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QUANTIZATION OF THE MONOTONE POISSON CENTRAL LIMIT THEOREM

YUNGANG LU*

ABSTRACT. In the present paper, we quantize the monotone (as well as anti–monotone) Poisson central limit theorem. One constructs a sequence of monotone independent binomial random variables in terms of the creation–annihilation operators on a specific interacting Fock space. By using these random variables, one sets up a quantization of the monotone Poisson central limit theorem with respect to the convergence both in mixed–moments and in law, which includes the monotone Laplace–de Moivre CLT as a part. Moreover, one represents the above limit in terms of creation–annihilation operators on the continuous monotone Fock space over $\mathbf{L}^2([0,1])$.

1. Introduction

In the present paper, as a continuation of [8], [9] and [10], we give a **quantization** of the monotone Poisson central limit theorem (CLT in short).

The monotone Poisson CLT, in terms of algebraic random variables and the monotone independence, can be formulated as follows (see [13]): *Let* $(\mathcal{X},\psi)$ *be an algebraic probability space and* $\{\xi_{n,k} : n\in\mathbb{N}^* \text{ and } k\le n\}$ *be a family of algebraic random variables, let* $\{p_n\}_{n=1}^\infty \subset [0,1]$. *If*

- $\psi\left(\xi_{n,k}^m\right)=\psi\left(\xi_{n,k}\right)=p_n$ *for any* $m,n\in\mathbb{N}^*$ *and* $k\le n$ *(in this case, one says that the* $\psi-$*distribution of* $\xi_{n,k}$ *is* $b\left(1,p_n\right)$:=*the binomial distribution with the parameter* $\left(1,p_n\right)$ and writes $\xi_{n,k}\overset{\psi}{\sim} b\left(1,p_n\right)$ *in short)*;
- *with respect to* ψ, $\{\xi_{n,1},\dots,\xi_{n,n}\}$ *is a monotone independent family for any* $n\ge 2$,

the $\psi-$*distribution of* $\sum_{k=1}^n \xi_{n,k}$ *goes, as* $n\to\infty$, *to* $P_{mo}(\lambda)$:=*the monotone Poisson distribution with the parameter* λ *(will be assumed strictly positive all over) whenever* $np_n\to\lambda$. Moreover, for convenience, one denotes $P_{mo}(0):=\delta_0$ and hereinafter, δ_x is the Dirac measure centred on x for any $x\in\mathbb{R}$.

In terms of the monotone convolution "$\rhd$" (see [5], [6] and references within), one can reformulate the monotone Poisson CLT as follows: *If* $\{p_n\}_{n=1}^\infty\subset[0,1]$

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verifies $np_n \to \lambda$, *then in the weak convergence*

$$\lim_{n\to\infty}\left((1-p_n)\,\delta_0+p_n\delta_1\right)^{\rhd n}=P_{mo}(\lambda)$$

Where, one recalls that

1) the monotone convolution is defined by means of the *reciprocal Cauchy transform*: for any $\mu,\nu\in\mathcal{P}:=\{$probability measure on $(\mathbb{R},\mathcal{B})\}$ with the reciprocal Cauchy transforms H_μ and H_ν respectively, the monotone convolution $\mu\rhd\nu$ is the element of $\mathcal{P}$ with the reciprocal Cauchy transform $H_{\mu\rhd\nu}=H_\mu\circ H_\nu$,

2) for any $\{\mu_n\}_{n=1}^\infty\subset\mathcal{P}$ and $\mu\in\mathcal{P}$, $\mu_n\overset{w}{\longrightarrow}\mu$ if and only if $\operatorname{Im}H_{\mu_n}(x+iy)\longrightarrow\operatorname{Im}H_\mu(x+iy)$ for **some** $y>0$ and **any** $x\in\mathbb{R}$.

Remark Recall that, instead of using the reciprocal Cauchy transform to study the monotone convolution, one uses conventionally

- the Fourier transform to study the *classical* convolution;
- the Cauchy transform to study the *free* convolution (see, e.g. [3], [12], [14], [16]);
- the *self–energy* function to study the *Boolean* convolution (see [15]).

