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A Jump-Diffusion Process for Asset Price with Non-Independent Jumps

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A JUMP-DIFFUSION PROCESS FOR ASSET PRICE WITH NON-INDEPENDENT JUMPS

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Abstract. A market recovery model, defined as a jump-diffusion model for the asset price where the jumps and the diffusion are not independent, is proposed. In this model a jump will be triggered when there is an unusually large downward movement over a certain time interval, and the jump size is correlated to this downward drop. We show that the market data supports such a model and parameter estimates based on market data is discussed. An explicit formula for the risk-neutral drift will be presented so that the option prices based on this model can be computed through Monte-Carlo simulation of the asset price. The characteristic function for the asset price is derived, through which the option prices can be computed by numerical integration. The volatility of asset classes in this model, defined by the variance swap (VIX) equation, is analyzed. A sensitivity study of the volatility with respect to jump parameters is performed. Results are compared to other well-known jump models.

1. Introduction

In the celebrated Black-Scholes model [3] for pricing options, the asset price is governed by the stochastic differential equation

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW \tag{1.1}$$

In this formulation the volatility σ is assumed constant, and the equation implies that the log return of the stock $\ln(S_t) - \ln(S_0)$ is normally distributed. However, from market data the return on asset price exhibit stylized facts that are not supported by the Black-Scholes model, among these are the heavy and non-symmetric tails. Moreover, the market data on option prices across different strikes implied different volatilities, commonly refered to as the volatility smile. Over the years various jump-diffusion models were proposed to address these issues.

One such model was proposed by Merton [17] shortly after the Black-Scholes model appeared. Merton was able to produce an analytic solution to the the option prices for the jump-diffusion model when the jump size was assumed to be lognormally distributed. Subsequently, other jump models were proposed, notably the double exponential jump-diffusion model of Kou [14]. The parameters for the double exponential distribution allow one to address the heavy-tails of the return of the asset price. Amin [1] considered a discrete verions of the model to allow pricing

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of American options. Hanson and Westman [12] considered a model where jump returns are log-uniformly distributed and dealt with parameter estimate problem to fit S&P data. More elaborate models include Andersen and Andreasen [2] with time-dependent volatility to study the volatility surface when fitting the S&P data with the model, and Dufflie et al [10] in which a model where both the asset return and the volatility undergo jump processes. Realizing a jump-diffusion process more realistically represents the market data, advances in mathematical techniques were developed for these models (cf. the book by Cont and Tankov [5]). Of note is the work of Cont and Voltchkova [6] who considered a finite difference scheme to solve the resulting partial integro-differential equation that is stable under very general conditions. Jump-diffusion models were used in pricing options other than European style call options; these include pricing barrier option [6] and Parisian options [13], to list but a few.

In all these models, the jump size and the diffusion terms are assumed independent.

Market data suggests that jumps depend on the market activities over a time interval prior to the jump. We identify jumps in the S&P data using the jump test of Lee and Mykland [15] and analyse the market activity over a short time period $\tau = 5$ days prior to the jumps. We notice clearly that the conditional distribution of the returns, conditioned on having a jump following the period, is distinct from the distribution of the returns without the jump. It is therefore desirable to have a jump-diffusion model where jump and diffusion are not independent. Careful analysis of the market data shows four different jump scenarios: a down market over τ followed by either an upward or downward jump, and an up market over τ followed by either jumps. The probability of each type are different, and they are different depending on the general market trend over which we analyse the data.

For this presentation we will focus on the case when an upward jump follows an unusually large drop in the stock price. To be specific, we propose a model in which an upward jump will occur if during a period prior to the jump, the diffusion term results in an unusually large drop in the asset price. Moreover, the expected size of the jump depends on the magnitude of this drop. Thus our model can be viewed as a market recovery model. Our model is different from the CIR model [7], in which a trendline is built into the drift term in eq (1.1). In [10] the authors considered a model where the instantaneous jump rate λ_t is proportional to the asset price S_t. In our case, the jump rate is a constant parameter of the model, it is used to determine what size of the downward movement is deemed unusual which in turn triggers an upward recovery jump. In assuming that the market will recover, we are creating arbitrage opportunities in our model. This problem can be eliminated if we also allow a probability of having a downward jump followed by the downward prior moverment. In fact, when all four types of jumps are considered, it is easy to assign probabilities so that no arbitrage will result. However, for the sake of clarity, we will consider only one type of jump in this presentation.

We have derived the explicit formula for the characteristic function of the return of our stock model, from which the option prices can be computed by the integral formula due to Madan and Bakshi [16]. Fast Fourier transform techniques were developed by Carr and Madan [4] to handle this computation efficiently. The characteristic function also allows us to compute the skewness of the return and study the relation of the skewness with the jump parameters used in the model. Most importantly, it allow us to obtain the explicit formula for the "risk-neutral drift" in the sense of Gatheral [11] for our model and option prices computed through Monte-Carlo simulation.

The significance of the risk-neutral drift is seen as follows: given a particular model, there is a probability distribution of the stock price at a specified future time T. To price a European option with expiration T, one needs to find a measure Φ so that the expected future stock price satisfies the risk-neutral condition with respect to this measure

$$\mathbb{E}_\Phi(S_t) = e^{rt} S_0 \qquad \forall t > 0 \tag{1.2}$$

and the option price is the present value of the expected payoff with respect to this risk neutral measure. For a call option C with strike K and expiration T,

$$C(S_0, K, T) = e^{-rT} \mathbb{E}_\Phi(\max(S_T - K, 0)) \tag{1.3}$$

This point is illustrated succintly in the standard Black-Scholes model (1.1): by setting $\mu = r$, S_t is risk neutral, the distribution of S_T is lognormal

$$\Phi(s \,|S_0, r, T, \sigma\,) = \frac{1}{s\sigma\sqrt{2\pi T}} \exp\left(-\frac{1}{2\sigma^2 T}\left[\ln(s) - \Big(\ln(S_0) + (r - \tfrac{1}{2}\sigma^2)T\Big)\right]^2\right) \tag{1.4}$$

and despite the ususal explicit formula for the option prices, the European call option can also be expressed as an expected value

$$C(K|S_0, r, T, \sigma) = e^{-rT} \int_K^\infty (s - K)\Phi(s)\, ds \tag{1.5}$$

Other exotic options can be priced, at least in principle, through simulation of the stock price once μ is calibrated with the risk free rate. The expected value in equation (1.5) becomes the sample mean of the call option payoff.

Merton's jump-diffusion model is defined with an extra jump term

$$\frac{dS}{S} = \mu\, dt + \sigma\, dW + dJ \tag{1.6}$$

where one can similarly calibrate μ so that the expected future price is $e^{rt}S_0$. Gatheral [11] refers to this μ as the "risk neutral drift". Instead of switching to a risk-neutral measure and compute option prices using the Radon-Nikodym derivative (cf. [19]), one could replace the asset price model with the risk neutral model. Once this is achieved, other exotic options, for instance Asian options, can be priced through simulating the stock paths using the risk-neutral model. It also allows us to analyze this stock in conjunction with other asset classes for the purpose of hedging or similar risk managements. As pointed out in [8], the expected returns of all asset classes should be the risk-free return.

