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TRANSFORMATION OF QUANTUM LÉVY PROCESSES ON HOPF ALGEBRAS

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Abstract. A quantum Lévy process is given by its generator, a conditionally positive linear functional on the underlying Hopf algebra or bialgebra. A transformation between two bialgebras, in the sense of this paper, is a counit preserving algebra homomorphism. We show that transformation on the level of the corresponding quantum Lévy processes is given by convolution product integrals. This general result is applied to a bialgebra and its 'generator Hopf algebra' as well as to its 'Weyl bialgebra'. It follows that a quantum Lévy process can be realized on Bose Fock space as a convolution product integral of its generator process such that the vacuum vector is cyclic. At the same time, it can be reconstructed from its Weyl process. A further application are Trotter product formulae for quantum Lévy processes.

1. Introduction

A stochastic process $X_t \colon E \to G$, $t \geq 0$, over some probability space E taking values in a topological group G is called a (stationary) Lévy process on G if the increments $X_{s,t} = X_s^{-1} X_t$, $0 \leq s \leq t$, of disjoint intervals are independent, if the distribution of $X_{s,t}$ only depends on $t - s$, and if, for $t \to 0$ from the right, we have that X_t converges in law to the Dirac measure concentrated at the unit element of G. This can be generalized to stochastic processes $(X_{s,t})_{0 \leq s \leq t}$ taking values in a monoid G if the evolution equations $X_{r,s} X_{s,t} = X_{r,t}$ hold. Classical Lévy processes are commutative in the following sense. If we replace G and E by suitable $*$–algebras of functions (on G and E; e.g. replace G by $\mathrm{L}^\infty(G)$ and E by $\mathrm{L}^\infty(E)$) then $X_{s,t} \colon E \to G$ will give the $*$–algebra homomorphism $j_{s,t}$ mapping a function f on G to the function $j_{s,t}(f) = f \circ X_{s,t}$ on E. The $j_{s,t}$ form a commutative process in the sense that they are defined on a commutative $*$–algebra. Replacing the monoid G by a $*$–bialgebra and the classical probability space E by what is called a quantum probability space, the notion of a quantum Lévy process (QLP) on a $*$–bialgebra over a quantum probability space has been introduced in L. Accardi, M. Schürmann, and W. von Waldenfels [1].

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A representation theorem for such processes, M. Schürmann [11, Theorem 2.5.3], says that they can always be realized on a Boson Fock space as solutions to quantum stochastic differential equations in the sense of R.L. Hudson and K.R. Parthasarathy [4]. As pointed out in M. Skeide [14], QLPs can also be viewed as tensor product systems of type I in the sense of W. Arveson [2]. They are (up to stochastic equivalence) uniquely determined by their generators which are precisely the hermitian, normalized conditionally positive linear functionals on the underlying $*$–bialgebra. In this paper we are mainly interested in the following situation. If there are given two bialgebras and an algebra homomorphism between them with the additional property that the homomorphism preserves the counits, then generators are transformed into generators. The question arises how the two QLPs given by the two generators can be transformed into each other. Using infinitesimal convolution products which can be regarded as convolution product integrals, we establish a transformation on the level of the QLPs.

We describe very briefly what we do in a simplified setting. (For a precise description of the general situation see Sections 2 and 3.) In this simplified setting the situation is as follows. Suppose $(\mathcal{B}, \Delta, \delta)$ is a $*$–bialgebra. Then the comultiplication Δ induces a convolution $\star$ for algebra-valued linear mappings on $\mathcal{B}$; see Section 2. A QLP $j = \left(j_{s,t}\right)_{0 \leq s \leq t}$ satisfies

$$j_{s,t}(b) = j_{t_0,t_1} \star \cdots \star j_{t_{n-1},t_n}(b)$$

for all $s = t_0 < t_1 < \ldots < t_{n-1} < t_n = t$. Suppose on $\mathcal{B}$ there is a second comultiplication Δ'. We shall show that, in the canonical representation of j on a pre-Hilbert space D with cyclic vector Ω, the expressions

$$j_{t_0,t_1} \star' \cdots \star' j_{t_{n-1},t_n}(b)\Omega$$

(with the convolution with respect to Δ replaced by the convolution with respect to Δ') form a Cauchy net over the partitions of the interval $[s,t]$. From this it is easy to show that their limits, which we denote by $k_{s,t}(b)\Omega$ determine on their linear hull a unique QLP k over $(\mathcal{B}, \Delta', \delta)$, the *tranform* of j. Moreover, we shall show that under suitable cyclicity conditions this procedure can be reversed. See Theorems 3.4, 3.6 and 3.7 for a precise formulation in a more general context.

The transformation has various applications. For example, there are two QLPs associated with a given QLP in a natural way. One is the QLP's Weyl operator type process, the other is the generator process of the QLP which is composed of annihilation, preservation and creation processes on Boson Fock space. The Weyl type process can be used to show in a nice way why the result of M. Skeide [3] holds which says that the vacuum vector is always cyclic for the QLP. The generator process allows for a construction of the QLP as a product system by infinitesimal convolution products as a kind of convolution product stochastic integral. Both types of processes admit direct realizations on the Boson Fock space. Writing down the backwards transformations provides two different new proofs of the fact that every QLP may be realized as a (cyclic) process on a Boson Fock space. Other applications are the approximation of the Azéma martingales by infinitesimal convolution products of the Wiener process (and vice versa), and Trotter product formulae for QLPs.

In Section 2 we repeat the necessary definitions that, in Section 3, are used to formulate the transformation theorems. Section 3 also provides the constructions of several related $*$–bialgebras and applications of the theory. Section 4 presents the proof of the transformation theorems.

There is work in progress (for a first step into this direction see M. Schürmann and S. Voss [12]) to generalize the results of this article to QLPs on Dual Groups in the sense of D. Voiculescu [15] and to a more general notion of non-commutative independence.

Our method relies on finite-dimensional arguments based on the Fundamental Theorem on (algebraic) Coalgebras, which says that a coalgebra is the union of its finite-dimensional sub-coalgebras. A natural question is as to whether the theory presented in this paper can be generalized to the topological context of C^*-bialgebras (cf. J.M. Lindsay and A. Skalski [7]). For compact quantum groups there should not be a problem, because these unital C^*–bialgebras always have a dense $*$–subalgebra which is a proper algebraic coalgebra. For the noncompact case, this is an open problem.

2. Preliminaries

A $*$*–vector space* is a vector space V with an involution, i.e. an anti-linear mapping $v \mapsto v^*$ on V satisfying $(v^*)^* = v$. A $*$*–algebra* is an algebra $\mathcal{A}$ which is also a $*$–vector space such that $(ab)^* = b^*a^*$ for all $a, b \in \mathcal{A}$. If $\mathcal{A}$ is a $*$–algebra, then so is $\mathcal{A} \otimes \mathcal{A}$ with involution defined by $(a_1 \otimes a_2)^* = a_1^* \otimes a_2^*$.

A complex vector space $\mathcal{C}$ is a *coalgebra* if there are linear maps $\Delta\colon \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ and $\delta\colon \mathcal{C} \to \mathbb{C}$, called *coproduct* and *counit* respectively, satisfying

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta \quad \text{(coassociativity)}$$
$$(\delta \otimes id) \circ \Delta = id = (id \otimes \delta) \circ \Delta \quad \text{(counit property).}$$

Following Sweedler we frequently use the notation $c_{(1)} \otimes c_{(2)}$ for $\Delta(c)$ suppressing both summation and indices. Let $\Delta_0 := \delta$, and for $n \geq 1$ define

$$\Delta_n = (\Delta_{n-1} \otimes \mathsf{id}) \circ \Delta.$$

Sweedler's notation extends to writing $c_{(1)} \otimes c_{(2)} \otimes \cdots \otimes c_{(n)}$ for $\Delta_n(c)$, $n \geq 1$.

Sometimes we shall need to equip also the *conjugate* vector space $\overline{\mathcal{C}}$ with a coalgebra structure. Note that the canonical bijection $i = i_1\colon c \mapsto \overline{c}$ from $\mathcal{C}$ to $\overline{\mathcal{C}}$ is an anti-linear isomorphism. The same is true for the canonical bijections i_n from the n–fold tensor power of $\mathcal{C}$ to the n–fold tensor power of $\overline{\mathcal{C}}$. We may write

$$i_n(c_1 \otimes \cdots \otimes c_n) = \overline{c_1 \otimes \ldots \otimes c_n} = \overline{c}_1 \otimes \ldots \otimes \overline{c}_n.$$

Note that $i_n \otimes i_m = i_{n+m}$ (where the tensor product of antilinear mappings is well-defined). By i_0 we denote complex conjugation of $\mathbb{C}$. It is, then, easy to convince oneself that $\overline{\delta} := i_0 \circ \delta \circ i_1^{-1}$ and $\overline{\Delta} := i_2 \circ \Delta \circ i_2^{-1}$ make $(\overline{\mathcal{C}}, \overline{\Delta}, \overline{\delta})$ a coalgebra.

We shall also need the *tensor product* $(\mathcal{C}_1 \otimes \mathcal{C}_2, \Delta, \delta)$ of two coalgebras $(\mathcal{C}_1, \Delta_1, \delta_1)$ and $(\mathcal{C}_2, \Delta_2, \delta_2)$ where $\delta := \delta_1 \otimes \delta_2$ and $\Delta := (\mathsf{id} \otimes \tau \otimes \mathsf{id}) \circ (\Delta_1 \otimes \Delta_2)$ and τ denotes the flip $c \otimes d \mapsto d \otimes c$.

A $*$*–bialgebra* $(\mathcal{B}, \Delta, \delta)$ is a coalgebra which is also a unital $*$–algebra, and in such a way that Δ and δ are $*$–algebra homomorphisms. If $\mathcal{A}$ is a unital $*$–algebra with the multiplication map $m\colon \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ defined by setting $m(a_1\otimes a_2)=a_1a_2$, then we define the *convolution* of two linear mappings $j,k\colon \mathcal{B}\to\mathcal{A}$ by $j\star k := m\circ(j\otimes k)\circ\Delta$. In particular, the convolution of two linear functionals φ and ψ on $\mathcal{B}$ is $\varphi\star\psi=(\varphi\otimes\psi)\circ\Delta$. Unitality for a bialgebra $(\mathcal{B},\Delta,\delta)$ means that it is unital as an algebra, i.e. there exists $\mathbf{1}\in\mathcal{B}$ such that $m(b\otimes\mathbf{1})=m(\mathbf{1}\otimes b)=b$ for all $b\in\mathcal{B}$ and the coproduct and counit are unital, i.e. $\Delta(\mathbf{1})=\mathbf{1}\otimes\mathbf{1}$ and $\delta(\mathbf{1})=1$. We only consider unital algebras. A $*$–bialgebra $\mathcal{B}$ with an antipode S (i.e. a mapping $S\colon\mathcal{B}\to\mathcal{B}$ such that $m\circ(S\otimes\mathsf{id})\circ\Delta=\delta\mathbf{1}=m\circ(\mathsf{id}\otimes S)\circ\Delta$) is called a *Hopf $*$–algebra*.

A linear functional Φ on a $*$–algebra $\mathcal{A}$ is called positive if $\Phi(a^*a)\geq 0$ for all $a\in\mathcal{A}$. Let $(\mathcal{A},\Phi)$ be a *quantum probability space*, that is, a unital $*$–algebra with a *state* (a normalized positive linear functional $\Phi\colon\mathcal{A}\to\mathbb{C}$). A *quantum stochastic process* $j=\big(j_i\big)_{i\in I}$, indexed by some index set I, is a family of *quantum random variables* j_i (that is, of unital $*$–algebra homomorphisms $j_i\colon\mathcal{B}\to\mathcal{A}$). By $\varphi_i:=\Phi\circ j_i$ we denote the *distribution* of j_i. The notion of independence used for quantum Lévy processes on $*$–bialgebras in this paper is the tensor independence.

A (stationary) *quantum Lévy process* on $\mathcal{B}$ over $\mathcal{A}$ is a quantum stochastic process $j=\big(j_{s,t}\big)$, indexed by $s,t\in\mathbb{R}_+$, $s\leq t$, satisfying the following four conditions.

(LP1) The increments $j_{s,t}$ of disjoint intervals are *tensor independent* in Φ, that is,

$$\Phi\big(j_{s_1,t_1}(b_1)\cdots j_{s_n,t_n}(b_n)\big)=\varphi_{s_1,t_1}(b_1)\cdots\varphi_{s_n,t_n}(b_n)$$

for all $n\in\mathbb{N}, b_k\in\mathcal{B}$, $(s_1,t_1],\ldots,(s_n,t_n]$ mutually disjoint intervals of $\mathbb{R}_+$, and

$$[j_{s_k,t_k}(b_1),j_{s_l,t_l}(b_2)]=0$$

for all $k\neq l$ and all $b_1,b_2\in\mathcal{B}$

(LP2) The increments are *stationary*, that is, $\varphi_{s,t}=\varphi_{0,t-s}$ for all $0\leq s\leq t$.

(LP3) The process is *weakly continuous* in Φ, that is, $\lim_{t\to 0}\varphi_{0,t}(b)=\delta(b)$ for all $b\in\mathcal{B}$.

(LP4) The $j_{s,t}$ are increments under convolution, that is, $j_{r,s}\star j_{s,t}=j_{r,t}$ for all $0\leq r\leq s\leq t$ and $j_{t,t}(b)=\delta(b)\mathbf{1}$ for all $0\leq t<\infty$.

