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E-SEMIGROUPS SUBORDINATE TO CCR FLOWS

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Abstract. The subordinate E-semigroups of a fixed E-semigroup α are in one-to-one correspondence with local projection-valued cocycles of α. For the CCR flow we characterise these cocycles in terms of their stochastic generators, that is, in terms of the coefficient driving the quantum stochastic differential equation of Hudson-Parthasarathy type that such cocycles necessarily satisfy. In addition various equivalence relations and order-type relations on E-semigroups are considered, and shown to work especially well in the case of those semigroups subordinate to the CCR flows by exploiting our characterisation.

1. Introduction

Two strands of noncommutative analysis developed contemporaneously in the 1980's: within the field of quantum probability there was great interest in quantum stochastic calculi, and especially successful was the bosonic calculus of Hudson and Parthasarathy ([6]). Meanwhile, Arveson and Powers considered the problem of understanding *endomorphism* or *E-semigroups* on $B(\mathsf{H})$, the algebra of all bounded operators on a Hilbert space H, thereby generalising Wigner's well-known work for automorphism groups from the 1930's ([3]). Significant relations between the two were explored by Bhat some time later ([5]), exploiting the fact that a central object for the quantum probabilists is an archetypal example of an E-semigroup, namely the right shift semigroup on (symmetric) Fock space, otherwise known as the *CCR flow.* Bhat in particular initiated the study of various order structures, looking at *local positive contraction cocycles* for the CCR flow since these characterise the completely positive (CP) semigroups subordinate to the given E-semigroup. In this paper we shall continue our study of cocycles of the CCR flow from [14], in particular studying the order relation on orthogonal projection-valued cocycles, and determining which of these cocycles are equivalent in a natural sense.

To make matters more concrete, we give some definitions.

Definition 1.1. Let H be a Hilbert space. An *E-semigroup* on $B(\mathsf{H})$ is a family of maps $\alpha = (\alpha_t : B(\mathsf{H}) \to B(\mathsf{H}))_{t\geqslant 0}$ that satisfy:

(i) $\alpha_0 = \mathrm{id}_{B(\mathsf{H})}$ and $\alpha_{s+t} = \alpha_s \circ \alpha_t$ for all $s, t \geqslant 0$;
(ii) α_t is an endomorphism of $B(\mathsf{H})$, i.e. a *-homomorphism, for each $t \geqslant 0$;

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(iii) the map $S \mapsto \alpha_t(S)$ is normal for each $t \geqslant 0$, and the map $t \mapsto \alpha_t(S)$ is continuous in the weak operator topology for each $S \in B(\mathsf{H})$.

If, in addition, we have

(iv) $\alpha_t(I) = I$ for all t, where I is the identity operator on H,

then α is called an E_0*-semigroup*.

The theory of these has proved to be significantly more tricky than the subclass of *automorphism* semigroups; dropping the requirement of Wigner that $\operatorname{Ran}\alpha_t = B(\mathsf{H})$ is a nontrivial business. Recent work of Powers et al. ([12, 1, 11]) has established a new route to the construction of *spatial* E_0-semigroups through the study of CP flows, and as with Bhat's work, local cocycles and subordination are a key ingredient.

Definition 1.2. Let H be a Hilbert space and let $\alpha, \beta : B(\mathsf{H}) \to B(\mathsf{H})$ be a pair of completely positive maps. We say that α *dominates* β, or, equivalently, that β *is subordinate to* α, if their difference $\alpha - \beta$ is a completely positive map, and denote this $\alpha \geqslant \beta$, or $\beta \leqslant \alpha$.

Similarly, if instead $\alpha = (\alpha_t)_{t \geqslant 0}$ and $\beta = (\beta_t)_{t \geqslant 0}$ are E-semigroups, then β is a *subordinate* of α, written $\beta \leqslant \alpha$, if $\alpha_t - \beta_t$ is completely positive for all $t \geqslant 0$.

The notion of domination or subordination goes back at least as far as work of Arveson. More recent generalisations include subordination in the theory of completely positive definite kernels (i.e. maps $k : S \times S \to B(\mathcal{A}; \mathcal{B})$ satisfying a suitable positivity requirement, where S is a set and $\mathcal{A}$, $\mathcal{B}$ are C^*-algebras) and semigroups of these ([4], Section 3.3). A characterisation of the subordinates of a given CP map α is the content of Arveson's Radon-Nikodým Theorem ([2], Theorem 1.4.2). In this paper we will look only at subordination of one E-semigroup by another, which gives the opportunity to provide the following short proof of Arveson's result in this setting.

Lemma 1.3. *Let* H *be a Hilbert space and let* α, β *be a pair of endomorphisms on* $B(\mathsf{H})$. *The following are equivalent*:

(i) $\alpha \geqslant \beta$;
(ii) $\beta(I) \in \alpha\big(B(\mathsf{H})\big)'$ *and* $\beta(S) = \beta(I)\alpha(S)$ *for all* $S \in B(\mathsf{H})$.

Furthermore, for any projection $P \in \alpha\big(B(\mathsf{H})\big)'$, *the map* $\gamma : S \mapsto P\alpha(S)$ *is a subordinate of* α.

Remark 1.4. Throughout, unless otherwise specified, or obvious, all projections on Hilbert spaces will be *orthogonal* projections.

Proof. Suppose that $\alpha \geqslant \beta$. Let $S \in B(\mathsf{H})$. Then

$$0 \leqslant \begin{bmatrix} I & S^* \end{bmatrix} \begin{bmatrix} I \\ S \end{bmatrix} = \begin{bmatrix} I & S \\ S^* & S^*S \end{bmatrix}$$

and since the difference $\alpha - \beta$ is 2-positive we have

$$0 \leqslant \begin{bmatrix} \beta(I) & 0 \\ 0 & I \end{bmatrix}^* (\alpha - \beta)^{(2)} \left(\begin{bmatrix} I & S \\ S^* & S^*S \end{bmatrix} \right) \begin{bmatrix} \beta(I) & 0 \\ 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} \beta(I)\alpha(I)\beta(I) - \beta(I)^3 & \beta(I)\big(\alpha(S) - \beta(S)\big) \\ \big(\alpha(S^*) - \beta(S^*)\big)\beta(I) & \alpha(S^*S) - \beta(S^*S) \end{bmatrix} =: R.$$

But $\alpha(I) \geqslant \beta(I)$, and these operators are projections, hence

$$\beta(I)\alpha(I)\beta(I) = \beta(I) = \beta(I)^3.$$

Thus the top-left hand corner of the positive 2×2 operator matrix R is zero. Writing this as $R = T^*T$ for $T = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, where $A, B, C, D \in B(H)$, it follows that $A = C = 0$, and so the off-diagonal entries of R are also zero. Hence $\beta(I)\alpha(S) = \beta(S) = \alpha(S)\beta(I)$ and thus (i) implies (ii).

