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EFFICIENT ESTIMATION OF SPECTRAL FUNCTIONALS FOR GAUSSIAN STATIONARY MODELS

MAMIKON S. GINOVYAN*

ABSTRACT. The paper considers a problem of construction of asymptotically efficient estimators for spectral functionals, and bounding the minimax mean square risks. We consider the efficiency concepts of estimators, based on the variants of Hájek-Ibragimov-Khas'minskii convolution theorem (H-efficiency) and Hájek-Le Cam local asymptotic minimax theorem (IK-efficiency), and show that the simple "plug-in" statistic $\Phi(I_T)$, where $I_T = I_T(\lambda)$ is the periodogram of the underlying stationary Gaussian process $X(t)$ with an unknown spectral density $\theta(\lambda)$, is H- and IK-asymptotically efficient estimator for a linear functional $\Phi(\theta)$, while for a nonlinear smooth functional $\Phi(\theta)$ an H- and IK-asymptotically efficient estimator is the statistic $\Phi(\widehat{\theta}_T)$, where $\widehat{\theta}_T$ is a sequence of "undersmoothed" kernel estimators of the unknown spectral density $\theta(\lambda)$. Exact asymptotic bounds for minimax mean square risks of estimators of linear functionals are also obtained.

1. Introduction

The problem of efficient nonparametric estimation of different kind of functionals for various statistical models has been extensively discussed in the literature (see, for instance, Ibragimov and Khas'minskii [25], Pfanzagl and Wefelmeyer [40], Taniguchi and Kakizawa [46], Kutoyants [30], and references therein).

This paper is concerned with the problem of asymptotically efficient nonparametric estimation of spectral functionals, and bounding the minimax mean square risks for stationary Gaussian models.

Let $\{X(u),\ u \in \mathbb{U}\}$ be a centered real-valued Gaussian stationary process possessing a spectral density (s.d.) $f(\lambda)$, $\lambda \in \mathbb{Q}$. We consider simultaneously the continuous-time (c.t.) case, where $\mathbb{U} = \mathbb{R} := (-\infty, \infty)$, and the discrete-time (d.t.) case, where $\mathbb{U} = \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$. The domain $\mathbb{Q}$ of the frequency variable λ is $\mathbb{Q} = \mathbb{R}$ in the c.t. case, and $\mathbb{Q} = \mathbb{T} := [-\pi.\pi]$ in the d.t. case. In the c.t. case the process $X(u)$ is assumed measurable and mean square continuous.

The problem. Suppose we observe a realization $\mathbf{X}_T = \{X(u),\ 0 \le u \le T$ (or $u = \overline{1,T}$ in the d.t. case)$\}$ of the process $X(t)$ with an *unknown* spectral

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density function $\theta(\lambda)$, $\lambda \in \mathbb{Q}$. We assume that $\theta(\lambda)$ belongs to a given (infinite-dimensional) class $\mathbf{\Theta} \subset L^p := L^p(\mathbb{Q})$ $(p \geq 1)$ of spectral densities possessing some smoothness properties. Let $\Phi(\cdot)$ be some *known* functional, the domain of definition of which contains $\mathbf{\Theta}$. The distribution of the process $X(t)$ is completely determined by the spectral density, and we consider $\theta(\lambda)$ as an infinite-dimensional "parameter" on which the distribution of $X(t)$ depends. The problem is to estimate the value $\Phi(\theta)$ of the functional $\Phi(\cdot)$ at an unknown point $\theta \in \mathbf{\Theta}$ on the basis of observation $\mathbf{X}_T$, and investigate the asymptotic (as $T \to \infty$) properties of the suggested estimators. The main objective is construction of asymptotically efficient estimators for $\Phi(\theta)$.

This problem for discrete-time Gaussian stationary processes has been considered in a number of articles. We cite merely the papers Millar [36], Has'minskii and Ibragimov [23], Ibragimov and Khas'minskii [27], Ginovyan [11], [18], and Dahlhaus and Wefelmeyer [7].

Notice that in Millar [36], and Dahlhaus and Wefelmeyer [7] were considered efficiency concept based on a nonparametric version of Hájek convolution theorem, while in Has'minskii and Ibragimov [23], Ibragimov and Khas'minskii [27], and Ginovyan [11] the efficiency is based on a nonparametric version of Hájek-Le Cam local asymptotic minimax theorem. In Ginovyan [18] were considered both efficiency concepts in a class of spectral densities possessing singularities.

For continuous-time processes the problem was partially studied in Ginovyan [12]–[14], where efficient nonparametric estimators for linear spectral functionals were constructed and asymptotic upper bounds for minimax mean square risks of these estimators were obtained.

The objective of the present paper is construction of asymptotically efficient nonparametric estimators for linear and some nonlinear smooth spectral functionals and bounding the minimax mean square risks of suggested estimators in the case where the underlying model is a discrete- or continuous-time Gaussian stationary process with possibly unbounded or vanishing spectral density function.

Note that solution of the above estimation problem essentially depends on the type of the estimand functional $\Phi(f)$ - linear or non-linear. Examples of important linear functionals provide the covariance $(r(u))$ and spectral $(F(\mu))$ functions of the process $X(u)$:

$$\Phi_r(f) = r(u) := \frac{1}{2\pi} \int_{\mathbb{Q}} e^{iu\lambda} f(\lambda)\, d\lambda,$$

$$\Phi_F(f) = F(\mu) := \int_a^{\mu} f(\lambda)\, d\lambda,$$

where $a = -\infty$ in the c.t. case, and $a = -\pi$ in the d.t. case.

A brief catalog of important non-linear functionals would include:

1. L^p-norms of spectral density f or derivatives $f^{(k)}$:

$$\Phi_{p,k}(f) = ||f^{(k)}||_p := \begin{cases} \left(\int_{\mathbb{Q}} |f^{(k)}(\lambda)|^p d\lambda\right)^{1/p}, & \text{for } 1 \leq p < \infty, \\ \sup_{\lambda \in \mathbb{Q}} |f^{(k)}(\lambda)|, & \text{for } p = \infty. \end{cases}$$

2. The Shannon negentropy functional $\Phi_S(f)$:

$$\Phi_S(f) = \int_{\mathbb{Q}} f(\lambda) \ln f(\lambda)\, d\lambda.$$

3. The Burg (or spectral) entropy functional $\Phi_B(f)$:

$$\Phi_B(f) = \int_{\mathbb{Q}} \ln f(\lambda)\, d\lambda.$$

4. The spectral indeterminism functional $\Phi_I(f)$:

$$\Phi_I(f) = \int_{\mathbb{Q}} f^{-1}(\lambda)\, d\lambda.$$

The functionals $\Phi_B(f)$ and $\Phi_I(f)$ arise in prediction and interpolation of stationary processes (see, e.g., Rozanov [42]). For instance, by the well-known Kolmogorov-Szegö formula, the one-step prediction error σ^2 when predicting $X(1)$ on the entire past $\{X(u),\, u \le 0\}$ is given by

$$\sigma^2 = 2\pi \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln f(\lambda)\, d\lambda \right\} = 2\pi \exp\left\{ \frac{1}{2\pi} \Phi_B(f) \right\}.$$

For construction of asymptotically efficient estimators we use a general powerful method developed in Has'minskii and Ibragimov [23], and Ibragimov and Khas'minskii [27] (see, also, Goldstein and Khas'minskii [19]). Our plan will be as follows:

- We define the concept of local asymptotic normality (LAN) in the spirit of Ibragimov and Khas'minskii [27], and derive conditions under which the underlying family of Gaussian distributions is LAN at a point $\theta \in \mathbf{\Theta}$.
- Using LAN we state variants of Hájek-Le Cam local asymptotic minimax theorem and Hájek-Ibragimov-Khas'minskii convolution theorem.
- We define the concepts of H- and IK-efficiency of estimators, based on the variants of Hájek-Ibragimov-Khas'minskii convolution theorem and Hájek-Le Cam local asymptotic minimax theorem, respectively, and show that the simple "plug-in" statistic $\Phi(I_T)$, where $I_T = I_T(\lambda)$ is the periodogram of the underlying Gaussian stationary process $X(t)$ with an unknown spectral density $\theta(\lambda)$, $\lambda \in \mathbb{Q}$, is H- and IK-asymptotically efficient estimator for a linear functional $\Phi(\theta)$, while for a nonlinear smooth functional $\Phi(\theta)$ an H- and IK-asymptotically efficient estimator is the statistic $\Phi(\widehat{\theta}_T)$, where $\widehat{\theta}_T$ is a suitable sequence of the so-called "undersmoothed" kernel estimators of the unknown spectral density $\theta(\lambda)$.
- We obtain exact asymptotic bounds for minimax mean square risks of estimators of linear functionals.

The rest of the paper is organized as follows. In Section 2 we describe the model. In Section 3 we state some preliminary results: LAN, variants of Hájek-Ibragimov-Khas'minskii convolution theorem and Hájek-Le Cam local asymptotic minimax theorem, and define the concepts of H- and IK-efficiency of estimators. Section 4 is devoted to the construction of asymptotically efficient estimators for linear and nonlinear smooth spectral functionals, and bounding the minimax mean square risks. In Section 5 we outline the proofs of results stated in Section 4.