We now summarize our presentation. In section 2 we review Merton's model, and compare the option price using Merton's formula and through Monte Carlo simulation. We will use the result of this well known example as a bench mark for what is the accepted error in numerical simulations in our model. In section 3 we will analyze the market data, the S&P index from year 2000 to 2016, and show that market data supports our assumption that a positive correction often follows a downward movement. Here we used the criterion of Lee and Mykland [15] to detect the jumps in the market. In section 4 we detail our jump model, we present the characteristic function for $\ln(S_t)$ in closed form, and derives the explicit formula for μ which will put the asset price in the risk-neutral world. Based on this formula, we will simulate in section 5 the stock price and show that it is indeed risk neutral. We will further compute the European option prices for this stock model, we will show that these option prices are significantly different when compared to those of the Black-Scholes and Merton's models. We investigate in section 6 the volatility of our stock model and compare with the volatility of stocks from Merton's jump model. Here volatility of a stock is defined using the option prices via the variance swap equation [9], which is the equation used to compute the VIX index. In section 7 we perform sensitivity analysis of this volatility on the parameters that control the jump statistics of our model. Our formula for μ is exact when we assume the time lag τ, defined as the time interval for the downward movement before the jump, is fixed. We will present in section 8 two examples showing that even if τ is a random variable, there is an approximation formula for μ which is statistically acceptable. In this case Monte-Carlo simulation is the only method to compute option prices since the characteristic function is no longer available. We record in the Appendix an approximation formula for the neutral drift using a Poisson process and present scenarios in which such an approximation becomes useful.

2. Merton's Jump Diffusion Model

Merton's jump diffusion model for the price of a stock S with lognormal jumps is given by the stochastic differential equation

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW + (Y-1)dN_t \tag{2.1}$$

In this equation

N_t is a Poisson process with pararmeter λ :

$$dN_t = \begin{cases} 0 & \text{with probability } 1-\lambda dt \\ 1 & \text{with probability } \lambda dt \end{cases} \tag{2.2}$$

Y is lognormal with parameters ν and δ so that $\mathbb{E}(Y) = e^{\nu+\frac{1}{2}\delta^2}$

$\mu = \alpha - \lambda\left(e^{\nu+\frac{1}{2}\delta^2} - 1\right)$

where α is the instantaneous expected return on the stock. As in Merton we will denote by

$$\begin{aligned} k &= e^{\gamma} - 1 \\ \gamma &= \nu + \tfrac{1}{2}\delta^2 \end{aligned} \tag{2.3}$$

The solution to equation (2.1) is

$$S_t = S_0 \exp\left[(\mu - \tfrac{1}{2}\sigma^2)t + \sigma\sqrt{t}X\right]\prod_{i=1}^{N_t} Y_i \tag{2.4}$$

with expected future value $\mathbb{E}(S_t|S_0) = e^{\alpha t}S_0$. As noted in Merton, by pretending the instanteneous return rate α is the riskless rate r,

$$\mu = r - \lambda k = r - \lambda(e^{\nu+\frac{1}{2}\delta^2} - 1) \tag{2.5}$$

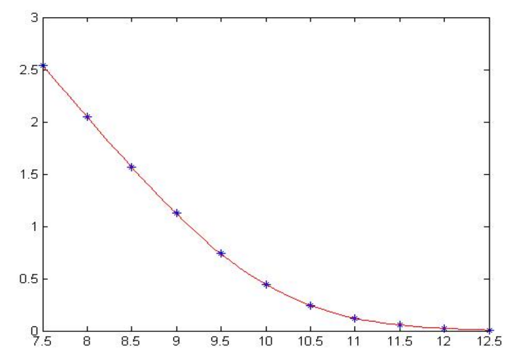
the discounted stock price $e^{-rt}S_t$ is a Martingale. This allows the computation of option prices as expected return of the option, μ in equation (2.5) is the risk-neutral drift in the sense of [11]. The explicit formula for the option price is given in terms of a weighted sum in [17]. For instance the European call option is

$$C(K, S_0, T) = e^{-\lambda' T} \sum_{n=0}^{\infty} \frac{(\lambda' T)^n}{n!} C_{BS}(S_0, K, r_n, T, \sigma_n) \tag{2.6}$$

$$\begin{aligned} \text{where } \lambda' &= \lambda k \\ r_n &= r - \lambda k + \tfrac{n\gamma}{T} \\ \sigma_n^2 &= \sigma^2 + \tfrac{n\delta^2}{T} \end{aligned} \tag{2.7}$$

Here k and γ given by (2.3) and C_{BS} is the Black-Scholes call option price with r_n and σ_n as riskless rate and volatility. Meanwhile, one can simply simulate a sample of the stock prices according to the solution (2.4) with the calibrated μ and compute the sample mean of the payoff. The result is indistinguishable from the explicit formula. To illustrate this point, if we use the parameters for the SDE with $S_0 = \$10$, $\sigma = .2$, $T = .25$ and $r = .02$ the following graph (figure 1) is produced

FIGURE 1. Option price theoretical verses Monte Carlo, case 1.

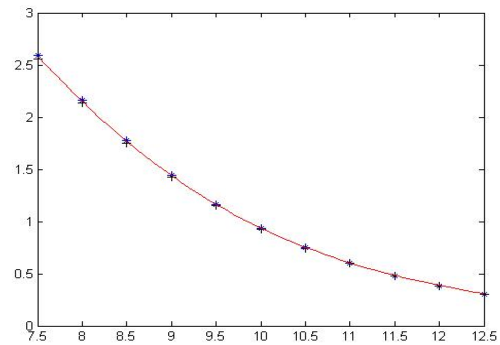


$\nu = .03$, $\delta = .01$, $\lambda = 4$, option price from formula (2.6) in red, simulated sample mean in blue

The blue points are from simulating 10,000 sample prices S_T using (2.4), then the payoff $\max(S_T - K, 0)$ of the call options with strikes K from \$7.50 to \$12.50 are computed. The price of each option is the discounted sample mean. The red curve is from Merton's formula (2.6), where we sum the first 20 terms.

Even with extreme parameters this method gives very acurate results. Figure (2) compares the simulation using (2.4) with Merton's formula (2.6) with $\lambda = 8$, $\nu = .1$ and $\delta = .1$. These parameters implies a 20% return due to the jumps over the $T = 3-$month period. We have also computed the option prices by simulating the stock path using the stochastic differential equation (2.1) by brute force, without any variance reduction techniques, and the results are remarkably accurate.

FIGURE 2. Option price theoretical verses Monte Carlo, case 2.



$\nu = .1$, $\delta = .1$, $\lambda = 8$, red=formula (2.6), blue=simulation using (2.1), black=simulation using (2.4)

The simplicity of this Monte Carlo approach makes clear the importance of being able to calibrate the parameters in the stochastic differential equation to the risk-free return.

3. Market Data Analysis

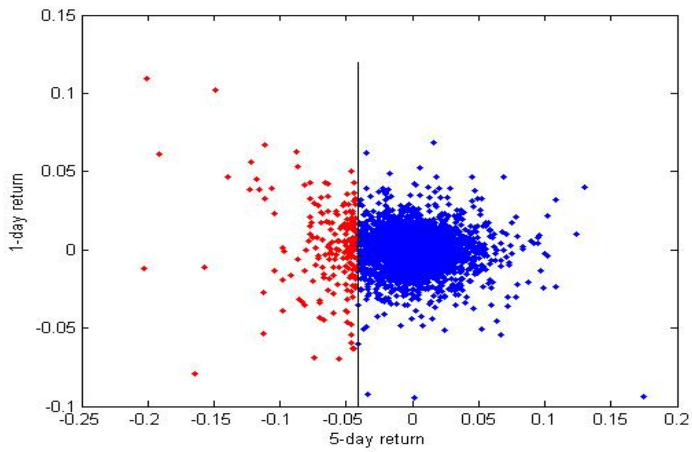
In this section we present results from statistical analysis of S&P data from the first day of year 2000 to the last day of year 2016. The goal of the analyses was to explore whether the assumption, about independence between the jump term and the geometric Brownian motion term in eq.(2.1), holds for real data (-in our case, S&P data).