(For a topological extension of this notion of a QLP to compact quantum groups and operator space coalgebras see [7].) We observe that by (LP1) and (LP4) every Lévy process fulfills the condition

(LP4') $\varphi_{r,s}\star\varphi_{s,t}=\varphi_{r,t}$ for all $0\leq r\leq s\leq t$ and $\varphi_{t,t}=\delta$.

Therefore, by (LP2) and (LP3) the states $\varphi_t:=\varphi_{0,t}$ form a weakly continuous semigroup under convolution. By (LP1), (LP2) and (LP4) this convolution semigroup determines all *joint moments* (that is exactly all expressions of the form of the left-hand side of the first equation of (LP1), even if we drop the condition that the $(s_k,t_k]$ are mutually disjoint). In other words, two Lévy processes are *stochastically equivalent*, if and only if they have the same convolution semigroup. We can associate a *generator* ψ with a convolution semigroup through $\varphi_t=\mathrm{e}_\star^{t\psi}$ for all $t\geq 0$. Essentially, this follows from the Fundamental Theorem on Coalgebras;

see [1] and Section 4.1. Then ψ is a linear functional on $\mathcal{B}$, satisfying $\psi(\mathbf{1}) = 0$, and it is conditionally positive and hermitian which means that $\psi(b^*b) \geq 0$ for all b in the kernel of the counit δ and that $\psi(b^*) = \overline{\psi(b)}$ for all $b \in \mathcal{B}$. Thus, Lévy processes on $*$–bialgebras can also be characterized (up to equivalence) by their generator.

If $\mathcal{B}$ is a Hopf $*$–algebra a quantum stochastic process j_t on $\mathcal{B}$ indexed by time $t \in \mathbb{R}_+$ is called a QLP if the 'increments' $j_{s,t} = (j_s \circ S) \star j_t$, $0 \leq s \leq t$, form a QLP in the above sense.

Let D be a pre-Hilbert space and denote by $\mathcal{L}^a(D)$ the $*$–algebra of adjointable operators on D, i.e. $\mathcal{L}^a(D)$ consists of all mappings T on D for which there is a mapping T^* on D such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in D$. If Ω is a unit vector in D, then $\big(\mathcal{L}^a(D), \langle \Omega, \cdot\, \Omega\rangle\big)$ is a quantum probability space. We call it a *concrete* quantum probability space and write it as (D, Ω). If a Lévy process j takes values in a concrete quantum probability space, then we say j is a *concrete* Lévy process. By GNS-construction every quantum probability space $(\mathcal{A}, \Phi)$ gives rise to a concrete quantum probability space (D, Ω), determined uniquely by the properties that there is a $*$–representation $\pi\colon \mathcal{A} \to \mathcal{L}^a(D)$ such that $\Phi(\cdot) = \langle \Omega, \pi(\cdot)\Omega\rangle$ and that Ω is cyclic for $\mathcal{A}$, that is $\pi(\mathcal{A})\Omega = D$.

Consequently, every Lévy process gives rise to a concrete Lévy process *over* (D, Ω). We will say the Lévy process is *cyclic*, if Ω is cyclic for the $*$–subalgebra

$$\mathcal{A}_j \ := \ \mathsf{span}\Big\{ j_{t_0,t_1}(b_1) \cdots j_{t_{n-1},t_n}(b_n) \colon n \in \mathbb{N}, b_k \in \mathcal{B}, 0 = t_0 \leq \cdots \leq t_n \Big\}$$

of $\mathcal{L}^a(D)$. Notice that by (LP1) this space does not change, if we allow that the disjoint intervals are not consecutive, and by (LP4) it also does not change if we allow for arbitrary intervals. By restricting to the invariant subspace $\mathcal{A}_j\Omega$ of D that is generated by the process from Ω, we obtain from every Lévy process over D a cyclic Lévy process on $\mathcal{A}_j\Omega = D_j$.

By a GNS-type construction applied to a generator ψ on $\mathcal{B}$ we obtain a pre-Hilbert space K, a surjective mapping $\eta\colon \mathcal{B} \to K$ and a $*$–representation $\rho\colon \mathcal{B} \to \mathcal{L}^a(K)$ such that

$$\eta(ab) = \rho(a)\eta(b) + \eta(a)\delta(b)$$

and

$$-\langle \eta(a^*), \eta(b)\rangle = \delta(a)\psi(b) - \psi(ab) + \psi(a)\delta(b) \tag{2.1}$$

for all $a, b \in \mathcal{B}$. The specified triple (ρ, η, ψ) is called a *surjective Lévy triple*. There is a one-to-one correspondence between Lévy processes (modulo equivalence) on $\mathcal{B}$, convolution semigroups of states on $\mathcal{B}$, generators on $\mathcal{B}$ and surjective Lévy triples on $\mathcal{B}$ (modulo unitary equivalence).

For every convolution semigroup $\varphi = \big(\varphi_t\big)_{t\in\mathbb{R}_+}$ there is (up to unitary equivalence) at most one cyclic Lévy process. (Unitary equivalence is much stronger than stochastic equivalence.) Effectively, if j is a cyclic process on (D, Ω) which fulfills (LP1) - (LP3) and (LP4'), then it is not difficult to show that also (LP4) holds. M. Schürmann [11, Proposition 1.9.5] shows that for every convolution semigroup of states on a $*$–bialgebra there is a (unique up to unitary equivalence) cyclic Lévy process (even without continuity). This construction involves the GNS-construction of all φ_t, their tensor products and an inductive limit over the interval partitions of $\mathbb{R}_+$. However, it is completely algebraic and does not involve analytic tools. On the contrary, [11, Theorem 2.5.3] constructs a Lévy process as the solution of quantum stochastic differential equations in the sense of R.L. Hudson and K.R. Parthasarathy [4] of the form

$$\mathrm{d}j_{s,t} = j_{s,t} \star \mathrm{d}I_t;\ j_{t,t} = \delta \tag{2.2}$$

where

$$I_t = A_t(\eta(b^*)) + \Lambda_t(\rho(b) - \delta(b)) + A^*_t(\eta(b)) + \psi(b)t \tag{2.3}$$

denotes the 'generator process' of $j_{s,t}$ with A_t, Λ_t, A^*_t the annihilation, preservation and creation processes on Boson Fock space over $\Gamma_s(L^2(\mathbb{R}_+, \overline{K})) = \Gamma_s(L^2(\mathbb{R}_+) \otimes \overline{K}$ where $\overline{K}$ denotes the completion of the index space K. For quite a long time it was an open problem, to decide whether Fock space and differential equation can be set in such a way that the Fock vacuum is cyclic for the resulting Lévy process. Only quite recently and simultaneously, U. Franz, M. Schürmann, and M. Skeide came up, not with just one, but with a whole bunch of proofs for the affirmative answer.

The proof due to M. Skeide (see U. Franz [3, Theorem 1.21]) uses in an essential way the representation on the Fock space and equation (2.2) and shows that for every $b \in \mathcal{B}$ with $\delta(b) = 1$ the vectors

$$j_{t_0,t_1}(b) \cdots j_{t_{n-1},t_n}(b)\Omega, \tag{2.4}$$

$s = t_0 \leq t_1 \leq \ldots \leq t_{n-1} \leq t_n = t$, converge over the interval partitions of $(s,t]$ to an exponential vector of the form $\exp(k\mathbf{1}_{(s,t]})$ where $k \in K$ is a vector depending on b. (Cyclicity is, then, a simply consequence of Skeide's proof in [13] of a result due to K.R. Parthasarathy and V.S. Sunder [9].) Immediately, from this construction, the idea emerged to construct an explicit isomorphism from the space of the abstract Lévy process of [11, Proposition 1.9.5] to the Fock space of the Lévy process obtained via [11, Theorem 2.5.3]. Namely, if in (2.4) we replace j and Ω with the abstract process $\widetilde{j}$ and its cyclic vector $\widetilde{\Omega}$, we know from [3, Theorem 1.21] that they converge. Sending the limit to $\exp(k\mathbf{1}_{(s,t]})$ establishes a unitary from the abstract representation space $\widetilde{D}$ to the Fock space. If we can manage to do this without using [3, Theorem 1.21], then we will obtain a direct proof of representability of the Lévy process as a cyclic process on the Fock space.

The idea for a transformation of a (cyclic) Lévy process originates in the following observation. Let us put $\mathcal{B}_1 := \{b \in \mathcal{B} \colon \delta(b) = 1\}$. Suppose the element $b \in \mathcal{B}_1$ is *group-like*, that is, $\Delta(b) = b \otimes b$. (Note that $b \in \mathcal{B}$ being group-like, the counit property forces $b = 0$ or $b \in \mathcal{B}_1$.) Then

$$j_{t_0,t_1}(b) \cdots j_{t_{n-1},t_n}(b) = j_{t_0,t_1} \star \cdots \star j_{t_{n-1},t_n}(b) = j_{s,t}(b)$$

so that the limit is over a constant and gives back what $j_{s,t}(b)$ does to the cyclic vector. In general, there need not be group-like elements in $\mathcal{B}_1$, and if, then they need not generate $\mathcal{B}$. However, if we were able to define a different comultiplication on $\mathcal{B}$ for which all elements in $\mathcal{B}_1$ are group-like, then

$$k_{s,t}(b)\Omega = \lim j_{t_0,t_1}(b) \cdots j_{t_{n-1},t_n}(b)\Omega$$

would define a family of homomorphisms $k_{s,t}$ that form a Lévy process with respect to the group-like comultiplication. In other words, we transformed one Lévy process into another.

It is easy to give a direct realization of such a group-like process on a suitable Fock space; see Section 4.1. Thus, provided that the process k acts cyclicly on Ω, we would find the representation theorem. The easiest way to establish cyclicity is to reconstruct j from k by a reverse transformation. Recall that the construction of k involved replacing the original comultiplication with one that makes all $b \in \mathcal{B}_1$ into group-like elements so that $j_{t_0,t_1}(b) \cdots j_{t_{n-1},t_n}(b)$ is nothing but $j_{t_0,t_1} \star' \cdots \star' j_{t_{n-1},t_n}$ with respect to the new comultiplication. Now we do just the opposite and look at the limit of

$$k_{t_0,t_1} \star \cdots \star k_{t_{n-1},t_n}(b)\Omega \tag{2.5}$$

for the original comultiplication. If this reverse transformation gives back j, then, knowing that the representation space of the intermediate group-like process k is isomorphic to a Fock space, we will know that also the representation space of j is a Fock space. Technically, in general, it is not possible to equip $\mathcal{B}$ directly with a comultiplication that makes the elements of $\mathcal{B}_1$ group-like.

However, it is possible to associate with every $*$–bialgebra $\mathcal{B}$ its *Weyl bialgebra* $\mathbb{C}\mathcal{B}_1$. The vector space $\mathbb{C}\mathcal{B}_1$ contains the set $\mathcal{B}_1$ as a basis consisting entirely of group-like elements. And the $k_{s,t}(b)\Omega$ defined on elements of $\mathcal{B}_1$ determine a unique Lévy process on $\mathbb{C}\mathcal{B}_1$. But now the k_{st} do no longer define a linear mapping $\mathcal{B} \to \mathcal{L}^a(D)$. (They do define a linear mapping $\mathbb{C}\mathcal{B}_1 \to \mathcal{L}^a(\widetilde{D})$ where $\widetilde{D}$ is the linear span in $\overline{D}$ of what the $k_{s,t}(b)$ generate from Ω.) So the convolutions in (2.5) with respect to the comultiplication of $\mathcal{B}$ do no longer have a meaning. The problem is solved if we associate again with $\mathcal{B}$ a special kind of $*$–bialgebra; see example 3.2. We will equip this tensor $*$–bialgebra with a certain comultiplication, so that the convolutions in (2.5) are defined with respect to this comultiplication.

3. Statement of Results and Applications

In Section 3.1 we state the main result of this paper (Theorem 3.4) on the transformation of QLPs. We introduce two $*$–bialgebra structures on the tensor $*$–algebra over the kernel of the counit of a $*$–bialgebra. Moreover, we show that lifted generators give rise to QLPs on the tensor $*$–bialgebra which by restriction lead back to a version of the original QLP (Proposition 3.3). Section 3.2 is on the reversion of the transformation which is always possible if the transformation is surjective. In Section 3.3 we treat applications of our results to classical Lévy processes, to realizations of QLPs by their Weyl process and by their generator process, to the passage from the Wiener process to Azéma martingales, and to Trotter product formulae for QLPs.

3.1. Transformation of QLPs. Let $(\mathcal{B}, \Delta, \delta)$ and $(\mathcal{C}, \Lambda, \lambda)$ be two $*$–bialgebras. A *transformation* of $\mathcal{B}$ is a unital $*$–algebra homomorphism $\varkappa \colon \mathcal{C} \to \mathcal{B}$ satisfying

$$\delta \circ \varkappa = \lambda. \tag{3.1}$$

This means that $\varkappa$ *preserves the counit.* Since $\varkappa(\mathbf{1}) = \mathbf{1}$ it is easy to see that (3.1) is equivalent to the condition $\varkappa(\mathcal{C}_0) \subset \mathcal{B}_0$ where $\mathcal{C}_0 = \ker \lambda$, $\mathcal{B}_0 = \ker \delta$. In the sequel, if we have such a situation $\varkappa\colon \mathcal{C} \to \mathcal{B}$, we should warn the reader that we call $\mathcal{B}$ the *first* and $\mathcal{C}$ the *second* $*$–bialgebra.