Throughout the paper the letters C, C_k, $C(\cdot)$, c and c_k are used to denote positive constants. Also, we use the letters θ and f to denote the spectral density of the model $X(t)$.

2. The Model

Statistical analysis of Gaussian stationary processes usually requires two type of conditions imposed on the spectral density $f(\lambda)$. The first type of these conditions controls the singularities (zeros and poles) of function $f(\lambda)$, and describes the dependence structure of the underlying process $X(t)$, while the second type conditions requires smoothness of spectral density $f(\lambda)$. Much of statistical inferences (parametric and non-parametric) is concerned with the so-called *short-memory* stationary models, in which case the spectral density $f(\lambda)$ of the model $X(t)$ is assumed to be separated from zero and infinity, that is, with some constants C_1 and C_2

$$0 < C_1 \leq f(\lambda) \leq C_2 < \infty. \tag{2.1}$$

However, the data in many fields of science (e.g. in economics, engineering, finance, hydrology, etc.) occur in the form of a realization of a stationary process $X(t)$ with possibly unbounded (*long-memory* model) or vanishing (*anti-persistent* model) spectral density (see, for instance, Beran [3], and references therein).

In the discrete context, a basic long-memory model (or a model for representing long-range dependence) is the ARFIMA$(0, d, 0)$ process $X(t)$:

$$(1-B)^d X(t) = \varepsilon(t), \quad 0 < d < \frac{1}{2}, \tag{2.2}$$

where B is the backshift operator $BX(t) = X(t-1)$ and $\varepsilon(t)$ is a discrete-time white noise. The spectral density of $X(t)$ is given by

$$f(\lambda) = (2\sin(\lambda/2))^{-2d}, \quad 0 < \lambda \leq \pi. \tag{2.3}$$

In the continuous context, a basic process which has commonly been used to model long-range dependence is fractional Brownian motion (fBm) B_H with Hurst index H. This is a Gaussian process which has stationary increments and spectral density of the form

$$f(\lambda) = \frac{c}{|\lambda|^{2H+1}}, \quad c > 0, \quad 0 < H < 1, \quad \lambda \in \mathbb{R}, \tag{2.4}$$

where the form (2.4) can be understood in the sense of time-scale analysis or in a limiting sense since the fBm B_H is a nonstationary process (see, e.g., Anh et al. [1] and Gao et al. [10]).

A proper stationary model in lieu of fBm is the fractional Riesz-Bessel motion (fRBm), introduced in Anh et al. [2], and then extensively discussed in a number of papers (see, e.g., Anh et al. [1], Gao et al. [10], and references therein), is defined to be a continuous-time Gaussian process $X(t)$ with spectral density function of the form

$$f(\lambda) = \frac{c}{|\lambda|^{2\alpha}(1+\lambda^2)^{\beta}}, \quad \lambda \in \mathbb{R}, \tag{2.5}$$

where $0 < c < \infty$, $0 < \alpha < 1$ and $\beta > 0$.

It is noted that the process $X(t)$ is stationary if $0 < \alpha < 1/2$ and is nonstationary with stationary increments if $1/2 \le \alpha < 1$. Observe that the spectral density (2.5) behaves as $O(|\lambda|^{-2\alpha})$ as $|\lambda| \to 0$ and as $O(|\lambda|^{-2(\alpha+\beta)})$ as $|\lambda| \to \infty$. Thus, under the conditions $0 < \alpha < 1/2$, $\beta > 0$ and $\alpha + \beta > 1/2$, the function $f(\lambda)$ in (2.5) is well-defined for both $|\lambda| \to 0$ and $|\lambda| \to \infty$ due to the presence of the component $(1 + \lambda^2)^{-\beta}$, $\beta > 0$, which is the Fourier transform of the Bessel potential. Note that in the spacial case $0 < \alpha < 1/2$, $\beta > 1/2$ the condition $\alpha + \beta > 1/2$ holds automatically. The exponent α determines the long-range dependence, or self-similarity of fRBm, while the exponent β indicates the second-order intermittency of the process (see, e.g., [1], [10]).

Comparing (2.4) and (2.5), we observe that the spectral density of fBm is the limiting case as $\beta \to 0$ that of fRBm with Hurst index $H = \alpha - 1/2$. Thus, the form (2.5) means that fRBm may exhibit both LRD and second-order intermittency.

Finally, the data can also occur in the form of a realization of a "mixed" short-long-memory stationary process $X(t)$ with spectral density given by $f(\lambda) = f_L(\lambda) f_S(\lambda)$, where $f_L(\lambda)$ and $f_S(\lambda)$ are the long- and short-memory components, respectively.

So, it is important to consider a model that will include the above discussed cases. To specify the model we need the following definition (see, e.g., Hunt, Muckenhoupt and Wheeden [24], Stein [44], Sec. 5.1).

Definition 2.1 (Muckenhoupt condition (A_2))**.** We say that a nonnegative locally integrable function $f(\lambda)$ $(\lambda \in \mathbb{Q})$ satisfies the *Muckenhoupt condition* (A_2) (or has Muckenhoupt type singularities), if

$$\sup \frac{1}{|J|^2} \int_J f(\lambda)\, d\lambda \int_J \frac{1}{f(\lambda)}\, d\lambda < \infty, \qquad (A_2)$$

where the supremum is over all intervals J, and $|J|$ stands for the length of J. The class of functions $f(\lambda)$ satisfying condition (A_2) we denote by $\mathcal{A}_2$.

Remark 2.2. It is clear that the spectral densities of short-memory processes belong to $\mathcal{A}_2$. The class $\mathcal{A}_2$ also contains spectral densities possessing singularities. In particular, it can be shown (see, e.g., Böttcher and Karlovich [5], Sec. 2.1) that functions of the form

$$f_0(\lambda) := \begin{cases} \prod_{k=1}^n |e^{i\lambda} - e^{i\lambda_k}|^{2\alpha_k}, & \lambda_k \in \mathbb{T} \ \text{(for d.t. case)}, \\ \prod_{k=1}^n |\lambda - \lambda_k|^{2\alpha_k}, & \lambda_k \in \mathbb{R} \ \text{(for c.t. case)}, \end{cases} \qquad (2.6)$$

belong to $\mathcal{A}_2$ if and only if $-1/2 < \alpha_k < 1/2$ for all $k = \overline{1,n}$. Notice also that for any slowly varying at $\lambda = 0$ function $l(\lambda)$, the function $f_0(\lambda) l(\lambda) \in \mathcal{A}_2$, where $f_0(\lambda)$ is as in (2.6).

Remark 2.3. Condition (A_2) controls the singularities of the spectral density $f(\lambda)$, and describes the dependence structure of the underlying process $X(t)$. It is a regularity condition for $X(t)$, and means that the maximal coefficient of correlation between the past and future of $X(t)$ is less than 1, or equivalently, the minimal angle between the past and future of $X(t)$ is positive (see Ibragimov and Rozanov [28], Ch. 6). Indeed, let $L^2(\Omega) := \left\{\xi : \mathbb{E}[\xi] = 0,\ ||\xi||^2 := \mathbb{E}[\xi^2] < \infty\right\}$. For $a, b \in$

$\mathbb{R}$, $-\infty \le a \le b \le \infty$ define $H_a^b(X)$ to be the linear closed subspace of $L^2(\Omega)$ spanned by the random variables $X(t,\omega)$, $t \in [a,b]$:

$$H_a^b(X) := \overline{sp}\{X(t),\ a \le t \le b\}_{L^2(\Omega)}.$$

The space $H := H_{-\infty}^{\infty}(X)$ is called the *time-domain* of the process $X(u)$, and the subspaces $H^- := H_{-\infty}^{0}(X)$ and $H^+ := H_1^{\infty}(X)$ are called the *past* (or history) and the *future* of $X(u)$, respectively (for simplicity, we assume that $X(u)$ is a d.t. process). The *minimal angle* $u(H^-, H^+)$ $(0 \le u(H^-, H^+) \le \frac{\pi}{2})$ between the subspaces H^- and H^+ is defined by

$$\cos u(H^-, H^+) = \sup_{\xi \in H^-,\ \eta \in H^+} \frac{|(\xi,\eta)|}{||\xi||\,||\eta||}.$$

Then condition (A_2) is equivalent to the following: the minimal angle $u(H^-, H^+)$ between H^- and H^+ is positive, that is,

$$\rho_1(f) := \cos u(H^-, H^+) < 1.$$

Remark 2.4. Let $L^2(f)$ be the *frequency-domain* of the process $X(t)$ with spectral density $f(\lambda)$, i.e., the space of complex-valued functions $\varphi(\lambda)$, $\lambda \in \mathbb{Q}$, defined by

$$L^2(f) = \left\{ \varphi(\lambda) :\ ||\varphi||_f^2 := \int_{\mathbb{Q}} |\varphi(\lambda)|^2 f(\lambda) d\lambda < \infty \right\},$$

and let $H_a^b(f)$, $a, b \in \mathbb{R}$, $-\infty \le a \le b \le \infty$ be the closed linear subspace of $L^2(f)$ spanned by the exponents $e^{it\lambda}$, $t \in [a,b]$, that is, $H_a^b(f) = \overline{sp}\{e^{it\lambda},\ a \le t \le b\}_{L^2(f)}$. Then the following assertions are equivalent (see, e.g., Nikol'skii [37]):

(a) $f(\lambda) \in \mathcal{A}_2$.
(b) The minimal angle between $H_{-\infty}^{0}(f)$ and $H_1^{\infty}(f)$ is positive.
(c) $\ln f(\lambda) = u(\lambda) + \widetilde{v}(\lambda)$, where $u(\lambda)$ and $v(\lambda)$ are real bounded functions, and $||v||_\infty < \pi/2$ ($\widetilde{v}$ is the harmonic conjugate of v: $\widetilde{v} = \Im h$, $v = \Re h$ for some $h \in H^{2+}$, where H^{2+} is the Hardy space).
(d) The family $\{z^n :\ n \in \mathbb{Z}\}$ forms a basis in $L^2(f)$.
(e) The Riesz projector $P^+ : L^2 \to H^{2+}$ is a bounded operator in $L^2(f)$, that is, $||P^+||_f \le C < \infty$.

