[Journal of Stochastic Analysis](https://digitalcommons.lsu.edu/josa)

[Volume 4](https://digitalcommons.lsu.edu/josa/vol4) | [Number 1](https://digitalcommons.lsu.edu/josa/vol4/iss1) [Article 3](https://digitalcommons.lsu.edu/josa/vol4/iss1/3)

March 2023

Random Variables with Overlapping Number and Weyl Algebras I

Ruma Dutta
Missouri State University, Springfield, MO 65897, U.S.A., rd5e@missouristate.edu

Gabriela Popa
Ohio University, Zanesville, OH 43701, U.S.A., popag@ohio.edu

Aurel Stan
The Ohio State University at Marion, Marion, OH 43302 USA, stan.7@osu.edu

Recommended Citation

Dutta, Ruma; Popa, Gabriela; and Stan, Aurel (2023) "Random Variables with Overlapping Number and Weyl Algebras I," *Journal of Stochastic Analysis*: Vol. 4: No. 1, Article 3.
DOI: 10.31390/josa.4.1.03
Available at: https://digitalcommons.lsu.edu/josa/vol4/iss1/3

Journal of Stochastic Analysis
Vol. 4, No. 1 (2023) Article 3 (17 pages)
DOI: 10.31390/josa.4.1.03

RANDOM VARIABLES WITH OVERLAPPING NUMBER AND WEYL ALGEBRAS I

RUMA DUTTA, GABRIELA POPA, AND AUREL I. STAN*

Abstract. For any random variable X, having finite moments of all orders, and any polynomial function f, if N denotes the number operator of X, then $f(N)$ can be written uniquely as an infinite series of terms of the form $A_k(X)D^k$, where A_k is a polynomial of degree at most k and D denotes the differentiation operator. We study the random variables for which this infinite series, called the position-momentum decomposition of X, is a finite sum meaning that after a while all the position coefficients, $A_k(X)$, vanish. Thus, $f(N)$ belongs to the Weyl algebra of X. A simple method, for recovering the probability distribution of X, from the given finite sum position-momentum decomposition of $f(N)$ is presented first. We apply this method to the case when f is linear and $f(N)$ is quadratic in D, recovering the Gaussian and Gamma distributions, and their Szegö-Jacobi parameters.

1. Introduction

Given a random variable, X, having finite moments of all orders, using the Gram-Schmidt orthogonalization procedure, we can define its sequence of orthogonal polynomials $\{f_n(X)\}_{n\geq 0}$ with leading coefficient 1. The terms of this sequence satisfy the classic three-term recursive relation giving rise to the two sequences of the so-called Szegö-Jacobi parameters. There are some classic tools for finding these parameters given the probability distribution of X, and backwards for recovering the probability distribution of X given the Szegö-Jacobi parameters. These methods involve, among others, the use of a generating function, continued fractions, and Cauchy-Stieltjes transform, see [7], [9], and [18]. Another important tool in solving these problems is the method of multiplicative renormalization, see [4], [5], and [10].

In the last twenty years, using an operatorial approach that emphasizes more the quantum operators: creation, preservation, and annihilation or the semi-quantum operators: semi-creation and semi-annihilation, and the commutators between any two of these operators, a new method for recovering the probability distribution of X from its Szegö-Jacobi parameters has been developed, see [17] and [11] for the one-dimensional case, and [12] and [14] for the multi-dimensional case. The commutator method was extended to the q-commutator method in [8].

Received 2023-2-9; Accepted 2023-2-25; Communicated by Hui-Hsiung Kuo.
2020 *Mathematics Subject Classification.* Primary 62P05; Secondary 05E35.
Key words and phrases. Quantum operators, number operator, Szegö-Jacobi parameters, position-momentum decomposition.
* Corresponding author.

Unfortunately, the commutator method had been successful mostly in recovering the Meixner class, since for the Meixner class, the vector space spanned by the quantum operators has a nice Lie algebra structure, which the other classic random variables do not possess. Few years ago, a position-momentum decomposition of the quantum operators has been introduced in [15] and it has been observed that knowing the quantum operators is equivalent with knowing the number operator, N. The position-momentum decomposition of the number operator contains the complete information about the quantum operators. In this decomposition, every linear operator, from the space of polynomial random variables in X to itself, is written as an infinite sum of compositions of powers of the differentiation operator $D = \partial/\partial x$ (interpreted as the momentum operator) and powers of the multiplication operator by X (interpreted as the position operator), with the position factors being placed to the left of the momentum factors. This infinite sum decomposition extends naturally the Weyl algebra which is made of finite sums of the same terms.

In this paper, we show first a direct method of recovering the probability distribution of a random variable in the case when there exists a polynomial function, $f(N)$, of the number operator, N, such that $f(N)$ belongs to the Weyl algebra, that means, its position-momentum decomposition contains only finitely many non-zero terms. This method recovers the probability distribution of X directly, without the use of a transform (like Cauchy-Stieltjes, Laplace, or Fourier transforms) that needs to be inverted in the end. We apply this method to the class of Gaussian and Gamma distributions, in the paper, and to the beta distributions is a following paper.

The paper is divided as follows. In section 2, we give a brief background of the Szegö-Jacobi parameters, quantum operators, and position-momentum decomposition. In section 3, we present a theoretical method of recovering the probability distribution of a random variable X, when the position-momentum decomposition of a function, $f(N)$, of the number operator, has only finitely-many non-zero terms (that means $f(N)$ belong to the Weyl algebra). Finally, we apply this theoretical method to the concrete case when f is a linear function, and the position-momentum decomposition of $f(N)$ is quadratic in D, recovering the Gaussian and Gamma distributions, whose orthogonal polynomials are the Hermite and Laguerre polynomials, respectively. We also show how this particular form of $f(N)$ is connected to the Szegö-Jacobi parameters of these classes of random variables.