Example 3.1. *(Generator Hopf algebra associated with a $*$–bialgebra)*
For a vector space V the (unital) *tensor algebra* $\mathcal{T}(V)$ over V is the vector space

$$\mathcal{T}(V) = \bigoplus_{n\in\mathbb{N}} V^{\otimes n}$$

where $V^{\otimes n}$ denotes the n-fold tensor product of V with itself, $V^{\otimes 0} = \mathbb{C}$, with unit element $(1, 0, \dots)$ and the multiplication given by

$$(v_1 \otimes \cdots \otimes v_n)(w_1 \otimes \cdots \otimes w_m) = v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m$$

for $n, m \in \mathbb{N}, v_1, \dots, v_n, w_1, \dots, w_m \in V$. The tensor algebra satisfies the following universal property. There exists a vector space embedding $\iota\colon V \to \mathcal{T}(V)$ of V into $\mathcal{T}(V)$ such that for any linear mapping f from V into an algebra $\mathcal{A}$ there is a unique algebra homomorphism $\mathcal{T}(f)\colon \mathcal{T}(V) \to \mathcal{A}$ such that $\mathcal{T}(f)\circ\iota(v) = f(v)$ for all $v \in V$. Then, any algebra homomorphisms $g\colon \mathcal{T}(V) \to \mathcal{A}$ is uniquely determined by its restriction to V. In a similar way, an involution on V gives rise to a unique extension as an involution on $\mathcal{T}(V)$. Thus, for a $*$–vector space V we can form the tensor $*$–algebra $\mathcal{T}(V)$ over V. This $*$–algebra becomes a $*$–bialgebra if we extend the mappings $\Lambda\colon V \to \mathcal{T}(V) \otimes \mathcal{T}(V), v \mapsto v \otimes \mathbf{1} + \mathbf{1} \otimes v$ and $\mathcal{T}(0)\colon V \to \mathcal{C}, v \mapsto 0$ as $*$–algebra homomorphisms to define the comultiplication and the counit on $\mathcal{T}(V)$. The elements $v \in V$ are so-called primitive elements of the coalgebra $\mathcal{T}(V)$. By extending $v \mapsto -v$ as an algebra anti-homomorphism, we obtain an antipode so that $\mathcal{T}(V)$ becomes a Hopf $*$–algebra. We call $\mathcal{T}(V)$ the tensor Hopf $*$–algebra over V.

Let $(\mathcal{B}, \Delta, \delta)$ be any $*$–bialgebra. The set $\mathcal{B}_0 = \{b \in \mathcal{B}\colon \delta(b) = 0\}$ is a $*$–ideal of $\mathcal{B}$. The tensor Hopf $*$–algebra $(\mathcal{T}(\mathcal{B}_0), \Lambda, \mathcal{T}(0))$ is called the *generator Hopf algebra* of $\mathcal{B}$.

We obtain a pair of $*$–bialgebras by taking for the first $*$–bialgebra $\mathcal{B}$ itself, and for the second one the generator Hopf algebra of $\mathcal{B}$. The role of $\varkappa$ is played by the counit preserving $*$–algebra homomorphism $\varkappa$ defined by $\varkappa(b_1 \otimes \cdots \otimes b_n) = b_1 \cdots b_n$ for $b_1, \dots, b_n \in \mathcal{B}_0$. We call $\varkappa$ the multiplication map and denote it by M.

Example 3.2. *(Induced tensor $*$–bialgebra associated with a $*$–bialgebra)*
Let $(\mathcal{B}, \Delta, \delta)$ and $(\mathcal{T}(\mathcal{B}_0), \Lambda, \mathcal{T}(0))$ be as in example 3.1. We can define another coalgebra structure on $\mathcal{T}(\mathcal{B}_0)$. Denote by

$$E\colon \mathcal{B}_0 \oplus \mathcal{B}_0 \oplus (\mathcal{B}_0 \otimes \mathcal{B}_0) \to \mathcal{T}(\mathcal{B}_0) \otimes \mathcal{T}(\mathcal{B}_0)$$

the canonical embedding coming from the identifications of $\mathcal{B}_0$ with $\mathcal{B}_0 \otimes \mathbf{1}$ and $\mathbf{1} \otimes \mathcal{B}_0$ respectively and of $\mathcal{B}_0 \otimes \mathcal{B}_0 \subset \mathcal{T}(\mathcal{B}_0) \otimes \mathcal{T}(\mathcal{B}_0)$. Moreover, consider the restriction Δ_0 of Δ to $\mathcal{B}_0$. Then

$$\Delta_0\colon \mathcal{B}_0 \to \mathcal{B}_0 \oplus \mathcal{B}_0 \oplus (\mathcal{B}_0 \otimes \mathcal{B}_0)$$

and $(\mathcal{T}(\mathcal{B}_0), \mathcal{T}(E \circ \Delta_0), \mathcal{T}(0))$ is a $*$–bialgebra. We can understand this $*$–bialgebra as a kind of big version of $\mathcal{B}$ and $(\mathcal{T}(\mathcal{B}_0), \mathcal{T}(\Delta_0), \mathcal{T}(0))$ is called the *induced tensor $*$–bialgebra* associated with $\mathcal{B}$. In the context of the algebraic set-up the first $*$–bialgebra is $(\mathcal{T}(\mathcal{B}_0), \Lambda, \mathcal{T}(0))$ and the second $*$–bialgebra is $(\mathcal{T}(\mathcal{B}_0), \mathcal{T}(\Delta_0), \mathcal{T}(0))$.

The identity on $\mathcal{T}(\mathcal{B}_0)$ is an example of a counit preserving $*$–algebra homomorphism $\varkappa$.

The following result will frequently be needed. If $\psi \in \mathcal{B}'$, $\mathcal{B}'$ the algebraic dual space of the vector space $\mathcal{B}$, is a generator on $\mathcal{B}$ we have that $\psi \circ M$ is a generator on the induced tensor $*$–bialgebra $\mathcal{T}(\mathcal{B}_0)$. By the bialgebra property of $\mathcal{B}$ we have

$$\Delta \circ m = (m \otimes m) \circ (\Delta \otimes \Delta)$$

where $m\colon \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$ again denotes the multiplication of $\mathcal{B}$. This implies

$$(\varphi_1 \star \varphi_2) \circ M = (\varphi_1 \circ M) \star (\varphi_2 \circ M) \tag{3.2}$$

for all $\varphi_1, \varphi_2 \in \mathcal{B}'$ where the first $\star$ in (3.2) is with respect to Δ and the second with respect to $\mathcal{T}(\Delta_0)$.

Proposition 3.3. *Let ψ be a generator on the $*$–bialgebra $\mathcal{B}$ and let $J_{s,t}$ be a cyclic QLP on the induced $*$–bialgebra $\mathcal{T}(\mathcal{B}_0)$, over (D_J, Ω) with generator $\psi \circ M$. Then the restriction of $J_{s,t}$ to $\mathcal{B} \subset \mathcal{T}(\mathcal{B}_0)$ is a QLP on $\mathcal{B}$ with generator ψ.*

Proof. We show that $J_{s,t}$ satisfies

$$J_{s,t}(b_1) \cdots J_{s,t}(b_n) = J_{s,t}(b_1 \cdots b_n) \tag{3.3}$$

for all $n \in \mathbb{N}$, $b_1, \dots, b_n \in \mathcal{B}_0$, $0 \le s \le t$. By equation (3.2) we have for $\Phi_t(\cdot) := \langle \Omega, J_{0,t}(\cdot)\Omega\rangle$

$$\begin{aligned} \langle \Omega, J_{s,t}(b_1) \cdots J_{s,t}(b_n)\Omega\rangle &= \Phi_{t-s}(b_1 \otimes \cdots \otimes b_n) \\ &= \mathrm{e}_\star^{(t-s)\psi}(b_1 \cdots b_n) \\ &= \langle \Omega, J_{s,t}(b_1 \cdots b_n)\Omega\rangle \end{aligned}$$

Using the properties of a QLP, we obtain from that

$$\langle \zeta, J_{s,t}(b_1) \cdots J_{s,t}(b_n)\xi\rangle = \langle \zeta, J_{s,t}(b_1 \cdots b_n)\xi\rangle$$

for all $\zeta, \xi \in D_J$ which proves (3.3). □

A generator ψ of a Lévy process on $\mathcal{B}$ is lifted via $\varkappa$ to a generator $\psi \circ \varkappa$ of a Lévy process on $\mathcal{C}$. The question arises, what is the relationship between the two Lévy processes? We will show how the second process can be computed from the first one and vice versa.

In the sequel, $\mathfrak{Z}_{st}$ denotes the set of all *partitions* of an interval $[s,t] \subset \mathbb{R}_+$. Let $\alpha = \{s = t_0 < t_1 < \cdots < t_{n-1} < t_n = t\}$ be a partition of $[s,t]$ and define

$$\|\alpha\| = \max\{t_{j+1} - t_j \mid 0 \le j \le n-1\}.$$

We turn $\mathfrak{Z}_{st}$ into a directed set by writing $\alpha_1 \prec \alpha_2 :\Leftrightarrow \alpha_1 \subset \alpha_2$.

Theorem 3.4. *Let $(\mathcal{B}, \Delta, \delta)$ be a $*$–bialgebra and let $(j_{s,t})_{0 \le s \le t}$ be the unique cyclic Lévy process over (D_j, Ω) whose convolution semigroup is given by a generator ψ. Let $(\mathcal{C}, \Lambda, \lambda)$ be another $*$–bialgebra and let $\varkappa\colon \mathcal{C} \to \mathcal{B}$ be a transformation of $\mathcal{B}$. Denote by H_k the Hilbert subspace of $\overline{D_j}$ defined by*

$$\begin{aligned} H_k \ := \ \overline{\mathsf{span}}\Big\{&(j_{t_0,t_1} \circ \varkappa)(c_1) \cdots (j_{t_{n-1},t_n} \circ \varkappa)(c_n)\Omega\colon \\ &n \in \mathbb{N}, c_1, \dots, c_n \in \mathcal{C}, 0 \le s \le t, s = t_0 \le t_1 \le \cdots \le t_{n-1} \le t_n = t\Big\}. \end{aligned}$$

Then for every $c \in \mathcal{C}$ and $0 \le s \le t$ the net $\big(\vartheta_\alpha(c)\big)_{\alpha \in \mathfrak{Z}_{st}}$ converges in norm to an element in H_k where

$$\vartheta_\alpha(c) = (j_{t_0,t_1} \circ \varkappa) \star \cdots \star (j_{t_{n-1},t_n} \circ \varkappa)(c)\Omega. \tag{3.4}$$

Moreover, setting

$$k_{s,t}(c)\Omega \ := \ \lim_\alpha \vartheta_\alpha\,(c)$$

determines a unique cyclic Lévy process $k = \big(k_{s,t}\big)_{0 \le s \le t}$ on $\mathcal{C}$ over a dense subspace (D_k, Ω) of H_k. The convolution semigroup of this process has generator $\psi \circ \varkappa$.

The proof will be given in Section 4. We call k the *transform* of j.

We formally will describe the construction of $k_{s,t}$ out of $j_{s,t}$ in the above theorem by the short hand writing

$$\overrightarrow{\prod}^{\star}_{\Lambda}(j_{s,t} \circ \varkappa) = k_{s,t}. \tag{3.5}$$

We may call $k_{s,t}$ the infinitesimal convolution product of $j_{s,t} \circ \varkappa$.

Remark 3.5. In more detail, Theorem 3.4 says that, under the assumptions of the theorem, the following L^2-type construction holds. If $\widetilde{k}_{s,t}$ is a concrete cyclic Lévy process over $(\widetilde{D}, \widetilde{\Omega})$, $\widetilde{D}$ a pre-Hilbert space with cyclic vector $\widetilde{\Omega}$, on $\mathcal{C}$ with generator $\psi \circ \varkappa$, then

$$k_{s,t}(c)\Omega \mapsto \widetilde{k}_{s,t}(c)\widetilde{\Omega}$$

defines a unitary mapping

$$\mathcal{U} \colon D_k \to \widetilde{D}$$

such that

$$\widetilde{k}_{s,t}(c)\,\mathcal{U} = \mathcal{U}\,k_{s,t}(c) \tag{3.6}$$

for all $c \in \mathcal{C}$. Notice that the formal writing (3.5) can always be given a mathematical meaning by (3.6). Of course, when the process $\widetilde{k}_{s,t}$ consists of bounded operators, equations (3.6) make sense on H_k and $\widetilde{H}$ where $\widetilde{H}$ denotes the Hilbert space which is the completion of $\widetilde{D}$. However, boundedness does not always hold in the applications; cf. Examples 3.10, 3.12, 3.13.