Hölder classes. Given numbers $0 < \alpha \le 1$, $p \ge 1$, and $r \in \mathbb{N}_0$, where $\mathbb{N}_0$ stands for the set of nonnegative integers. We put $\beta = r + \alpha$, and denote by $\mathbf{H}_p(\beta)$ the Hölder class of functions, that is, the class of functions $\psi(\lambda) \in L^p := L^p(\mathbb{Q})$, which have r-th derivatives in L^p and satisfy

$$||\psi^{(r)}(\cdot + u) - \psi^{(r)}(\cdot)||_p \le C|u|^\alpha,$$

where $||h||_p$ denotes the L^p-norm of a function h, and C is a positive constant. Also, by $\mathbf{\Sigma}_p(\beta)$ we denote the set of all spectral densities which belong to $\mathbf{H}_p(\beta)$.

Now we are in a position to specify our model. The basic assumption on the observed process $X(t)$ is the following.

Assumption 2.1. $\{X(t),\ t \in \mathbb{U}\}$ is a centered real-valued (mean square continuous in the c.t. case) Gaussian stationary process with a spectral density $\theta(\lambda)$ satisfying Muckenhoupt condition (A_2) and belonging to a Hölder class $\mathbf{H}_p(\beta)$. Thus, $\theta(\lambda) \in \mathbf{\Theta} \subseteq \mathcal{A}_2 \cap \mathbf{\Sigma}_p(\beta)$, where $0 < \alpha \le 1$, $p \ge 1$, $\beta = r + \alpha$ and $r \in \mathbb{N}_0$.

3. Preliminaries: LAN and Efficiency Concepts

In this section we establish local asymptotic normality of families of distributions generated by a discrete- or continuous-time Gaussian stationary process, then state variants of Hájek-Ibragimov-Khas'minskii convolution theorem and Hájek-Le Cam local asymptotic minimax theorem, and define the concepts of H- and IK-efficiency of estimators.

3.1. Local asymptotic normality. The notion of local asymptotic normality (LAN) of families of distributions, introduced by Le Cam in 1960 (see Le Cam [31]), plays an important role in asymptotic estimation theory. Le Cam, Hájek, Ibragimov and Khas'minskii and others have shown (see, for instance, Le Cam [32], Hájek [21], [22], Ibragimov and Khas'minskii [25], [27], Kutoyants [30], and references therein), that many important properties of statistical estimators (characterization of limiting distributions, lower bounds on the accuracy, asymptotic efficiency, etc.) follow in fact from LAN condition. The importance of LAN concept for nonparametric estimation problems has been emphasized by Levit [33], [34], Ibragimov and Khas'minskii [25], [27], Millar [36], Kutoyants [30], and others. The LAN for families of distributions generated by discrete-time Gaussian stationary processes has been studied by Davies [8], Dzhaparidze [9], Ginovyan [16], for continuous-time processes sufficient conditions for LAN were obtained in Solev and Zerbet [43].

Following Ibragimov and Khas'minskii [27], where a definition of LAN for the case where the underlying parametric set is a subset of a normed space or a smooth infinite-dimensional manifold was suggested, we define LAN for our model.

Definition 3.1. Let $\mathbb{P}_{T,\theta}$ be the probability distribution of the observation $\mathbf{X}_T = \{X(u),\ 0 \le u \le T \text{ (or } u = \overline{1,T} \text{ in the d.t. case)}\}$ with spectral density $\theta(\lambda)$. A family of distributions $\{\mathbb{P}_{T,\theta},\ \theta \in \boldsymbol{\Theta}\}$ is called *locally asymptotically normal (LAN)* at a point $\theta_0 \in \boldsymbol{\Theta}$ in the direction $L^2 := L^2(\mathbb{Q})$ with norming factors $A_T := A_T(\theta_0)$, if there exist a linear manifold $H := H(\theta_0) \subset L^2$ with closure $\overline{H} = L^2$, and a family $\{A_T\}$ of linear operators $A_T : L^2 \to L^2$ that satisfy:

1) for any $h \in H$, $||A_T h||_2 \to 0$ as $T \to \infty$, where $||\cdot||_2$ denotes the L^2–norm;
2) for any $h \in H$ there is a natural $T(h)$ such that $\theta_0 + A_T h \in \boldsymbol{\Theta}$ for all $T > T(h)$;
3) for any $h \in H$ and $T > T(h)$

$$\ln \frac{d\mathbb{P}_{T,\theta_0+A_T h}}{d\mathbb{P}_{T,\theta_0}}(\mathbf{X}_T) = \Delta_T(h,\theta_0) - \frac{1}{2}||h||_2^2 + \phi(T,h,\theta_0), \tag{3.1}$$

where $\Delta_T(h) := \Delta_T(h,\theta_0)$ is a random linear function on H, and for any $h \in H$ the variable $\Delta_T(h)$ is asymptotically $N(0,\|h\|_2^2)$–normally distributed, while $\phi(T,h,\theta_0) \to 0$ in $\mathbb{P}_{T,\theta_0}$–probability as $T \to \infty$.

Note that the presence of LAN property depends on the point θ_0, the space L^2 and the family of operators $\{A_T\}$. The choice of $H = H(\theta_0)$ may be rather arbitrary. We need only that $\overline{H} = L^2$.

Definition 3.2 (Condition $(\mathcal{H})$)**.** We say that a pair of functions (f, g) satisfies *condition* $(\mathcal{H})$, if $f \in \mathbf{\Sigma}_p(\beta)$ for $1 \le p \le 2$ and $\beta > 1/p$, and $g \in L^q$, where q is the conjugate of p: $1/p + 1/q = 1$.

The parametric set $\mathbf{\Theta}$ we will always assume to be a subset of the space L^p $(p \ge 1)$ consisting of spectral densities satisfying Muckenhoupt condition (A_2) and belonging to the Hölder class $\mathbf{\Sigma}_p(\beta)$. Define $H = H(\theta)$ to be the linear manifold consisting of bounded functions $h(\lambda)$ such that the pair $(\theta, h\theta^{-1})$ satisfies the condition $(\mathcal{H})$. We also define $A_T : L^2 \to L^2$ by $A_T h = [T^{-1/2}\theta] \cdot h$, that is, A_T is the operator of multiplication by function $T^{-1/2}\theta(\lambda)$. The next theorem contains sufficient conditions for LAN.

Theorem 3.3. *Let $\mathbf{\Theta}$, H and A_T be defined as above. Then the family of distributions $\{\mathbb{P}_{T,\theta}, \theta \in \mathbf{\Theta}\}$ satisfies LAN condition at any point $\theta \in \mathbf{\Theta}$ in the direction L^2 with norming factors A_T and*

$$\Delta_T(h) \;=\; \frac{T^{1/2}}{4\pi} \int_{\mathbb{Q}} \left[\frac{I_T(\lambda)}{\theta(\lambda)} - 1\right] h(\lambda)\, d\lambda, \tag{3.2}$$

where

$$I_T(\lambda) := \begin{cases} \frac{1}{2\pi T} \left|\sum\limits_{u=1}^{T} X(u) e^{-iu\lambda}\right|^2, & \text{for d.t. case,} \\ \frac{1}{2\pi T} \left|\int_0^T X(u) e^{-iu\lambda}\, du\right|^2, & \text{for c.t. case,} \end{cases} \tag{3.3}$$

is the periodogram of the process $X(t)$.

Remark 3.4. For d.t. case Theorem 3.3 was proved in Ginovyan [16]. For c.t. case the proof is similar to that of d.t. case (see also Solev and Zerbet [43], where the LAN for c.t. case was proved under different smoothness conditions).