2. Background

Let X be a real valued random variable, defined on a probability space (Ω, $\mathcal{F}$, P), and having finite moments of all orders, that means:

$$\begin{aligned} E\left[|X|^r\right] &:= \int_\Omega |X(\omega)|^r dP(\omega) \\ &< \infty, \end{aligned}$$

for all $r > 0$. We define the space F of all polynomial random variables in X as:

$$F := \cup_{n=0}^{\infty} F_n,$$

where, for all non-negative integers n, F_n denotes the space of all polynomial random variables of degree at most n in X, defined as:

$$F_n \quad := \quad \left\{a_n X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0 \mid \forall 1 \leq i \leq n, a_i \in \mathbb{C}\right\}.$$

It is clear that F is a vector space, and for all $n \in \mathbb{N} \cup \{0\}$, F_n is a vector subspace of F. Moreover, because X has finite moments of all orders, we have:

$$\mathbb{C} \equiv F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F \subseteq L^2(\Omega, \mathcal{F}, P).$$

Since each space F_n, for $n \in \mathbb{N} \cup \{0\}$, is finitely generated, being spanned by the monomial random variables 1, X, X^2, ..., X^n, F_n is a closed subspace of $L^2(\Omega,$ $\mathcal{F}$, $P)$. Thus, we can orthogonalize the spaces $\{F_n\}_{n\geq 0}$, and define, for all $n \geq 0$, the space:

$$G_n \quad := \quad F_n \ominus F_{n-1},$$

where "$F_n \ominus F_{n-1}$" denotes the orthogonal complement of F_{n-1} into F_n, and for $n = 0$, $F_{n-1} = F_{-1} := \{0\}$ is the null space (that means, $G_0 = F_0 \equiv \mathbb{C}$, which is called the *vacuum space*). For each non-negative integer n, we call the space G_n, the *$n-th$ homogenous chaos space generated by* X, and each polynomial random variable $f(X)$, from G_n, a *homogenous polynomial random variable of degree* n. We also define the Hilbert space:

$$H \quad = \quad G_0 \oplus G_1 \oplus G_2 \oplus \cdots,$$

and call it the *homogenous chaos space generated by* X.

We identify the random variable X with the multiplication operator $X : F \to F$, that maps:

$$f(X) \quad \mapsto \quad Xf(X),$$

for all polynomial random variables $f(X)$. In this way, the space of all polynomial random variables in X, F, can be identified with the unital algebra generated by the multiplication operator X.

The following lemma is well known, see [3] and [1]:

Lemma 2.1. *For all $n \in \mathbb{N}$, if we apply the multiplication operator X to the n-th homogenous chaos space G_n, generated by X, then we have:*

$$XG_n \quad \perp \quad G_k,$$

for all $k \in \mathbb{N} \cup \{0\} \setminus \{n-1,\, n,\, n+1\}$, where "$\perp$" means "orthogonal to".

This lemma implies that if $f(X) \in G_n$, for some $n \in \mathbb{N} \cup \{0\}$, then there exist and are unique three polynomial random variables $f^+(X) \in G_{n+1}$, $f^0(X) \in G_n$, and $f^-(X) \in G_{n-1}$, such that:

$$f(X) \quad = \quad f^+(X) + f^0(X) + f^-(X).$$

We define the operators:

- $$D_n^+ : G_n \to G_{n+1},$$
  $$D_n^+ f(X) \quad = \quad f^+(X),$$

and call it the *creation operator* since it increases the degree of a homogenous polynomial random variable by 1 unit.

- $$D_n^0 : G_n \to G_n,$$
$$D_n^0 f(X) = f^0(X),$$
and call it the *preservation operator* since it preserves the degree of a homogenous polynomial random variable.

- $$D_n^- : G_n \to G_{n-1},$$
$$D_n^- f(X) = f^-(X),$$
and call it the *annihilation operator* since it decreases the degree of a homogenous polynomial random variable by 1 unit.

So far, the creation, preservation, and annihilation operators have been defined separately on each homogenous chaos space G_n, for $n \geq 0$. We can easily extend their definitions to the space F of all polynomial random variables in X, since each polynomial random variable $f(X)$ can be written uniquely as an orthogonal sum:

$$f(X) = f_0(X) \oplus f_1(X) \oplus f_2(X) \oplus \cdots ,$$

where for each $n \geq 0$, $f_n(X) \in G_n$, and only finitely many of the terms $f_0(X)$, $f_1(X)$, $f_2(X)$, $\ldots$, of this orthogonal sum, are different from zero. We define:

- The creation operator $a^+ : F \to F$ by:
$$a^+ f(X) = D_0^+ f_0(X) + D_1^+ f_1(X) + D_2^+ f_2(X) + \cdots .$$
- The preservation operator $a^0 : F \to F$ by:
$$a^0 f(X) = D_0^0 f_0(X) + D_1^0 f_1(X) + D_2^0 f_2(X) + \cdots .$$
- The annihilation operator $a^- : F \to F$ by:
$$a^- f(X) = D_0^- f_0(X) + D_1^- f_1(X) + D_2^- f_2(X) + \cdots .$$

The operators a^+, a^0, and a^- are called the *quantum operators of* X. Lemma 2.1 can now be restated as:

Theorem 2.2. *As a multiplication operator, X can be written as the sum of its three quantum operators, that means,*

$$X = a^+ + a^0 + a^-. \tag{2.1}$$

In this equation, the domain of X, a^+, a^0, and a^- is understood to be the space F of all polynomial random variables in X.