That (ii) implies (i) follows from the final part of the lemma concerning the map γ, and this follows since for such projections P we have

$$\alpha(S) - \gamma(S) = P^{\perp}\alpha(S)P^{\perp},$$

so that $\alpha - \gamma$ is CP. $\square$

Remark 1.5. (a) In proving that (i) implies (ii) we do not use the full force of the assumption that $\alpha - \beta$ is CP, but in fact only require that this difference be 2-positive (and can then conclude that the difference is in fact CP). Such a feature is also found in Bhat's generalisation of this lemma ([5], Proposition 4.2) where he starts with a unital endomorphism α and a CP map β, and, assuming only that $\alpha - \beta$ is positive, shows that the difference is CP, with $\beta(S) = \beta(I)\alpha(S)$ where $\beta(I) \in \alpha\big(B(H)\big)'$ is a positive contraction.

(b) In the above, if α and β are endomorphisms with $\alpha \geqslant \beta$, then β is determined by the value of the projection $\beta(I)$. On the other hand, unless $\alpha(I) = I$, different projections in $\alpha\big(B(\mathsf{H})\big)'$ can yield the same subordinate of α. Indeed, if $\gamma_1(S) = P_1\alpha(S)$ and $\gamma_2(S) = P_2\alpha(S)$ are subordinates of α defined by projections $P_1, P_2 \in \alpha\big(B(\mathsf{H})\big)'$, then $\gamma_1 = \gamma_2$ if and only if there are projections $p \leqslant \alpha(I)$ and $q_1, q_2 \leqslant \alpha(I)^{\perp}$ such that $P_1 = p + q_1$ and $P_2 = p + q_2$.

Example 1.6. Let H be a Hilbert space and $L \in B(\mathsf{H})$ an isometry. Let $\alpha(S) := LSL^*$, an endomorphism of $B(\mathsf{H})$. Projections $P \in \alpha\big(B(\mathsf{H})\big)'$ are either of the form $LL^* + Q$ or Q where Q is a projection with $Q \leqslant \alpha(I)^{\perp}$. The corresponding subordinates of α are α and 0 respectively.

To apply the lemma to E-semigroups in Theorem 1.9 below (which is a special case of Theorem 4.3 of [5]) we require a little more terminology.

Definition 1.7. Let H be a Hilbert space and α an E-semigroup on $B(\mathsf{H})$. A family of contraction operators $X = (X_t)_{t \geqslant 0}$ on H is a *left α-cocycle* if

(i) $X_0 = I$;
(ii) $X_{s+t} = X_s\alpha_s(X_t)$ for all $s, t \geqslant 0$;
(iii) the maps $t \mapsto X_t$ and $t \mapsto X_t^*$ are continuous in the strong operator topology.

If, instead, (ii) is replaced by

(ii)′ $X_{s+t} = \alpha_s(X_t)X_s$ for all $s, t \geqslant 0$

then we speak of a *right α-cocycle.* A *local α-cocycle* is a cocycle for which $X_t \in \alpha_t\big(B(\mathsf{H})\big)'$ for each $t \geqslant 0$ and so is both a left and right cocycle.

Remark 1.8. (a) Contractivity of each X_t is not always assumed; in [10] we considered cocycles for which the X_t can be *unbounded* (in which case the continuity condition (iii) is altered appropriately). However, the contractivity restriction here is appropriate for this paper.

(b) Typically it is only strong continuity of $t \mapsto X_t$ that is assumed, and the class of cocycles is restricted (e.g. to unitary-valued cocycles). The continuity required above is automatically true for weak operator measurable isometry-valued cocycles ([3], Proposition 2.3.1) and weak operator measurable positive contraction cocycles ([5], Appendix A) on separable Hilbert spaces; also for local cocycles of the CCR flow that are weakly continuous at $t = 0$ ([14], Proposition 2.1) on arbitrary Hilbert spaces.

(c) Although a local cocycle is both a left and right cocycle, the converse is not true as can easily be seen from Proposition 3.1 of [14].

Theorem 1.9. *Let H be a Hilbert space and let α and β be E-semigroups on $B(\mathsf{H})$. The following are equivalent*:

(i) $\alpha \geqslant \beta$;

(ii) $B := (B_t = \beta_t(I))_{t\geqslant 0}$ *is a projection-valued, local α-cocycle such that* $\beta_t(S) = B_t\alpha_t(S)$ *for all* $S \in B(\mathsf{H})$.

Moreover, if $C = (C_t)_{t\geqslant 0}$ is any projection-valued, local α-cocycle, then $\gamma_t(S) := C_t\alpha_t(S)$ defines an E-semigroup subordinate to α.

Proof. Using Lemma 1.3, the proof is now just a matter of checking that definitions hold. For example if (i) holds, then each $B_t = \beta_t(I) \in \alpha_t\big(B(\mathsf{H})\big)'$ and is projection-valued, with

$$B_{s+t} = \beta_{s+t}(I) = \beta_s\big(\beta_t(I)\big) = \beta_s(I)\alpha_s\big(\beta_t(I)\big) = B_s\alpha_s(B_t).$$

Since $B_s \leqslant B_t$ whenever $s \geqslant t$, and $t \mapsto B_t$ is weak operator continuous, it is strong operator continuous and thus a local α-cocycle. □

Remark 1.10. (a) The lack of injectivity for single maps noted after Lemma 1.3 does not arise for E-semigroups and cocycles: if P is a projection-valued, local α-cocycle, then

$$P_t = P_t\alpha_t(P_0) = P_t\alpha_t(I) \quad \text{so} \quad P_t \leqslant \alpha_t(I).$$

(b) It follows from this result that the only E_0-semigroup subordinate to an E_0-semigroup α is α itself.

Example 1.11. Let H be a Hilbert space and $L = (L_t)_{t\geqslant 0}$ a strongly continuous semigroup of isometries on H. Put $\alpha_t(S) := L_tSL_t^*$, then α is an E-semigroup. The only projection-valued, local α-cocycle is $(\alpha_t(I))_{t\geqslant 0}$, so that the only subordinate E-semigroups of α is α itself.