3.2. Characterization of limiting distribution: H-efficiency. We now consider the problem of estimating the value $\Phi(\theta)$ of a known functional $\Phi(\cdot)$ at an unknown point $\theta \in \mathbf{\Theta}$ on the basis of an observation $\mathbf{X}_T$, which has distribution $\mathbb{P}_{T,\theta}$. We assume that the family $\{\mathbb{P}_{T,\theta},\, \theta \in \mathbf{\Theta}\}$ satisfies the LAN condition at a point $\theta_0 = f \in \mathbf{\Theta}$ in the direction L^2 with norming factors A_T. We also assume the functional $\Phi(\theta)$ to be Fréchet differentiable at $f \in L^2$ with derivative $\Phi'(f) := \Phi'(f; \lambda)$, that is, there exists a linear continuous functional $\Phi'_f : L^2 \to \mathbb{R}$:

$$\Phi'_f(\psi) = \int_{\mathbb{Q}} \Phi'(f; \lambda)\psi(\lambda) d\lambda, \quad \psi \in L^2 \tag{3.4}$$

such that for $f, g \in L^2$ we have

$$|\Phi(g) - \Phi(f) - \Phi'_f(g - f)| = o\left(||g - f||_2\right) \quad \text{as} \quad ||g - f||_2 \to 0. \tag{3.5}$$

Furthermore, we assume that the derivative $\Phi'(f)$ satisfies the condition: uniformly for $f \in \mathbf{\Theta}$

$$0 < ||\Phi'(f) f||_2 < \infty. \tag{3.6}$$

We need a version of Hájek-Ibragimov-Khas'minskii convolution theorem for regular estimators. Recall that (see, e.g., Ibragimov and Khas'minskii ([25], Sec.

2.9) an estimator $\widehat{\Phi}_T := \widehat{\Phi}_T(\mathbf{X}_T)$ of $\Phi(\theta)$ is called H–regular at $\theta_0 \in \mathbf{\Theta}$, if for any $h \in H$ there exists not depending on h a proper limit distribution function F of the normed difference $T^{1/2}\left(\widehat{\Phi}_T - \Phi(\theta_h)\right)$, where $\theta_h = \theta_0 + A_T h$, in the sense of weak convergence

$$\mathcal{L}\left\{T^{1/2}\left(\widehat{\Phi}_T - \Phi(\theta_h)\right)\Big|\, \mathbb{P}_{T,\theta_h}\right\} \Longrightarrow F \quad \text{as} \quad T \to \infty.$$

The next theorem follows from Theorem 3.1 in Ibragimov and Khas'minskii [27], and Theorem 3.3.

Theorem 3.5. *Let $\widehat{\Phi}_T$ be a H-regular estimator of $\Phi(\theta)$ at $f \in \mathbf{\Theta}$. Assume that the pair $(f, \Phi'(f))$ satisfies (3.6). Then under the assumptions of Theorem 3.3 the limit distribution F of $T^{1/2}\left(\widehat{\Phi}_T - \Phi(f)\right)$ is a convolution of a probability distribution G and a centered normal distribution with variance $\sigma^2 := 4\pi||\Phi'(f)f||_2^2$:*

$$F = N(0, \sigma^2) * G. \tag{3.7}$$

By a well-known lemma of Anderson (see, e.g., Ibragimov and Khas'minskii [25], Sec. 2.10), the distribution F in (3.7) is less concentrated in symmetric intervals than the normal distribution $N(0, \sigma^2)$. This justifies the following definition of H-efficiency (cf. Millar [36], Dahlhaus and Wefelmeyer [7], Kutoyants [29], Sec. 2.1, Ginovyan [18]).

Definition 3.6. Let the family $\{\mathbb{P}_{T,\theta},\, \theta \in \mathbf{\Theta}\}$ be LAN at a point $f \in \mathbf{\Theta}$. An estimator $\widehat{\Phi}_T$ of $\Phi(\theta)$ is called *H–asymptotically efficient* at f (in the class of H-regular estimators) with asymptotic variance $\sigma^2 := 4\pi||\Phi'(f)f||_2^2$, if

$$\mathcal{L}\left\{T^{1/2}\left(\widehat{\Phi}_T - \Phi(\theta_h)\right)\Big|\, \mathbb{P}_{T,\theta_h}\right\} \Longrightarrow N(0, \sigma^2) \quad \text{as} \quad T \to \infty,$$

that is, the distribution G in (3.7) is degenerate.

Remark 3.7. This efficiency concept is a nonparametric version of Hájek-efficiency, and admits the same intuitive interpretation: the asymptotic distribution of any regular estimator $\widehat{\Phi}_T$ of $\Phi(\theta)$ is always "more spread out" than the centered normal distribution with variance $4\pi||\Phi'(f)f||_2^2$ (cf. Hájek [21], Millar [36]).

Remark 3.8. We also have the following characterization of H-regular and H-asymptotically efficient estimators (cf. Dahlhaus and Wefelmeyer [7], Taniguchi and Kakizawa [46], Ch.6, Ginovyan [18]): if the family $\{\mathbb{P}_{T,\theta},\, \theta \in \mathbf{\Theta}\}$ is LAN at $f \in \mathbf{\Theta}$, then an estimator $\widehat{\Phi}_T$ of $\Phi(f)$ is H-regular and H-asymptotically efficient at f with asymptotic variance $4\pi||f\Phi'(f)||_2^2$ if and only if it admits the following stochastic approximation:

$$T^{1/2}[\widehat{\Phi}_T - \Phi(f)] - \Delta_T(f\Phi'(f)) \,=\, o_P(1) \quad \text{as} \quad T \to \infty, \tag{3.8}$$

where $\Delta_T(f\Phi'(f))$ is defined by (3.2) with $\theta = f$ and $h = f\Phi'(f)$.

3.3. A lower bound for the asymptotic minimax risks: IK–efficiency. Denote by $\mathbf{\Phi}_T$ the set of all estimators of $\Phi(\theta)$ constructed on the basis of an observation $\mathbf{X}_T$, and let $\mathbf{W}$ denote the set of all loss functions $w : \mathbb{R} \to \mathbb{R}$, which are symmetric and non-decreasing on $\mathbb{R}^+ := (0, \infty)$, and satisfy $w(x) \geq 0, \; w(0) = 0$.

The next theorem, which is a consequence of Theorem 4.1 in Ibragimov and Khas'minskii [27], and Theorem 3.3, contains a minimax lower bound for risks of all possible estimators $\widehat{\Phi}_T$ of $\Phi(\cdot)$ in the neighborhood of a point $f \in \mathbf{\Theta}$ (cf. Has'minskii and Ibragimov [23], Ginovyan [13], [18]).

Theorem 3.9. *Assume that the pair $(f, \Phi'(f))$ satisfies (3.6). Then under the assumptions of Theorem 3.3, for all $w \in \mathbf{W}$*

$$\liminf_{\delta \to 0} \lim_{T \to \infty} \inf_{\widehat{\Phi}_T \in \mathbf{\Phi}_T} \sup_{||\theta - f||_2 < \delta} \mathbb{E}_\theta \{ w(T^{1/2}(\widehat{\Phi}_T - \Phi(f))) \} \geq \mathbb{E} w(\xi), \tag{3.9}$$

where ξ is a centered normal random variable with variance $4\pi ||\Phi'(f) f||_2^2$, and $\mathbb{E}_\theta\{\cdot\}$ stands for the expectation with respect to measure corresponding to spectral density $\theta(\lambda)$.

Basing on Theorem 3.9, we define the notion of asymptotically efficient estimators in the spirit of Ibragimov and Khas'minskii (IK-efficiency) (see Has'minskii and Ibragimov [23], and Ibragimov and Khas'minskii [27]).

Definition 3.10. Let the family $\{\mathbb{P}_{T,\theta}, \, \theta \in \mathbf{\Theta}\}$ be LAN at a point $f \in \mathbf{\Theta}$. An estimator $\widehat{\Phi}_T$ of $\Phi(\theta)$ is called *IK-asymptotically efficient* at f for the loss function $w(x) \in \mathbf{W}$, with asymptotic variance $\sigma^2 = 4\pi ||\Phi'(f) f||_2^2$, if

$$\liminf_{\delta \to 0} \lim_{T \to \infty} \sup_{||\theta - f||_2 < \delta} \mathbb{E}_\theta \{ w(T^{1/2}(\widehat{\Phi}_T - \Phi(f))) \} = \mathbb{E} w(\xi), \tag{3.10}$$

where ξ is as in Theorem 3.9.

Remark 3.11. An estimator $\widehat{\Phi}_T$ of $\Phi(\theta)$ satisfying (3.10) is also called *locally asymptotically minimax (LAM)* estimator (see, e.g., Levit [33], Kutoyants [29], Sec. 2.1).

Remark 3.12. Both definitions of efficiency – H- and IK-efficiency – roughly speaking, require from an asymptotically efficient estimator $\widehat{\Phi}_T$ the local uniformity of the convergence of the normed difference $T^{1/2}\big(\widehat{\Phi}_T - \Phi(f)\big)$ to the centered normal random variable ξ with variance $4\pi ||\Phi'(f) f||_2^2$, and for bounded loss functions $w(\cdot)$ they are rather close. An attraction of the definition of IK-efficiency over that of H-efficiency is that it compares all the estimators constructed on the basis of an observation $\mathbf{X}_T$, rather than only the regular estimators, while an attraction of the definition of H-efficiency over that of IK-efficiency is that it is concerned with limiting distributions, rather than limits of expectations (cf. Beran [4], Kutoyants [29], Sec. 2.1). For detailed discussion of definitions and relationships of various efficiency concepts we refer to Ibragimov and Khas'minskii [25], Ch.2.

4. Efficient Estimation and Asymptotic Bounds

In this section we construct asymptotically efficient estimators for linear and nonlinear smooth spectral functionals, and obtain exact asymptotic bounds for minimax mean square risks of estimators of linear functionals.