Formula (2.1) is called the *quantum decomposition* of X. It must also be mentioned that the creation operator a^+ is the polynomial dual of the annihilation operator a^-, while the preservation operator a^0 is polynomially self-dual. That means, for all $f(X)$ and $g(X)$ polynomial random variables, we have:

$$\langle a^+ f(X), g(X) \rangle = \langle f(X), a^- g(X) \rangle$$

and

$$\langle a^0 f(X), g(X)\rangle = \langle f(X), a^0 g(X)\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product of the space $L^2(\Omega, \mathcal{F}, P)$. Moreover, since for all $n \geq 0$, we have:

$$F_n = F_{n-1} + \mathbb{C}X^n,$$

the algebraic codimension of F_{n-1} into F_n is at most 1. Thus, the dimension of the orthogonal complement, G_n, of F_{n-1} into F_n, is at most 1. Therefore, if G_n is not the null space, then the dimension of G_n is 1, and so, there exists a unique polynomial f_n, of degree n, with the leading coefficient 1, such that $f_n(X) \in G_n$. We call f_n the *monic orthogonal polynomial of degree n*. It is easy to see that, if the support of the probability measure μ of X is a finite set of cardinality k, then:

$$F_0 \underset{\neq}{\subset} F_1 \underset{\neq}{\subset} \cdots \underset{\neq}{\subset} F_{k-1} = F_k = F_{k+1} = \cdots = F = L^2(\Omega, \sigma(X), P),$$

where $\sigma(X)$ denotes the smallest sub-sigma algebra of $\mathcal{F}$ with respect to which X is measurable. Therefore, in this case, for all $0 \leq n \leq k-1$, G_n has dimension 1, while for all $k \geq n$, G_n is the null space.

If the probability distribution, μ, of X, has an infinite support, then for all $n \geq 0$, G_n has dimension 1.

Since $Xf_n(X) \in G_{n+1} \oplus G_n \oplus G_{n-1}$ and $Xf_n(X)$ has the leading coefficient (of X^{n+1}) equal to 1, there must exist α_n and ω_n real numbers, such that:

$$Xf_n(X) = f_{n+1}(X) + \alpha_n f_n(X) + \omega_n f_{n-1}(X). \tag{2.2}$$

For $n = 0$, since $f_{n-1}(X) = f_{-1}(X) = 0$ (the zero polynomial), the number ω_0 can be chosen arbitrarily. The other numbers: $\{\alpha\}_{n\geq 0}$ and $\{\omega_n\}_{n\geq 1}$, in the case when the probability distribution, μ, of X has an infinite support, are uniquely defined and called the *Szegö-Jacobi parameters* of X. It is known, that for all $n \geq 1$, the square of the L^2–norm of $f_n(X)$ is:

$$E\left[f_n^2(X)\right] = \omega_1 \cdot \omega_2 \cdots \omega_n,$$

from which is follows inductively that if the probability distribution, μ, of X has an infinite support, then $\forall n \geq 1$, $\omega_n > 0$, while in the case when μ has a finite support of cardinality k, then for all $1 \leq n \leq k-1$, $\omega_n > 0$, and $\omega_k = 0$.

Comparing the quantum decomposition of X, (2.1), and formula (2.2), we can see that for all $n \geq 0$, we have:

$$a^+ f_n(X) = f_{n+1}(X),$$

$$a^0 f_n(X) = \alpha_n f_n(X),$$

and

$$a^- f_n(X) = \omega_n f_{n-1}(X).$$

The *number operator* is the linear map $N : F \to F$, such that:

$$Nf_n(X) := nf_n(X),$$

for all monic orthogonal polynomial random variables $f_n(X)$, with $n \geq 0$. That means, if $f(X) \in F$, then there exist and are unique c_0, c_1, c_2, $\ldots$ complex numbers, with only finitely many of them being different from 0, such that:

$$f(X) = \sum_{n=0}^{\infty} c_n f_n(X).$$

We define:

$$Nf(X) := \sum_{n=0}^{\infty} n c_n f_n(X).$$

The number operator is polynomially self-dual and satisfies the following universal commutation relationships, see Theorem 3.1, from [15]:

$$\left[a^-, N\right] = a^-, \tag{2.3}$$

$$\left[N, a^0\right] = 0, \tag{2.4}$$

and

$$\left[N, a^+\right] = a^+. \tag{2.5}$$

It follows from (2.1), (2.3), (2.4), and (2.5), that:

$$[N, X] = a^+ - a^-.$$

We close this section by reviewing the position-momentum decomposition of the linear operators defined on F with values in F.

As shown in [15], every linear operator $T : F \to F$ can be written uniquely as:

$$T = \sum_{n=0}^{\infty} A_n(X) D^n,$$

where for each $n \geq 0$, $A_n(X)$ is the multiplication operator by the polynomial random variable $A_n(X)$, and $D = \partial/\partial x$ denotes the formal differentiation operator.

Moreover, if there exists an integer k, such that for all $n \geq 0$, we have $TF_n \subseteq F_{n+k}$, then for each $n \geq 0$, we can take A_n to be a polynomial of degree at most $n + k$. That means:

(1) If $T = a^+$ is the creation operator, since for all $n \geq 0$, $a^+ F_n \subseteq F_{n+1}$, we have $k = 1$, and so, for all $n \geq 0$, A_n is a polynomial of degree at most $n + 1$.
(2) If $T = a^0$ is the preservation operator, since for all $n \geq 0$, $a^0 F_n \subseteq F_n$, we have $k = 0$, and so, for all $n \geq 0$, A_n is a polynomial of degree at most n.
(3) If $T = a^-$ is the annihilation operator, since for all $n \geq 0$, $a^- F_n \subseteq F_{n-1}$, we have $k = -1$, and so, for all $n \geq 0$, A_n is a polynomial of degree at most $n - 1$.
(4) If $T = N$ is the number operator, since for all $n \geq 0$, $NF_n \subseteq F_n$, we have $k = 0$, and so, for all $n \geq 0$, A_n is a polynomial of degree at most n. Moreover, since $N1 = 0$, we have $A_0(X) = 0$. It is also not hard to prove that $A_1(X) = X - \mu$, where $\mu := E[X]$ is the expectation of X, and for all $n \geq 2$, $A_n(X)$ has degree at most $n - 1$. See [16] for more details.

Finally, the Weyl algebra is the algebra of linear operators of the form:

$$T = \sum_{n=0}^{N} A_n(X)D^n,$$

for some non-negative integer N, that means, the linear operators $T : F \to F$ whose position-momentum decomposition has only finitely many non-zero terms.