From the S&P data we have a sample of 4271 5-day returns and the follow up 1-day returns, plotted in figure 3. Here d-day returns ($d = 1$ or 5) were calculated as

$$\frac{SP(t_i) - SP(t_{i-d})}{SP(t_{i-d})} \tag{3.1}$$

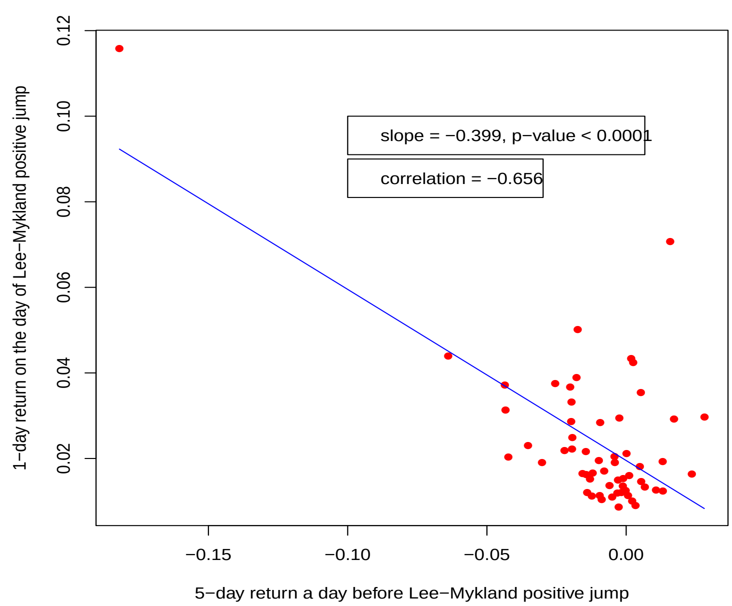
where $SP(t_i)$ denotes the value of the S&P data on day t_i. Suppose we pick a subset of N sample points consisting of the lowest 5-day returns, depicted as the points to the left of the vertical line in figure 3. Consider the vertical coordinates of the whole sample and the sub-sample. The hypothesis that these 2 samples are identically distributed is rejected by the Kolmogorov-Smirnoff test with $p < 10^{-5}$ for N ranges from 30 to 1000 points. The rejection persists even when outliers (50 top 1-day returns and 50 bottom 1-day returns) are removed, with N ranges from 100 to 1000. This implies that the distribution for the follow-up 1-day returns for the whole sample is different from the (conditional) distribution of the follow-up 1-day returns just after a low 5-day return. We postulate that this difference is caused by some type of jump process.

FIGURE 3. Data from S&P index.



5-day verses follow-up 1-day return, left of vertical line consists of $N = 200$ points

FIGURE 4. 5-day verses 1-day return among days with jumps.



Lee-Mykland test with a window-size of $K = 15$ was used to identify the jumps. We considered only jumps for which the Lee-Mykland statistic was above 2.97, so that the jumps identified were all upward jumps. The threshold 2.97 corresponded to 0.05 significance level for rejecting the null hypothesis of no jump at each time point. In the figure 4, we plotted the 1-day return on the day of a jump on the y-axis and 5-day return a day before the jump on x-axis. The blue line in the figure is the regression line fit for the scatter plot; the slope of the regression line was estimated to be -0.399, which was significantly different from zero (p-value < 0.0001). As clearly seen from the figure, the 1-day returns on the day of the jump are highly dependent on the 5-day return the day prior to the jump, overall. If we consider 1-day return on the day of the jump as a proxy-measure for jump size, then this analysis provides good support for the fact that for upward jumps, the jump-size is correlated (negatively) to the drop just prior to it. These figures support a model where the jumps, and their jump size, are correlated with the market activities prior to the jump.

In the analyses so far, we focused on the relation between 1-day return and the prior 5-day returns because 5 days correspond to a week. A typical scenario would be to imagine that if there was drop from Monday to Friday of a week, then the week's activity will be pondered and digested over the weekend, and the market correction occurs the Monday following the weekend. Of course the market does not behave in such a compartmental form. Hence we made further exploration by changing the length of the prior interval to 1 day. We considered the lowest 1% and the lowest 5% 1-day returns and plotted the distributions of the 1-day returns a day after such low returns (top two panels in figure 5). The distributions in the top two panels were different from the distribution of 1-day return after a down-day in general (plotted in the bottom panel). (The distributions were compared based on kurtosis, skewness and Kolmogorov-Smirnoff test.) The difference in these distributions show that the behavior of 1-day returns after a large drop in S&P values is different from that of 1-day returns after just any down-day. Thus S&P data shows similar pattern when we change the length of the prior interval, as illustrated in the panels above. This indicates that, irrespective of the length of the prior window, the jumps and brownian motion prior to jumps may not be completely independent.

4. The Market Recovery Model

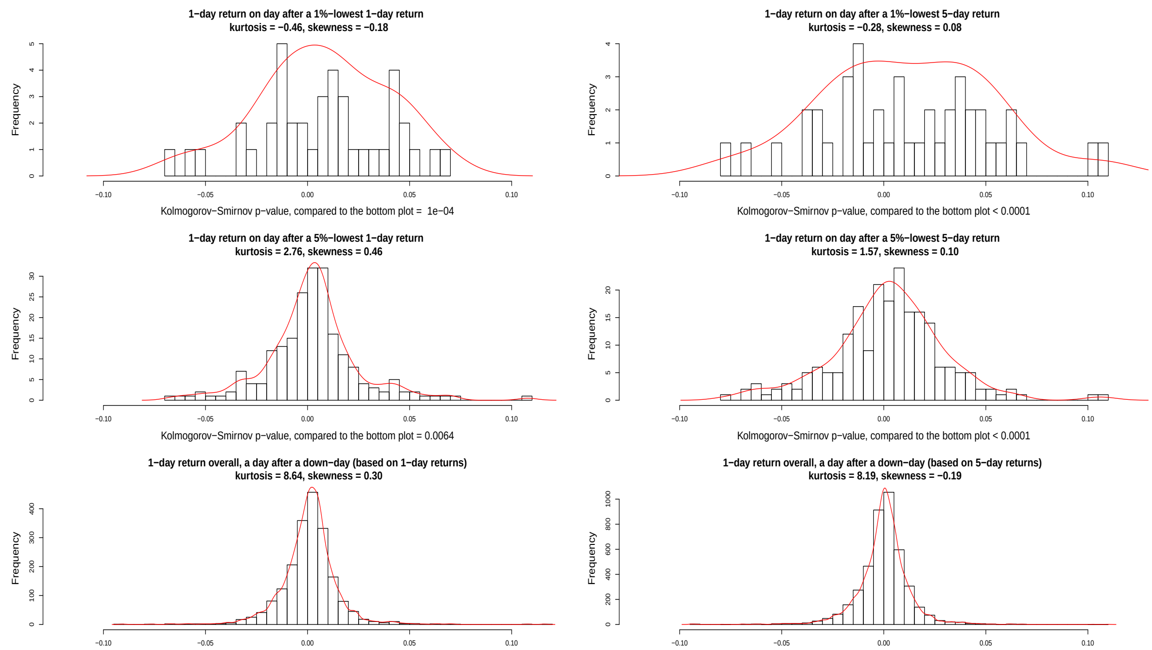
In this section we will present our asset price model and derive the characteristic function in closed form. This is used to find the risk-neutral drift parameter.