3.2. Reversion of the transformation. The reverse transformation of a Lévy process on $(\mathcal{C}, \Lambda, \lambda)$ into a Lévy process on $(\mathcal{B}, \Delta, \delta)$ requires a counit preserving $*$–algebra homomorphism $\widetilde{\varkappa}$ which, roughly speaking, is the inverse of $\varkappa$. The construction of $\widetilde{\varkappa}$ assumes the surjectivity of $\varkappa$. This is equivalent to $\varkappa(\mathcal{C}_0) = \mathcal{B}_0$ and the existence of an injective linear $*$–mapping

$$\upsilon \colon \mathcal{B}_0 \to \mathcal{C}_0 \quad \text{such that} \quad \varkappa \circ \upsilon = id_{\mathcal{B}}.$$

The linear $*$–mapping υ is not unique. Its existence follows from the existence of a self-adjoint basis $\big(b_i\big)_{i \in I}$ of the $*$–vector space $\mathcal{B}_0$, I some index set. Choose $c_i \in \mathcal{C}$ self-adjoint such that $\varkappa(c_i) = b_i$. This is possible since $\varkappa$ is surjective. Define the linear $*$–map υ by $\upsilon(b_i) = c_i$. In view of the universal property of tensor algebras we extend the linear $*$–map υ to a $*$–algebra homomorphism

$$\widetilde{\varkappa} = \mathcal{T}(\upsilon) \colon \mathcal{T}(\mathcal{B}_0) \to \mathcal{C}$$

to the induced tensor $*$–bialgebra $\mathcal{T}(\mathcal{B}_0)$. The coalgebra structure on $\mathcal{T}(\mathcal{B}_0)$ is that of the induced tensor $*$–algebra of Example 3.2. Indeed, the $*$–algebra homomorphism $\widetilde{\varkappa}$ preserves the counits. It is sufficient to show this for the generators of $\mathcal{T}(\mathcal{B}_0)$. For all $b \in \mathcal{B}_0$ we have

$$\lambda \circ \upsilon(b) = \delta \circ \varkappa \circ \upsilon(b) = \delta \circ id_{\mathcal{B}_0}(b) = 0 = \delta(b).$$

The above situation is described by

$$(\mathcal{T}(\mathcal{B}_0), \mathcal{T}(\Delta_0), \mathcal{T}(0)) \xrightarrow{\widetilde{\varkappa}} (\mathcal{C}, \Lambda, \lambda) \xrightarrow{\varkappa} (\mathcal{B}, \Delta, \delta).$$

An application of Theorem 3.4 to this setting gives

Theorem 3.6. *Let $\varkappa$ be a* surjective *transformation of the $*$–bialgebra $\mathcal{B}$. For a generator ψ on $\mathcal{B}$ let $\left(k_{s,t}\right)_{0\leq s\leq t}$ be a cyclic Lévy process on $(\mathcal{C}, \Lambda, \lambda)$ over (D_k, Ω) with generator $\psi \circ \varkappa$.*

For every $b \in \mathcal{B}$ and $0 \leq s \leq t$ the net $\left(\zeta_\alpha\right)_{\alpha \in \mathfrak{Z}_{st}}$ converges in norm to an element in $\overline{D_k}$ where

$$\zeta_\alpha(b) := (k_{t_0,t_1} \circ \widetilde{\varkappa}) \star_{\mathcal{T}(\Delta_0)} \cdots \star_{\mathcal{T}(\Delta_0)} (k_{t_{n-1},t_n} \circ \widetilde{\varkappa})(b)\Omega.$$

Moreover, setting

$$j_{s,t}(b)\Omega \ := \ \lim_{\alpha} \zeta_\alpha\,(b)$$

determines a unique cyclic Lévy process $j = \left(j_{s,t}\right)_{0\leq s\leq t}$ on $\mathcal{B}$ over a dense subspace (D_j, Ω) of $\overline{D_k}$. The convolution semigroup of this process has generator ψ.

The following result says that the transform j in Theorem 3.6 is the reversion of the transform k in Theorem 3.4.

Theorem 3.7. *Let $\varkappa$ be a* surjective *transformation of the $*$–bialgebra $\mathcal{B}$. and let ψ be a generator on $\mathcal{B}$. An application of Theorem 3.6 to the process $\left(k_{s,t}\right)_{0\leq s\leq t}$ of Theorem 3.4 gives back the original Lévy process on $(\mathcal{B}, \Delta, \delta)$. Moreover, we have $H_k = \overline{D_j} = \overline{D_k}$.*

For a proof of Theorems 3.6 and 3.7 see Section 4.

3.3. Applications.

Example 3.8. *(Group-like $*$–bialgebras)*
For a set S the vector space generated by S is the vector space $\mathbb{C}S$ consisting of all functions $f\colon S \to \mathbb{C}$ with finite support. Assume in addition that S is a monoid with identity $e \in S$. Since S is a basis, the multiplication map $S \times S \to S$ induces a map $m\colon \mathbb{C}S \otimes \mathbb{C}S \to \mathbb{C}S$ that turns $\mathbb{C}S$ into an algebra with identity element $e \in S \subset \mathbb{C}S$. Since S is a basis of $\mathbb{C}S$ the mapping m induces an algebra structure on $\mathbb{C}S$ with unit element e.

The vector space generated by a set satisfies the following universal property. There exists an embedding $\iota\colon S \to \mathbb{C}S$ such that any mapping ϕ from S to some vector space V can be uniquely extended to a linear mapping $\widetilde{\phi}\colon \mathbb{C}S \to V$ such that $\phi = \widetilde{\phi} \circ \iota$. This can be used to define a coalgebra structure on $\mathbb{C}S$. We understand S as a set of group-like elements. We extend the mappings $\Lambda\colon S \to \mathbb{C}S \otimes \mathbb{C}S$, $\Lambda(s) = s \otimes s$ and $\lambda\colon S \to \mathbb{C}$, $\lambda(s) = 1$ to linear mappings on $\mathbb{C}S$. We will denote the comultiplication and the counit on $\mathbb{C}S$ again by Λ and λ. If S is a monoid Λ and λ are algebra homomorphism since $\Lambda(xy) = xy \otimes xy = (x \otimes x)(y \otimes y) = \Lambda(x)\Lambda(y)$ and $\lambda(xy) = 1 = \lambda(x)\lambda(y)$ for all $x, y \in S$. An involution on S can also be uniquely extended to an involution on $\mathbb{C}S$. Thus, for a $*$–monoid S we can form the *group-like $*$–bialgebra* $(\mathbb{C}S, \Lambda, \lambda)$ over S.

Let $(\mathcal{B}, \Delta, \delta)$ be a $*$–bialgebra. The set $\mathcal{B}_1 = \{b \in \mathcal{B} \colon \delta(b) = 1\}$ is a $*$–monoid with multiplication and involution of the $*$–algebra $\mathcal{B}$. Hence, $(\mathbb{C}\mathcal{B}_1, \Lambda, \lambda)$ is a $*$–bialgebra which will be called the *Weyl bialgebra* of $\mathcal{B}$; see also the next example.

In the sequel, we write $\widehat{b}$ for the element b in $\mathcal{B}_1 \subset \mathbb{C}\mathcal{B}_1$. The comultiplication Λ and the counit λ on $\mathbb{C}\mathcal{B}_1$ are defined by $\Lambda(\widehat{b}) = \widehat{b} \otimes \widehat{b}$ and $\lambda(\widehat{b}) = 1$ for $\widehat{b} \in \mathbb{C}(\mathcal{B}_1)$. $\mathcal{B}_1$ is equal to the set of group-like elements in $\mathbb{C}\mathcal{B}_1$, i.e. $\mathcal{B}_1 = \{0 \neq \widehat{b} \in \mathbb{C}B_1 \colon \Lambda(\widehat{b}) = \widehat{b} \otimes \widehat{b}\}$. The situation is described by

$$(\mathcal{T}(\mathcal{B}_0), \mathcal{T}(\Delta_0), \mathcal{T}(0)) \xrightarrow{\widetilde{\varkappa}} (\mathbb{C}\mathcal{B}_1, \Lambda, \lambda) \xrightarrow{\varkappa} (\mathcal{B}, \Delta, \delta)$$

where the counit preserving $*$–algebra homomorphisms $\varkappa$ and $\widetilde{\varkappa}$ are defined by $\varkappa(\widehat{b}) = b$ for $b \in \mathcal{B}_1$ and $\widetilde{\varkappa}(b) = \widehat{b + \mathbf{1}} - \widehat{\mathbf{1}}$ for $b \in \mathcal{B}_0$. Now we are able to express the reverse transformation (2.5) by $(k_{t_0,t_1} \circ \widetilde{\varkappa}) \star_{\mathcal{T}(\Delta_0)} \cdots \star_{\mathcal{T}(\Delta_0)} (k_{t_{n-1},t_n} \circ \widetilde{\varkappa})(b)\Omega$ for $b \in \mathcal{B}_0$.

Example 3.9. *(Construction of quantum Lévy processes I)*
We apply Theorem 3.4 to Example 3.8. Let $(\mathcal{B}, \Delta, \delta)$ be some $*$–bialgebra and let $\left(j_{s,t}\right)_{0 \le s \le t}$ be a cyclic Lévy process on $\mathcal{B}$ over (D_j, Ω) with generator ψ. In view of Theorem 3.4 we have that

$$k_{s,t}(\hat{b})\Omega := \lim_{\alpha} j_{t_0,t_1}(b) \ldots j_{t_{n-1},t_n}(b)\Omega,$$

$b \in \mathcal{B}_1$, defines a cyclic Lévy process $\left(k_{s,t}\right)_{0 \le s \le t}$ on the Weyl bialgebra $(\mathbb{C}\mathcal{B}_1, \Lambda, \lambda)$ of $\mathcal{B}$ over (D_k, Ω) where D_k is a linear subspace of $\overline{D_j}$. Thus, for each pair $k_{s,t}(\hat{b})\Omega$, $k_{s,t}(\hat{c})\Omega$ for $b, c \in \mathcal{B}_1$ and $0 \le s \le t < \infty$ we have

$$\langle k_{s,t}(\hat{b})\Omega, k_{s,t}(\hat{c})\Omega\rangle = e^{(t-s)\psi(b^*c)}.$$

The generator ψ defines a coboundary by (2.1). Thus, we compute

$$\begin{aligned}\langle e^{-(t-s)\psi(b)} k_{s,t}(\hat{b})\Omega, e^{-(t-s)\psi(c)} k_{s,t}(\hat{c})\Omega\rangle &= e^{(t-s)(-\psi(b^*)-\psi(c)+\psi(b^*c))} \\ &= e^{(t-s)\langle \eta(b), \eta(c)\rangle} \\ &= \langle E(\eta(b) \otimes \mathbf{1}_{[s,t]}), E(\eta(c) \otimes \mathbf{1}_{[s,t]})\rangle\end{aligned}$$

where $\eta \colon \mathcal{B}_1 \to K$ is the canonical mapping to a dense linear subspace K of a Hilbert space $\overline{K}$ and $E(\eta(\cdot) \otimes \mathbf{1}_{[s,t]})$ denotes the exponential vector of $\eta(\cdot) \otimes \mathbf{1}_{[s,t]}$ in the Boson Fock space $\Gamma_s(\mathrm{L}^2([s,t], \overline{K}))$. Here $\eta(\cdot) \otimes \mathbf{1}_{[s,t]}$ denotes the function in $\mathrm{L}^2([s,t], \overline{K})$ which is a constant equal to $\eta(\cdot)$ on the interval $[s,t]$ and 0 elsewhere. The space K is obtained from ψ by the GNS construction mentioned in Section 2. Hence,

$$k_{s,t}(b)\Omega \cong e^{(t-s)\psi(b)} E(\eta(b) \otimes \mathbf{1}_{[s,t]}) \in \Gamma_s(\mathrm{L}^2([s,t], \overline{K}))$$

where $b \in \mathcal{B}_1, \psi(b) \in \mathbb{C}$ and $\eta(b) \in K$. In other words, the vectors $k_{s,t}(b)\Omega$ behave like exponential vectors in the Boson Fock space $\Gamma_s(\mathrm{L}^2([s,t], \overline{K}))$ and the $k_{s,t}(b)$ act on these exponential vectors like Weyl operators. Moreover, the vectors $k_{s,t}(b)\Omega$ 'generate' the Hilbert subspace $D_{k_{s,t}}$ of $\overline{D_k}$ where

$$D_{k_{s,t}} = \mathsf{span}\Big\{k_{t_0,t_1}(c_1)\cdots k_{t_{n-1},t_n}(c_n)\Omega\colon \\ n\in\mathbb{N},\, s=t_0\le t_1\le\cdots\le t_{n-1}\le t_n=t,\, c_1,\ldots,c_n\in\mathcal{C}\Big\}.$$

Therefore, we have $\overline{D_{k_{s,t}}} \cong \Gamma_s(\mathrm{L}^2([s,t],\overline{K}))$ and thus $\overline{D_k} \cong \Gamma_s(\mathrm{L}^2(\mathbb{R}^+,\overline{K}))$. Theorem 3.6 states that the vectors $k_{s,t}(b)\Omega$, $b\in\mathcal{B}_1$, are total in $\overline{D_{j_{s,t}}}\subset\overline{D_j}$ as well, i.e.

$$\overline{D_j} = \overline{D_k} \cong \Gamma_s(\mathrm{L}^2(\mathbb{R}^+,\overline{K})).$$

We proved that each cyclic quantum Lévy process on a $*$–bialgebra can be realized on a Boson Fock space $\Gamma_s(\mathrm{L}^2(\mathbb{R}^+,\overline{K}))$.

