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A quadratic nonconforming vector finite element for $H(\text{curl};\Omega)\cap H(\text{div};\Omega)$ ☆

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ABSTRACT

We present a quadratic nonconforming vector finite element for problems posed on the space $H(\text{curl};\Omega)\cap H(\text{div};\Omega)$, where $\Omega\subset\mathbb{R}^2$. Generalizations to higher order and higher dimension are also discussed.

Let Ω be a domain in $\mathbb{R}^d$ for $d = 2, 3$. The space $H(\text{curl};\Omega)\cap H(\text{div};\Omega)$ of square integrable vector fields whose weak curl and weak divergence are also square integrable plays an important role in many applications, such as the cavity resonance problem and the acoustic fluid–structure interaction problem. However, a finite element vector field belonging to $H(\text{curl};\Omega)\cap H(\text{div};\Omega)$ is necessarily continuous across element boundaries and hence actually belongs to $[H^1(\Omega)]^d$. Furthermore, it is known [1,6] that under certain boundary conditions the H^1 vector fields form a closed subspace of $H(\text{curl};\Omega)\cap H(\text{div};\Omega)$. Consequently, unless the solution of the continuous problem belongs to $[H^1(\Omega)]^d$ (which is not always the case), it can never be captured by $H(\text{curl};\Omega)\cap H(\text{div};\Omega)$ conforming finite element vector fields.

On the other hand, problems involving $H(\text{curl};\Omega)\cap H(\text{div};\Omega)$ can be solved numerically by nonconforming finite element methods. For example, the source problem (deterministic problem) and the eigenproblem (cavity resonance problem) for the time-harmonic Maxwell equations in two dimensions were solved in [4,3,2,5] using the Crouzeix–Raviart [7] weakly continuous P_1 vector fields.

In this note we present a two-dimensional quadratic nonconforming vector finite element that can be used to solve problems involving $H(\text{curl};\Omega)\cap H(\text{div};\Omega)$. This vector element is fundamentally different from the quadratic element of Fortin and Soulie [8], which can be applied to the Laplace equation and the Stokes problem. We also discuss general nonconforming vector finite elements for $H(\text{curl};\Omega)\cap H(\text{div};\Omega)$ in two and three dimensions in terms of two conjectures.

First we introduce some notation. We will use boldfaced letters to represent vectors. The space of polynomials of total degree $\le k$ in d variables will be denoted by $P_k(\mathbb{R}^d)$, and the space of homogeneous harmonic polynomials of degree k in d variables will be denoted by $\mathcal{H}_k(\mathbb{R}^d)$. It is well-known [9, page 445] that the dimension of $\mathcal{H}_k(\mathbb{R}^2)$ is 2 for $k\ge 1$, while the dimension of $\mathcal{H}_k(\mathbb{R}^3)$ is $2k+1$ for $k\ge 1$. Let T be a d-dimensional simplex (triangle if $d=2$ and tetrahedron if $d=3$). The barycentric coordinates of T will be denoted by λ_i for $1\le i\le d+1$, i.e., $\lambda_j\in P_1(\mathbb{R}^d)$ and $\lambda_i(p_j)=\delta_{ij}$, where p_j is the j-th vertex of T, $\delta_{ij}=1$ if $i=j$ and $\delta_{ij}=0$ if $i\ne j$. The canonical basis vectors of $\mathbb{R}^d$ will be denoted by $\mathbf{e}_i$ for $1\le i\le d$, where the j-th component of $\mathbf{e}_i$ equals δ_{ij}. Note that the subscript i (or j) is to be interpreted as an integer modulo $d+1$, i.e., $d+k+1$ is treated the same as k.

The two-dimensional quadratic vector finite element has a triangle T as the element domain,

$$\mathcal{P}_2 = [P_2(\mathbb{R}^2)]^2 \oplus \nabla\mathcal{H}_4(\mathbb{R}^2)$$

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as the space of shape vector fields, and $\mathcal{N}_2 = \{M_\ell : 1 \le \ell \le 14\}$ as the set of degrees of freedom, where the linear functionals $M_\ell : \mathcal{P}_2 \longrightarrow \mathbb{R}$ are defined as follows.