2. Cocycles of CCR Flows

The most understood class of E-semigroups are the CCR and CAR flows; these comprise the type I examples, with the work of Powers et al. ([12, 1, 11]) designed to yield new examples of type II semigroups. Here we revert to a consideration of the CCR flow, bringing stochastic methods to bear.

Let $\mathfrak{h}$ and k be a pair of Hilbert spaces, called the *initial space* and *noise dimension space* respectively. For each measurable set $I \subset [0,\infty[$ let $\mathcal{F}_I$ denote the symmetric/bosonic Fock space over $L^2(I;\mathsf{k})$, with $\mathcal{F}_{\mathbb{R}_+}$ abbreviated to $\mathcal{F}$, so that

$$\mathcal{F} \cong \mathcal{F}_I \otimes \mathcal{F}_{I^c}.$$

This isomorphism is conveniently effected via the correspondence

$$\varepsilon(f) \longleftrightarrow \varepsilon(f|_I) \otimes \varepsilon(f|_{I^c})$$

defined in terms of the useful total set of exponential vectors:

$$\mathcal{E}_S := \{\varepsilon(f) : f \in S\}, \text{ where } \varepsilon(f) := (1, f, (2!)^{-1/2} f \otimes f, (3!)^{-1/2} f \otimes f \otimes f, \ldots),$$

and where S is any sufficiently large subset of $L^2(\mathbb{R}_+;\mathsf{k})$. For the purposes of this paper we shall write $\mathcal{E}$ for $\mathcal{E}_S$ in the case that S is the set of (right-continuous) step functions in $L^2(\mathbb{R}_+;\mathsf{k})$. The natural unitary isomorphism $L^2(\mathbb{R}_+;\mathsf{k}) \cong L^2([t,\infty[;\mathsf{k})$ gives rise to the identification

$$\mathcal{F} \cong \varepsilon(0) \otimes \mathcal{F}_{[t,\infty[} \subset \mathcal{F}_{[0,t[} \otimes \mathcal{F}_{[t,\infty[} \cong \mathcal{F},$$

and thence the right-shift map σ_t^{k} on $B(\mathcal{F})$ that has image $I_{[0,t[} \otimes B(\mathcal{F}_{[t,\infty[})$. This map is a normal *-homomorphism. The CCR flow determined by $\mathfrak{h}$ and k is then

$$\sigma^{\mathfrak{h},\mathsf{k}} = \big(\sigma_t^{\mathfrak{h},\mathsf{k}} := \mathrm{id}_{B(\mathfrak{h})} \overline{\otimes} \sigma_t^{\mathsf{k}}\big)_{t \geqslant 0}.$$

[We now clarify tensor product notation: $\underline{\otimes}$ denotes the algebraic tensor product, $\otimes$ is used for the tensor product of Hilbert spaces and the spatial tensor product, whereas $\overline{\otimes}$ denotes the ultraweak tensor product.]

In general we will drop the dependence of the CCR flow on $\mathfrak{h}$ and k in what follows, referring, for example, to σ-cocycles.

Definition 2.1. A σ-cocycle X is:

(a) *adapted* if $X_t \in B(\mathfrak{h} \otimes \mathcal{F}_{[0,t[}) \otimes I_{[t,\infty[}$ for all $t \geqslant 0$,

(b) *Markov-regular* if the expectation semigroup $T_t := \mathbb{E}[X_t]$ on $\mathfrak{h}$ is norm-continuous, where the vacuum conditional expectation $\mathbb{E}$ is defined by

$$\langle u, \mathbb{E}[S]v\rangle = \langle u \otimes \varepsilon(0), Sv \otimes \varepsilon(0)\rangle, \quad u, v \in \mathfrak{h}.$$

Remark 2.2. A σ-cocycle X is local if $X_t \in I_{\mathfrak{h}} \otimes B(\mathcal{F}_{[0,t[}) \otimes I_{\mathcal{F}_{[t,\infty[}}$ for each t, which is a strictly stronger condition than adaptedness whenever $\dim \mathfrak{h} > 1$.

The following combines Theorem 6.6 of [10] with Theorem 7.5 of [9]. We define $\widehat{\mathsf{k}} := \mathbb{C} \oplus \mathsf{k}$, and $\Delta := I_{\mathfrak{h}} \otimes P$, where P is the projection $\widehat{\mathsf{k}} \to \{0\} \oplus \mathsf{k}$.

Theorem 2.3. *There is a one-to-one correspondence between the set of Markov-regular, contraction, left, adapted σ-cocycles and*

$$C_0(\mathfrak{h},\mathsf{k}) := \{F \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}}) : F + F^* + F^*\Delta F \leqslant 0\},$$

under which cocycles X are associated to their stochastic generatorF through $X \longleftrightarrow X^F$, where X^F is the unique solution of the left Hudson-Parthasarathy quantum stochastic differential equation

$$X_t = I + \int_0^t \widehat{X_s}(F \otimes I_{\mathcal{F}})\, d\Lambda_s, \tag{2.1}$$

with $\widehat{X_s}$ denoting the ampliation of $X_s \in B(\mathfrak{h} \otimes \mathcal{F})$ to all of $B(\mathfrak{h} \otimes \widehat{\mathsf{k}} \otimes \mathcal{F})$.

The theorem above says that every such cocycle satisfies the equation (2.1), and that all (contractive) solutions of this equation are indeed left cocycles. In the case when $\mathfrak{h}$ is finite dimensional or the cocycle is local, the fact that we assumed that $t \mapsto X_t$ is strongly continuous is enough to guarantee that the cocycle is Markov-regular, but this condition is a nontrivial requirement otherwise. Our basic reference for quantum stochastic calculus is [8], where the same notations are used.

Locality of σ-cocycles is also easily characterised.

Lemma 2.4. *Let $F \in C_0(\mathfrak{h}, \mathsf{k})$ and let $X = X^F$ be the associated left σ-cocycle. Then X is local if and only if $F \in I_{\mathfrak{h}} \otimes B(\widehat{\mathsf{k}})$.*

Proof. Since $\sigma_t\big(B(\mathfrak{h} \otimes \mathcal{F})\big)' = I_{\mathfrak{h}} \otimes B(\mathcal{F}_{[0,t[}) \otimes I_{[t,\infty[}$, the result follows from Corollary 6.5 of [10]. $\square$

Hudson and Parthasarathy focused on determining those $F \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ that give rise to unitary solutions of (2.1). (Co)isometry and contractivity were considered later, with positive contraction-valued and projection-valued adapted σ-cocycles studied in [14] (although for the former see [5] for the case when $\mathfrak{h} = \mathbb{C}$). In Section 5.2 of [4] the morphisms of time ordered Fock modules are characterised; Theorem 4.4.8 of the same paper gives a one-to-one correspondence between endomorphisms of product systems of Hilbert modules and local cocycles of an E_0-semigroup on a related C^*-algebra. These results contain, as a special case, an alternative characterisation of the local cocycles of the CCR flow.