4.1. Asymptotically efficient estimators for linear functionals. Let the estimand functional $\Phi(f)$, $f \in \mathbf{\Theta}$ be linear and continuous in $L^p(\mathbb{Q})$, $p \geq 1$. It is well-known (see, e.g., Riesz and Nagy [41]) that $\Phi(f)$ admits the representation

$$\Phi(f) = \int_{\mathbb{Q}} f(\lambda)g(\lambda)d\lambda, \tag{4.1}$$

where $g(\lambda) \in L^q,\ 1/p+1/q = 1$. As an estimator for $\Phi(f)$ we consider the averaged periodogram statistic, that is, the simple "plug-in" statistic:

$$\widehat{\Phi}_T := \Phi(I_T) = \int_{\mathbb{Q}} I_T(\lambda)g(\lambda)d\lambda, \tag{4.2}$$

where $I_T(\lambda)$ is the periodogram of $X(t)$ defined by (3.3).

Let $\mathbf{W}_e$ denote the subset of loss functions $w \in \mathbf{W}$ which for some constants $C_1 > 0$ and $C_2 > 0$ satisfy the condition $w(x) \leq C_1 \exp\{C_2|x|\}$.

Theorem 4.1. *Let $\Phi(f)$ and $\widehat{\Phi}_T$ be defined by (4.1) and (4.2). Assume that the pair of functions (f, g) satisfies the conditions $(\mathcal{H})$ and $0 < ||fg||_2 < \infty$ uniformly for $f \in \mathbf{\Theta}$. Then the statistic $\widehat{\Phi}_T$ is:*

(a) *H-regular and H-asymptotically efficient estimator of $\Phi(f)$ with asymptotic variance $4\pi||fg||_2^2$;*
(b) *IK-asymptotically efficient estimator of $\Phi(f)$ for $w(x) \in \mathbf{W}_e$ with asymptotic variance $4\pi||fg||_2^2$.*

Example 4.2. (Estimation of covariance function) Assume that $X(t)$ is a c.t. process, and let $g(\lambda) = \frac{1}{2\pi}e^{iu\lambda}$, then

$$\Phi(f) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iu\lambda}f(\lambda)\, d\lambda := r(u).$$

Thus, in this special case our problem becomes to the estimation of the the covariance function $r(u) = \mathbb{E}[X(t+u)X(t)]$ of the process $X(t)$. By Theorem 4.1 the simple "plug-in" statistic

$$\widehat{\Phi}_T = \widehat{r}_T(u) = \int_{-\infty}^{\infty} e^{iu\lambda}I_T(\lambda)\, d\lambda = \frac{1}{T}\int_0^{T-|u|} X(t)X(t+u)dt$$

is H- and IK-asymptotically efficient estimator for $r(u)$ with asymptotic variance

$$\sigma_u^2 \,=\, 2\pi\int_{-\infty}^{\infty} \cos^2(u\lambda)f^2(\lambda)\, d\lambda.$$

Example 4.3. (Estimation of spectral function) Assume that $X(t)$ is a d.t. process, and let $g(\lambda) = \chi_{[0,\mu]}(\lambda)$ be the indicator of an interval $[0, \mu]$, then

$$\Phi(f) = \int_{-\pi}^{\pi} \chi_{[0,\mu]}(\lambda)f(\lambda)\, d\lambda = \int_0^{\mu} f(\lambda)\, d\lambda := F(\mu).$$

Thus, in this case the estimand functional is the spectral function $F(\mu)$ of the process $X(u)$, and by Theorem 4.1 the simple "plug-in" statistic

$$\widehat{\Phi}_T = \widehat{F}_T(\mu) = \int_0^{\mu} I_T(\lambda)\, d\lambda = \frac{1}{2\pi T}\int_0^{\mu}\left|\sum_{t=1}^{T} X(t)e^{-i\lambda t}\right|^2 \tag{4.3}$$

is H- and IK-asymptotically efficient estimator for $F(\mu)$ with asymptotic variance

$$\sigma^2(\mu) = 2\pi \int_0^\mu f^2(s)\, ds.$$

Remark 4.4. For discrete-time processes the problem of estimation of spectral function $F(\mu)$ has been considered in Millar [36], and in Levit and Samarov [35]. In particular, in [35] was shown that for a broad class of loss functions the statistic $\widehat{F}_T(\mu)$ given by (4.3) is a locally asymptotically minimax, that is, IK-efficient estimator for $F(\mu)$.

4.2. Asymptotically efficient estimators for smooth nonlinear functionals. The problem of asymptotically efficient estimation becomes somewhat more complicated for non-linear functionals. In this case the simple "plug-in" statistic $\Phi(I_T)$ is not necessary a consistent estimator for the functional $\Phi(f)$, and hence instead of the periodogram $I_T(\lambda)$, we need to use a suitable sequence of consistent estimators $\{\widehat{f}_T\}$ of f (cf. Dahlhaus and Wefelmeyer [7], Ginovyan [18], Has'minskii and Ibragimov [23], Taniguchi and Kakizawa [46]).

Thus, in this case, we have two estimation problems: estimation of unknown spectral density f, and estimation of functional $\Phi(f)$. There are two approaches of construction of asymptotically efficient estimators for $\Phi(f)$.

In the first approach, called "independent" estimation method, first we estimate f, by taking as an estimator of f an arbitrary *optimal* point-wise estimator $\widehat{f}_T$, that is, the one with optimal bandwidth, and then consider the "plug-in" estimator $\Phi(\widehat{f}_T)$ for $\Phi(f)$. In this approach, the "plug-in" statistic $\Phi(\widehat{f}_T)$ will be "very bad estimator" for $\Phi(f)$, in the sense that $\Phi(\widehat{f}_T)$ will converge to $\Phi(f)$ too slowly to be asymptotically efficient. However, it is possible to improve $\Phi(\widehat{f}_T)$ to make it asymptotically efficient, by adding an extra linear term, and by imposing extra restrictions on the parametric set (see Has'minskii and Ibragimov [23]).

In the second approach, called "dependent" or "undersmoothing" estimation method, we estimate f and $\Phi(f)$ simultaneously, and instead of optimal estimator of f, we use the so-called "undersmoothed" kernel estimator. In this approach, the "plug-in" statistic $\Phi(\widehat{f}_T)$ will provide an asymptotically efficient estimator for $\Phi(f)$.

Remark 4.5. By "undersmoothed" kernel estimator $\widehat{f}_T$ of f we mean the following: the bandwidth used in the kernel estimator $\widehat{f}_T$ is not optimal for the estimation of unknown spectral density f; rather, we take advantage of the smooth, integral nature of the derivative of the estimand functional and undersmooth it. By choosing a small bandwidth, that is, by undersmoothing, the bias term becomes negligible and the behavior of the estimator is determined by a random term, which, with an appropriate normalization, obeys a central limit theorem. A similar approach was applied in Dahlhaus and Wefelmeyer [7], Ginovyan [18], and Taniguchi [45] for discrete-time processes, and in Goldstein and Khas'minskii [19], and Goldstein and Messer [20] for efficient estimation of smooth functionals defined on a set of probability density functions.

We consider a sequence $\{\widehat{f}_T\}$ of "undersmoothed" kernel estimators of the unknown spectral density $f(\lambda)$, and derive conditions under which the "plug-in" statistic $\Phi(\widehat{f}_T)$ is asymptotically efficient estimator for $\Phi(f)$, provided that the estimand functional $\Phi(f)$ is smooth enough.

We assume that $f \in \mathbf{\Sigma}_p(\beta)$ ($p \geq 1$, $\beta = r + \alpha$, $0 < \alpha \leq 1$, $r \in \mathbb{N}_0$), and as an estimator for unknown spectral density f we take the statistic (cf. Dahlhaus and Wefelmeyer [7], Ginovyan [18], Parzen [39], Taniguchi [45]):

$$\widehat{f}_T(\lambda) = \int_{\mathbb{Q}} W_T(\lambda - \mu) I_T(\mu)\, d\mu, \tag{4.4}$$

where $I_T(\lambda)$ is the periodogram of $X(t)$ defined by (3.3). For the kernel $W_T(\lambda)$ we set down the following assumptions.

Assumption 4.1. $W_T(\lambda) = M_T W(M_T \lambda)$, where $M_T = O(T^\gamma)$, and $b_T := M_T^{-1}$ is the bandwidth. The choice of the number γ ($0 < \gamma < 1$) will depend on the appriori knowledge about f and Φ.

Assumption 4.2. $W(\lambda)$ is bounded, even, nonnegative function with $W(\lambda) \equiv 0$ for $|\lambda| > 1$ and

$$\int_{-1}^{1} W(\lambda)\, d\lambda = 1, \quad \int_{-1}^{1} \lambda^k W(\lambda)\, d\lambda = 0, \quad k = 1, 2, \dots, r,$$

where $r = [\beta]$ is the integer part of β.