Merton's model assumes that the Poisson process N_t and the Wiener process W_t are independent. As we have seen from the S&P500 data, this independence assumption is too simplistic. We will build in a tendency of an upward jump when the stock suffers from an unusually large negative return over a period prior to the jump. We will analyse the effect on option prices when such a correlation between the jump and diffusion terms is included in the asset price model.

More precisely, we fix a τ and a bound bd; a jump is triggered at time $t+\tau$ if $W_{t+\tau} - W_t < bd\sqrt{\tau}$. That is, if the drop is worse than a pre-determined amount, the jump mechanism will be triggered. Since $W_{t+\tau} - W_t = Z\sqrt{\tau}$ where Z is normally distributed with mean 0 and variance 1, we can compute the probability of having a jump. Define λ by

$$\lambda = \frac{1}{\tau\sqrt{2\pi}} \int_{-\infty}^{bd} e^{-\frac{1}{2}z^2}\, dz, \tag{4.1}$$

FIGURE 5. Statistical analysis on jump returns.



For the figure in the top-left panel, the days for which 1-day returns were in the lowest 1% percentile were identified, and the histogram of 1-day returns on the day after such days was plotted. The red curve is the kernel density estimate. The figure in the middle-left panel is the similar to the top-left panel, except that we expanded the number of days to include the days for which 1-day returns were in the lowest 5% percentile. The figure in the bottom-left panel includes all days for which the 1-day return denoted a drop. The figures in the right panel mimic those in the left panel - the only difference is that the days considered are based on the lowest percentiles of 5-day returns instead of lowest percentiles of 1-day returns.

then $\lambda\tau$ is the probability of having a jump over the time interval of length τ, so we expect λ jumps per year. We will use λ and τ as parameters for our jump model, and bd is determined once λ and τ are fixed. Duffie et al [10] considered an affine jump-diffusion model in which the instantaneous jump rate λ_t depends linearly on the asset price. In our case the jump rate is a constant parameter of our model that is estimated based on market data. We also postulate, similar to Merton's model that the jump $Y = e^y$ is lognormally distributed, where y has mean and standard deviation depending linearly on the size of the prior drop. So over the time interval

$[t, t+\tau]$ in which a jump occurs, we have

$$S_{t+\tau} = S_t \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}Z\right]\exp(y) \tag{4.2}$$

$$\text{where } y \sim N(\nu, \delta), \quad \nu = \alpha\sigma\sqrt{\tau}Z, \quad \delta = \beta\sigma\sqrt{\tau}Z, \tag{4.3}$$

and Z has density function of a normal distribution conditioned on the set $\{z < bd\}$. Explicitly, denote by I the indicator function, the pdf of Z takes the form

$$\frac{1}{\lambda\tau\sqrt{2\pi}}\, e^{-z^2/2} I(z < bd). \tag{4.4}$$

Here ν and δ depend linearly on Z for convenience, we will be able to tune the parameters so that the stock price obeys the risk-neutral condition. Since $Z < bd$, is always negative, β must be negative also. As a market recovery model, since bd is negative, an appropriate choice of α and β will recover a desired portion of the prior drop, and a negative α will cause a positive jump.

Over a time interval $[t, t+\tau]$ in which there is no jump we have

$$S_{t+\tau} = S_t \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}X\right] \tag{4.5}$$

$$\text{where } X \text{ is conditional normal with pdf } \frac{1}{(1-\lambda\tau)\sqrt{2\pi}}\, e^{-x^2/2},\ x > bd \tag{4.6}$$

Equations (4.2) and (4.5) allow us to analyze the future stock price S_T in a binomial distribution framework in which $\lambda\tau$ is the probability of success. In the event where the time interval τ is small compared to a future time T, for instance the expiration time of an option, the jumps are rare and can be approximated by a Poisson process with parameter λT. Our results indicate that even if we choose $\tau = 1-$week and $T = 4-$months, Poisson approximation is very acurate, with risk-neutral drift parameter correct to 4-decimal places. In the appendix we will derive the risk-neutral drfit under Poisson approximation and list its benefits.

Assume there are n jumps in the time interval $[0, T]$, we divide the time interval into $N = (m+n)$ blocks of length τ so that $N\tau = T$. The return on each block is independent to one another. The price at time T is a product

$$S_T \mid n \text{ jumps } = S_0\, e^{(\mu - \frac{1}{2}\sigma^2)T} \times \exp\left(\sigma\sqrt{\tau}\sum_{j=1}^{m} X_j\right) \times \prod_{i=1}^{n} \exp\left(\sigma Z_i\sqrt{\tau} + y_i\right) \tag{4.7}$$

involving the returns over m blocks with no jump and n blocks with jumps y_i.

The characteristic function $\phi_T(\xi)$, defined as the expected value of $\exp(i\xi\ln(S_T))$, is

$$\phi_T(\xi) = \sum_{n=0}^{N} B_n \mathbb{E}(\exp(i\xi \ln(S_T)) \mid n \text{ jumps}) \tag{4.8}$$

$$= \sum_{n=0}^{N} B_n \mathbb{E}(\exp(i\xi\Theta_n)) \tag{4.9}$$

$$\text{where } B_n = \binom{n+m}{n} (\lambda\tau)^n (1-\lambda\tau)^m \text{ is the binomial distribution} \tag{4.10}$$

$$\text{and } \Theta_n = \ln(S_0) + \left(\mu - \tfrac{1}{2}\sigma^2\right) T + \left(\sigma\sqrt{\tau}\sum_{j=1}^{m} X_j\right) + \sum_{i=1}^{n} \left(\sigma Z_i \sqrt{\tau} + y_i\right) \tag{4.11}$$

is from equation (4.7)

The expected value of $\exp(i\xi\Theta_n)$ is a product, using the pdf of Z, y and X from equations (4.4), (4.3) and (4.6) we have

$$\mathbb{E}(\exp(i\xi\Theta_n)) = S_0^{i\xi} e^{i\xi\left(\mu - \frac{1}{2}\sigma^2\right)T} A^m B^n \tag{4.12}$$

$$\begin{aligned} A &= \mathbb{E}(\exp(i\xi\sigma\sqrt{\tau}X) \\ &= \tfrac{1}{2(1-\lambda\tau)} \exp\left(-\tfrac{1}{2}\xi^2\sigma^2\tau\right) \left[1 - \operatorname{erf}\left(\tfrac{bd - i\xi\sigma\sqrt{\tau}}{\sqrt{2}}\right)\right] \end{aligned} \tag{4.13}$$

$$\begin{aligned} B &= \mathbb{E}(\exp i\xi(\sigma\sqrt{\tau}Z + y)) \\ &= \tfrac{1}{2Q\lambda\tau} \exp\left(-\tfrac{P^2}{2Q^2}\right) \left[1 + \operatorname{erf}\left(\tfrac{bd\,Q^2 - iP}{\sqrt{2}\,Q}\right)\right] \end{aligned} \tag{4.14}$$

$$\text{where } P = \xi\sigma\sqrt{\tau}(\alpha+1) \text{ and } Q = \sqrt{\xi^2\sigma^2\beta^2\tau + 1} \tag{4.15}$$