The following is Proposition 3.2 of [14]. Throughout, when writing elements of $B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ in a 2×2 block form, we use the identification

$$\mathfrak{h} \otimes \widehat{\mathsf{k}} \cong \mathfrak{h} \oplus (\mathfrak{h} \otimes \mathsf{k}),$$

so that

$$F = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

for $A \in B(\mathfrak{h})$, $B \in B(\mathfrak{h} \otimes \mathsf{k}; \mathfrak{h})$, $C \in B(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$ and $D \in B(\mathfrak{h} \otimes \mathsf{k})$.

Proposition 2.5. *Let $F \in C_0(\mathfrak{h}, \mathsf{k})$. The following are equivalent:*

(i) *$X = X^F$ is projection-valued;*
(ii) *$F \in \mathcal{N} \overline{\otimes} B(\widehat{\mathsf{k}})$ for some commutative von Neumann algebra $\mathcal{N}$, and $F + F^*\Delta F = 0$;*
(iii) *$F = \left[\begin{smallmatrix} -L^*L & L^* \\ L & P-I \end{smallmatrix}\right] \in \mathcal{N} \overline{\otimes} B(\widehat{\mathsf{k}})$ for some commutative von Neumann algebra $\mathcal{N}$, where $P \in \mathcal{N} \overline{\otimes} B(\mathsf{k})$ is a projection and $PL = 0$.*

Remark 2.6. (a) The fact that X is, in particular, self-adjoint means that it must be a left and right cocycle, but not necessarily a local cocycle. Indeed, using Lemma 2.4, an example is constructed by taking $\mathfrak{h}$ with $\dim \mathfrak{h} > 1$, $\mathsf{k} = \mathbb{C}$, $L = 0$, and $P \in B(\mathfrak{h})$ any nontrivial projection.

(b) When $\mathfrak{h} = \mathbb{C}$, we have $L \in B(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k}) = B(\mathbb{C}; \mathsf{k}) = |\mathsf{k}\rangle$, the column operator space associated to k, where $|u\rangle \in |\mathsf{k}\rangle$ denotes the map $\lambda \mapsto \lambda u$, and $\langle u| := |u\rangle^*$. In this case projection-valued, adapted σ-cocycles, which are now all local, are indexed by pairs (P, u), where $P \in B(\mathfrak{h})$ is a projection, and $u \in \operatorname{Ker} P$ is a vector.

The new result in this section is the following.

Theorem 2.7. *Let X^F and X^G be a pair of Markov-regular, projection-valued, adapted σ-cocycles with generators*

$$F = \begin{bmatrix} -L^*L & L^* \\ L & P - I \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} -M^*M & M^* \\ M & Q - I \end{bmatrix}. \tag{2.2}$$

(a) *The following are equivalent*:

(i) $X_t^F \leqslant X_t^G$ *for all* $t \geqslant 0$, *equivalently* $X_t^F X_t^G = X_t^F$ *for all* $t \geqslant 0$.

(ii) $G + G\Delta F = 0$.

(iii) $P \leqslant Q$ *and* $M = Q^\perp L$.

(b) *Suppose that $F, G \in \mathcal{N} \overline{\otimes} B(\widehat{\mathsf{k}})$ for some commutative von Neumann algebra $\mathcal{N}$. Then the following are equivalent*:

(i) *There is some $H \in \big(\mathcal{N} \overline{\otimes} B(\widehat{\mathsf{k}})\big) \cap C_0(\mathfrak{h}, \mathsf{k})$ such that*

$$X_t^F = (X_t^H)^* X_t^H \quad \text{and} \quad X_t^G = X_t^H (X_t^H)^* \quad \text{for all } t \geqslant 0. \tag{2.3}$$

(ii) *The projections P and Q are equivalent in $\mathcal{N} \overline{\otimes} B(\mathsf{k})$, that is, there is some $D \in \mathcal{N} \overline{\otimes} B(\mathsf{k})$ such that $P = D^*D$ and $Q = DD^*$.*

In this case any H for which (2.3) *holds has the form $H = \left[\begin{smallmatrix} A & B \\ C & D-I \end{smallmatrix}\right]$ with D as above and where*

$$C = M + E, \quad B = L^* - E^*D, \quad \text{and} \quad A = -\tfrac{1}{2}(E^*E + BB^* + C^*C) + iK$$

for some $E \in \mathcal{N} \overline{\otimes} |\mathsf{k}\rangle$ satisfying $Q^\perp E = 0$ and some $K = K^* \in \mathcal{N}$.

Remark 2.8. The condition $E \in \mathcal{N} \overline{\otimes} |\mathsf{k}\rangle$ effectively means that it can be written in block form as a column with entries taken from $\mathcal{N}$.

Proof. (a) Let $\xi = \sum_i u_i \otimes \varepsilon(f_i), \zeta = \sum_j v_j \otimes \varepsilon(g_j) \in \mathfrak{h} \underline{\otimes} \mathcal{E}$. Then the first and second fundamental formulae of quantum stochastic calculus ([8], Theorems 3.13 and 3.15) give that

$$\langle X_t^F \xi, X_t^G \zeta \rangle - \langle \xi, X_t^F \zeta \rangle$$

is equal to

$$\begin{aligned} &\int_0^t \Big\{ \langle \widehat{X_s^F} \xi(s), (G \otimes I_{\mathcal{F}}) \widehat{X_s^G} \zeta(s) \rangle + \langle (F \otimes I_{\mathcal{F}}) \widehat{X_s^F} \xi(s), \widehat{X_s^G} \zeta(s) \rangle \\ &\qquad + \langle (F \otimes I_{\mathcal{F}}) \widehat{X_s^F} \xi(s), (\Delta G \otimes I_{\mathcal{F}}) \widehat{X_s^G} \zeta(s) \rangle - \langle \xi(s), (F \otimes I_{\mathcal{F}}) \widehat{X_s^F} \zeta(s) \rangle \Big\} \, ds \end{aligned} \tag{2.4}$$

$$= \int_0^t \Big\{ \langle \xi(s), \widehat{X_s^F} [(F + G + F\Delta G) \otimes I_{\mathcal{F}}] \widehat{X_s^G} \zeta(s) \rangle - \langle \xi(s), (F \otimes I_{\mathcal{F}}) \widehat{X_s^F} \zeta(s) \rangle \Big\} \, ds$$

where $\xi(s) = \sum_i u_i \otimes \widehat{f_i(s)} \otimes \varepsilon(f_i)$ and similarly for $\zeta(s)$, with $\widehat{d} := \binom{1}{d}$ for any $d \in \mathsf{k}$. Thus if (i) holds, then the integral above is zero, and since the integrand is continuous in a neighbourhood of 0 we get

$$\langle \xi(0), [(F + G + F\Delta G) \otimes I_{\mathcal{F}}]\zeta(0)\rangle - \langle \xi(0), (F \otimes I_{\mathcal{F}})\zeta(0)\rangle = 0$$

from which (ii) follows since $\xi(0)$ and $\zeta(0)$ range over a total subset of $\mathfrak{h} \otimes \widehat{\mathsf{k}} \otimes \mathcal{F}$.