We assume the functional $\Phi(\cdot)$ to be Fréchet differentiable in L^2 with derivative $\Phi'(f) := \Phi'(f;\lambda)$ satisfying (3.6) and a Hölder condition: there exist constants $C > 0$ and δ ($0 < \delta \leq 1$) such that for any $f_1, f_2 \in L^2$,

$$||\Phi'(f_1) - \Phi'(f_2)|| \leq C ||f_1 - f_2||_2^\delta. \tag{4.5}$$

Theorem 4.6. *Let the spectral density $f(\cdot)$ and the functional $\Phi(\cdot)$ be such that: (i) the pair $(f, \Phi'(f))$ satisfies the conditions $(\mathcal{H})$ and (3.6) uniformly for $f \in \mathbf{\Theta}$; (ii) $\Phi(\cdot)$ satisfies the condition (4.5) with $\delta \geq (2\beta - 1)^{-1}$.*
Let the estimator $\widehat{f}_T$ for f be defined by (4.4) with the kernel $W_T(\lambda)$ satisfying Assumptions 4.1 and 4.2 with $\frac{1}{2\beta} < \gamma < \frac{\delta}{\delta+1}$. Then the "plug-in" statistic $\Phi(\widehat{f}_T)$ is:

(a) *H-regular and H-asymptotically efficient estimator of $\Phi(f)$ with asymptotic variance $4\pi||\Phi'(f)f||_2^2$;*
(b) *IK-asymptotically efficient estimator of $\Phi(f)$ for $w(x) \in \mathbf{W}_e$ with asymptotic variance $4\pi||\Phi'(f)f||_2^2$.*

Example 4.7. Consider the problem of estimation of the integrated squared spectral density functional $\Phi(f)$:

$$\Phi(f) := ||f||_2^2 = \int_{-\infty}^{\infty} f^2(\lambda)\, d\lambda. \tag{4.6}$$

In this case $\Phi'(f) = 2f$, and it follows from Theorems 4.6 that the "plug-in" statistic

$$\widehat{\Phi}_T = \Phi(\widehat{f}_T) = \int_{-\infty}^{\infty} [\widehat{f}_T(\lambda)]^2\, d\lambda,$$

where $\widehat{f}_T(\lambda)$ is as in (4.4), is H- and IK-asymptotically efficient estimator for functional (4.6) with asymptotic variance $\sigma^2 = 16\pi||f^2||_2^2$.

Remark 4.8. For discrete-time processes Theorems 4.1 and 4.6 were proved in Ginovyan [18]. For short-memory discrete-time models H-asymptotically efficient estimators were constructed by Dahlhaus and Wefelmeyer [7], and Millar [36], while IK-asymptotically efficient estimators were constructed by Has'minskii and Ibragimov [23], Ibragimov and Khas'minskii [27], and Ginovyan [11].

4.3. Exact asymptotic bounds for the minimax mean square risk. We return to the problem of estimation of a linear, L^p-continuous functional $\Phi(f)$. If $1 \le p \le 2$ the functional $\Phi(f)$ is continuous in L^2, and so we can apply Theorem 4.1 to construct an asymptotically efficient estimator for $\Phi(f)$. If $p > 2$ we no longer have an efficient estimator, and it becomes of interest to estimate the rate of decrease (as $T \to \infty$) of the minimax risk

$$\inf_{\widehat{\Phi}_T \in \mathbf{\Phi}_T} \sup_{f \in \mathbf{\Sigma}} \mathbb{E}_f\{w(\widehat{\Phi}_T - \Phi(f))\},$$

where $\mathbf{\Sigma}$ is a given class of spectral densities and $\mathbf{\Phi}_T$ is the set of all estimators of $\Phi(f)$ constructed on the basis of an observation $\mathbf{X}_T$. It is clear that the bounds will depend on the number p and the smoothness properties of functions from $\mathbf{\Sigma}$. Below we obtain exact asymptotic bounds for the minimax mean square risk:

$$\Delta_T^2 := \sup_{||\Phi||=1} \inf_{\widehat{\Phi}_T} \sup_{f \in \mathbf{\Sigma}_p(\beta)} \mathbb{E}_f|\widehat{\Phi}_T - \Phi(f)|^2. \tag{4.7}$$

More precisely, we show that $\Delta_T^2 \asymp T^{-a}$ $(a > 0)$, where the number a is determined by the parameters p and β. (Here and below the notation $a_T \asymp b_T$ means that the ratio a_T/b_T is asymptotically (as $T \to \infty$) bounded away from 0 and ∞).

Theorem 4.9. *Let $\Phi(f)$ be a linear L^p-continuous functional, and let Δ_T^2 be as in (4.7). The following assertions hold:*

(a) *If $p \ge 2$ and $\beta > 1/p$, then $\Delta_T^2 \asymp T^{-\frac{2p\beta}{p+2p\beta-2}}$.*
(b) *If either $p \ge 2$ and $\beta \le 1/p$ or $1 \le p \le 2$ and $\beta \le 1/2$, then $\Delta_T^2 \asymp T^{-2\beta}$.*
(c) *If $1 \le p \le 2$ and $\beta \ge 1/2$, then $\Delta_T^2 \asymp T^{-1}$.*

Remark 4.10. A similar result for probability density functionals was proved in Ibragimov and Has'minski [26]. For discrete-time processes asymptotically exact bounds were obtained in Ginovyan [17]. For continuous-time processes asymptotically upper bounds were obtained in Ginovyan [13].

Remark 4.11. Asymptotic bounds can be proved for a more broad class of loss functions $\mathbf{W}_e := \{w \in \mathbf{W} : \ w(x) \le C_1 \exp\{C_2|x|\}\}$. For instance, if $\mathbf{\Sigma} = \mathbf{\Sigma}_p(\beta)$, $p \ge 2$, $\beta > 1/p$, and $\Phi(f)$ is a linear L^p-continuous functional, then for any loss function $w(x) \in \mathbf{W}_e$

$$\limsup_{T\to\infty} \sup_{f \in \Sigma} \mathbb{E}_f \left\{ w\left(T^{p\beta/(p-2+2p\beta)}|\widehat{\Phi}_T - \Phi(f)| \right) \right\} < \infty.$$

(cf. Ibragimov and Has'minski [26], Ginovyan [15]).

Remark 4.12. Using well-known embedding theorems (see, e.g., Nikol'skii [38], Sec. 6.2), it can be shown that the results remain valid for Sobolev and Besov classes (instead of Hölder classes).

5. Proofs

In this section we outline the proofs of Theorems 4.1, 4.6 and 4.9. First we state three lemmas. The proof of the next lemma is similar to that of Theorem 3 in Ginovyan [13].

Lemma 5.1. *Let $\Phi(f)$ and $\widehat{\Phi}_T$ be defined by (4.1) and (4.2). Assume that the pair of functions (f, g) satisfies the conditions $(\mathcal{H})$ and $0 < ||fg||_2 < \infty$ uniformly for $f \in \mathbf{\Theta}$. Then for all $w(x) \in \mathbf{W}_e$ uniformly for f,*

$$\lim_{T\to\infty} \mathbb{E}_f\{w(T^{1/2}(\widehat{\Phi}_T - \Phi(f)))\} = \mathbb{E}w(\xi), \tag{5.1}$$

where ξ is a centered normal random variable with variance $4\pi||fg||_2^2$. In particular, the following asymptotic relations hold:

(a) $\lim_{T\to\infty} T^{1/2}[\mathbb{E}_f(\widehat{\Phi}_T - \Phi(f))] = 0$.
(b) $\lim_{T\to\infty} T\,\mathbb{E}_f(\widehat{\Phi}_T - \Phi(f))^2 = 4\pi||fg||_2^2$.
(c) $T^{1/2}(\widehat{\Phi}_T - \Phi(f)) \sim AN(0, 4\pi||fg||_2^2)$.

Lemma 5.2. *Let $f \in \mathbf{\Sigma}_p(\beta)$, and $\psi(\lambda)$ be a continuous even function such that the pair (f, ψ) satisfies the conditions $(\mathcal{H})$ and $0 < ||f\psi||_2 < \infty$. Let $\widehat{f}_T(\lambda)$ be as in (4.4) with kernel $W_T(\lambda)$ satisfying Assumptions 4.1 and 4.2, where $\frac{1}{2\beta} < \gamma < 1$. Then the distribution of the random variable*

$$\eta_T = T^{1/2} \int_{\mathbb{Q}} \psi(\lambda) \left[\widehat{f}_T(\lambda) - f(\lambda)\right] d\lambda \tag{5.2}$$

as $T \to \infty$ tends to the normal distribution $N(0, \sigma^2)$, where

$$\sigma^2 = 4\pi \int_{\mathbb{Q}} \psi^2(\lambda) f^2(\lambda)\, d\lambda. \tag{5.3}$$

Proof. It follows from Lemma 5.1 that the distribution of the random variable

$$\xi_T = T^{1/2} \int_{\mathbb{Q}} \psi(\lambda) \left[I_T(\lambda) - f(\lambda)\right] d\lambda \tag{5.4}$$

tends (as $T \to \infty$) to normal distribution $N(0, \sigma^2)$ with σ^2 as in (5.3). Therefore to complete the proof it is enough to show that

$$|\xi_T - \eta_T| = o_P(1) \quad \text{as} \quad T \to \infty, \tag{5.5}$$

We have

$$\begin{aligned} \eta_T \quad &= \quad T^{1/2} \int_{\mathbb{Q}} \psi(\lambda) \left[\int_{\mathbb{Q}} W_T(\lambda - \mu)[I_T(\mu) - f(\mu)]d\mu\right] d\lambda \\ &\quad + T^{1/2} \int_{\mathbb{Q}} \psi(\lambda) \left[\int_{\mathbb{Q}} W_T(\lambda - \mu) f(\mu) d\mu - f(\lambda)\right] d\lambda \\ &:= \quad \eta_T^{(1)} + S_T \quad \text{(say)}. \end{aligned} \tag{5.6}$$

Using the change of variable $M_T(\lambda - \mu) = t$, and the dominated convergence theorem it can be shown that (cf. Taniguchi [45])

$$|\eta_T^{(1)} - \xi_T| = o_P(1) \quad \text{as } T \to \infty. \tag{5.7}$$

Next, applying Hölder and Minkowski generalized inequalities we get

$$|S_T| \le T^{1/2}||\psi||_q(1 + ||W_T||_1)E_{M_T,p}(f), \tag{5.8}$$

where $E_{A,p}(f)$ is the best approximation (in the metric of L^p) of function f by entire analytic functions of exponential type A (or by polynomials of degree A in d.t. case).