The (non-unital) algebra generated by the number operator is:

$$\mathcal{A}_N := \left\{ \sum_{k=1}^{n} c_k N^k \mid n \in \mathbb{N}, \forall 1 \leq k \leq n, c_k \in \mathbb{R} \right\}.$$

3. Random Variables for Which the Number Operator Algebra and Weyl Algebra Overlaps

Let us suppose now that X is a random variable, having finite moments of all orders, such that the intersection of the (non-unital) algebra generated by the number operator and Weyl algebra is not empty. That means, there exist two natural numbers p and q, real constants c_1, c_2, $\dots$, c_p, and polynomials A_1, A_2, $\dots$, A_q, such that:

$$f(N) = A_1(X)D + A_2(X)D^2 + \cdots + A_q(X)D^q, \tag{3.1}$$

where:

$$\begin{aligned} f(N) &:= c_1N + c_2N^2 + \cdots + c_pN^p \\ &= g(N)N, \end{aligned}$$

with

$$g(N) := c_1 + c_2N + \cdots + c_pN^{p-1}.$$

Moreover, for all $i \in \{1, 2, \dots, q\}$, the degree of A_i satisfies $deg(A_i) \leq i$, since for all non-negative integers n, $f(N)$ maps F_n into F_n.

In formula (3.1), $A_0(X) = 0$ due to the fact that $f(N)1 = g(N)N1 = g(N)0 = 0$.

We have the following proposition:

Proposition 3.1. *For any two operators S and T, we have:*

1. *If S and T are both self-adjoint or both anti-self-adjoint, then their commutator:*

$$[S, T] := ST - TS$$

is anti-self-adjoint.

2. *If S is anti-self-adjoint and T is self-adjoint, then their commutator, $[S, T]$, is self-adjoint.*

Proof. If S and T are both self-adjoint operators, then:

$$\begin{aligned} [S,T]^* &= (ST-TS)^* \\ &= T^*S^* - S^*T^* \\ &= TS - ST \\ &= [T,S] \\ &= -[S,T]. \end{aligned}$$

Similarly, if both S and T are anti-self-adjoint, we can prove that $[S,\, T]$ is anti-self-adjoint.

If S is anti-self-adjoint and T is self-adjoint, then:

$$\begin{aligned} [S,T]^* &= (ST-TS)^* \\ &= T^*S^* - S^*T^* \\ &= T(-S) - (-S)T \\ &= ST - TS \\ &= [S,T]. \end{aligned}$$

□

Because N is polynomially self-adjoint, for every polynomial function f with real coefficients, $f(N)$ is a polynomially self-adjoint operator. Since X is also polynomially self-adjoint, it follows, from the previous proposition, that $[f(N),\, X]$ is polynomially anti-self-adjoint. Applying again the previous proposition, $[[f(N),\, X],\, X]$ is polynomially self-adjoint. This further implies that $[[[f(N),\, X],\, X],\, X]$ is polynomially anti-self-adjoint. Applying one more time the proposition, we conclude that: $[[[[f(N),\, X],\, X],\, X],\, X]$ is polynomially self-adjoint, and so on. In general:

(1) If the number of nested commutators involved is even, then:
$$[[\cdots[f(N),X],\cdots,X],X]$$
is polynomially self-adjoint.

(2) If the number of nested commutators involved is odd, then:
$$[[\cdots[f(N),X],\cdots,X],X]$$
is polynomially anti-self-adjoint.

We introduce the following: **Notation:**

- *We say that an operator is* $+1$ *self-adjoint if it is self-adjoint.*
- *We say that an operator is* -1 *self-adjoint if it is anti-self-adjoint.*

A method of finding the probability density $\varphi(x)$

Suppose that, for a certain polynomial function f, with $f(0) = 0$, $f(N)$ belongs to the Weyl algebra. That means, there exist $n_0 \in \mathbb{N}$, A_{n_0} a polynomial of degree at most n_0, A_{n_0-1} a polynomial of degree at most $n_0 - 1$, ..., A_1 a polynomial of degree at most 1, such that:

$$f(N) = A_{n_0}(X)D^{n_0} + A_{n_0-1}(X)D^{n_0-1} + \cdots + A_1(X)D,$$

and $A_{n_0}(X) \neq 0$.

We have the following:

Claim: The number n_0 must be even.

Indeed, since $[f(N),\, X]$ is polynomially anti-self-adjoint, using Leibniz commutator rule, and the fact that for all $1 \leq k \leq n_0$, $[D^k,\, X] = kD^{k-1}$, we obtain:

$$\begin{aligned}
[f(N), X] &= \left[\sum_{k=1}^{n_0} A_k(X)D^k, X\right] \\
&= \sum_{k=1}^{n_0} \left[A_k(X)D^k, X\right] \\
&= \sum_{k=1}^{n_0} \left\{[A_k(X), X]\, D^k + A_k(X)\left[D^k, X\right]\right\}.
\end{aligned}$$

Since for every polynomial A, $A(X)$ commutes with X, we have:

$$\begin{aligned}
[f(N), X] &= \sum_{k=1}^{n_0} \left\{0D^k + A_k(X)kD^{k-1}\right\} \\
&= \sum_{k=1}^{n_0} kA_k(X)D^{k-1} \\
&= \sum_{k=0}^{n_0-1} (k+1)A_{k+1}(X)D^k
\end{aligned}$$

is polynomially anti-self-adjoint.

Using now the fact that $[[f(N),\, X],\, X]$ is polynomial self-adjoint, we obtain:

$$\begin{aligned}
[[f(N), X], X] &= \sum_{k=0}^{n_0-1} (k+1)A_{k+1}(X)\left[D^k, X\right] \\
&= \sum_{k=1}^{n_0-1} (k+1)A_{k+1}(X)kD^{k-1} \\
&= \sum_{k=0}^{n_0-2} (k+1)(k+2)A_{k+2}(X)D^k
\end{aligned}$$

is polynomially self-adjoint. And so on, we keep commuting with X, $n_0 - 1$ times, obtaining in the end that:

$$\sum_{k=0}^{n_0-(n_0-1)} (k+1)(k+2)\cdots(k+n_0-1)\, A_{k+n_0-1}(X)D^k$$

is polynomially $(-1)^{n_0-1}$ self-adjoint. That means,

$$(n_0-1)!A_{n_0-1}(X) + n_0!A_{n_0}(X)D$$

is polynomially $(-1)^{n_0-1}$ self-adjoint. Dividing by $(n_0-1)!$, we conclude that:

$$A_{n_0-1}(X) + n_0A_{n_0}(X)D$$

is polynomially $(-1)^{n_0-1}$ self-adjoint.