Let $\mathcal{N}$ be a commutative von Neumann algebra as in Proposition 2.5. To see that (ii) implies (i), note that in (2.4) the term in square brackets equals $F \otimes I_{\mathcal{F}} \in (\mathcal{N} \overline{\otimes} B(\widehat{\mathsf{k}})) \otimes I_{\mathcal{F}}$, which commutes with $\widehat{X_s^F} \in (\mathcal{N} \otimes I_{\widehat{\mathsf{k}}}) \overline{\otimes} B(\mathcal{F})$. Consequently $X^F X^G$ and X^F are both solutions of the right version of the Hudson-Parthasarathy QSDE (2.1) for the same coefficient F, and so uniqueness of solutions gives (i).

Finally, (ii) is equivalent to the following four equations being satisfied:

$$M^*L = M^*M, \quad M^*P = 0, \quad M = Q^{\perp}L \ \text{ and } \ PQ = P.$$

That all four together are equivalent to just the last two is a consequence of the fact that $QM = 0$, equivalently $Q^{\perp}M = M$, and also $PL = 0$.

(b) This time, for any $H \in \big(\mathcal{N} \overline{\otimes} B(\widehat{\mathsf{k}})\big) \cap C_0(\mathfrak{h}, \mathsf{k})$, it follows that $(X^H)^*X^H$ and $X^H(X^H)^*$ are both left and right cocycles, with stochastic generators $H + H^* + H^*\Delta H$ and $H + H^* + H\Delta H^*$ respectively. (The argument follows similar lines to part (a); see also Lemma 3.1 of [14].) Uniqueness of generators implies that an H exists such that (i) holds if and only if

$$H + H^* + H^*\Delta H = F \ \text{ and } \ H + H^* + H\Delta H^* = G.$$

Writing H in block form as in the statement, this becomes

$$\begin{bmatrix} A + A^* + C^*C & B + C^*D \\ B^* + D^*C & D^*D - I \end{bmatrix} = \begin{bmatrix} -L^*L & L^* \\ L & P - I \end{bmatrix} \tag{2.5}$$

and

$$\begin{bmatrix} A + A^* + BB^* & C^* + BD^* \\ C + DB^* & DD^* - I \end{bmatrix} = \begin{bmatrix} -M^*M & M^* \\ M & Q - I \end{bmatrix}. \tag{2.6}$$

This shows immediately that (i) implies (ii).

To see that (ii) implies (i), we have to satisfy a total of eight equations when comparing components in (2.5) and (2.6). However, (ii) is merely the statement that it is possible to find $D \in \mathcal{N} \overline{\otimes} B(\mathsf{k})$ such that the bottom-right components are equal in both equations. Comparing bottom-left components we need

$$B^* + D^*C = L \ \text{ and } \ C + DB^* = M. \tag{2.7}$$

But note that $DD^* = Q$ and $D^*D = P$, so D is a partial isometry with initial projection P and final projection Q. Since $PL = 0$, it follows that $DL = 0$ as well; similarly $D^*M = 0$. Eliminating B from (2.7) yields a necessary condition on C:

$$(I - DD^*)C = M - DL \ \text{ so } \ Q^{\perp}C = M.$$

Since $Q^\perp M = M$, all solutions to this equation have the form $C = M + E$ with $E \in \mathcal{N}\,\overline{\otimes}|\mathsf{k}\rangle$ satisfying $Q^\perp E = 0$. But now putting this back into the first equation in (2.7) we get

$$B^* = L - D^*M - D^*E \quad \text{so} \quad B = L^* - E^*D.$$

Now the bottom-left components of (2.5) are equal by construction, hence so are the top-right components. For the off-diagonal components in (2.6) we have

$$C + DB^* = M + E + DL - DD^*E = M + DL + Q^\perp E = M$$

as required.

Finally, to ensure that the top-left components in (2.5) and (2.6) are equal, it is clear that twice the real part of A must equal both $-L^*L - C^*C$ and $-M^*M - BB^*$, with the imaginary part unconstrained. That these two conditions can be satisfied simultaneously follows since

$$\begin{aligned} -L^*L - C^*C &= -L^*L - (M+E)^*(M+E) \\ &= -L^*L - M^*M - E^*E - M^*E - E^*M \\ &= -L^*L - M^*M - E^*E, \end{aligned}$$

since $M^*E = (Q^\perp M)^*E = M^*Q^\perp E = 0$, and similarly

$$\begin{aligned} -M^*M - BB^* &= -M^*M - (L^* - E^*D)(L^* - E^*D)^* \\ &= -M^*M - L^*L - E^*QE + E^*DL + (DL)^*E \\ &= -L^*L - M^*M - E^*E. \end{aligned}$$

Thus, if (ii) holds, that is if P and Q are equivalent in $\mathcal{N}\,\overline{\otimes}B(\mathsf{k})$, then it is possible to fill out the matrix for H in a way such that (2.5) and (2.6) both hold, hence (i) holds. Moreover, the computations above reveal all possible solutions to the problem. $\square$

Remark 2.9. (a) Part (a) above is in some sense more satisfactory that part (b) since we must have $F \in \mathcal{M}\overline{\otimes}B(\widehat{\mathsf{k}})$ and $G \in \mathcal{N}\,\overline{\otimes}B(\widehat{\mathsf{k}})$ for a pair of commutative von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$, but no relation between these algebras is imposed. Indeed, take $\mathsf{k} = \mathbb{C}^2$, and $\mathfrak{h}$ of dimension at least 2 so that we can pick noncommuting projections $p, q \in B(\mathfrak{h})$. Then let

$$F = \begin{bmatrix} 0 & 0 \\ 0 & P - I \end{bmatrix}, \text{ and } G = \begin{bmatrix} 0 & 0 \\ 0 & Q - I \end{bmatrix} \text{ where } P = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} I_{\mathfrak{h}} & 0 \\ 0 & q \end{bmatrix}.$$

Since $G + G\Delta F = 0$, we have $X^F \leqslant X^G$. However, any von Neumann algebra $\mathcal{N}_1$ that satisfies $F, G \in \mathcal{N}_1\overline{\otimes}B(\widehat{\mathsf{k}})$ must contain both p and q. The assumption in (b) about the existence of the common commutative von Neumann algebra is to facilitate pushing the coefficient H past the cocycle X^H in the proof; when $\mathfrak{h} = \mathbb{C}$, however, this is not actually a restriction.