The assumption $f \in \mathbf{\Sigma}_p(\beta)$ implies $E_{A,p}(f) \le CA^{-\beta}$ (see, e.g., Nikol'skii [38], Sec. 8.10). Hence in view of (5.8)

$$S_T = O\left(T^{1/2}M_T^{-\beta}\right) = O\left(T^{1/2-\gamma\beta}\right) \to 0 \tag{5.9}$$

as $T \to \infty$ because by assumption $\gamma > \frac{1}{2\beta}$.
A combination of (5.6), (5.7) and (5.9) yields (5.5). This completes the proof of Lemma 5.2. □

Lemma 5.3. *Let $f \in \mathbf{\Sigma}_p(\beta)$, and let Let $\widehat{f}_T(\lambda)$ be as in (4.4) with kernel $W_T(\lambda)$ satisfying Assumptions 4.1 and 4.2 with $\frac{1}{2\beta} < \gamma < \frac{\delta}{\delta+1}$, $\delta > 0$, then*

$$T^{1/2}||\widehat{f}_T - f||_2^{1+\delta} = o_P\,(1) \qquad \text{as} \quad T \to \infty.$$

Proof. Along the lines of the proof of Theorem 4 in Ginovyan [13] it can be shown that

$$E|\widehat{f}_T(\lambda) - f(\lambda)|^2 = O\left(M_T T^{-1}\right) + O\left(M_T^{-2\beta}\right) \tag{5.10}$$

uniformly in λ. Using Fubini's theorem from (5.10) we have

$$||\widehat{f}_T - f||_2 = O_P\left(M_T^{1/2}T^{-1/2}\right) + O_P\left(M_T^{-\beta}\right). \tag{5.11}$$

Hence taking into account that $M_T = O\left(T^\gamma\right)$ from (5.11) we obtain

$$T^{1/2}||\widehat{f}_T - f||_2^{1+\delta} = O_P\left(T^{\frac{1}{2}+\left(\frac{\gamma}{2}-\frac{1}{2}\right)(1+\delta)}\right) + O_P\left(T^{\frac{1}{2}-\gamma\beta(1+\delta)}\right). \tag{5.12}$$

The assumptions imply $\frac{1}{2} + \left(\frac{\gamma}{2} - \frac{1}{2}\right)(1+\delta) < 0$ and $\frac{1}{2} - \gamma\beta(1+\delta) < 0$. Hence both terms in (5.12) are $o_P(1)$ as $T \to \infty$, and the result follows. □

Proof of Theorem 4.1. Since in this case $\Phi'(f) = g$, it follows from (3.2), (4.1) and (4.2) that

$$T^{1/2}[\widehat{\Phi}_T - \Phi(f)] = T^{1/2}\int_{\mathbb{Q}}[I_T(\lambda) - f(\lambda)]\,g(\lambda)\,d\lambda = \Delta_T(fg). \tag{5.13}$$

So, the assertion (a) of the theorem follows from (5.13) and Remark 3.8, while the assertion (b) follows from Lemma 5.1. □

Proof of Theorem 4.6. It follows from (3.4), (3.5) and (4.5) that

$$\left|\Phi(\widehat{f}_T) - \Phi(f) - \int_{\mathbb{Q}}\Phi'(f;\lambda)(\widehat{f}_T(\lambda) - f(\lambda))\,d\lambda\right|$$

$$\le ||\widehat{f}_T - f||\sup_{0\le\theta\le1}||\Phi'(f + \theta(\widehat{f}_T - f)) - \Phi'(f)|| \le C||\widehat{f}_T - f||^{1+\delta}.$$

Therefore

$$T^{1/2}[\Phi(\widehat{f}_T) - \Phi(f)] = T^{1/2} \int_{\mathbb{Q}} \Phi'(f;\lambda) \left[\widehat{f}_T(\lambda) - f(\lambda)\right] d\lambda + O_P\left(T^{1/2}||\widehat{f}_T - f||^{1+\delta}\right). \quad (5.14)$$

Using the arguments of the proof of Lemma 5.2 with $\psi(\lambda) = \Phi'(f;\lambda)$, we conclude that as $T \to \infty$

$$\int_{\mathbb{Q}} \Phi'(f;\lambda) \left[\widehat{f}_T(\lambda) - f(\lambda)\right] d\lambda = \int_{\mathbb{Q}} \Phi'(f;\lambda) \left[I_T(\lambda) - f(\lambda)\right] d\lambda + o_P(T^{-1/2}).$$

Hence, by Lemma 5.3 and (5.14)

$$T^{1/2}[\Phi(\widehat{f}_T) - \Phi(f)] = T^{1/2} \int_{\mathbb{Q}} \Phi'(f;\lambda) \left[I_T(\lambda) - f(\lambda)\right] d\lambda + o_P(1). \quad (5.15)$$

By (3.2) and (5.15) we have

$$T^{1/2}[\Phi(\widehat{f}_T) - \Phi(f)] - \Delta_T(f\Phi'(f)) = o_P(1) \quad \text{as} \quad T \to \infty. \quad (5.16)$$

Comparing (3.8) and (5.16), we conclude that $\Phi(\widehat{f}_T)$ is an H-regular and H-asymptotically efficient estimator for $\Phi(f)$ with asymptotic variance $4\pi||f\Phi'(f)||_2^2$. The assertion (b) of the theorem follows from (5.16) and Lemma 5.1. □

Proof of Theorem 4.9. The corresponding upper bounds have been obtained in Ginovyan [13] and [15]. So, we need only to establish the corresponding lower bounds for the risk Δ_T^2, which are collected in the next proposition.

Proposition 5.4. *Under the conditions of Theorem 4.9:*

(a) *If $p \ge 2$ and $\beta > 1/p$, then $\Delta_T^2 \ge c\,T^{-\frac{2p\beta}{p+2p\beta-2}}$;*
(b) *If either $p \ge 2$ and $\beta \le 1/p$ or $1 \le p \le 2$ and $\beta < 1/2$, then $\Delta_T^2 \ge c\,T^{-2\beta}$;*
(c) *If $1 \le p \le 2$ and $\beta \ge 1/2$, then $\Delta_T^2 \ge c\,T^{-1}$.*

To prove Proposition 5.4, we use Stein-Levit method (see, e.g., Ibragimov and Has'minskii [26] and [25], Ch.6). As an estimator of the linear functional $\Phi(f)$ we take the statistic $\widehat{\Phi}_{T,A}$:

$$\widehat{\Phi}_{T,A} = \int_{\mathbb{Q}} I_T(\lambda) g_A(\lambda) d\lambda, \quad (5.17)$$

where $A = A(T) \le T$, $A(T) \to \infty$ as $T \to \infty$, $I_T(\lambda)$ is the periodogram given by (3.3), and $g_A(\lambda)$ is the Dirichlet singular integral corresponding to the function $g(\lambda)$ defined by

$$g_A(\lambda) := \begin{cases} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin A(\lambda - x)}{\sin(\lambda - x)} g(x) dx, & \text{for } \lambda \in [-\pi, \pi], \\ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin A(\lambda - x)}{\lambda - x} g(x) dx, & \text{for } \lambda \in \mathbb{R}. \end{cases} \quad (5.18)$$

Note that $g_A(\lambda)$ is an entire analytic function of exponential type A (or a trigonometric polynomial of degree A).

We set $\mathbf{\Sigma}'_p(\beta) = \{f \in \mathbf{\Sigma}_p(\beta);\ f(\lambda) \ge c > 0\}$, and

$$h(\lambda) = h_A(\lambda) := \frac{g_A(\lambda)}{\sqrt{T}||fg_A||_2^2}. \tag{5.19}$$

Observe that if $f(\lambda)$ is some spectral density function from the class $\mathbf{\Sigma}'_p(\beta)$ such that $f(\lambda)h_A(\lambda) \in \mathbf{\Sigma}_p(\beta)$, then for sufficiently large values of T the function

$$f_{T,h}(\lambda) := f(\lambda)\left(1 + h_A(\lambda)\right), \quad A = A(T)$$

will be a spectral density from the class $\mathbf{\Sigma}'_p(\beta)$.