Combining equations (4.9), (4.10) and (4.12), together with $N = T/\tau$ we obtain the characteristic function in closed form

$$\begin{aligned} \phi_T(\xi) &= S_0^{i\xi}\, e^{i\xi(\mu - \frac{1}{2}\sigma^2)T} \left((1-\lambda\tau)A + \lambda\tau B\right)^{T/\tau} \\ &= S_0^{i\xi}\, e^{i\xi(\mu - \frac{1}{2}\sigma^2)T} \left(\tilde{A} + \tilde{B}\right)^{T/\tau} \end{aligned} \tag{4.16}$$

$$\begin{aligned} \text{where } \tilde{A} &= \tfrac{1}{2} \exp\left(-\tfrac{1}{2}\xi^2\sigma^2\tau\right) \left[1 - \operatorname{erf}\left(\tfrac{bd - i\xi\sigma\sqrt{\tau}}{\sqrt{2}}\right)\right] \\ \text{and } \tilde{B} &= \tfrac{1}{2Q} \exp\left(-\tfrac{P^2}{2Q^2}\right) \left[1 + \operatorname{erf}\left(\tfrac{bd\,Q^2 - iP}{\sqrt{2}\,Q}\right)\right] \end{aligned} \tag{4.17}$$

The expected future value of the stock price is easily obtained by $\mathbb{E}(S_T) = \phi_T(-i)$. Note that the future time T appears linearly in the exponent on ϕ_T, thus it is possible to solve the risk neutral equation for μ

$$\mathbb{E}(S_T) = S_0 e^{rT} \tag{4.18}$$

$$\mu = r + \frac{1}{2}\sigma^2 - \frac{1}{\tau}\ln\left[\overline{A} + \overline{B}\right] \tag{4.19}$$

where $\overline{A} = \frac{1}{2}\exp\left(\frac{1}{2}\sigma^2\tau\right)\left[1 - \text{erf}\left(\frac{bd-\sigma\sqrt{\tau}}{\sqrt{2}}\right)\right]$

$$\text{and } \overline{B} = \frac{1}{2\sqrt{1-\sigma^2\beta^2\tau}}\exp\left(\frac{\sigma^2\tau(\alpha+1)^2}{2(1-\sigma^2\beta^2\tau)}\right)\left[1 + \text{erf}\left(\frac{bd(1-\sigma^2\beta^2\tau)-\sigma\sqrt{\tau}(\alpha+1)}{\sqrt{2(1-\sigma^2\beta^2\tau)}}\right)\right] \tag{4.20}$$

$$\tau = \frac{1}{2\lambda}\left[1 + \text{erf}\left(\frac{bd}{\sqrt{2}}\right)\right]$$

This is the drift rate with which our asset price model is risk-neutral. With this drift rate, we can compute the European style option prices by Monte-Carlo simulation.

The characteristic function also allows us to express the European call option price $C(K, S_0, T)$ as an integral [16].

$$C(K, S_0, T) = S_0\Pi_1 - Ke^{-rT}\Pi_2 \tag{4.21}$$

$$\Pi_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \text{Re}\left(\frac{e^{-i\xi\ln(K)}\phi_T(\xi - i)}{i\xi\phi_T(-i)}\right)d\xi \tag{4.22}$$

$$\Pi_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \text{Re}\left(\frac{e^{-i\xi\ln(K)}\phi_T(\xi)}{i\xi}\right)d\xi \tag{4.23}$$

These integrals will be evaluated numerically once the parameters of our model are chosen.

5. Monte-Carlo Simulation and Option Price Comparisons

The expression for μ in (4.19) gives the stock price that satisfies the risk-neutral condition, therefore we can compute the call option prices through Monte-Carlo simulation. We will measure the accuracy of the Monte-Carlo option prices against the theoretical computation resulted in (4.21). We are interested in a variation of our model in which explicit form of the characteristic function is not available, and Monte-Carlo simulation is the only method to compute option prices.

For our stock price simulation, we will let $r = .02$, $\sigma = .2$, $T = 1$ year, $\lambda = 4$ jumps per year, time step $dt = 1/2520$, which is 10 steps per day, and initial stock price $S_0 = 10$. To simulate our "market recovery" model, we will let $\tau = \frac{5}{252}$, 5 trading days. With $\lambda = 4$ we can compute $bd = -1.4094$ from equation (4.1). We let $\alpha = -.57$ and $\beta = -.19$ in equation (4.3). So we are recovering 57% of the 5−day drop prior to the jump on average, and occasionally (2.26 standard deviations to the right) we recover the whole drop. With $bd = -1.4094$, the expected value of Z in equation (4.4) is $\mathbb{E}(Z) = -.1862$, thus $\mathbb{E}(\nu) = \alpha\sigma\sqrt{\tau}\mathbb{E}(Z) = .03$ and $\mathbb{E}(\delta) = \beta\sigma\sqrt{\tau}\mathbb{E}(Z) = .01$, and $y \sim N(.03, .01)$. We will compare our results with the results of Merton's when the jumps have the same statistics. Thus the jumps in both models are statistically indistinguishable, except in Merton's case the jumps occur randomly, and in the market recovery model the jumps are strategically placed.

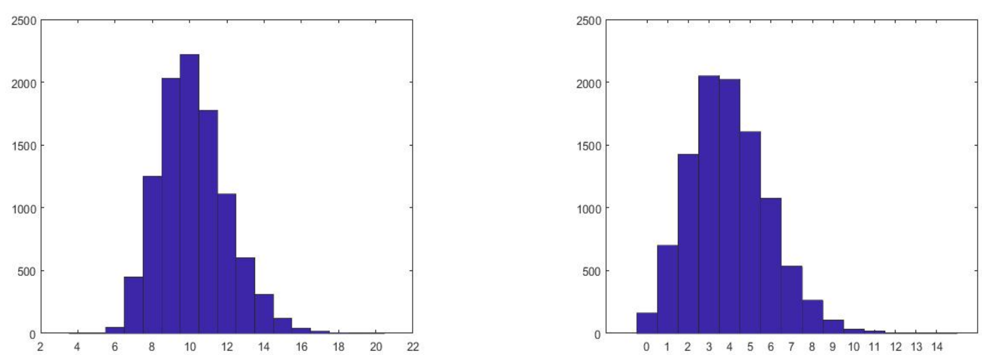
With these parameters, $\mu = -.0950$ computed according to equation(4.19). We simulate a stream of 2520 normally distributed random numbers, and look at blocks of 50 which corresponds to a 5−day activites. If the sum of a block is less than $bd\sqrt{50}$ we will add an appropriately generated random $Y = e^y$ at the end of the block:

$$S_{i+1} = S_i(1 + \mu\, dt + \sigma X_i \sqrt{dt} + e^y - 1) \tag{5.1}$$

$$\begin{aligned} \text{where } Z &= \frac{1}{\sqrt{50}} \sum_{j=i-49}^{i} X_j < bd = -1.4094 \\ y &\sim N(\alpha\sigma\sqrt{\tau}Z, \beta\sigma\sqrt{\tau}Z) = N(-.0161Z, -.0054Z) \end{aligned} \tag{5.2}$$

We simulate 10000 run, the present value of the sample mean is 9.9746 with standard deviation 1.7938. This is quite accurate compared to the theoretical mean of $S_0 = 10$. We also record the average number of jumps for our sample is 3.9671 per year, which is about 1% off from the theoretical $\lambda = 4$. Here are the histograms of the stock prices S_T and the number of jumps of the sample (figure 6).