(b) In part (b), since X^F and X^G are projection-valued cocycles, it follows that X^H must be partial isometry-valued. A necessary and sufficient condition on $H \in \mathcal{N}\,\overline{\otimes}B(\widehat{\mathsf{k}})$ for it to be the generator of a partial isometry-valued cocycle ([14], Proposition 3.3) is that

$$H + H^* + H^*\Delta H + H\Delta H + H\Delta H^* + H\Delta H^*\Delta H = 0.$$

The keen reader is invited to check directly that an H satisfying the structure relations in part (b) does indeed satisfy this equation.

3. Relations on the Subordinates of an E-semigroup

For a given E-semigroup α on some $B(\mathsf{H})$, let $\mathrm{Sub}(\alpha)$ denote the set of E-semigroups that are subordinate to α. We now turn our attention to possible natural relations that can be defined on the set of all E-semigroups on $B(\mathsf{H})$, or perhaps just on $\mathrm{Sub}(\alpha)$ for some such semigroup α.

The most obvious is clearly the relation $\leqslant$ of subordination itself. It is immediate from the definition that it is a partial order on the set of all E-semigroups, hence on each subset $\mathrm{Sub}(\alpha)$.

Lemma 3.1. *Let α be an E-semigroup and $\beta, \gamma \in \mathrm{Sub}(\alpha)$. Then $\gamma \leqslant \beta$ if and only if $\gamma_t(I) \leqslant \beta_t(I)$ for all $t \geqslant 0$.*

Proof. If $\gamma \leqslant \beta$, then in particular $(\beta_t - \gamma_t)(I) \geqslant 0$, and so $\gamma_t(I) \leqslant \beta_t(I)$. Conversely, suppose that we have $\gamma_t(I) \leqslant \beta_t(I)$ for each t. Now by Theorem 1.9, the families $(\beta_t(I))_{t\geqslant 0}$ and $(\gamma_t(I))_{t\geqslant 0}$ are projection-valued, local α-cocycles, and also each $\beta_t(I) - \gamma_t(I)$ is a projection, with

$$\beta_t(S) - \gamma_t(S) = \big(\beta_t(I) - \gamma_t(I)\big)\alpha_t(S) = \big(\beta_t(I) - \gamma_t(I)\big)\alpha_t(S)\big(\beta_t(I) - \gamma_t(I)\big)$$

for all $S \in B(\mathsf{H})$, so that $\beta_t - \gamma_t$ is CP as required. □

Remark 3.2. The temptation to adjust the hypotheses above and deal with all E-semigroups rather than those from $\mathrm{Sub}(\alpha)$ should be resisted. Whilst it is true that given *any* two E-semigroups β and γ, if $\beta \geqslant \gamma$, then $\beta_t(I) \geqslant \gamma_t(I)$, the converse is not true: if β and γ are E_0-semigroups, then obviously $\beta_t(I) = \gamma_t(I)$, but we need not have $\beta = \gamma$.

Let us apply this to the set $\mathrm{Sub}(\sigma)$ for the CCR flow $\sigma = \sigma^{\mathfrak{h},\mathsf{k}}$. By Theorem 1.9 any E-semigroup α subordinate to σ is determined by a projection-valued, local σ-cocycle $(X_t^\alpha = \alpha_t(I))_{t\geqslant 0}$, and from Lemma 2.4, Theorem 2.3 and Proposition 2.5, this in turn is specified uniquely by a pair (P_α, u_α) where $P_\alpha \in B(\mathsf{k})$ is a projection, and $u_\alpha \in \operatorname{Ker} P_\alpha$, through

$$X^\alpha = X^F \quad \text{for} \quad F = I_{\mathfrak{h}} \otimes \begin{bmatrix} -\|u_\alpha\|^2 & \langle u_\alpha| \\ |u_\alpha\rangle & -P_\alpha^\perp \end{bmatrix}. \tag{3.1}$$

So, from part (a) of Theorem 2.7, we get the following.

Proposition 3.3. *Suppose that $\alpha, \beta \in \mathrm{Sub}(\sigma)$ with associated projection-valued, local σ-cocycles X^α and X^β, whose stochastic generators have the form* (3.1)*. Then $\alpha \leqslant \beta$ if and only if*

(i) *$P_\alpha \leqslant P_\beta$ and*
(ii) *$u_\alpha = u_\beta + u_{\alpha\beta}$ for some $u_{\alpha\beta} \in P_\beta(\mathsf{k}) \cap P_\alpha^\perp(\mathsf{k})$.*

Example 3.4. Let $\mathfrak{h} = \mathbb{C}$ and $\mathsf{k} = L^2[0,1]$. For each $r \in [0,1]$ let P_r denote the projection $P_r f = f\mathbf{1}_{[0,r]}$ on k. Then define $\alpha^{(r)}$ to be the E-semigroup associated to the projection-valued, local σ-cocycle with stochastic generator

$$F_r = \begin{bmatrix} 0 & 0 \\ 0 & P_r - I \end{bmatrix}.$$

It follows that for all $r \leqslant s$ in $[0,1]$ we have $\alpha^{(r)} \leqslant \alpha^{(s)}$, that is, $\mathrm{Sub}(\sigma)$ contains an uncountable, linearly ordered subset. This can only happen since we have taken an infinite dimensional k, although the noise dimension space k is separable. In the case of a finite dimensional k the maximum number of distinct semigroups in a chain in $\mathrm{Sub}(\sigma)$ is $1 + \dim \mathsf{k}$.

In the theory of E-semigroups, a trick to overcome such features of relations is to consider semigroups up to some form of equivalence.

Definition 3.5. E-semigroups α and β on $B(\mathsf{H})$ are *cocycle conjugate* if there is a left α-cocycle U such that

(i) $\beta_t(S) = U_t \alpha_t(S) U_t^*$ for all $t \geqslant 0, S \in B(\mathsf{H})$, and
(ii) $\alpha_t(I) = U_t^* U_t$.