Let e_i, $1 \le i \le 3$, be the edge of T opposite the vertex p_i. We define, for $1 \le i \le 3$ and $\boldsymbol{v} \in \mathcal{P}_2$,

$$M_{4(i-1)+1}(\boldsymbol{v}) = \int_{e_i} \lambda_{i+1}(\boldsymbol{v}\cdot\mathbf{e}_1)\mathrm{d}s, \qquad M_{4(i-1)+2}(\boldsymbol{v}) = \int_{e_i} \lambda_{i+2}(\boldsymbol{v}\cdot\mathbf{e}_1)\mathrm{d}s,$$
$$M_{4(i-1)+3}(\boldsymbol{v}) = \int_{e_i} \lambda_{i+1}(\boldsymbol{v}\cdot\mathbf{e}_2)\mathrm{d}s, \qquad M_{4(i-1)+4}(\boldsymbol{v}) = \int_{e_i} \lambda_{i+2}(\boldsymbol{v}\cdot\mathbf{e}_2)\mathrm{d}s,$$

and

$$M_{13}(\boldsymbol{v}) = \int_T \boldsymbol{v}\cdot\mathbf{e}_1\mathrm{d}x, \qquad M_{14}(\boldsymbol{v}) = \int_T \boldsymbol{v}\cdot\mathbf{e}_2\mathrm{d}x.$$

In other words, $M_{4(i-1)+k}$ for $1 \le k \le 4$ are the linear functionals that define the moments of order ≤ 1 on edge $\mathcal{E}_i$ for the two components of $\boldsymbol{v}$, and M_{13}, M_{14} are the linear functionals that define the moments of order 0 on the triangle for the two components of $\boldsymbol{v}$.

Note that the definition of $\mathcal{P}_2$ implies that for any $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathcal{P}_2$ we have

$$\nabla\times\boldsymbol{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \in P_1(\mathbb{R}^2) \quad \text{and} \quad \nabla\cdot\boldsymbol{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \in P_1(\mathbb{R}^2). \tag{1}$$

We want to establish that $(T, \mathcal{P}_2, \mathcal{N}_2)$ is a finite element. Since $\dim \mathcal{P}_2 = 14$, we only need to show that $\mathcal{P}_2$ is determined by $\mathcal{N}_2$ (or $\mathcal{P}_2$ is unisolvent under $\mathcal{N}_2$). The following lemma is useful for this purpose.

Lemma 1. *Let $\phi \in P_3(\mathbb{R}^2)$. If $\Delta\phi = 0$,*

$$\mathcal{M}_{2(i-1)+k}(\phi) := \int_{e_i} \lambda_{i+k}\phi\mathrm{d}s = 0 \quad \text{for } 1 \le i \le 3,\ 1 \le k \le 2,$$

and

$$\mathcal{M}_7(\phi) = \int_T \phi\mathrm{d}x = 0,$$

then $\phi = 0$.

Proof. Observe first that the moments $\mathcal{M}_\ell$ ($1 \le \ell \le 7$) are linearly independent as functionals on $P_3(\mathbb{R}^2)$. Indeed, we can construct seven polynomials $q_\ell \in P_3(\mathbb{R}^2)$ such that $\mathcal{M}_\ell(q_k) = \delta_{\ell k}$ as follows.

Consider the polynomial

$$q = \lambda_{i+1}\lambda_{i+2}(\alpha\lambda_{i+1} + \beta\lambda_{i+2}) + \gamma\lambda_1\lambda_2\lambda_3.$$

Clearly q vanishes on the two edges defined by $\lambda_{i+1} = 0$ and $\lambda_{i+2} = 0$. On the edge e_i defined by $\lambda_i = 0$, by choosing appropriate α and β, we can ensure either

$$\int_{e_i} \lambda_{i+1} q\mathrm{d}s = 1 \quad \text{and} \quad \int_{e_i} \lambda_{i+2} q\mathrm{d}s = 0$$

or

$$\int_{e_i} \lambda_{i+1} q\mathrm{d}s = 0 \quad \text{and} \quad \int_{e_i} \lambda_{i+2} q\mathrm{d}s = 1,$$

and by choosing an appropriate γ, that

$$\int_T q\mathrm{d}x = 0.$$

We can therefore construct $q_1, \ldots, q_6$ such that $\mathcal{M}_\ell(q_k) = \delta_{\ell k}$ for $1 \le \ell \le 7$ and $1 \le k \le 6$. Furthermore, by taking $\alpha = \beta = 0$ and choosing an appropriate γ, we can construct q_7 such that $\mathcal{M}_\ell(q_7) = \delta_{\ell 7}$ for $1 \le \ell \le 7$.