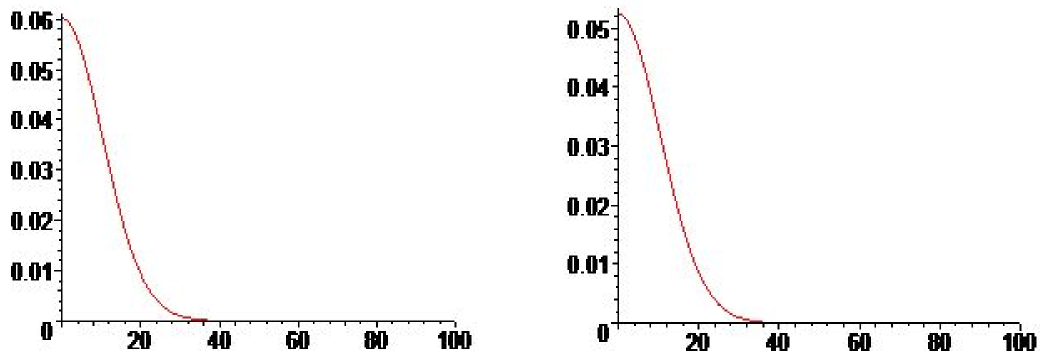
FIGURE 6. Jump model simulations.



price simulation jumps per run

For the option prices we let $T = 1/4$ to capture one jump on average, with strikes ranging from \$9.00 to \$11.00. In computing the integrals in (4.21) we notice that the integrands are very tame, and taper off at $\xi \sim 50$, as depicted in figure (7).

FIGURE 7. Madan Bakshi integrand.



integrand in Π_1, $K = 9.5$

integrand in Π_2, $K = 9.5$

A simple Riemann sum with $\Delta\xi = .01$ and ξ from 0 to 50 is used to produce the option prices. Simulation is done with a 10000 sample of prices at expiration and sample means of the option payoff at each strike are discounted to present. The result is tabulated in table 1:

TABLE 1. Option prices: simulation verse integration

Strike	from Monte-Carlo	from eq (4.21)	% error
9.00	1.0698	1.0816	-1.0896
9.25	0.8587	0.8691	-1.1940
9.50	0.6700	0.6780	-1.1758
9.75	0.5069	0.5124	-1.0768
10.00	0.3713	0.3747	-0.9009
10.25	0.2630	0.2651	-0.7998
10.50	0.1808	0.1814	-0.3418
10.75	0.1206	0.1200	0.5084
11.00	0.0784	0.0768	2.0801

We see that the simulated results are, when the strikes are not too far from the initial stock price, accurate to within 2%. One will need some importance sampling techniques to price options when the strikes are way out of the money.

Comparing the option prices from our model with those of Merton's, the difference is significant. Table 2 lists the prices from the Black-Scholes model, Merton's model and our market recovery model. The % differences are computed against the Black-Scholes prices.

TABLE 2. Option prices comparison

Strike	Black-Scholes	Merton	% difference	Market Recovery	% difference
9.00	1.1093	1.1185	0.8285	1.0698	-3.5602
9.25	0.9055	0.9179	1.3719	0.8587	-5.1636
9.50	0.7214	0.7368	2.1396	0.6700	-7.1168
9.75	0.5600	0.5779	3.1898	0.5069	-9.4911
10.00	0.4232	0.4426	4.5802	0.3713	-12.2613
10.25	0.3111	0.3309	6.3714	0.2630	-15.4623
10.50	0.2223	0.2415	8.6256	0.1808	-18.6859
10.75	0.1545	0.1721	11.4028	0.1206	-21.9274
11.00	0.1044	0.1198	14.7576	0.0784	-24.9023

% differences are differences from the Black-Scholes prices

Notice that when jumps are introduced as in Merton's model, it has the effect of increasing the volatility, thus the option prices are higher. As was mentioned earlier, our market recovery model reduces the volatility, and the option prices are impacted in a very significant way even when only one jump is anticipated. We have deliberately chosen the parameters and a one-sided jump to illustrate the impact. The effect on option prices will not be this pronounced when jumps in both directions are incorporated into the model. Results on a 2-sided jump model will be reported in a separate publication.

6. Volatility Comparison

In this section we will compare the volatility of the of our stock model and that of Merton's. Here volatility of a stock σ_{op} is defined by the variance swap equation [9] using the option prices, the equation relevant for our purpose is

$$\sigma_{op}^2 = \frac{2}{T} e^{rT} \left[\int_0^{S_f} \frac{1}{K^2} P(K)\, dK + \int_{S_f}^{\infty} \frac{1}{K^2} C(K)\, dK \right] \tag{6.1}$$

where K is the strike price, $S_f = e^{rT} S_0$ is the expected forward stock price, $P(K)$ and $C(K)$ are the put and call prices at strike K. This equation is applicable when the option prices are available at $K = S_f$, which is rarely the case in the options market. A slightly more involved equation is used to deal with this lack of market data, resulting in the formula that is used for the VIX index computation.

Table 3 compares the result of the volatility computation for Merton's Jump model and the market recovery mode. Here $S_0 = 10$, $r = .02$ and $\sigma = .2$ as before, and $\alpha = -.57$ and $\beta = -.19$ corresponding to $\nu = .03$ and $\delta = .01$ in Merton. We integrate over the interval $K \in [2.5, 25]$ with $dK = .1$ in a straight forward rectangle method. The standard Black-Scholes model is included to show the integrity of this numerical approximation.

Table 3. Volatility comparison

T	Black-Scholes	Merton	Market Recovery
0.25	0.2002	0.2100	0.1766
0.50	0.2001	0.2100	0.1765
0.75	0.2001	0.2099	0.1764
1.00	0.2000	0.2099	0.1768
1.25	0.2000	0.2099	0.1772
1.50	0.2000	0.2099	0.1771
1.75	0.2000	0.2099	0.1769
2.00	0.2000	0.2099	0.1772

We see that the Black-Scholes column yield very accurately the theoretical result $\sigma = .2$. The Merton option prices are computed using the weighted sum in eq (2.6). The volatility $\sigma_{op} = .21$ is independent of the expiration T, as it ought to, the jump rule is time independent. The option prices for the market recovery model is from a simulation of 100,000 stock paths and record the stock prices at each expiration time, and the expected return of the options computed from this sample. Similar to Merton's model, the volatility $\sigma_{op} = .1768$ ought to be $T-$independent. Table 4 compares the two jump model option prices reported earlier in table 2 with the Black-Scholes option prices using their corresponding volatilities.

Table 4. option prices verses Black-Scholes prices with σ_{op}

Strike	Merton model option price	Black-Scholes price with $\sigma = .21$	Market Recovery option price	Black-Scholes price with $\sigma = .1768$
9.00	1.1185	1.1199	1.0581	1.0873
9.25	0.9179	0.9192	0.8485	0.8754
9.50	0.7368	0.7380	0.6609	0.6838
9.75	0.5779	0.5788	0.4996	0.5168
10.00	0.4426	0.4431	0.3666	0.3772
10.25	0.3309	0.3308	0.2604	0.2654
10.50	0.2415	0.2409	0.1783	0.1799
10.75	0.1721	0.1711	0.1186	0.1175
11.00	0.1198	0.1185	0.0767	0.0739

Numerical results in table 4 shows that a good estimate of the "effective volatility" σ_{op} will allow an estimate of option prices to within 1% for Merton's model, and to within 3% for the Market Recovery Model. Unfortunately, we do not at present have a simple formula for σ_{op}. In Merton's case, even though the option price is given by a weighted sum of Black-Scholes prices, the risk-free rate r_n in each summand is different. As a result the forward prices are different and the volatility σ_{op} is not a weighted sum of the σ_n.