Example 3.10. *(Construction of quantum Lévy processes II)*
In the situation of Example 3.9, an application of Theorem 3.6 allows to reconstruct $j_{s,t}$ from the process $k_{s,t}$ on the Weyl bialgebra of $\mathcal{B}$. The realization of the latter on Fock space can simply be written down. In the present example we describe a realization of a Lévy process on a Bose Fock space that parallels the construction in [4] and [11] with the help of quantum stochastic calculus in the sense of R.L. Hudson and K.R. Parthasarathy.

Applying our result to the situation of Example 3.1 and 3.2 with $\varkappa=\mathsf{id}$ there are two possibilities. If we put the first bialgebra $\mathcal{B}$ equal to the induced $*$–bialgebra and $\mathcal{C}$ equal to the generator Hopf algebra, then for $b\in\mathcal{B}_0$ we have

$$\vartheta_\alpha(b) = \sum_{i=1}^{n} j_{t_{i-1},t_i}(b)\Omega \tag{3.7}$$

and Theorem 3.4 tells us that 3.7 converges to

$$I_{s,t}(b)\Omega = \big(A_{s,t}(\eta(b^*)) + \Lambda_{s,t}(\rho(b)) + A^*_{s,t}(\eta(b)) + \psi(b)\,(t-s)\big)\Omega$$

in norm where $A_{s,t},\Lambda_{s,t},A^*_{s,t}$ denote the annihilation, preservation and creation operators of the interval $[s,t]$ on Boson Fock space $\Gamma_s(\mathrm{L}^2(\mathbb{R}_+,\overline{K}))$; see the preceding section. For arbitrary $b\in\mathcal{B}$ we find

$$I_{s,t}(b) = \delta(b)\mathbf{1} + A_{s,t}(\eta(b^*)) + \Lambda_{s,t}(\rho(b)-\delta(b)) + A^*_{s,t}(\eta(b)) + \psi(b)\,(t-s).$$

$I_{s,t}$ is called the *generator process* of the Lévy process $j_{s,t}$.

We may construct $j_{s,t}$ out of $I_{s,t}$ if we take the generator Hopf algebra for the first bialgebra $\mathcal{B}$ and the induced one for the second bialgebra $\mathcal{C}$. Then by Theorem 3.4 we obtain a QLP $J_{s,t}$ on $\mathfrak{T}(\mathcal{B}_0)$ as the limit

$$J_{s,t} = \overrightarrow{\prod}^{\star}_{\mathfrak{T}(\Delta_0)} \mathfrak{T}(I_{s,t}-\delta)$$

of the convolution products of the generator process where now, of course, convolution is with respect to the original comultiplication Δ of $\mathcal{B}$. The limit is to be understood in the sense of the remark after Theorem 3.4; see equations (3.5) and (3.6).

The QLP $J_{s,t}$ has the generator $\psi\circ M$ and by Proposition 3.3 the restriction of its cyclic version to $\mathcal{B}$ is a version of a QLP with generator ψ. So our procedure allows, like quantum stochastic calculus (see equation (2.2)), a construction of the Lévy process $j_{s,t}$ from the elementary processes $A_{s,t}, \Lambda_{s,t}, A^*_{s,t}$ on Boson Fock space. In fact, if $\mathrm{d}t$ is 'small', then in all relevant formula one may substitute $j_{t,t+\mathrm{d}t}$ with $I_{t,t+\mathrm{d}t}$. We find, in a heuristic sense,

$$j_{s,t+\mathrm{d}t} - j_{s,t} \;=\; j_{s,t} \star j_{t,t+\mathrm{d}t} - j_{s,t} \;\simeq\; j_{s,t} \star I_{t,t+\mathrm{d}t} - j_{s,t} \;=\; j_{s,t} \star (I_{t,t+\mathrm{d}t} - \delta\mathbf{1}).$$

If we put $\mathrm{d}I_t = I_{s,t+\mathrm{d}t} - I_{s,t}$ (independent of $s < t$), this gives a heuristical meaning to

$$j_{s,t} \;=\; \delta\mathbf{1} + \int_s^t j_{s,r} \star \mathrm{d}I_r$$

as a quantum stochastic integral. We remark that this interpretation as an integral is not limited to the above choice. Whenever k is a transformed process obtained from j via (3.5), we formally may write

$$k_{s,t} \;=\; \delta\mathbf{1} + \int_s^t k_{s,r} \star (\mathrm{d}j_t \circ \varkappa),$$

where $\mathrm{d}j_t := j_{t,t+\mathrm{d}t} - \delta\mathbf{1}$.

Example 3.11. *(Classical Lévy processes and unitary evolutions)*
Let G be a topological group and denote by $\mathcal{R}(G)$ the space of all coefficient functions of continuous finite-dimensional representations of G. Then $f \in \mathcal{R}(G)$ iff there are $n \in \mathbb{N}$ and continuous complex-valued functions $f_1, \dots f_n, g_1, \dots g_n$ on G such that

$$f(xy) = \sum_{i=1}^n f_i(x)\, g_i(y) \;\forall x, y \in G.$$

$\mathcal{R}(G)$ is a commutative $*$–algebra. By setting

$$\Delta f = \sum_{i=1}^n f_i \otimes g_i, \quad \delta f = f(e)$$

$\mathcal{R}(G)$ becomes a commutative Hopf $*$–algebra. In various cases (e.g., when G is compact or locally compact abelian) the group G is uniquely determined by $\mathcal{R}(G)$. Let us assume that G is compact. Then $\mathcal{R}(G)$ is the Kreîn algebra of G.

A classical Lévy process X_t on G gives rise to a quantum Lévy process j_t on $\mathcal{R}(G)$ by putting $j_t(f) = f \circ X_t$. Here $j_t = j_{0.t}$ and $j_{s,t} = (j_s \circ S) \star j_t$ where S is the antipode of $\mathcal{R}(G)$. Let us specialize further to the case when G is the group $\mathcal{U}_d$ of unitary $d \times d$-matrices. Then $\mathcal{R}(G)$ equals the Hopf $*$–algebra $\mathbb{C}[x_{kl}, x^*_{kl}; k, l = 1, \dots d]$ divided by the $*$–ideal generated by the elements which are the entries of the matrices $x\,x^* - \mathbf{1}$ and $x^*\,x - \mathbf{1}$ where we put $x = (x_{kl})_{k,l=1,\dots d}$. The comultiplication is given by $\Delta x_{kl} = \sum_{i=1}^d x_{ki} \otimes x_{il}$ and the counit by $\delta x_{kl} = \delta_{kl}$. The antipode is given by $S(x_{kl}) = x^*_{lk}$. By replacing the commuting indeterminates x_{kl} by non-commuting indeterminates, we define a non-commutative $*$–bialgebra

$$\mathbb{C}\langle x_{kl}, x^*_{kl}; k, l = 1, \dots d\rangle / x\,x^* = \mathbf{1}, x^*\,x = \mathbf{1}$$

which we denote by $\mathcal{U}\langle d\rangle$; cf. [11]. (It is easy to see that $\mathcal{U}\langle d\rangle$ is not a Hopf algebra.)

Lévy triples on $\mathcal{U}\langle d\rangle$ are given by a Hilbert space $\overline{K}$, a unitary operator W on $\mathbb{C}^d \otimes \overline{K}$, a matrix $L \in M_d(\mathbb{C}) \otimes \overline{K}$ and a self-adjoint matrix $H \in M_d(\mathbb{C})$ via the equations

$$\begin{aligned}\rho(x_{kl}) &= W_{kl} \in \mathcal{B}(\overline{K})\\ \eta(x_{kl}) &= L_{kl}\\ \psi(x_{kl}) &= -\frac{1}{2}(LL^*)_{kl} + \mathrm{i}\,H_{kl};\end{aligned}$$

cf. [11] and [3]. The generator process (cf. the previous example) is given by matrices $I_{s,t} \in M_d(\mathbb{C}) \otimes \Gamma(\mathrm{L}^2(\mathbb{R}_+, \overline{K}))$ with

$$(I_{s,t})_{ij} = -A_{s,t}((W^*L)_{ji}) + \Lambda_{s,t}((W-\mathbf{1})_{ij}) + A^*_{s,t}(L_{ij}) + (\mathrm{i}\,H - \frac{1}{2}(LL^*))_{ij}\,(t-s)$$

Theorem 3.4 says that

$$I_{t_0,t_1} I_{t_1,t_2} \dots I_{t_{n-1},t_n}$$

converges, again in the sense of the remark after Theorem 3.4, to the Lévy process $U_{s,t}$ which is the unitary process on $\mathbb{C}^d \otimes \Gamma(\mathrm{L}^2(\mathbb{R}_+, \overline{K}))$ given by $(U_{s,t})_{ij} = j_{s,t}(x_{ij})$. This is a generalization of a construction already given by W. von Waldenfels [16]. A classical Lévy process on $\mathcal{U}_d$ is a special case of a QLP on $\mathcal{U}\langle d\rangle$.

Example 3.12. *(Azéma martingales)*
Consider the $*$–algebra $\mathbb{C}\langle x, x^*, y\rangle$ generated by x and a self-adjoint y. For $q \in \mathbb{R}$ divide $\mathbb{C}\langle x, x^*, y\rangle$ by the $*$–ideal generated by the element $xy - q\,yx$ to obtain a $*$–algebra $\mathcal{A}$.

On $\mathcal{A}$ we consider two $*$–bialgebra structures. The first is the one with x (and x^*) primitive and with y group-like, the second is given by

$$\begin{aligned}\Delta x &= x \otimes y + \mathbf{1} \otimes x \text{ and } \ \delta x = 0\\ \Delta y &= y \otimes y \text{ and } \ \delta y = 1\end{aligned}$$

and is called the *Azéma $*$–bialgebra of parameter q*. Again we apply our results to these two $*$–bialgebras with $\varkappa = \mathsf{id}$. If we choose for generator

$$\psi(M(x,x^*)\,y^k) = \begin{cases} 1 & \text{if } M(x,x^*) = xx^* \\ 0 & \text{otherwise}\end{cases}$$

$M(x,x^*) \in \mathcal{A}$ a monomial in x and x^*, $k \in \mathbb{N}_0$, then $K = \mathbb{C}$, $\eta(x^*) = 1$, $\eta(x) = 0$, $\rho(x) = 0$ and $\rho(y) = q$. The linear functional ψ is the generator of the quantum q-Azéma martingale (X_t, X_t^*, Y_t) if we consider the Azéma $*$–bialgebra, and it generates the process (A_t, A_t^*, Y_t) in the case of the primitive/group-like structure of $\mathcal{A}$ where Y_t is the second quantization of multiplication by $q\,\mathbf{1}_{[0,t]}$. The process X_t satisfies the quantum stochastic differential equation

$$\mathrm{d}X_t = (q-1)X_t\,\mathrm{d}\Lambda_t + \mathrm{d}A_t,\ X_0 = 0;$$

see [8, 10].

An application of Theorem 3.4 yields the formula

$$W_t = \lim \sum_{j=0}^{n-1} Z_{t_j,t_{j+1}}$$

with

$$Z_t = \lim\big(W_{t_0,t_1}\, Y_{t_1,t_2} \dots Y_{t_{n-1},t_n} + W_{t_1 t_2}\, Y_{t_2,t_3} \dots Y_{t_{n-1},t_n} + \dots \\ \dots + W_{t_{n-2},t_{n-1}}\, Y_{t_{n-1},t_n} + W_{t_{n-1},t_n}\big)$$

where W_t and Z_t denote the Wiener process and the q-Azéma martingale on Boson Fock space respectively. Again convergence is in the sense of the remark after 3.4.

Example 3.13. *(Trotter formulae for QLPs)*
This example can be regarded as a motivation of the whole paper. Trotter formulae were considered by V. Liebscher and M. Skeide [5] and initiated the theory of transformations as presented in this paper.

In the case when the initial space is finite dimensional, the following is a generalization to arbitrary $*$–bialgebras of a formula by J.M. Lindsay and K.B. Sinha [6] for unitary quantum stochastic processes on a Hilbert space which satisfy a quantum stochastic differential equation of the type (2.2) with constant bounded coefficients (see [4] and cf. Example 3.11).

Let $\mathcal{B}$ be a $*$–bialgebra and let ψ_1, ψ_2 be two generators on $\mathcal{B}$. Then it is immediate that $\psi_1 + \psi_2$ is again a generator on $\mathcal{B}$. One would like to construct the process $j_{s,t}$ given by $\psi_1 + \psi_2$ from the processes $j^{(1)}_{s,t}$ and $j^{(2)}_{s,t}$ given by ψ_1 and ψ_2 respectively. This can be done in the framework of transformation in at least three different ways.