Let $\mathbb{P}_{T,\theta}$ be the distribution of $\mathbf{X}_T = \{X(u),\ 0 \le u \le T$ (or $u = \overline{1,T}$ in the d.t. case)$\}$ with spectral density $\theta(\lambda)$. By Theorem 3.3 the family of Gaussian distributions $\{\mathbb{P}_{T,\theta},\ \theta \in \mathbf{\Sigma}'_p(\beta)\}$ is LAN at a point $f \in \mathbf{\Sigma}'_p(\beta)$ in the direction of the space L^2. Therefore, we can apply Theorem 3.9 for loss function $w(x) = x^2$, to obtain

$$\sup_{\theta \in \mathbf{\Sigma}_p(\beta)} \inf_{\widehat{\Phi}_T} \mathbb{E}_\theta |\widehat{\Phi}_T - \Phi(\theta)|^2 \ge \frac{c}{T} \int_{\mathbb{Q}} f^2(\lambda)\, g_A^2(\lambda)\, d\lambda, \tag{5.20}$$

where $\widehat{\Phi}_T$ is an arbitrary estimator of functional $\Phi(\theta)$, constructed on the basis of $\mathbf{X}_T$ and c is some positive constant.

So, to complete the proof of Proposition 5.4 we need to choose $A = A(T)$ to satisfy:

1) the function $f(\lambda)\, h_A(\lambda)$ belongs to the class $\mathbf{\Sigma}_p(\beta)$;
2) the right-hand side of inequality (5.20) has the form T^{-a}, where the number a is specified by proposition.

We outline only the proof of assertion (a), the assertions (b) and (c) can be proved similarly. Assume that $f(\lambda) \in \mathbf{\Sigma}_p(\beta)$, where $p \ge 2$ and $\beta > 1/p$. We show that $f(\lambda)h(\lambda) \in \mathbf{\Sigma}_p(\beta)$, where $h(\lambda) = h_A(\lambda)$ is as in (5.19). Let $\beta = \alpha + r$, where $r \in \mathbb{N}_0$ and $0 < \alpha \le 1$. Applying Leibnitz formula to compute the derivative $(fh_A)^{(r)}$, we find

$$\begin{aligned} J_r : &= \left\|(fh_A)^{(r)}(\cdot + \delta) - (fh_A)^{(r)}(\cdot)\right\|_p \\ &\le C \sum_{k=0}^{r} \left\|f^{(k)}(\cdot + \delta) h_A^{(r-k)}(\cdot + \delta) - f^{(k)}(\cdot) h_A^{(r-k)}(\cdot)\right\|_p . \end{aligned} \tag{5.21}$$

First consider the case where $r \ge 1$. Applying Hölder and Minkowski inequalities from (5.21) we obtain

$$\begin{aligned} J_r \quad \le \quad & C\, ||f||_\infty \left\|h_A^{(r)}(\cdot + \delta) - h_A^{(r)}(\cdot)\right\|_p + ||h_A^{(r)}||_\infty \, \|f(\cdot + \delta) - f(\cdot)\|_p \\ & + C \sum_{k=1}^{r} ||f^{(k)}||_p \left\|h_A^{(r-k)}(\cdot + \delta) - h_A^{(r-k)}(\cdot)\right\|_\infty \\ & + C \sum_{k=1}^{r} ||h_A^{(r-k)}||_\infty \left\|f^{(k)}(\cdot + \delta) - f^{(k)}(\cdot)\right\|_p . \end{aligned} \tag{5.22}$$

Since $f(\lambda) \in \mathbf{\Sigma}_p(\beta), \quad \beta = \alpha + r$, we have (see, e. g., Nikol'skii [38], Sec. 8.10)

$$\|f^{(k)}\|_p \le C < \infty, \qquad k = 1, 2, \ldots, r. \tag{5.23}$$

Because $r \ge 1$, we have $\beta > 1/p$, and hence by a Hardy-Littlewood embedding theorem for Hölder classes (see, e. g., Nikol'skii [38], Sec. 6.3) we obtain

$$\|f\|_\infty \le C < \infty. \tag{5.24}$$

The function $h(\lambda) = h_A(\lambda)$ is an entire function of exponential type A (or a trigonometric polynomial of degree A), hence by Bernstein inequality (see, e. g., Butzer and Nessel [6], Sec. 3.5)

$$\|h_A^{(k)}\|_s \le 2^k A^k \|h_A\|_s, \qquad 1 \le s \le \infty, \tag{5.25}$$

and the inequality (see Lemma 3 in Ibragimov and Has'minskii [26])

$$\|h_A\|_s \le C\, A^{1/t-1/s} \|h_A\|_t, \qquad t < s \le \infty, \tag{5.26}$$

we have

$$\|h_A^{(k)}\|_\infty \le 2^k A^k \|h_A\|_\infty \le 2^k A^{k+1/q} \|h_A\|_q. \tag{5.27}$$

From the inequalities (5.25) - (5.27) for $q_1 > q$ and $k \le r$, we obtain

$$\begin{aligned} \left\| h_A^{(k)}(\cdot + \delta) - h_A^{(k)}(\cdot) \right\|_{q_1} &= \left\| \int_x^{x+\delta} h_A^{(k+1)}(y)\, dy \right\|_{q_1} \\ &\le \min\left(\delta \left\| h_A^{(k+1)} \right\|_{q_1}, \, 2 \left\| h_A^{(k)} \right\|_{q_1} \right) \\ &\le \min\left(\delta A^{k+1+1/q-1/q_1} \|h_A\|_q, \, A^{k+1/q-1/q_1} \|h_A\|_q \right). \end{aligned} \tag{5.28}$$

Therefore, for $q_1 > q$ and $k \le r$

$$\left\| h_A^{(k)}(\cdot + \delta) - h_A^{(k)}(\cdot) \right\|_{q_1} \le C \cdot \delta^\alpha A^{\beta+1/q-1/p} \|h_A\|_q. \tag{5.29}$$

In view of (5.22) – (5.24), (5.27) and (5.29), we find for $r \ge 1$

$$J_r = \left\| (fh_A)^{(r)}(\cdot + \delta) - (fh_A)^{(r)}(\cdot) \right\|_p \le C \cdot \delta^\alpha A^{\beta+1/q-1/p} \|h_A\|_q. \tag{5.30}$$

Similarly, for $r = 0$ we obtain

$$\begin{aligned} J_0 : \; &= \; \|(fh_A)(\cdot + \delta) - (fh_A)(\cdot)\|_p \\ &\le \; C \cdot \{ \|h_A(\cdot + \delta) - h_A(\cdot)\|_p + \delta^\alpha \|h_A\|_\infty \} \\ &\le \; C \cdot \delta^\alpha A^{\beta+1/q-1/p} \|h_A\|_q. \end{aligned} \tag{5.31}$$

It follows from (5.19), (5.30) and (5.31) that for all $r \in \mathbb{N}_0, \quad p \ge 2$ and $\beta > 1/p$

$$J_r := \left\| (fh_A)^{(r)}(\cdot + \delta) - (fh_A)^{(r)}(\cdot) \right\|_p \le C \cdot M_{A,T} \delta^\alpha, \tag{5.32}$$

where

$$M_{A,T} := T^{-1/2} \cdot A^{\beta+1/q-1/p} \|fg_A\|_2^{-2}. \tag{5.33}$$

Therefore, the assertion $f(\lambda)h_A(\lambda) \in \mathbf{\Sigma}_p(\beta)$ will be fulfilled, if we can choose $A = A(T)$ to make $M_{A,T}$ given by (5.33), as small as needed. We set

$$g_A(\lambda) := \begin{cases} A^{-1/p} \cdot \frac{\sin(A\lambda)}{\sin\lambda}, & \text{for } \lambda \in [-\pi, \pi], \\ \\ A^{-1/p} \cdot \frac{\sin(A\lambda)}{\lambda}, & \text{for } \lambda \in \mathbb{R}. \end{cases} \tag{5.34}$$

Since $\int_{\mathbb{Q}} g_A^2(\lambda)\, d\lambda = \pi A^{1-2/p}$, and by assumption $f(\lambda) \geq c > 0$, we have

$$\int_{\mathbb{Q}} g_A^2(\lambda) f^2(\lambda)\, d\lambda \geq \pi c \cdot A^{1-2/p}. \tag{5.35}$$

Choosing $A = T^{p/(p-2+2p\beta)}$, and taking into account that $\frac{1}{q} = 1 - \frac{1}{p}$, from (5.33) and (5.35) we find

$$M_{A,T} \leq C \cdot \frac{T^{-1/2} \cdot A^{\beta+1/q-1/p}}{A^{1-2/p}} = T^{\frac{2-p}{2(2p\beta+p-2)}}. \tag{5.36}$$

By assumption $p \geq 2$, so from (5.32) and (5.36) we conclude $f(\lambda)h_A(\lambda) \in \Sigma_p(\beta)$. Finally, for $A = T^{p/(p-2+2p\beta)}$, from (5.20) and (5.35) we find

$$\Delta_T^2 \geq \frac{c}{T} \int_{\mathbb{Q}} g_A^2(\lambda) f^2(\lambda)\, d\lambda \geq c \cdot T^{-1} A^{1-2/p} \geq T^{-\frac{2p\beta}{2p\beta+p-2}}.$$

This completes the proof of the assertion (a). Theorem 4.9 is proved. □

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