Consequently, the dimension of the space

$$\Phi = \{\phi \in P_3(\mathbb{R}^2) : \mathcal{M}_\ell(\phi) = 0 \text{ for } 1 \le \ell \le 7\}$$

is 3. In fact, Φ is spanned by the three polynomials

$$\phi_i = \lambda_i(10\lambda_i^2 - 12\lambda_i + 3) \quad 1 \le i \le 3.$$

That ϕ_i belongs to Φ can be checked directly using the standard formulas

$$\int_{e_k} \lambda_i^\ell \mathrm{d}s = \frac{|e_k|}{\ell+1} \quad \text{for } k \neq i \quad \text{and} \quad \int_T \lambda_i^\ell \mathrm{d}x = \frac{2|T|}{(\ell+1)(\ell+2)},$$

where $|e_k|$ (resp. $|T|$) is the length of $|e_k|$ (resp. area of T). The linear independence of ϕ_1, ϕ_2 and ϕ_3 follows from the fact that $\phi_i(p_j) = \delta_{ij}$, where p_1, p_2 and p_3 are the vertices of T.

The invariance of the Laplace operator Δ under rigid motions and the dilation relation

$$\Delta\,(\phi(\delta x)) = \delta^2 (\Delta\phi)(\delta x)$$

imply that, in terms of spanning sets,

$$\Delta\Phi = \langle \Delta\phi_1, \Delta\phi_2, \Delta\phi_3 \rangle = \langle 5\lambda_1 - 2, 5\lambda_2 - 2, 5\lambda_3 - 2 \rangle = \langle \lambda_1, \lambda_2, \lambda_3 \rangle,$$

Hence $\dim(\Delta\Phi) = 3$ and $\Delta : \Phi \longrightarrow P_1(\mathbb{R}^2)$ is a one-to-one map. Therefore $\phi \in \Phi$ and $\Delta\phi = 0$ imply that $\phi = 0$. □

We can now prove the main result of this note.

Theorem. *$(T, \mathscr{P}_2, \mathscr{N}_2)$ is a finite element.*

Proof. Suppose $\boldsymbol{v} \in \mathscr{P}_2$ and $M_\ell(\boldsymbol{v}) = 0$ for $1 \leq \ell \leq 14$. We must show that $\boldsymbol{v} = \mathbf{0}$. In view of (1), Green's theorem and the vanishing moments of $\boldsymbol{v}$, we have

$$\int_T (\nabla \times \boldsymbol{v})(\nabla \times \boldsymbol{v})\mathrm{d}x = \int_{\partial T} (\boldsymbol{n} \times \boldsymbol{v})(\nabla \times \boldsymbol{v})\mathrm{d}s + \int_T \boldsymbol{v} \cdot \nabla \times (\nabla \times \boldsymbol{v})\mathrm{d}x = 0$$

where $\boldsymbol{n}$ is the outer unit normal along ∂T. Similarly, we have

$$\int_T (\nabla \cdot \boldsymbol{v})(\nabla \cdot \boldsymbol{v})\mathrm{d}x = \int_{\partial T} (\boldsymbol{n} \cdot \boldsymbol{v})(\nabla \cdot \boldsymbol{v})\mathrm{d}s - \int_T \boldsymbol{v} \cdot \nabla(\nabla \cdot \boldsymbol{v})\mathrm{d}x = 0.$$

Since $\nabla \times \boldsymbol{v} = 0 = \nabla \cdot \boldsymbol{v}$ and $\boldsymbol{v} \in [P_3(\mathbb{R}^2)]^2$, each component of $\boldsymbol{v}$ is a harmonic polynomial in $P_3(\mathbb{R}^2)$ that has vanishing moments up to order 1 on the edges of T and vanishing moment of order 0 on T. We can then apply Lemma 1 to conclude that the components of $\boldsymbol{v}$ must vanish and hence $\boldsymbol{v} = \mathbf{0}$. □

Given a triangulation $\mathcal{T}_h$ of a two-dimensional polygonal domain Ω, the finite element space V_h defined by this quadratic finite element is nonconforming because $V_h \not\subset H(\text{curl}; \Omega)$ and $V_h \not\subset H(\text{div}; \Omega)$. Nonetheless it can be used to solve problems posed on $H(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$ because of the existence of a good interpolation operator $\Pi : [H^s(T)]^2 \longrightarrow \mathscr{P}_2$ defined by, for any $s > 1/2$,

$$M_\ell(\Pi\boldsymbol{\zeta}) = M_\ell(\boldsymbol{\zeta}) \quad \text{for } 1 \leq \ell \leq 14,\ \boldsymbol{\zeta} \in [H^s(T)]^2. \tag{2}$$