7. Sensitivity Analysis

We now report on a sensitivity study of σ_{op} with respect to the jump parameters. In Merton's case, the jump parameters are ν and δ representing the mean and standard deviation of the jump returns. σ_{op} increases as either ν or δ increases. However, σ_{op} decreases as δ increases in the market recovery model. Tables 5 and 6 illustrate our observations. We set $T = \frac{1}{4}$ and $\lambda = 4$ to capture one jump on average within the expiration period, and we let $\tau = \frac{5}{252}$ and $\lambda\tau$ determines the bound *bd* for the market recovery model. Letting $\sigma = .2$ as before the relations (5.2) are explicitly

$$\nu = -.0525\alpha, \qquad \delta = -.0525\beta$$

We choose 5 values of α and β centered around $\alpha = -.57$ and $\beta = -.19$, which were the parameters used in the previous tabulations. The corresponding values of ν are

$\alpha =$	$-.37$	$-.47$	$-.57$	$-.67$	$-.77$
$\nu =$	.019425	.024675	.029925	.035175	.040425

so the jump magnitude is roughly 2 to 4 percent. The corresponding values for δ are

$\beta =$	$-.09$	$-.14$	$-.19$	$-.24$	$-.29$
$\delta =$	.00472	.007350	.009975	.012600	.015225

As before for each pair (ν_i, δ_j) of parameters with compute option prices for Merton's Jump model from the sum (2.6).

TABLE 5. σ_{op} for Merton's model

	ν_1	ν_2	ν_3	ν_4	ν_5
δ_1	0.2041	0.2045	0.2049	0.2055	0.2062
δ_2	0.2064	0.2067	0.2072	0.2078	0.2085
δ_3	0.2092	0.2095	0.2100	0.2105	0.2113
δ_4	0.2125	0.2128	0.2133	0.2138	0.2145
δ_5	0.2163	0.2166	0.2170	0.2176	0.2183

For each pair of parameters (α_i, β_j) we simulate 100,000 sample stock prices to compute option prices for the Market Recovery model. We have used the common random number method to reduce the variance in the computation of σ_{op}. With $dt = \frac{1}{2520}$ we need 630 random numbers for each stock path to expiration T for the diffusion term. We generate 100,000 streams of 630 random numbers and this same set of random numbers are used in each of the 25 cases in the tabulation, thus the randomness is in generating the jump sizes only.

TABLE 6. σ_{op} for Market Recovery model

	α_1	α_2	α_3	α_4	α_5
β_1	0.1825	0.1829	0.1834	0.1840	0.1847
β_2	0.1789	0.1793	0.1797	0.1804	0.1811
β_3	0.1758	0.1762	0.1766	0.1774	0.1780
β_4	0.1732	0.1735	0.1739	0.1746	0.1755
β_5	0.1710	0.1713	0.1718	0.1724	0.1733

We see that as the variance of the jump size increases (i.e., as β becomes more negative), the σ_{op} decreases for the Market Recovery model, the opposite behaviour occurs in Merton's model.

8. Models with a Random Jump Time

Among the four parameters that we have introduced into our jump model, three of them, λ which governs the average number of jumps per year, α and β which govern the size of the jumps, can be estimated based on market data. This is in spirit similar to the volatility σ, which is assumed constant over a short period of time. The fourth parameter τ, which governs the duration of the downward movement before a jump occurs, should vary from jump to jump. The assumption that τ is a constant is quite unreasonable. Meanwhile the expression for μ requires a constant value for τ. The purpose of this section is to present a scenario in which the expression for μ gives the stock price that satisfies the risk-neutral condition even if τ is a random variable, as long as its expected value is known.

As before, let τ be the duration of a time interval in which the stock drops by an unusual amount and the jump mechanism is triggered. Instead of simulating a jump right at the end of the time interval, we consider another time window of length τ_1 during which a jump will occur. To simulate the stock path we need a probability density function $p(t)$, $t \in [0, \tau_1]$ for the random variable T_j, the time that a jump occurs once it is realized that a correction is in order. There is a family of functions we may choose from based on the notion of a survival function [18]; Define

$$S(t) = \left(\frac{\tau_1 - t}{\tau_1} \right)^{\kappa} \tag{8.1}$$

$$h(t) = -\frac{S'}{S} = \frac{\kappa}{\tau_1 - t} \tag{8.2}$$

Here $S(t) = P(T_j > t)$ and it is zero by the time $t = \tau_1$, $h(t)$ is the hazard rate, $h(t)dt = P(t < T_j < t + dt \,|\, T_j > t)$. The hazard rate function is used to simulate the stock path. The probabilty density function of T_j is $S(t)h(t)$ and the expected value is easily computed as $\frac{\tau_1}{\kappa+1}$. Thus the expected time from the start of the market drop to the time of the jump is

$$\tau_{mean} = \tau + \frac{\tau_1}{\kappa + 1} \tag{8.3}$$

κ is a gauge of the delay in response to the drop, we can fix τ_{mean} at any point within the allowed interval by choosing κ appropriately.

For the purpose of our simulation, we will set $\tau_1 = 1/252$, within 1 day after an unusual drop over the period $\tau = 5/252$ as before, and $\kappa = 4$ so that $\tau_{mean} = 5.2$ days according to equation (8.3), $bd = -1.3882$ is computed with respect to τ_{mean} by equation (4.1). We will use $\alpha = -\frac{2}{3}$ and $\beta = -\frac{1}{9}$ so that on rare occasions, 3 standard deviations to the right, we will recover the whole drop. With these parameters $\mu = -.1160$ by equation (4.19). We inspect blocks of 50 to determine when a jump is triggered as in the previous situation, and we generate a stream of uniformly distributed random numbers to compare against the hazard rate function (8.2) to determine the exact time of the jump. The jump size is based on the actual drop through this simulated jump time. With a sample of 10000, the present value of the expected future stock price is \$9.9848, with standard deviation 1.7513. This is consistent with the results we have obtained in our previous examples. We note that the jump process we are presenting here is not formally a binomial process, and the risk-neutral drift formula based on a Poisson process is more appropriate. However, μ computed by equation (A.12) using the Poisson approximation produces identical value to 4-decimal places. Since we are using a bound bd which is slightly higher than what is called for for a 5-day interval, the average number of jumps is higher at $\lambda^* = 4.1219$. Here λ^* should obey the formula

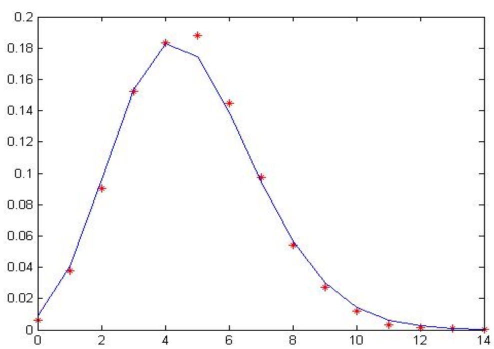
$$\lambda^* = \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{bd} e^{-x^2/2}\, dx \tag{8.4}$$

$$= 4.1599 \quad \text{when } bd = -1.3882 \text{ and } \tau = 5/252 \tag{8.5}$$

As in the previous case with constant τ, the sample λ^* is off from the theoretical value by about 1%. Similar results are obtained when κ is decreased.