1. For two $*$–bialgebras $\mathcal{B}_1$ and $\mathcal{B}_2$ we can form the tensor product $\mathcal{B}_1 \otimes \mathcal{B}_2$ as the $*$–bialgebra with comultiplication coming from the coalgebra tensor product (cf. Section 2) and the usual $*$–algebra structure on $\mathcal{B}_1 \otimes \mathcal{B}_2$ (i.e. $(b_1 \otimes b_2)(c_1 \otimes c_2) = b_1 c_1 \otimes b_2 c_2$ and $(b \otimes c)^* = b^* \otimes c^*$). For generators ψ_1, ψ_2 on $\mathcal{B}_1, \mathcal{B}_2$ we form the generator

$$\widetilde{\psi} := \psi_1 \otimes \delta_2 + \delta_1 \otimes \psi_2$$

on $\mathcal{B}_1 \otimes \mathcal{B}_2$. Indeed, ψ is hermitian with $\psi(\mathbf{1}) = 0$, and if $\tilde{b} \in \mathcal{B}_1 \otimes \mathcal{B}_2$, $(\delta_1 \otimes \delta_2)(\tilde{b}) = 0$

$$\begin{aligned} \widetilde{\psi}(\tilde{b}^* \tilde{b}) &= (\psi_1 \otimes \delta_2)(\tilde{b}^* \tilde{b}) + (\delta_1 \otimes \psi_2)(\tilde{b}^* \tilde{b}) \\ &= \psi_1\big((\mathsf{id} \otimes \delta_2)(\tilde{b}^*)(\mathsf{id} \otimes \delta_2)(\tilde{b})\big) + \psi_2\big((\delta_1 \otimes \mathsf{id})(\tilde{b}^*)(\delta_1 \otimes \mathsf{id})(\tilde{b})\big) \\ &\geq 0 \end{aligned}$$

since

$$\delta_1(\mathsf{id} \otimes \delta_2)(\tilde{b}) = 0 = \delta_2(\delta_1 \otimes \mathsf{id})(\tilde{b})$$

so that $\widetilde{\psi}$ is conditionally positve.

For QLPs $j^{(1)}_{s,t}, j^{(2)}_{s,t}$ on $\mathcal{B}_1, \mathcal{B}_2$ over $(D_1, \Omega_1), (D_2, \Omega_2)$ with generators ψ_1, ψ_2 we form the process $\widetilde{j}_{s,t}$ on $\mathcal{B}_1 \otimes \mathcal{B}_2$

$$\widetilde{j}_{s,t}(b_1 \otimes b_2) = j^{(1)}_{s,t}(b_1) \otimes j^{(2)}_{s,t}(b_2).$$

Then $\widetilde{j}_{s,t}$ is a QLP on $\mathcal{B}_1 \otimes \mathcal{B}_2$ over $(D_1 \otimes D_2, \Omega_1 \otimes \Omega_2)$ and consists of two independent components $j^{(1)}_{s,t}$ and $j^{(2)}_{s,t}$. Moreover, the convolution semigroup of $\widetilde{j}_{s,t}$ is given by

$$\varphi_t(b_1 \otimes b_2) = \varphi^{(1)}_1(b_1)\varphi^{(2)}_2(b_2)$$

$b_1 \in \mathcal{B}_1$, $b_2 \in \mathcal{B}_2$, if $\varphi_t^{(1)}$, $\varphi_t^{(2)}$ denote the convolution semigroups of $j_{s,t}^{(1)}$, $j_{s,t}^{(2)}$. The generator of $\widetilde{j}_{s,t}$ is given by $\widetilde{\psi}$, which shows once more that $\widetilde{\psi}$ is conditionally positive.

In the case $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}$ we consider the induced tensor $*$–bialgebra $(\mathcal{T}(\mathcal{B}_0), \mathcal{T}(\Delta_0), \mathcal{T}(0))$ of $\mathcal{B}$ and define the $*$–algebra homomorphism $\varkappa\colon \mathcal{T}(\mathcal{B}_0) \to \mathcal{B} \otimes \mathcal{B}$ by setting

$$\varkappa(b) = b \otimes \mathbf{1} + \mathbf{1} \otimes b,\ b \in \mathcal{B}_0.$$

Then $\varkappa$ is a transformation of $\mathcal{B} \otimes \mathcal{B}$ and we have for $b_1, \dots, b_n \in \mathcal{B}_0$

$$\begin{aligned}(\widetilde{\psi} \circ \varkappa)(b_1 \otimes \cdots \otimes b_n) &= \widetilde{\psi}\big((b_1 \otimes \mathbf{1} + \mathbf{1} \otimes b_1) \cdots (b_n \otimes \mathbf{1} + \mathbf{1} \otimes b_n)\big) \\ &= \sum_{A \subset \{1,\dots,n\}} \widetilde{\psi}(b_A \otimes b_{A^c})\end{aligned} \tag{3.8}$$

where for a subset $\{i_1 < \cdots < i_l\}$ of $\{1, \dots, n\}$ we put $b_A = b_{i_1} \cdots b_{i_l}$, $b_\emptyset = \mathbf{1}$. Since $b_1, \dots, b_n$ are in the kernel of the counit δ we have that (3.8) is equal to $\psi_1(b_1 \cdots b_n) + \psi_2(b_1 \cdots b_n)$ and $\widetilde{\psi} \circ \varkappa = (\psi_1 + \psi_2) \circ M$. Denote by $J_{s,t}$ the QLP on $\mathcal{T}(\mathcal{B}_0)$ with generator $(\psi_1 + \psi_2) \circ M$. By Proposition 3.3 the restriction $j_{s,t}$ of $J_{s,t}$ to $\mathcal{B} \subset \mathcal{T}(\mathcal{B}_0)$ is a QLP on $\mathcal{B}$ with generator $\psi_1 + \psi_2$. Equation (3.5) becomes $(b \in \mathcal{B}_0)$

$$\begin{aligned} j_{s,t}(b) &= \lim_\alpha \big(j_{t_0,t_1}^{(1)}(b_{(1)}) \otimes \mathbf{1} + \mathbf{1} \otimes j_{t_0,t_1}^{(2)}(b_{(1)}) - \delta(b_{(1)})\mathbf{1} \otimes \mathbf{1}\big) \cdots \\ &\qquad \cdots \big(j_{t_{n-1},t_n}^{(1)}(b_{(n)}) \otimes \mathbf{1} + \mathbf{1} \otimes j_{t_{n-1},t_n}^{(2)}(b_{(n)}) - \delta(b_{(1)})\mathbf{1} \otimes \mathbf{1}\big) \\ &= \lim_\alpha \big((j_{t_0,t_1}^{(1)} - \delta)(b_{(1)}) \otimes \mathbf{1} + \mathbf{1} \otimes j_{t_0,t_1}^{(2)}(b_{(1)})\big) \cdots \\ &\qquad \cdots \big((j_{t_{n-1},t_n}^{(1)} - \delta)(b_{(n)}) \otimes \mathbf{1} + \mathbf{1} \otimes j_{t_{n-1},t_n}^{(2)}(b_{(n)})\big) \\ &= \lim_\alpha \big(j_{t_0,t_1}^{(1)}(b_{(1)}) \otimes \mathbf{1} + \mathbf{1} \otimes (j_{t_0,t_1}^{(2)} - \delta)(b_{(1)})\big) \cdots \\ &\qquad \cdots \big(j_{t_{n-1},t_n}^{(1)}(b_{(n)}) \otimes \mathbf{1} + \mathbf{1} \otimes (j_{t_{n-1},t_n}^{(2)} - \delta)(b_{(n)})\big). \end{aligned}$$

If we take equidistant partitions we obtain

$$\begin{aligned}\langle \Omega, j_{s,t}(b)\Omega \rangle &= \lim_{n \to \infty} \big(\varphi^{(1)}_{\frac{t-s}{n}} + \varphi^{(2)}_{\frac{t-s}{n}} - \delta\big)^{\star n}(b) \\ &= \mathrm{e}_\star^{(t-s)(\psi_1 + \psi_2)}(b)\end{aligned}$$

which, of course, also follows from Theorem 3.4.

2. There is a 'multiplicative' version of the above construction. Define the $*$–algebra homomorphism $\beta\colon \mathbb{C}\mathcal{B}_1 \to \mathcal{B} \otimes \mathcal{B}$ by

$$\beta(b) = b \otimes b$$

for $b \in \mathcal{B}_1$. We use the notation of Example 3.8. One proves that $\varkappa = \widetilde{\varkappa} \circ \beta\colon \mathcal{T}(\mathcal{B}_0) \to \mathcal{B} \otimes \mathcal{B}$ is a transformation of $\mathcal{B} \otimes \mathcal{B}$, and that $\widetilde{\psi} \circ \varkappa = \psi \circ M$. This time equation (3.5) can be written $(b \in \mathcal{B}_1)$

$$j_{s,t}(b) = \lim_\alpha \sum_i \lambda_i \big(j_{t_0,t_1}^{(1)}(b_{1,i}) \otimes j_{t_0,t_1}^{(2)}(b_{1,i}) \cdots j_{t_{n-1},t_n}^{(1)}(b_{n,i}) \otimes j_{t_{n-1},t_n}^{(2)}(b_{n,i})\big)$$

with $\Delta_n\, b = \sum_i \lambda_i\, b_{1,i} \otimes \cdots \otimes b_{n,i}$, $b_{l,i} \in \mathcal{B}_1$, $\lambda_i \in \mathbb{C}$.

3. Formula (3.7) of [6] can also be obtained by transformation. In the transformation theorem 3.4 take for the second $*$–bialgebra $\mathcal{C}$ the original $*$–bialgebra $\mathcal{B}$ itself and for the $*$–bialgebra $\mathcal{B}$ in Theorem 3.4 take $\mathcal{B} \otimes \mathcal{B}$. Then the comultiplication $\Delta \colon \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$ is a transformation (notice that Δ in general is not a coalgebra homomorphism!). Of course, $\widetilde{\psi} \circ \Delta = \psi$, and we obtain a realisation of our QLP on $\mathcal{B}$ with generator ψ as

$$j_{s,t}(b) = \lim_{\alpha} \; j^{(1)}_{t_0,t_1}(b_{(1)}) \otimes j^{(2)}_{t_0,t_1}(b_{(2)}) \cdots j^{(1)}_{t_{n-1},t_n}(b_{(2n-1)}) \otimes j^{(2)}_{t_{n-1},t_n}(b_{(2n)}).$$

In the case of Example 3.11 this gives exactly the formula of [6] for a finite dimensional initial space. If $\mathcal{B}$ is cocommutative (in which case Δ is a coalgebra homomorphism!) we obtain $j_{s,t} = (j^{(1)}_{s,t} \otimes j^{(2)}_{s,t}) \circ \Delta$, and the Trotter formula becomes trivial.

The three constructions of a quantum Lévy process with generator $\psi_1 + \psi_2$ depend on the choice of the first bialgebra and the transformation $\varkappa$. The most natural seems to be the one of 3. where $\varkappa$ is the comultiplication Δ itself whereas in cases 1. and 2. the transformation $\varkappa$ does not depend on Δ. In 1. and 2. the two original processes are first put together in an additive and group-like way respectively. Case 3. is a 'real' Trotter formula for exponentials given by the comultiplication Δ.

4. Proof of Theorems

In principle, Theorem 3.4 is proved if we show that the nets in (3.4) are Cauchy. To show this, in Section 4.1 we prove a lemma about infinitesimal products in Banach algebras (an extension of ideas in V. Liebscher and M. Skeide [5]) and a coalgebra version, appealing to the Fundamental Theorem on Coalgebras; see L. Accardi, M. Schürmann, and W. von Waldenfels [1]. These lemmas plus the algebraic Proposition 4.3 allow to prove Proposition 4.4, which is the analytic heart of the proof of Theorem 3.4.

4.1. Preparatory lemmas. We start with a lemma that imitates, like in [5], proofs of the Trotter product formula.

Lemma 4.1. *Let $\mathcal{A}$ be a Banach algebra. Suppose we have a constant $R > 0$ and a family $\big(A^{(\mu)}\big)_{\mu \in M}$ of functions (M being some index set)*

$$r \;\longmapsto\; A^{(\mu)}_r = I + rG + \mathfrak{S}^{(\mu)}_r \;\in\; \mathcal{A}$$

on $\mathbb{R}_+$ where $G \in \mathcal{A}$ and $\mathfrak{S}^{(\mu)}_r$ satisfies $\big\|\mathfrak{S}^{(\mu)}_r\big\| \le r^2 \frac{C^2}{2}$ for some constant C not depending on $\mu \in M$ and all $r \le R$. Then for all intervals $[s,t] \subset \mathbb{R}_+$, all partitions $\alpha = \{s = t_0 < t_1 < \cdots < t_{n-1} < t_n = t\}$ ($n \in \mathbb{N}$) of $[s,t]$ with $\|\alpha\| \le R$, and an arbitrary choice of elements $\mu_1, \ldots, \mu_n$ of M, we have

$$\Big\|A^{(\mu_1)}_{t_1 - t_0} \cdots A^{(\mu_n)}_{t_n - t_{n-1}} - e^{(t-s)G}\Big\| \;\le\; \|\alpha\|\,(t-s) e^{(t-s)\max(\|G\|,C)} \frac{C^2 + \|G\|^2 e^{\|\alpha\|\,\|G\|}}{2}.$$

Proof. By assumption $\Big\|A^{(\mu_k)}_r\Big\| \le 1 + r\,\|G\| + r^2 \frac{C^2}{2} \le e^{r \max(\|G\|,C)}$, and thus

$$\Big\|A^{(\mu_\ell)}_{t_\ell - t_{\ell-1}} \cdots A^{(\mu_k)}_{t_k - t_{k-1}}\Big\| \le e^{(t_k - t_{\ell-1})\max(\|G\|,C)}$$

for all intervals $[s,t] \subset \mathbb{R}_+$, all partitions α_n of $[s,t]$, and all $1 \le \ell < k \le n$. The next calculation (cf. [5] proof of Proposition 3.3) is essential for the proof. We compute

$$\begin{aligned} A^{(\mu_1)}_{t_1-t_0} \cdots A^{(\mu_n)}_{t_n-t_{n-1}} - e^{(t-s)G} &= A^{(\mu_1)}_{t_1-t_0} \cdots A^{(\mu_n)}_{t_n-t_{n-1}} - e^{(t_1-t_0)G} \cdots e^{(t_n-t_{n-1})G} \\ &= \sum_{j=1}^{n} A^{(\mu_1)}_{t_1-t_0} \cdots A^{(\mu_{j-1})}_{t_{j-1}-t_{j-2}} \left(A^{(\mu_j)}_{t_j-t_{j-1}} - e^{(t_j-t_{j-1})G} \right) \\ &\qquad \cdot e^{(t_{j+1}-t_j)G} \cdots e^{(t_n-t_{n-1})G}. \end{aligned}$$