Since $[P_2(\mathbb{R}^2)]^2 \subset \mathscr{P}_2$, we can deduce from the Bramble–Hilbert lemma and the invariance of $\mathscr{P}_2$ under rigid motions and dilation that

$$\|\boldsymbol{\zeta} - \Pi\boldsymbol{\zeta}\|_{L_2(T)} + h_T^{\min(1,s)} |\boldsymbol{\zeta} - \Pi\boldsymbol{\zeta}|_{H^{\min(1,s)}(T)} \leq C h_T^s |\boldsymbol{\zeta}|_{H^s(T)} \tag{3}$$

for all $\boldsymbol{\zeta} \in [H^s(T)]^2$ and $1/2 < s \leq 3$, where $h_T = \operatorname{diam} T$ and the positive constant C depends only on the minimum angle of T if s is bounded away from 1/2.

Moreover, the interpolation operator Π enjoys special properties with respect to curl and divergence.

Lemma 2. *The following commutative relations hold for the interpolation operator:*

$$Q_1(\nabla \times \boldsymbol{\zeta}) = \nabla \times (\Pi\boldsymbol{\zeta}) \qquad \forall\, \boldsymbol{\zeta} \in [H^s(T)]^2 \text{ and } s > 1/2, \tag{4}$$

$$Q_1(\nabla \cdot \boldsymbol{\zeta}) = \nabla \cdot (\Pi\boldsymbol{\zeta}) \qquad \forall\, \boldsymbol{\zeta} \in [H^s(T)]^2 \text{ and } s > 1/2, \tag{5}$$

where Q_1 is the orthogonal projection from $L_2(T)$ onto $P_1(T) = P_1(\mathbb{R}^2)|_T$.

Proof. Let $\psi \in P_1(T)$ be arbitrary. In view of (2) and Green's theorem, we have

$$\begin{aligned} \int_T (\nabla \times \Pi\boldsymbol{\zeta})\psi \,\mathrm{d}x &= \int_{\partial T} (\boldsymbol{n} \times \Pi\boldsymbol{\zeta})\psi \,\mathrm{d}s + \int_T (\Pi\boldsymbol{\zeta}) \cdot (\nabla \times \psi)\mathrm{d}x \\ &= \int_{\partial T} (\boldsymbol{n} \times \boldsymbol{\zeta})\psi \,\mathrm{d}s + \int_T \boldsymbol{\zeta} \cdot (\nabla \times \psi)\mathrm{d}x = \int_T (\nabla \times \boldsymbol{\zeta})\psi \,\mathrm{d}x. \end{aligned}$$

Since $\nabla \times \Pi\boldsymbol{\zeta} \in P_1(T)$ by (1), we have proved (4). The proof of (5) is completely analogous. □

It follows from Lemma 2 that the quantities

$$\nabla \times (\boldsymbol{\zeta} - \Pi \boldsymbol{\zeta}) = (\nabla \times \boldsymbol{\zeta}) - Q_1(\nabla \times \boldsymbol{\zeta}),$$
$$\nabla \cdot (\boldsymbol{\zeta} - \Pi \boldsymbol{\zeta}) = (\nabla \cdot \boldsymbol{\zeta}) - Q_1(\nabla \cdot \boldsymbol{\zeta}),$$

which are parts of the $H(\mathrm{curl};\,\Omega)$ and $H(\mathrm{div};\,\Omega)$ norms for $\boldsymbol{\zeta} - \Pi\boldsymbol{\zeta}$, can be estimated in terms of the Sobolev norms of $\nabla \times \boldsymbol{\zeta}$ and $\nabla \cdot \boldsymbol{\zeta}$, i.e., we have good control over curl and divergence simultaneously.

These properties of the interpolation operator Π together with the continuity of the moments of order ≤ 1 across element boundaries make it possible to use V_h in the numerical solution of problems posed on $H(\mathrm{curl};\,\Omega) \cap H(\mathrm{div};\,\Omega)$.

Finally, we discuss two conjectures on general nonconforming vector finite elements for $H(\mathrm{curl};\,\Omega) \cap H(\mathrm{div};\,\Omega)$ in two and three dimensions. They are motivated by dimension counts and the analogs of (1), (4) and (5).