When we increased τ_1, that is, when we allow a longer time for the jump to take place, frequently the market has already recovered by the simulated jump time. This is in theory possible for our case when $\tau_1 = 1$ day but we have not observed it. To report on an extreme case, when we set $\tau_1 = 5/252$, then $\tau_{mean} = 6$ days, to offset the extra number of jumps being picked up by a much higher $bd = -1.3092$, we will not simulate a jump if the market has already recoverd by the simulated jump time. A sample of 10000 produces the value of \$10.0691, with standard deviation 1.7366. The average number of jumps $\lambda^* = 4.7701$, and the computed value from equation (8.4) is 4.7998. The difference is on par with those of the previous situations. There were 120 instances where jumps were not simulated due to the auto recovery of the market. The distribution of the number of jumps from our sample and the Poisson distribution with this sample mean is compared in the following figure:

FIGURE 8. Jump statistics.



Poisson distribution with $\lambda^* = 4.7701$ in blue, sample jump percentage in red

These examples show that the calibration formula (4.19) will produce risk neutral stock models in a variety of situations. We may also consider choosing a survival function $S(t)$ that does not go to 0, which will imply that there is a non-zero probability that a jump will not occur even when the jump process is triggered. This will require a formula for λ^* more elaborate than eq. (8.4) to estimate the average number of jumps per year.

Appendix A. Risk-neutral Drift for Poisson Jump Model

The model we have presented assumes a binomial probability framework in which the time interval is partitioned into subintervals of length τ. From a simulation perspective, this is quite restrictive. For instance it is more natural to have a jump triggered sooner, if a bigger drop has already occured by time τ' prior to τ so that the corresponding bd' is such that (τ', bd') obeys the relation (4.1):

$$\lambda\tau' = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{bd'} e^{-\frac{1}{2}z^2}\, dz$$

As a result, this jump process is a Poisson process.

There could be other rules for simulation in which a Poisson jump process is more desirable. For instance if the jump is triggered when the stock is compared to and index and the gap exceeds a given bound bd, then τ from eq. (4.1) does not correspond to any physical time.

In this appendix we present the risk-neutral drift formula for a Poisson jump process. From equation (4.7) the price at time T is given by

$$S_T \mid n \text{ jumps } = S_0\, e^{\left(\mu - \frac{1}{2}\sigma^2\right)T} \times \exp\left(\sigma\sqrt{\tau}\sum_{j=1}^{m} X_j\right) \times \prod_{i=1}^{n} \exp\left(\sigma Z_i \sqrt{\tau} + y_i\right)$$

where $T = (m+n)\tau$ and X_j normally distributed condition on $X_j > bd$. Thus the expected value is

$$\mathbb{E}(S_T \mid n \text{ jumps}) = S_0\, e^{\left(\mu - \frac{1}{2}\sigma^2\right)T} \times \mathbb{E}\left(\exp\Theta\right) \times B^n \tag{A.1}$$

$$\text{where } \Theta = \sigma\sqrt{\tau}\sum_{j=1}^{m} X_j = m\sigma\sqrt{\tau}\bar{X} \tag{A.2}$$

and $B = \mathbb{E}(\exp\left(\sigma Z_i\sqrt{\tau} + y_i\right)$ is given by equation (4.14):

$$B = \frac{1}{2\lambda\tau\sqrt{1-\sigma^2\beta^2\tau}} \exp\left(\frac{\sigma^2\tau(\alpha+1)^2}{2(1-\sigma^2\beta^2\tau)}\right)\left[1 + \operatorname{erf}\left(\frac{bd(1-\sigma^2\beta^2\tau) - \sigma\sqrt{\tau}(\alpha+1)}{\sqrt{2(1-\sigma^2\beta^2\tau)}}\right)\right]$$

Since X_j are identically distributed with

$$mean(X) = M = \frac{1}{1-\lambda\tau}\frac{1}{\sqrt{2\pi}}\int_{bd}^{\infty} x e^{\frac{1}{2}x^2}\, dx = \sqrt{\frac{2}{\pi}}\, \frac{e^{-\frac{1}{2}bd^2}}{1 - \operatorname{erf}\left(\frac{bd}{\sqrt{2}}\right)} \tag{A.3}$$

$$var(X) = V = \frac{1}{1-\lambda\tau}\frac{1}{\sqrt{2\pi}}\int_{bd}^{\infty} x^2 e^{-\frac{x^2}{2}}\, dx - mean(X)^2 = bd\, M + 1 - M^2 \tag{A.4}$$

Assuming m is large, $\bar{X}$, sample mean of size m, is normally distributed with mean M and variance $\frac{V}{m}$. Thus $\exp(\Theta) = \exp(m\sigma\sqrt{\tau}\bar{X})$ is lognormal

$$\mathbb{E}(\exp\Theta) = \exp\left[m\sigma\sqrt{\tau}M + \frac{1}{2}m\sigma^2\tau V\right] \tag{A.5}$$

$$= \exp\left[\left(\frac{T}{\tau} - n\right)\left(\sigma\sqrt{\tau}M + \frac{1}{2}\sigma^2\tau V\right)\right] \tag{A.6}$$

$$= \exp\left[T\left(\sigma\frac{M}{\sqrt{\tau}} + \frac{1}{2}\sigma^2 V\right)\right] \times G^n \tag{A.7}$$

$$\text{where } G = \exp\left[-\left(\sigma\sqrt{\tau}M + \frac{1}{2}\sigma^2\tau V\right)\right] \tag{A.8}$$

Combining equations (A.1), (4.14) and (A.8) the expected value of S_T is given by

$$\mathbb{E}(S_T|\ n \text{ jumps}) = S_0\ \exp\left[\left((\mu - \frac{1}{2}\sigma^2) + \frac{\sigma M}{\sqrt{\tau}} + \frac{1}{2}\sigma^2 V\right)T\right] \times G^n\, B^n \tag{A.9}$$

Finally, using the Poisson distribution we obtain

$$\mathbb{E}(S_T) = \sum_{n=0}^{\infty}\mathbb{E}(S_T|\ n \text{ jump})e^{-\lambda T}\frac{(\lambda T)^n}{n!} \tag{A.10}$$

$$= S_0\ \exp\left[\left((\mu - \frac{1}{2}\sigma^2) + \frac{\sigma M}{\sqrt{\tau}} + \frac{1}{2}\sigma^2 V + \lambda(G\,B - 1)\right)T\right] \tag{A.11}$$

Setting $\mathbb{E}(S_T) = e^{rT}S_0$ we have the risk neutral drift

$$\mu = r + \frac{1}{2}\sigma^2 - \frac{\sigma M}{\sqrt{\tau}} - \frac{1}{2}\sigma^2 V + \lambda(1 - G\,B) \tag{A.12}$$

Recall M and V are mean and variance of X given in (A.3) and (A.4), standard normal conditioned on $X > bd$, G and B are given in equations (A.8) and (4.14).

Equation (A.12) gives very accurate approximations in all cases we studied. Using the parameters from section 5, $\mu = -.0950$ and $-.0951$ from equations (4.19) and (A.12) respectively. Using parameters from section 8, $\mu = -0.11597$ and -0.11603 respectively.

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