We have

$$\begin{aligned} \left\| A^{(\mu_j)}_{t_j-t_{j-1}} - e^{(t_j-t_{j-1})G} \right\| &\le \left\| A^{(\mu_j)}_{t_j-t_{j-1}} - I - (t_j - t_{j-1})G \right\| \\ &\quad + \left\| I + (t_j - t_{j-1})G - e^{(t_j-t_{j-1})G} \right\| \\ &\le (t_j - t_{j-1})^2 \frac{C^2 + \|G\|^2 e^{(t_j-t_{j-1})\|G\|}}{2}. \end{aligned}$$

From this estimate, from the estimate preceding it, and from

$$\sum_{j=1}^{n} (t_j - t_{j-1})^2 \le \|\alpha\| \sum_{j=1}^{n} (t_j - t_{j-1}) = \|\alpha\| (t-s)$$

the statement follows. □

There is a coalgebra version of Lemma 4.1 deduced from the Fundamental Theorem on Coalgebras which yields that the coalgebra generated by a finite subset of a coalgebra is finite dimensional. In the sequel, $\mathbf{L}(V,W)$ denotes the vector space of linear maps between vector spaces V and W. We put $\mathbf{L}(V,V) = \mathbf{L}(V)$. Let $(\mathcal{C}, \Delta, \delta)$ be a coalgebra and let $\psi \in \mathbf{L}(\mathcal{C}, \mathbb{C}) = \mathcal{C}^*$ be a linear functional on $\mathcal{C}$. The map $T\colon \psi \mapsto (id \otimes \psi) \circ \Delta$ defines an injective unital algebra homomorphism from $(\mathbf{L}(\mathcal{C}, \mathbb{C}), \star)$ to $(\mathbf{L}(\mathcal{C}), \circ)$ with left inverse $\delta \circ \mathbf{1}$. Moreover, each $T(\psi)$ leaves every sub-coalgebra of $\mathcal{C}$ invariant. On an arbitrary finite-dimensional subcoalgebra $\mathcal{C}_c \ni c$ of $\mathcal{C}$ the series $e^{T(\psi)} \restriction \mathcal{C}_c := \sum_{n=0}^{\infty} \frac{T(\psi) \restriction \mathcal{C}_c}{n!}$ converges in any norm. By the Fundamental Theorem on Coalgebras for every $c \in \mathcal{C}$ such a $\mathcal{C}_c$ exists. We deduce that the series

$$e_\star^{\psi}(c) \;:=\; \sum_{n=0}^{\infty} \frac{\psi^{\star n}}{n!}(c) \;=\; \delta \circ e^{T(\psi)}(c) \tag{4.1}$$

converges for all $\psi \in \mathcal{C}^*$ and all $c \in \mathcal{C}$. Clearly, this limit of complex numbers cannot depend on the choice of $\mathcal{C}_c$; see [1].

We now prove the coalgebra version of Lemma 4.1.

Lemma 4.2. *Let $\mathcal{C}$ be a coalgebra. Suppose we have a constant $R > 0$ and a family $\left(f^{(\mu)}\right)_{\mu \in M}$ of functions*

$$r \;\longmapsto\; f_r^{(\mu)} = \delta + r\psi + \mathfrak{R}_r^{(\mu)} \;\in\; \boldsymbol{L}(\mathcal{C}, \mathbb{C})$$

on $\mathbb{R}_+$ where $\psi \in \boldsymbol{L}(\mathcal{C}, \mathbb{C})$ and $\mathfrak{R}_r^{(\mu)}(c)$ satisfies $\left|\mathfrak{R}_r^{(\mu)}(c)\right| \le r^2 D_c$ for some constant $D_c > 0$, depending on $c \in \mathcal{C}$ but not on μ, and all $r \le R$. Then there exist constants $C_c > 0$ and $\Psi_c > 0$ such that for all intervals $[s,t] \subset \mathbb{R}_+$, all partitions

$\alpha_n = \{s = t_0 < t_1 < \cdots < t_{n-1} < t_n = t\}$ $(n \in \mathbb{N})$ *of* $[s,t]$ *with* $\|\alpha\| \leq R$*, and an arbitrary choice of elements* $\mu_1, \ldots, \mu_n$ *of* M*, we have*

$$\left| f^{(\mu_1)}_{t_1-t_0} \star \cdots \star f^{(\mu_n)}_{t_n-t_{n-1}}(c) - e_\star^{(t-s)\psi}(c) \right| \\ \leq \|\alpha\| \, (t-s) e^{(t-s)\max(\Psi_c, C_c)} \frac{C_c^2 + \Psi_c^2 e^{\|\alpha\| \, \Psi_c}}{2}.$$

Proof. Choose $b \in \mathcal{C}$ and fix a finite-dimensional sub-coalgebra $\mathcal{C}_b$ of $\mathcal{C}$ containing b. Fix a norm on $\mathcal{C}_b$. From the weak estimates $\left|\mathfrak{R}^{(\mu)}_r(c)\right| \leq r^2 D_c$ we easily conclude the strong estimate $\left\|\mathfrak{R}^{(\mu)}_r\right\| \leq r^2 D$ for a suitable constant D for the linear functionals $\mathfrak{R}^{(\mu)}_r$ on $\mathcal{C}_b$. (Just take your favorite elementary proof of the Uniform Boundedness Principle for finite-dimensional Banach spaces.) Consider the linear operator

$$A^{(\mu)}_r := T(f^{(\mu)}_r) \restriction \mathcal{C}_b$$

on $\mathcal{C}_b$, so

$$A^{(\mu)}_r = I + rG + \mathfrak{S}^{(\mu)}_r$$

where $G := T(\psi) \restriction \mathcal{C}_b$ and $\mathfrak{S}^{(\mu)}_r = T(\mathfrak{R}^{(\mu)}_r) \restriction \mathcal{C}_b$.

$\mathbf{L}(\mathcal{C}_b)$ is a Banach algebra with respect to the operator norm. Since T is a bijection from $\mathcal{C}_b^*$ onto $T(\mathcal{C}_b^*) \subset \mathbf{L}(\mathcal{C}_b)$, and since all norms on finite-dimensional spaces are equivalent, $\mathfrak{S}^{(\mu)}_r$ satisfies $\left\|\mathfrak{S}^{(\mu)}_r\right\| \leq r^2 \frac{C^2}{2}$ for some constant C. In view of lemma 4.1 we obtain the claimed statement if we choose $C_c = C\sqrt{\|\delta\| \, \|c\|}$ and $\Psi_c = \|G\| \sqrt{\|\delta\| \, \|c\|}$. □

4.2. Proof of Theorem 3.4. Consider the Hilbert subspaces $(0 \leq s \leq t)$

$$H_{s,t} = \overline{\mathsf{span}}\Big\{ j_{t_0,t_1}(b_1) \cdots j_{t_{n-1},t_n}(b_n)\Omega \colon \\ n \in \mathbb{N},\, s = t_0 \leq t_1 \leq \cdots \leq t_n = t,\, b_1, \ldots, b_n \in \mathcal{B} \Big\}$$

of $\overline{D_j}$ where $H_0 = \mathbb{C}$. Put $H_t = H_{0,t}$. Using the shift and the unit vector Ω, we define mappings $U_{s,t} \colon H_s \otimes H_t \to H_{s+t}$ by

$$U_{s,t}(j_{s_0,s_1}(b_1) \cdots j_{s_{n-1},s_n}(b_n)\Omega \otimes j_{t_0,t_1}(c_1) \cdots j_{t_{m-1},t_m}(c_m)\Omega) \\ = j_{s_0,s_1}(b_1) \cdots j_{s_{n-1},s_n}(b_n) j_{t_0+s,t_1+s}(c_1) \cdots j_{t_{m-1}+s,t_m+s}(c_m)\Omega$$

where $U_{s,t}(\Omega \otimes \Omega) = \Omega$ and $b_1, \ldots, b_n, c_1, \ldots, c_m \in \mathcal{B}$, $n, m \in \mathbb{N}$. Indeed, the mappings $U_{s,t}$ are unitary. The shift is isometric and the unit vector Ω is cyclic which ensures surjectivity. Therefore, we may think of the family of Hilbert spaces $(H_t)_{t \geq 0}$ as a *tensor product system* in the sense of W. Arveson [2]; see M. Skeide [14]. In fact, it is of type I which means that it comes from a Boson Fock space; see Examples 3.10 and 3.9.

Let $0 \leq s = t_0 \leq t_1 \leq \cdots \leq t_{n-1} \leq t_n = t$. Using the unitary isomorphism $H_{t_0,t_1} \otimes H_{t_1,t_2} \otimes \cdots \otimes H_{t_{n-1},t_n} \cong H_{s,t}$, in the sequel, we identify

$$j_{t_0,t_1}(b_1) \cdots j_{t_{n-1},t_n}(b_n)\Omega = j_{t_0,t_1}(b_1)\Omega \otimes \cdots \otimes j_{t_{n-1},t_n}(b_n)\Omega. \tag{4.2}$$

In what follows we will often exploit in an essential way the coalgebra structure of $\overline{\mathcal{B}}\otimes\mathcal{C}$ (see Section 2) and its interplay with expressions like (4.2). The following proposition expresses the core of all such computations. It's proof is an easy verification and we omit it.

Proposition 4.3. *Let* $(\mathcal{B},\Delta,\delta)$ *and* $(\mathcal{C},\Lambda,\lambda)$ *be coalgebras. Let* D_i $(i = 1,2)$ *be two pre-Hilbert spaces and suppose we have linear mappings* $J_i\colon \mathcal{B}\to D_i$ *and* $K_i\colon \mathcal{C}\to D_i$*. Define the linear functionals* L_i *on the coalgebra* $\overline{\mathcal{B}}\otimes\mathcal{C}$ *by setting*

$$L_i(\overline{b}\otimes c) \ := \ \langle J_i(b), K_i(c)\rangle$$

and denote

$$\begin{aligned} J_1\star J_2 \ &:= \ (J_1\otimes J_2)\circ\Delta\colon \mathcal{B} \ \longrightarrow \ D_1\otimes D_2, \\ K_1\star K_2 \ &:= \ (K_1\otimes K_2)\circ\Lambda\colon \mathcal{C} \ \longrightarrow \ D_1\otimes D_2. \end{aligned}$$

Then

$$L_1\star L_2(\overline{b}\otimes c) \ = \ \langle J_1\star J_2(b), K_1\star K_2(c)\rangle.$$

Like in Proposition 4.3, in all what follows it is important to pay careful attention to the different comultiplications of the coalgebras $\mathcal{B}$, $\overline{\mathcal{B}}$, $\mathcal{C}$, $\overline{\mathcal{C}}$, $\overline{\mathcal{B}}\otimes\mathcal{C}$ and $\overline{\mathcal{C}}\otimes\mathcal{C}$ which lead to different convolutions.