This is denoted $\alpha \sim \beta$, or $\alpha \sim_U \beta$ if we want to highlight the particular cocycle.

Most of the literature in this area deals solely with E_0-semigroups, when a seemingly different definition of cocycle conjugacy is found: E_0-semigroups α and β are called cocycle conjugate if there is a unitary, left α-cocycle U such that condition (i) above holds. However, note that if α and β are E_0-semigroups above, then conditions (i) and (ii) together imply that the U_t are unitaries. Since we are forced to deal in this paper with general E-semigroups (recall Remark 1.10 (b)), we need the version of cocycle conjugacy given in Definition 3.5, where the U_t are partial isometries courtesy of (ii), but not necessarily unitaries.

Proposition 3.6. *The relation $\sim$ is an equivalence relation on the set of all E-semigroups on $B(\mathsf{H})$.*

Proof. We have $\alpha \sim_U \alpha$ for $U_t = \alpha_t(I)$. If $\alpha \sim_U \beta$, then

$$U_{s+t}^* = \alpha_s(U_t^*) U_s^* = \alpha_s(I)\alpha_s(U_t^*) U_s^* = U_s^* U_s \alpha_s(U_t^*) U_s^* = U_s^* \beta_s(U_t^*),$$

so that U^* is a left β-cocycle, with $\beta_t(I) = U_t \alpha_t(I) U_t^* = (U_t U_t^*)^2 = U_t U_t^*$, and $U_t^* \beta_t(S) U_t = \alpha_t(S)$, so that $\beta \sim_{U^*} \alpha$. Similar computations show that if $\alpha \sim_U \beta$ and $\beta \sim_V \gamma$, then $\alpha \sim_{VU} \gamma$, where $VU = (V_t U_t)_{t \geqslant 0}$ is a left α-cocycle. $\square$

The following relation was then discussed in [13].

Definition 3.7. Let H be a Hilbert space and let α, β be E-semigroups on $B(\mathsf{H})$. Write $\alpha \precsim \beta$ if $\alpha \sim \alpha'$ for some other E-semigroup α' for which $\alpha' \leqslant \beta$.

Proposition 3.8. *The relation $\precsim$ is reflexive and transitive.*

Proof. Reflexivity is immediate: $\alpha \sim \alpha$ and $\alpha \leqslant \alpha$, so that $\alpha \precsim \alpha$. To establish transitivity assume that $\alpha \precsim \beta$ and $\beta \precsim \gamma$. So there are E-semigroups α' and β' and cocycles U and V such that

$$\alpha \sim_U \alpha', \quad \alpha' \leqslant \beta, \quad \beta \sim_V \beta' \text{ and } \beta' \leqslant \gamma.$$

Set $V_t' = V_t \alpha_t'(I)$, then, using Theorem 1.9,

$$V_{s+t}' = V_s \beta_s(V_t) \alpha_s'\big(\alpha_t'(I)\big) = V_s \beta_s(V_t) \beta_s\big(\alpha_t'(I)\big) \alpha_s'(I) = V_s' \alpha_s'(V_t'),$$

so that V' is a left α'-cocycle. Moreover

$$(V_t')^* V_t' = \alpha_t'(I) V_t^* V_t \alpha_t'(I) = \alpha_t'(I) \beta_t(I) \alpha_t'(I) = \alpha_t'(I).$$

Consequently, if we define

$$\alpha''_t(S) = V'_t\alpha'_t(S)(V'_t)^*,$$

then α'' is an E-semigroup, with $\alpha' \sim_{V'} \alpha''$ by construction. Also

$$\beta'_t(S) - \alpha''_t(S) = V_t\beta_t(S)V_t^* - V'_t\alpha'_t(S)(V'_t)^* = V_t(\beta_t - \alpha'_t)(S)V_t^*,$$

so that $\alpha'' \leqslant \beta'$. But then $\alpha \sim \alpha' \sim \alpha''$ and $\alpha'' \leqslant \beta' \leqslant \gamma$, showing that $\alpha \precsim \gamma$. $\square$

However, as noted in [13], the problem is proving that $\precsim$ is antisymmetric on the set of all E-semigroups considered up to cocycle conjugacy. That is, if $\alpha \precsim \beta$ and $\beta \precsim \alpha$ do we necessarily have $\alpha \sim \beta$? New results of Liebscher suggests that it is *not* antisymmetric ([7]).

Inspired by the results of Section 2, we instead introduce an alternative relation on the subset $\mathrm{Sub}(\alpha)$, rather than deal with all E-semigroups simultaneously.

Definition 3.9. Let α be an E-semigroup on $B(\mathsf{H})$, and let $\beta, \gamma, \delta \in \mathrm{Sub}(\alpha)$. We define $\beta \sim^\alpha \gamma$ if there is a local α-cocycle U such that

$$\beta_t(I) = U_t^*U_t \quad \text{and} \quad \gamma_t(I) = U_tU_t^* \quad \text{for all } t \geqslant 0.$$

We write $\beta \sim_U^\alpha \gamma$ to highlight the particular α-cocycle U. In addition we define $\beta \precsim^\alpha \delta$ if there is some $\beta' \in \mathrm{Sub}(\alpha)$ such that $\beta \sim^\alpha \beta'$ with $\beta' \leqslant \delta$.

These relations behave similarly to the versions without the superscript α.

Proposition 3.10. *The relation $\sim^\alpha$ on $\mathrm{Sub}(\alpha)$ is an equivalence relation. The relation $\precsim^\alpha$ is reflexive and transitive.*

Proof. If $\beta \in \mathrm{Sub}(\alpha)$, then $U_t = \beta_t(I)$ is a local α-cocycle for which $\beta \sim_U^\alpha \beta$, so that $\sim^\alpha$ is reflexive. Symmetry is obvious, since if U is a local α-cocycle, then so is U^*. Finally, transitivity follows since if U and V are local α-cocycles, then so is the pointwise product $VU = (V_tU_t)_{t\geqslant 0}$, from which we quickly get that if $\beta \sim_U^\alpha \gamma$ and $\gamma \sim_V^\alpha \delta$, then $\beta \sim_{VU}^\alpha \delta$.