If we commute one more time with X, then we obtain that:

$$\begin{aligned}[A_{n_0-1}(X), X] + n_0 A_{n_0}(X)[D, X] &= 0 + n_0 A_{n_0}(X) I \\ &= n_0 A_{n_0}(X)\end{aligned}$$

is polynomially $(-1)^{n_0}$ self-adjoint. Thus, $A_{n_0}(X)$ is $(-1)^{n_0}$ polynomially self-adjoint. On the other hand, it is obvious that the multiplication operator by the real valued random variable $A_{n_0}(X)$ is polynomially self-adjoint. Since, by assumption $A_{n_0}(X) \neq 0$, we must have that k_0 is even (because the only operator that is both polynomially self-adjoint and polynomially anti-self-adjoint is the zero operator).

Theorem 3.2. *If X is a continuous random variable, as described above, of density φ, and φ is a differentiable function on the open interval (α, β) vanishing outside (α, β), such that for every polynomial function, h, we have:*

$$\lim_{x \to \alpha^+} [h(x) A_{n_0}(x) \varphi(x)] = \lim_{x \to \beta^-} [h(x) A_{n_0}(x) \varphi(x)] = 0, \tag{3.2}$$

then its density function satisfies the following differential equation:

$$n_0 A_{n_0}(x) \varphi'(x) + \left[n_0 A'_{n_0}(x) - 2A_{n_0-1}(x)\right] \varphi(x) = 0. \tag{3.3}$$

Proof. For all f and g polynomial functions, since $A_{n_0-1}(X) + n_0 A_{n_0}(X) D$ is polynomially $(-1)^{n_0-1}$ self-adjoint, that means, polynomially anti-self-adjoint (because $n_0 - 1$ is odd), we have:

$$\begin{aligned}&\langle [A_{n_0-1}(X) + n_0 A_{n_0}(X) D] f(X), g(X) \rangle \\ = \; & -\langle f(X), [A_{n_0-1}(X) + n_0 A_{n_0}(X) D] g(X) \rangle .\end{aligned}$$

This is equivalent to:

$$\begin{aligned}&\int_{\mathbb{R}} A_{n_0-1}(x) f(x) g(x) \varphi(x) dx + n_0 \int_{\mathbb{R}} A_{n_0}(x) f'(x) g(x) \varphi(x) dx \\ = \; & -\int_{\mathbb{R}} f(x) A_{n_0-1}(x) g(x) \varphi(x) dx - n_0 \int_{\mathbb{R}} f(x) A_{n_0}(x) g'(x) \varphi(x) dx.\end{aligned}$$

We can see from here that we must have:

$$\begin{aligned}&n_0 \int_{\mathbb{R}} A_{n_0}(x) [f'(x) g(x) + f(x) g'(x)] \varphi(x) dx \\ = \; & -2 \int_{\mathbb{R}} A_{n_0-1}(x) f(x) g(x) \varphi(x) dx.\end{aligned}$$

This is equivalent to, for all f and g polynomial functions:

$$n_0 \int_{\alpha}^{\beta} [f(x) g(x)]' A_{n_0}(x) \varphi(x) dx = -2 \int_{\alpha}^{\beta} A_{n_0-1}(x) f(x) g(x) \varphi(x) dx.$$

Integrating by parts in the left side, and using the assumption that, for all f and g polynomial functions, we have:

$$\lim_{x \to \alpha^+} [A_{n_0}(x) f(x) g(x) \varphi(x)] = \lim_{x \to \beta^-} [A_{n_0}(x) f(x) g(x) \varphi(x)] = 0,$$

we obtain:

$$\begin{aligned} & -n_0 \int_{\mathbb{R}} f(x)g(x) \left[A'_{n_0}(x)\varphi(x) + A_{n_0}(x)\varphi'(x)\right] dx \\ = & \quad -2 \int_{\mathbb{R}} A_{n_0-1}(x) f(x) g(x) \varphi(x) dx. \end{aligned}$$

For this to hold for all f and g polynomial functions, we must have:

$$n_0 A_{n_0}(x)\varphi'(x) + \left[n_0 A'_{n_0}(x) - 2A_{n_0-1}(x)\right] \varphi(x) \quad = \quad 0.$$

□

We will be calling the differential equation (3.3) the *generalized Pearson equation*, since, as we will see in the next section and in a following paper, when $f(N)$ is linear or quadratic in N and $n_0 = 2$, it will reduce to the classic Pearson equation producing the densities of the Gaussian, Gamma, and beta distributions.

4. The Gaussian and Gamma Cases

Let us suppose now that $f(N)$ is linear in N and $n_0 = 2$. Since $f(0) = 0$, without loss of generality, we may assume that $f(N) = N$. Then, we must have:

$$N \quad = \quad A_2(X)D^2 + A_1(X)D.$$

As it was mentioned before, we know that the first left-position coefficient of the number operator, N, of X must be:

$$A_1(X) \quad = \quad X - \mu,$$

where $\mu := E[X]$ is the expectation of X.