Proposition 4.4. *For* $c,d\in\mathcal{C}$ *and* $T>0$ *there exists a* $C>0$ *such that the following holds. For each* $[s,t]\subset[0,T]$ *and* $\alpha\in\mathfrak{Z}_{st}$ *and for each* $\beta\in\mathfrak{Z}_{st}$ *finer than* α *we have*

$$\left|\langle\vartheta_\alpha(c),\vartheta_\beta(d)\rangle - e_\star^{(t-s)\psi\circ\varkappa}(c^*d)\right| < \|\alpha\|\,(t-s)C. \tag{4.3}$$

Proof. Let the partitions α and β be given by

$$\alpha = \{s = s_0 < s_1 < \cdots < s_l = t\}$$

and

$$\begin{aligned} \beta = \{s = s_0 &= t_0^{(1)} < t_1^{(1)} < \cdots < t_{k_1-1}^{(1)} < t_{k_1}^{(1)} = s_1 \\ &= t_0^{(2)} < t_1^{(2)} < \cdots < t_{k_2-1}^{(2)} < t_{k_2}^{(2)} = s_2 \\ &\ \vdots \\ &= t_0^{(l)} < t_1^{(l)} < \cdots < t_{k_l-1}^{(l)} < t_{k_l}^{(l)} = s_l = t\}. \end{aligned}$$

Denote further

$$\alpha^{(n)} = \{s_{n-1} = t_0^{(n)} < t_1^{(n)} < \cdots < t_{k_n-1}^{(n)} < t_{k_n}^{(n)} = s_n\}$$

for $n = 1,\ldots,l$. For a pair of partitions γ,ζ of an interval $[s,t]$ define the linear functionals $L_{\gamma,\zeta}$ on $\overline{\mathcal{C}}\otimes\mathcal{C}$ by setting $L_{\gamma,\zeta}(\overline{c}\otimes d) := \langle\vartheta_\gamma(c),\vartheta_\zeta(d)\rangle$. Then, by Proposition 4.3,

$$L_{\alpha,\beta} \ = \ L_{\{s_0,s_1\},\alpha^{(1)}}\star\ldots\star L_{\{s_{l-1},s_l\},\alpha^{(l)}}.$$

In the expression for $L_{\{s_{n-1},s_n\},\alpha^{(n)}}$ we have

$$j_{s_{n-1},s_n}\circ\varkappa(c) = (j_{t_0^{(n)},t_1^{(n)}}\star\ldots\star j_{t_{k_n-1}^{(n)},t_{k_n}^{(n)}})\circ\varkappa(c)$$

since, by assumption, $(j_{s,t})_{0\leq s\leq t}$ is a Lévy process with respect to the comultiplication of $\mathcal{B}$. If for a partition

$$\gamma = \{s = t_0 < t_1 < \cdots < t_m = t\}$$

of an interval $[s,t]$ we define the linear functionals M_γ on $\overline{\mathcal{B}} \otimes \mathcal{C}$ by

$$M_\gamma(\overline{b} \otimes c) := \langle j_{t_0,t_1} \star \ldots \star j_{t_{m-1},t_m}(b)\Omega, \vartheta_\gamma(c)\rangle$$

then, again by Proposition 4.3,

$$L_{\{s_{n-1},s_n\},\alpha^{(n)}}(\overline{c} \otimes d) \;=\; M_{\{t_0^{(n)},t_1^{(n)}\}} \star \ldots \star M_{\{t_{k_n}^{(n)},t_{k_n-1}^{(n)}\}}(\overline{\varkappa(c)} \otimes d).$$

For $\rho \in [0, \|\alpha\|]$ we define $L_\rho^{(n)} := L_{\{s_{n-1},s_{n-1}+\rho\},\alpha^{(n)}(\rho)}$, where

$$\alpha^{(n)}(\rho) \;:=\; \Big([s_{n-1}, s_{n-1}+\rho] \cap \alpha^{(n)}\Big) \cup \big\{s_{n-1}+\rho\big\}.$$

(Roughly speaking, if $\rho \leq s_n - s_{n-1}$, then $\alpha^{(n)}(\rho)$ concides with the part of $\alpha^{(n)}$ up to $s_{n-1}+\rho$, and otherwise it adds another interval to the partition.)

We define the linear functionals $M_r := M_{\{\tau,\tau+r\}}$ on $\overline{\mathcal{B}} \otimes \mathcal{C}$. Note that these do not depend on $\tau \geq 0$. We find

$$\begin{aligned} &M_r(\overline{b} \otimes c) \;=\; M_{\{\tau,\tau+r\}}(\overline{b} \otimes c)\\ &=\; \langle j_{\tau,\tau+r}(b)\Omega, j_{\tau,\tau+r} \circ \varkappa(c)\Omega\rangle \;=\; \varphi_r(b^*\varkappa(c)) \;=\; \big((\overline{\delta} \otimes \lambda) + rG + \mathfrak{R}_r\big)(\overline{b} \otimes c),\end{aligned}$$

where $G(\overline{b} \otimes c) := \psi(b^*\varkappa(c))$ and $\mathfrak{R}_r$ fulfills the condition of Lemma 4.2. For fixed $[s,t]$, it follows that for every $\overline{c} \otimes d \in \overline{\mathcal{C}} \otimes \mathcal{C}$ there exists a constant $C_{c,d}$ such that

$$\left|L_\rho^{(n)}(\overline{c} \otimes d) - e_\star^{\rho G}(\overline{\varkappa(c)} \otimes d)\right| \;\leq\; \|\alpha^{(n)}(\rho)\|\, \rho C_{c,d} \;\leq\; \rho^2 C_{c,d}$$

for all partitions $\alpha^{(n)}$ of $[s_{n-1}, s_n]$. (The constant $C_{c,d}$ might depend on $[s,t]$.) From this it is routine to conclude that the $L_\rho^{(n)}$ fulfill the condition of Lemma 4.2 at least for all $\overline{c}\otimes d \in \overline{\mathcal{C}}\otimes\mathcal{C}$ with the linear first order functional $\overline{c}\otimes d \mapsto \psi \circ \varkappa(c^*d)$. By taking (finite!) linear combinations, we obtain suitable constants D_γ for every $\gamma \in \overline{\mathcal{C}} \otimes \mathcal{C}$. From this the statement follows. □

Corollary 4.5. *The net $\big(\vartheta_\alpha(c)\big)_{\alpha\in\mathfrak{Z}_{st}}$ is a Cauchy net.*

Proof. We have to show that for $\varepsilon > 0$ there is a γ such that $\alpha, \beta \in \mathfrak{Z}_{st}$, $\alpha \succ \gamma$ and $\beta \succ \gamma$, implies $\|\vartheta_\alpha(c) - \vartheta_\beta(c)\| < \varepsilon$. By Proposition 4.4 there is a γ such that for $\eta \in \mathfrak{Z}_{st}$ with $\eta \succ \gamma$, we have

$$\left|\langle \vartheta_\gamma(c), \vartheta_\eta(c)\rangle - e_\star^{(t-s)\psi\circ\varkappa}(c^*c)\right| < \frac{\varepsilon^2}{16}. \tag{4.4}$$

So, for $\alpha \succ \gamma$ we have

$$\begin{aligned}\|\vartheta_\eta(c) - \vartheta_\alpha(c)\|^2 &= \langle \vartheta_\eta(c), \vartheta_\eta(c)\rangle + \langle \vartheta_\alpha(c), \vartheta_\alpha(c)\rangle\\ &\quad - \langle \vartheta_\eta(c), \vartheta_\alpha(c)\rangle - \langle \vartheta_\alpha(c), \vartheta_\eta(c)\rangle\\ &\leq \frac{\varepsilon^2}{4}.\end{aligned}$$

Thus, for $\alpha \succ \gamma$ and $\beta \succ \gamma$

$$\|\vartheta_\alpha(c) - \vartheta_\beta(c)\| \leq \|\vartheta_\alpha(c) - \vartheta_\eta(c)\| + \|\vartheta_\beta(c) - \vartheta_\eta(c)\| \leq \varepsilon$$

which finishes the proof. □

The limit of the Cauchy net $\big(\vartheta_\alpha(c)\big)_{\alpha\in\mathfrak{Z}_{st}}$ in $\overline{D_j}$ will be denoted by $\vartheta_{s,t}(c)$.

Remark 4.6. Taking the limit of (4.3) over $\beta \succ \alpha$ for fixed α, we find the same estimate for $\langle\vartheta_\alpha(c), \vartheta_{s,t}(c)\rangle$. The fact that (4.3) does not depend on the precise form of α but only on its width $\|\alpha\|$ and computations similar to the proof of the corollary, show that $\|\vartheta_\alpha(c) - \vartheta_{s,t}(c)\|$ is small, whenever $\|\alpha\|$ is sufficiently small. In particular, it follows that

$$\lim_{n\to\infty} \vartheta_{\alpha_n}(c) = \vartheta_{s,t}(c)$$

for each sequence α_n in $\mathfrak{Z}_{st}$ with $\lim_{n\to\infty} ||\alpha_n|| = 0$.

To conclude the proof of Theorem 3.4, we start by observing that

$$\vartheta_{s,t}(c) \;=\; \vartheta_{t_0,t_1} \star \ldots \star \vartheta_{t_{n-1},t_n}. \tag{4.5}$$

(To see this, simply take the limit of ϑ_β over the subnet of partions $\beta \succ \alpha$.) For $\alpha = (s = t_0 < t_1 < \ldots < t_{n-1} < t_n) \in \mathfrak{Z}_{st}$ $(0 \leq s < t)$ we define

$$D_{k_\alpha} \;:=\; \mathsf{span}\Big\{\vartheta_{t_0,t_1}(c_1) \otimes \ldots \otimes \vartheta_{t_{n-1},t_n}(c_n)\colon c_1, \ldots, c_n \in \mathcal{C}\Big\}.$$

By (4.5), $\vartheta_{s,t}(c) = \vartheta_{t_0,t_1} \star \ldots \star \vartheta_{t_{n-1},t_n}$ it follows $\beta \succ \alpha \implies D_{k_\beta} \supset D_{k_\alpha}$. We put $D_{k_{s,t}} := \bigcup_\alpha D_{k_\alpha}$. Of course, $[s', t'] \supset [s, t] \implies D_{k_{s',t'}} \supset D_{k_{s,t}}$. We put $D_{k_{t,\infty}} := \bigcup_{t\leq r<s} D_{k_{r,s}}$ and $D_k := D_{k_{0,\infty}} \ni \Omega$. On $D_{k_{s,t}}$ we define an operator by setting

$$\vartheta_{t_0,t_1}(c_1) \otimes \cdots \otimes \vartheta_{t_{n-1},t_n}(c_n) \;\longmapsto\; \vartheta_{t_0,t_1}(c_{(1)}c_1) \otimes \cdots \otimes \vartheta_{t_{n-1},t_n}(c_{(n)}c_n).$$

To see that this is well-defined, we simply observe that the operator has a formal adjoint on that domain, namely, simply the operator whith c replaced by c^*. (By taking joint refinements, if necessary, we may assume that the two vectors we choose to check the adjoint condition are in the same D_{k_α}.)

We extend this operator by amplification to an operator $k_{s,t}(c)$ on $D_k = D_{k_{0,s}} \otimes D_{k_{s,t}} \otimes D_{k_{t,\infty}}$. Clearly, $c \mapsto k_{s,t}(c)$ is multiplicative, so that the $k_{s,t}$ define a family of $*$–homomorphisms.

A simple application of coassociativity (and, once more, (4.5)) shows that $k_{r,s} \star k_{s,t} = k_{r,t}$ for $r < s < t$. Therefore, the family of mappings $k_{s,t}$ forms a Lévy process on $\mathcal{C}$ over (D_k, Ω) with generator $\psi \circ \varkappa$. That D_k is dense in H_k, will follow from the proof of Theorem 3.6.

4.3. Proof of Theorems 3.6 and 3.7. By Theorem 3.4 we know that for $B \in \mathcal{T}(\mathcal{B}_0)$

$$\zeta_\alpha(B) = (k_{t_0,t_1} \circ \widetilde{\varkappa}) \star_{\mathcal{T}(\Delta_0)} \cdots \star_{\mathcal{T}(\Delta_0)} (k_{t_{n-1},t_n} \circ \widetilde{\varkappa})(B)\Omega$$

converges in norm and defines a cyclic QLP $J_{s,t}$ on D_J. We have $\varkappa \circ \widetilde{\varkappa}\lceil\mathcal{B} = \mathsf{id}$ and $(\psi \circ \varkappa) \circ \widetilde{\varkappa} = \psi \circ M$ is the generator of $J_{s,t}$. by Proposition 3.3 the restriction of $J_{s,t}$ to $\mathcal{B}$ is a QLP $\widetilde{j}_{s,t}$ with generator ψ. For the proof of $\widetilde{j} = j$ it suffices to show that $\|\zeta_\alpha - j_{s,t}(b)\Omega\|^2 \to 0$. So it remains to be shown

Proposition 4.7. *For all $b, d \in \mathcal{B}$ we have*

$$\lim_{\alpha} \langle \zeta_\alpha(b), j_{s,t}(d)\Omega \rangle \;=\; e_\star^{(t-s)\psi}(b^* d).$$

Proof. Let $\alpha = \{s = t_0 < t_1 < \ldots < t_{n-1} < t_n = t\}$ and write $j_{s,t} = j_{t_0,t_1} \star \ldots \star j_{t_{n-1},t_n}$. Then, as in the proof of Proposition 4.4, from Proposition 4.3 we find

$$\langle \zeta_\alpha(b), j_{s,t}(d)\Omega \rangle \;=\; L_{t_1-t_0} \star \ldots \star L_{t_n-t_{n-1}}(\bar{b} \otimes d),$$

where we define the linear functionals $L_r(\bar{b} \otimes d) := \langle k_{0,r} \circ \widetilde{\varkappa}(b)\Omega, j_{0,r}(d)\Omega \rangle$ on $\overline{\mathcal{B}} \otimes \mathcal{B}$.

We are done, if we show that the L_r fulfill the conditions of Lemma 4.2 with the correct linear term. In fact, if in (4.3) we insert $\alpha = \{0, r\}$ (so that $\|\alpha\| = r$) and perform the limit over β, the estimate remains valid for $\langle \vartheta_{\{0,r\}} \circ \widetilde{\varkappa}(d), k_{0,r} \circ \widetilde{\varkappa}(b)\Omega \rangle = \overline{L_r(\bar{b} \otimes d)}$. □

This ends also the proof of Theorem 3.7. We have

Corollary 4.8. *The vectors $k_{s,t}\Omega$, $c \in \mathcal{C}$, generate $\overline{D_j}$ in the sense that*

$$\overline{D_j} = \overline{D_k} = \overline{\mathsf{span}}\Big\{ k_{t_0,t_1}(c_1) \cdots k_{t_{n-1},t_n}(c_n)\Omega \colon \\ n \in \mathbb{N},\, 0 \le s \le t < \infty,\, s = t_0 \le t_1 \le \cdots \le t_{n-1} \le t_n = t,\, c_1, \ldots, c_n \in \mathcal{C} \Big\}.$$

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