For the putative partial order $\precsim^\alpha$, reflexivity follows as for $\precsim$. For transitivity, suppose that $\beta, \gamma, \delta \in \mathrm{Sub}(\alpha)$ with $\beta \precsim^\alpha \gamma$ and $\gamma \precsim^\alpha \delta$, through the use of local α-cocycles U and V for which

$$\beta \sim_U^\alpha \beta', \quad \beta' \leqslant \gamma, \quad \gamma \sim_V^\alpha \gamma' \quad \text{and} \quad \gamma' \leqslant \delta.$$

Set $V'_t = V_t\beta'_t(I)$, a pointwise product of local α-cocycles, hence itself a local α-cocycle. Moreover

$$\mathrm{Ran}\,\beta'_t(I) \subset \mathrm{Ran}\,\gamma_t(I) = \text{initial space of } V_t,$$

so that V' is partial isometry-valued with initial projection $\beta'_t(I)$. Let $P_t = V'_t(V'_t)^*$, a projection-valued, local α-cocycle, and let $\beta'' \in \mathrm{Sub}(\alpha)$ be the E-semigroup determined by this P through Theorem 1.9. Then $\beta''_t(I) = P_t = V'_t(V'_t)^*$, hence $\beta \sim_U^\alpha \beta' \sim_{V'}^\alpha \beta''$. Moreover,

$$\beta''_t(I) = V_t\beta'_t(I)V_t^* \leqslant V_tV_t^* = \gamma'_t(I)$$

so that $\beta'' \leqslant \gamma' \leqslant \delta$. Thus $\beta \precsim^\alpha \delta$ as required. $\square$

For the CCR flow the missing piece of the puzzle is provided by Theorem 2.7.

Proposition 3.11. *For the CCR flow σ, if $\alpha, \beta \in \mathrm{Sub}(\sigma)$ with $\alpha \precsim^\sigma \beta$ and $\beta \precsim^\sigma \alpha$, then $\alpha \sim^\sigma \beta$.*

Proof. Let $\alpha, \beta \in \mathrm{Sub}(\sigma)$ with associated projection-valued, local σ-cocycles X^α and X^β, whose stochastic generators are F_α and F_β, where

$$F_\alpha = I_{\mathfrak{h}} \otimes \begin{bmatrix} -\|u_\alpha\|^2 & \langle u_\alpha| \\ |u_\alpha\rangle & -P_\alpha^\perp \end{bmatrix}, \quad F_\beta = I_{\mathfrak{h}} \otimes \begin{bmatrix} -\|u_\beta\|^2 & \langle u_\beta| \\ |u_\beta\rangle & -P_\beta^\perp \end{bmatrix},$$

in the notation (3.1). By part (b) of Theorem 2.7 we have $\alpha \sim^\sigma \beta$ if and only if $P_\alpha \sim P_\beta$, that is if and only if P_α and P_β are equivalent projections.

Now by assumptions there are $\alpha', \beta' \in \mathrm{Sub}(\sigma)$ such that $\alpha \sim^\sigma \alpha'$ with $\alpha' \leqslant \beta$, and $\beta \sim^\sigma \beta'$ with $\beta' \leqslant \alpha$. Maintaining the same notation for the generators of $X^{\alpha'}$ and $X^{\beta'}$, we have, now using part (a) of Theorem 2.7 as well as Lemma 3.1, that

$$P_\alpha \sim P_{\alpha'}, \quad P_{\alpha'} \leqslant P_\beta, \quad P_\beta \sim P_{\beta'} \text{ and } P_{\beta'} \leqslant P_\alpha.$$

That is, $P_\alpha \precsim P_\beta$ and $P_\beta \precsim P_\alpha$, where $\precsim$ here denotes subequivalence of projections in $B(\mathsf{k})$. But this shows that $P_\alpha \sim P_\beta$, and so $\alpha \sim^\sigma \beta$ as required. □

Example 3.12. Let $\{\alpha^{(r)} : r \in [0,1]\} \subset \mathrm{Sub}(\sigma^{\mathfrak{h},\mathsf{k}})$ be the E-semigroups from Example 3.4 where $\mathfrak{h} = \mathbb{C}$, $\mathsf{k} = L^2[0,1]$. Now $P_r \sim I$ for each $r \in]0,1]$ (since $P_r = T_r^* T_r$ and $T_r T_r^* = I$ where $(T_r f)(s) = \sqrt{r} f(rs)$), and so it follows that $\alpha^{(r)} \sim^\sigma \sigma$.

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References

1. Alevras, A., Powers, R. T., and Price, G. L.: Cocycles for one-parameter flows of $B(H)$, *J. Funct. Anal.* **230** (2006) 1–64.
2. Arveson, W.: Subalgebras of C^*-algebras, *Acta Math.* **123** (1969) 141–224.
3. Arveson, W.: *Noncommutative Dynamics and E-semigroups*, Springer-Verlag, New York, 2003.
4. Barreto, S. D., Bhat, B. V. R., Liebscher, V., and Skeide, M.: Type I product systems of Hilbert modules, *J. Funct. Anal.* **212** (2004) 121–181.
5. Bhat, B. V. R.: Cocycles of CCR flows, *Mem. Amer. Math. Soc.* **149** no. 709 (2001).
6. Hudson, R. L. and Parthasarathy, K. R.: Quantum Itô's formula and stochastic evolutions, *Comm. Math. Phys.* **93** (1984) 301–323.
7. Liebscher, V.: A counter-Bernstein example for product systems, Presentation at *International Conference on Quantum Probability and Related Topics Bangalore* (2010).
8. Lindsay, J. M.: Quantum stochastic analysis — an introduction, in: *Quantum Independent Increment Processes*, volume 1865 of *Lecture Notes in Mathematics* (2005) 181–271, Springer, Heidelberg.
9. Lindsay, J. M. and Wills, S. J.: Existence, positivity, and contractivity for quantum stochastic flows with infinite dimensional noise, *Probab. Theory. Related Fields* **116** (2000) 505–543.
10. Lindsay, J. M. and Wills, S. J.: Markovian cocycles on operator algebras, adapted to a Fock filtration, *J. Funct. Anal.* **178** (2000) 269–305.
11. Markiewicz, D. and Powers, R. T.: Local unitary cocycles of E_0-semigroups, *J. Funct. Anal.* **256** (2009) 1511–1543.

12. Powers, R. T.: Continuous spatial semigroups of completely positive maps of $B(h)$, *New York J. Math.* **9** (2003) 165–269.
13. Powers, R. T.: Comparison theory for E-semigroups, Presentation at *Quantum Probability: Noncommutative and Stochastic Analysis, Nottingham* (2006).
14. Wills, S. J.: On the generators of quantum stochastic operator cocycles, *Markov Proc. Related Fields* **13** (2007) 191–211.

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