We also know that the second left-position coefficient of N must have the degree strictly less than 2. That means:

$$A_2(X) \quad = \quad bX + c,$$

where b and c are two real numbers. Thus, we have:

$$N \quad = \quad (bX + c)D^2 + (X - \mu)D. \tag{4.1}$$

We will first compute the Szegö-Jacobi parameters of X. We have the following:

Theorem 4.1. *A random variable X, having finite moments of all orders, has the number operator of the form:*

$$N \quad = \quad (bX + c)D^2 + (X - \mu)D,$$

where b and c are real numbers, and μ denotes the expectation of X, if and only if its Szegö-Jacobi parameters are, for all $n \geq 0$:

$$\alpha_n \quad = \quad -2bn + \mu$$

and

$$\omega_n \quad = \quad b^2 n^2 - \left(b^2 + b\mu + c\right) n.$$

Proof. Indeed, equation (4.1) holds if and only if N satisfies the following initial value double commutator problem:

$$\begin{cases} [[N,X],X] & = & 2(bX+c) \\ N1 & = & 0 \\ NX & = & X-\mu \end{cases}. \tag{4.2}$$

Let the Szegö-Jacobi parameters of X be $\{\alpha_n\}_{n\geq 0}$ and $\{\omega_n\}_{n\geq 1}$, that means, if $\{f_n(X)\}_{n\geq 0}$ denotes the sequence of monic orthogonal polynomials generated by X, then for all $n \geq 0$, we have:

$$Xf_n(X) \quad = \quad f_{n+1}(X)+\alpha_n f_n(X)+\omega_n f_{n-1}(X).$$

Since $f_{-1}(X)=0$, we can take ω_0 however we want. Let us choose $\omega_0 := 0$. We can identify the sequences $\{\alpha_n\}_{n\geq 0}$ and $\{\omega_n\}_{n\geq 0}$, with the operators $\alpha(N)$ and $\omega(N)$, defined as:

$$\alpha(N)f_n(X) \quad = \quad \alpha_n f_n(X)$$

and

$$\omega(N)f_n(X) \quad = \quad \omega_n f_n(X),$$

for all $n \geq 0$.

If a^+, a^0, and a^- denote the creation, preservation, and annihilation operators of X, then, using equation (4.2), we have:

$$\begin{aligned} 2(bX+c) &= [[N,X],X] \\ &= \left[a^+ - a^-, X\right] \\ &= \left[a^+ - a^-, a^+ + a^0 + a^-\right] \\ &= -\left[a^0, a^+\right] - 2\left[a^-, a^+\right] - \left[a^-, a^0\right]. \end{aligned}$$

Replacing X by $a^+ + a^0 + a^-$, we obtain:

$$2\left[b\left(a^+ + a^0 + a^-\right) + cI\right] \quad = \quad -\left[a^0, a^+\right] - 2\left[a^-, a^+\right] - \left[a^-, a^0\right],$$

which means:

$$2ba^+ + 2\left(ba^0 + cI\right) + 2ba^- \quad = \quad -\left[a^0, a^+\right] - 2\left[a^-, a^+\right] - \left[a^-, a^0\right]. \tag{4.3}$$

Let us observe that for all $n \geq 0$, we have:

(1) the operators $2ba^+$ and $-[a^0, a^+]$ are mapping G_n into G_{n+1},
(2) the operators $2ba^0 + 2cI$ and $-2[a^-, a^+]$ are mapping G_n into G_n,
(3) the operators $2ba^-$ and $-[a^-, a^0]$ are mapping G_n into G_{n-1}.

Thus, due to the fact that, for all $n \geq 0$, the homogenous chaos spaces G_{n+1}, G_n, and G_{n-1} are orthogonal, equation (4.3) is equivalent to the following system of three separate equations:

$$\begin{cases} 2ba^+ & = & -\left[a^0, a^+\right] \\ 2\left(ba^0 + cI\right) & = & -2\left[a^-, a^+\right] \\ 2ba^- & = & -\left[a^-, a^0\right] \end{cases}. \tag{4.4}$$

In fact, the third equation of the system (4.4) can be obtained from the first equation by taking the dual in both sides since $(a^+)^* = a^-$ and $[a^0, a^+]^* = [a^-, a^0]$.

Therefore, the above system of three equations is a system of only two equations.
Claim 1: We have:

$$\left[a^0, a^+\right] = a^+ \left[\alpha(N+1) - \alpha(N)\right]. \tag{4.5}$$

Indeed, for all $n \geq 0$, we have:

$$\begin{aligned}
\left[a^0, a^+\right] f_n(X) &= a^0 a^+ f_n(X) - a^+ a^0 f_n(X) \\
&= a^0 f_{n+1}(X) - a^+ \left(\alpha_n f_n(X)\right) \\
&= \alpha_{n+1} f_{n+1}(X) - \alpha_n a^+ f_n(X) \\
&= \alpha_{n+1} f_{n+1}(X) - \alpha_n f_{n+1}(X) \\
&= \left(\alpha_{n+1} - \alpha_n\right) a^+ f_n(X) \\
&= a^+ \left[\left(\alpha_{n+1} - \alpha_n\right) f_n(X)\right] \\
&= a^+ \left[\alpha(N+1) - \alpha(N)\right] f_n(X).
\end{aligned}$$

Claim 2: We have:

$$\left[a^-, a^+\right] = \omega(N+1) - \omega(N). \tag{4.6}$$

Indeed, for all $n \geq 1$, we have:

$$\begin{aligned}
\left[a^-, a^+\right] f_n(X) &= a^- a^+ f_n(X) - a^+ a^- f_n(X) \\
&= a^- f_{n+1}(X) - a^+ \left(\omega_n f_{n-1}(X)\right) \\
&= \omega_{n+1} f_n(X) - \omega_n a^+ f_{n-1}(X) \\
&= \omega_{n+1} f_n(X) - \omega_n f_n(X) \\
&= \left(\omega_{n+1} - \omega_n\right) f_n(X) \\
&= \left[\omega(N+1) - \omega(N)\right] f_n(X).
\end{aligned}$$