In the two-dimensional case, in order to have the following analogs:

$$Q_{k-1}(\nabla \times \boldsymbol{\zeta}) = \nabla \times (\Pi \boldsymbol{\zeta}) \qquad \forall\, \boldsymbol{\zeta} \in [H^s(T)]^2 \text{ and } s > 1/2$$
$$Q_{k-1}(\nabla \cdot \boldsymbol{\zeta}) = \nabla \cdot (\Pi \boldsymbol{\zeta}) \qquad \forall\, \boldsymbol{\zeta} \in [H^s(T)]^2 \text{ and } s > 1/2$$

of (4) and (5), where Q_{k-1} ($k \geq 3$) is the orthogonal projection from $L_2(T)$ to $P_{k-1}(T) = P_{k-1}(\mathbb{R}^2)|_T$, the degrees of freedom should be the moments of order $\leq k-1$ on the three edges and the moments of order $\leq k-2$ on T, for the two components of the vector field. The dimension of $\mathcal{P}_k$ should therefore be

$$2\left(3k + \frac{(k-1)k}{2!}\right) = k(k+5).$$

Since the dimension of $[P_k(\mathbb{R}^2)]^2$ is $(k+1)(k+2)$, and we need

$$\nabla \times \boldsymbol{v} \in P_{k-1}(\mathbb{R}^2) \quad \text{and} \quad \nabla \cdot \boldsymbol{v} \in P_{k-1}(\mathbb{R}^2) \quad \forall\, \boldsymbol{v} \in \mathcal{P}_k,$$

which is the analog of (1), we supplement $[P_k(\mathbb{R}^2)]^2$ with $2(k-1)$ many vector fields that are the gradients of homogeneous harmonic polynomials of degrees from $k+2$ to $2k$.

Conjecture 1. *Let T be a triangle,*

$$\mathcal{P}_k = [P_k(\mathbb{R}^2)]^2 \oplus \left(\nabla \mathcal{H}_{k+2}(\mathbb{R}^2) \oplus \cdots \oplus \nabla \mathcal{H}_{2k}(\mathbb{R}^2)\right),$$

and

$$\mathcal{N}_k = \{M_\ell :\ 1 \leq \ell \leq k(k+5)\}$$

be the set of the linear functionals defining the moments on the three edges of T up to order $k-1$ and the moments on T of order up to $k-2$, for the two components of the vector fields. Then $(T, \mathcal{P}_k, \mathcal{N}_k)$ is a finite element for $k \geq 3$.

In three dimensions, the degrees of freedom are the moments of order $\leq k-1$ on the four faces of the tetrahedron T and the moments of order $\leq k-2$ on T, for the three components of the vector fields. The dimension of $\mathcal{P}_k$ should therefore be

$$3\left(4\frac{k(k+1)}{2!} + \frac{(k-1)k(k+1)}{3!}\right) = \frac{k(k+1)(k+11)}{2}.$$

To preserve

$$\nabla \times \boldsymbol{v} \in [P_{k-1}(\mathbb{R}^3)]^3 \quad \text{and} \quad \nabla \cdot \boldsymbol{v} \in P_{k-1}(\mathbb{R}^3) \quad \forall\, \boldsymbol{v} \in \mathcal{P}_k,$$

which is the analog of (1), we supplement $[P_k(\mathbb{R}^3)]^3$ whose dimension is

$$\frac{(k+1)(k+2)(k+3)}{2}$$

with the gradients of homogeneous harmonic polynomials (in three variables) of degrees from $k+2$ to $2k$, whose dimensions add up to

$$[2(k+2)+1] + \cdots + [2(2k)+1] = 3(k-1)(k+1) = \frac{k(k+1)(k+11)}{2} - \frac{(k+1)(k+2)(k+3)}{2}.$$

Conjecture 2. *Let T be a tetrahedron*

$$\mathcal{P}_k = [P_k(\mathbb{R}^3)]^3 \oplus \left(\nabla \mathcal{H}_{k+2}(\mathbb{R}^3) \oplus \cdots \oplus \nabla \mathcal{H}_{2k}(\mathbb{R}^3)\right)$$

and

$$\mathcal{N}_k = \left\{M_\ell :\ 1 \leq \ell \leq \frac{k(k+1)(k+11)}{2}\right\}$$

be the set of the linear functionals defining the moments on the four faces of T up to order $k-1$ and the moments on T of order up to $k-2$, for the three components of the vector fields. Then $(T, \mathcal{P}_k, \mathcal{N}_k)$ is a finite element for $k \geq 2$.

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