded by the MEST (NRF-2022R1F1A1067601). KS is supported by JSPS KAKENHI Grant Number 23K03139.

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