For $n = 0$, we have:

$$\begin{aligned}
\left[a^-, a^+\right] f_0(X) &= a^- a^+ f_0(X) - a^+ a^- f_0(X) \\
&= a^- f_1(X) - a^+ 0 \\
&= \omega_1 f_0(X) \\
&= \left(\omega_1 - \omega_0\right) f_0(X),
\end{aligned}$$

since we have chosen $\omega_0 = 0$.
Because, for all $n \geq 0$, we have:

$$a^0 f_n(X) = \alpha_n f_n(X),$$

we conclude that:

$$a^0 = \alpha(N).$$

From (4.5), (4.6), and the first two equations of the system (4.4) it follows that:

$$\alpha(N+1) - \alpha(N) = -2bI \tag{4.7}$$

and

$$\omega(N+1) - \omega(N) = -b\alpha(N) - cI. \tag{4.8}$$

Applying equation (4.7) to $f_0(X)$, $f_1(X)$, $\dots$, $f_{n-1}(X)$, for a fixed natural number n, we obtain:

$$\begin{aligned} \alpha_1 - \alpha_0 &= -2b \\ \alpha_2 - \alpha_1 &= -2b \\ \vdots \quad & \vdots \quad \vdots \\ \alpha_n - \alpha_{n-1} &= -2b. \end{aligned}$$

Adding up all these equations, we get:

$$\alpha_n - \alpha_0 = -2bn,$$

which means

$$\alpha_n = -2bn + \alpha_0,$$

for all $n \geq 0$. Denoting:

$$\alpha := -2b,$$

we have, for all $n \geq 0$,

$$\alpha_n = \alpha n + \alpha_0. \tag{4.9}$$

Since $\alpha_0 = E[X]$, we must have $\alpha_0 = \mu$.

Having found α_n, for $n \geq 0$, we can find ω_n, for all $n \geq 1$. We apply equation (4.8) to f_0, f_1, $\dots$, f_{n-1}, for some fixed natural number n. We have:

$$\begin{aligned} \omega_1 - \omega_0 &= -b\alpha_0 - c \\ \omega_2 - \omega_1 &= -b\alpha_1 - c \\ \vdots \quad & \vdots \quad \vdots \\ \omega_n - \omega_{n-1} &= -b\alpha_{n-1} - c. \end{aligned}$$

Summing up from $k = 0$ to $n - 1$, we obtain:

$$\omega_n - \omega_0 = -b\left(\sum_{k=0}^{n-1} \alpha_k\right) - cn$$

which (since $\omega_0 = 0$) is equivalent to:

$$\omega_n = -b\left[\sum_{k=0}^{n-1} (-2bk + \alpha_0)\right] - cn.$$

This means:

$$\omega_n = b^2(n-1)n - b\alpha_0 n - cn,$$

which is equivalent to:

$$\omega_n = \beta n^2 + (t - \beta)n, \tag{4.10}$$

where

$$\begin{aligned} \beta &:= b^2 \\ &\geq 0 \end{aligned}$$

and

$$t \quad := \quad -b\mu - c.$$

Since $\omega_1 \geq 0$, we must have $b\mu + c \leq 0$.

If $t = -b\mu - c = 0$, then $\omega_1 = 0$, and so $f_1(X) = X - \mu = 0$, almost surely, since $E[f_1^2(X)] = \omega_1$. In this case, $X = \mu$ almost surely. This case is not interesting.

Let us assume now that $t = -b\mu - c > 0$. This implies, for all $n \geq 1$, we have:

$$\begin{aligned} \omega_n &= \beta\left(n^2 - n\right) + tn \\ &> 0, \end{aligned}$$

so, the probability distribution of X has an infinite support.

Because, $\alpha = -2b$ and $\beta = b^2$, we can see that:

$$\alpha^2 \quad = \quad 4\beta. \tag{4.11}$$

We recognize from formulas (4.9), (4.10), and (4.11), that X is Meixner random variable belonging to the critical case $\alpha^2 = 4\beta$. □

Since $n_0 = 2$, $A_1(X) = X - \mu$ and $A_2(X) = bX + c$, the generalised Pearson equation (3.3),

$$n_0 A_{n_0}(x)\varphi'(x) + \left[n_0 A'_{n_0}(x) - 2A_{n_0-1}(x)\right]\varphi(x) \quad = \quad 0,$$

becomes

$$2(bx + c)\varphi'(x) + [2b - 2(x - \mu)]\,\varphi(x) \quad = \quad 0.$$

This equation is equivalent to:

$$(bx + c)\varphi'(x) + (b + \mu - x)\varphi(x) \quad = \quad 0. \tag{4.12}$$

We distinguish between two cases:

Case 1. If $b = 0$, then since $t = -b\mu - c > 0$, we have $c < 0$. Therefore, we may define the positive the number:

$$\sigma \quad := \quad \sqrt{-c}.$$

Substituting $b = 0$ and $c = -\sigma^2$ in the differential equation (4.12), we obtain:

$$-\sigma^2\varphi'(x) + (\mu - x)\varphi(x) \quad = \quad 0,$$

which means:

$$\varphi'(x) + \frac{x - \mu}{\sigma^2}\varphi(x) \quad = \quad 0.$$

Multiplying both sides of this equation by the integrating factor:

$$\exp\left(\int \frac{x - \mu}{\sigma^2}dx\right) \quad = \quad \exp\left(\frac{(x - \mu)^2}{2\sigma^2}\right),$$

and using the product rule of differentiation, we obtain:

$$\frac{d}{dx}\left[\exp\left(\frac{(x - \mu)^2}{2\sigma^2}\right)\varphi(x)\right] \quad = \quad 0.$$

Thus, we have:

$$\exp\left(\frac{(x - \mu)^2}{2\sigma^2}\right)\varphi(x) \quad = \quad k,$$

from where we find:

$$\varphi(x) \quad = \quad k\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),$$

where k is a positive constant such that $\int_{\mathbb{R}} \varphi(x)dx = 1$. We can see that $\varphi(x)$ satisfies the condition (3.2) of Theorem 3.2 on the interval $(\alpha, \beta) := (-\infty, \infty)$. In this case X is a Gaussian random variable with mean μ and variance σ^2.

Case 2. If $b \neq 0$, then equation (4.12) is equivalent to:

$$\begin{aligned} \frac{\varphi'(x)}{\varphi(x)} &= \frac{x-b-\mu}{bx+c} \\ &= \frac{1}{b} - \frac{b\mu + c + b^2}{b(bx+c)}. \end{aligned}$$

Substituting $b\mu + c = -t < 0$ in the numerator of the last fraction from the right, we obtain:

$$\frac{\varphi'(x)}{\varphi(x)} \quad = \quad \frac{1}{b} - \frac{-t+b^2}{b(bx+c)}.$$

This equation is equivalent to:

$$\frac{d}{dx}\left[\ln(\varphi(x))\right] \quad = \quad \frac{d}{dx}\left[\frac{1}{b}x + \left(\frac{t}{b^2} - 1\right)\ln\left(|bx+c|\right)\right].$$

It follows from here that:

$$\varphi(x) \quad = \quad k|bx+c|^{(t/b^2)-1}e^{(1/b)x},$$

where k is a positive constant.

Since $b \neq 0$, we have two subcases:

Subcase 2.1. If $b < 0$, then we can take $(\alpha, \beta) := (-c/b, \infty)$ to make φ integrable on (α, β), and choose k the positive constant for which $\int_{-c/b}^{\infty} \varphi(x)dx = 1$. We can see that φ satisfies the hypotheses of Theorem 3.2, since for all polynomial functions $h(x)$, we have:

$$\begin{aligned} \lim_{x \to (-c/b)^+} \left[h(x)A_2(x)\varphi(x)\right] &= \lim_{x \to (-c/b)^+} \left[h(x)(bx+c)(-bx-c)^{(t/b^2)-1}e^{(1/b)x}\right] \\ &= -\lim_{x \to (-c/b)^+} \left[h(x)(-bx-c)^{t/b^2}e^{(1/b)x}\right] \\ &= 0, \end{aligned}$$

because $t/b^2 > 0$.

We also have:

$$\begin{aligned} \lim_{x \to \infty} \left[h(x)A_2(x)\varphi(x)\right] &= -\lim_{x \to \infty} \left[h(x)(-bx-c)^{t/b^2}e^{(1/b)x}\right] \\ &= 0, \end{aligned}$$

since $1/b < 0$. In this case, we can see that X is a shifted and re-scaled gamma-distributed random variable.

Subcase 2.2. If $b > 0$, then we can take $(\alpha, \beta) := (-\infty, -c/b)$, and in a similar way we can see that φ satisfies the hypotheses of Theorem 3.2, and again X is a shifted and re-scaled gamma-distributed random variable.

References

1. Accardi, L., Kuo, H.–H., and Stan, A. I.: Characterization of probability measures through the canonically associated interacting Fock spaces, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **7** (2004), no. 4, 485–505.
2. Accardi, L., Kuo, H.–H., and Stan, A. I.: Moments and commutators of probability measures, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **10** (2007), no. 4, 591–612.
3. Accardi, L. and Nahni, M.: Interacting Fock Spaces and Orthogonal Polynomials in several variables, in: *Non-Commutativity, Infinite-Dimensionality and Probability at the Crossroads* **16** (2002), 192–205, World Scientific Publishing.
4. Asai, N., Kubo, I., and Kuo, H.–H.: Multiplicative renormalization and generating functions I, *Taiwanese J. Math.* **7** (2003), 89–101.
5. Asai, N., Kubo, I., and Kuo, H.–H.: Multiplicative renormalization and generating functions II, *Taiwanese J. Math.* **8** (2004), 593–628.
6. Asai, N., Kubo, I., and Kuo, H.–H.: Generating functions of orthogonal polynomials and Szegö–Jacobi parameters, *Prob. Math. Stat.* **23** (2003), 273–291.
7. Chihara, T. S.: *An Introduction to Orthogonal Polynomials*, Gordon & Breach, New York, 1978.
8. Drożdżewicz, K. and Matysiak, W.: Moments and q-commutators of noncommutative random vectors, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **14** (2011), no. 4, 629–45.
9. Ismail, M.E.H.: *Classical and Quantum Orthogonal Polynomials in One Variable*, Encyclopedia of Mathematics and Its Applications, vol. 98, Cambridge University Press, 2008.
10. Kubo, I., Kuo, H.–H., and Namli, S.: Interpolation of Chebyshev polynomials and interacting Fock spaces, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **9** (2006), no. 3, 361–371.
11. Popa, G. and Stan, A. I.: Gamma distributed random variables and their semi–quantum operators, in: *J. Phys.: Conf. Ser.* 563 012029 (2014), doi:10.1088/1742-6596/563/1/012029.
12. Popa, G. and Stan, A. I.: Two–dimensional 1–Meixner random vectors and their semi–quantum operators, *Commun. Stoch. Anal.* **9** (2015), no. 4, 425–455.
13. Popa, G. and Stan, A. I.: A characterization of probability measures in terms of semi-quantum operators, *Infin. Dimens. Anal. Quantum Probab. and Relat. Top.* **22** (2019), no. 2,
DOI: 10.1142/S0219025719500097.
14. Stan, A. I. and Catrina, F.: 1-Meixner random vectors, *J. Theor. Probab.* **34** (2021), no. 4, 2033–2080.
15. Stan, A. I., Popa, G., and Dutta, R.: Position-momentum decomposition of linear operators defined on algebras of polynomials, *J. Math. Phys.* **62** (2021), 012101,
https://doi.org/10.1063/5.0008155.
16. Stan, A. I., Popa, G., and Dutta, R.: A study of random variables in terms of the number operator, *J. Math. Phys.* **64** (2023), 032102, https://doi.org/10.1063/5.0124172.
17. Stan, A. I. and Whitaker, J. J.: A study of probability measures through commutators, *J. Theor. Probab.* **22** (2009), no. 1, 123–145.
18. Szegö, M.: *Orthogonal Polynomials*, Coll. Publ., **23**, Amer. Math. Soc., 1975.

Ruma Dutta: Department of Physics, Astronomy & Material Science, Missouri State University, Springfield, MO 65897, U.S.A.

Email address: `rd5e@missouristate.edu`

Gabriela Popa: Department of Physics, Ohio University, Zanesville, OH 43701, U.S.A.

Email address: `popag@ohio.edu`

Aurel I. Stan: Department of Mathematics, The Ohio State University, Marion, OH 43302, U.S.A.

Email address: `stan.7@osu.edu`