Notice that the monotone Poisson CLT is independent of the specific construction of the algebraic probability space and random variables, if we take a **particular** algebraic probability space $(\mathcal{A},\phi)$ and such a family of random variables $\{X_{n,k}:n\in\mathbb{N}^*$ and $k\le n\}$ that, with respect to the state ϕ,

- the distribution of $X_{n,k}$ is $b(1,p_n)$, i.e. $X_{n,k}\overset{\phi}{\sim}b(1,p_n)$, for any $n\in\mathbb{N}^*$ and $k\le n$,
- $\{X_{n,1},\dots,X_{n,n}\}$ is a monotone independent family for any $n\ge 2$,

then the monotone Poisson CLT tells us that the *$\phi-$distribution of* $\sum_{k=1}^n X_{n,k}$ *goes to* $P_{mo}(\lambda)$ if $np_n\to\lambda$.

In this paper, we take the following particular algebraic probability space $(\mathcal{A},\phi)$ and random variables $\{X_{n,k}:n\in\mathbb{N}^*$ and $k\le n\}$: Let

- $\mathcal{H}$ be a (pre–)Hilbert space with an onb $\{e_k\}_{k=1}^\infty$;
- $\Gamma_{mo}(\mathcal{H})$ be a particular interacting Fock space (IFS in short) over $\mathcal{H}$, namely the (discrete) monotone Fock space, and its construction will be given in Section 2;
- for any $k\in\mathbb{N}^*$, a_k^+ (resp. a_k) be the creation (resp. annihilation) operator on $\Gamma_{mo}(\mathcal{H})$ with the test function e_k;
- $\mathcal{A}$ be the algebra generated by $\{\mathcal{A}_k\}_{k\in\mathbb{N}^*}$ and

$$\mathcal{A}_k:=\left\{\text{polynomial in } a_k \text{ and } a_k^+ \text{ with degree}\ge 1\right\},\quad \forall k\in\mathbb{N}^* \tag{1.1}$$

- ϕ :=the vacuum state, i.e. $\phi(\cdot):=\langle\Psi,\cdot\Psi\rangle$ and Ψ :=the vacuum vector of $\Gamma_{mo}(\mathcal{H})$;
- for any $n\in\mathbb{N}^*$, $k\le n$ and for any given $\{p_n\}_{n=1}^\infty\subset[0,1]$

$$X_{n,k}:=\sqrt{p_n(1-p_n)}\left(a_k+a_k^+\right)+p_na_ka_k^++(1-p_n)\,a_k^+a_k \tag{1.2}$$

In Section 2, we give first of all the definition of the above mentioned IFS $\Gamma_{mo}(\mathcal{H})$ and creation–annihilation operators. Then we prove that, with respect to the vacuum state ϕ, the algebras $\mathcal{A}_k$'s defined in (1.1) are monotone independent and each $X_{n,k}$ introduced in (1.2) is a projector and $\phi(X_{n,k}) = p_n$, in particular, $X_{n,k} \overset{\phi}{\sim} b(1, p_n)$. In additional, by denoting

$$a_k^{(\varepsilon)} := \begin{cases} a_k, & \text{if } \varepsilon = -1 \\ a_k^+, & \text{if } \varepsilon = +1 \\ a_k a_k^+, & \text{if } \varepsilon = 0 \\ a_k^+ a_k, & \text{if } \varepsilon = 2 \end{cases}, \quad \forall k \in \mathbb{N}^* \tag{1.3}$$

and

$$B_n^{(\pm 1)} := \sqrt{p_n(1-p_n)} \sum_{k=1}^n a_k^{(\pm 1)}; \quad B_n^{(0)} := p_n \sum_{k=1}^n a_k^{(0)}; \quad B_n^{(2)} := (1-p_n) \sum_{k=1}^n a_k^{(2)}$$

$$B_n := B_n^{(-1)} + B_n^{(+1)} + B_n^{(0)} + B_n^{(2)} = \sum_{k=1}^n X_{n,k} \tag{1.4}$$

we show that $\left\{ B_n^{(\varepsilon)} : n \in \mathbb{N} \text{ and } \varepsilon \in \{-1, 0, 1, 2\} \right\}$ is uniformly bounded. Consequently, the convergence in law (i.e., the weak convergence of the vacuum distribution) of this family is equivalent to that in mixed–moments.

Remark The terms $\sqrt{p_n(1-p_n)} a_k^{(-1)}$ and $\sqrt{p_n(1-p_n)} a_k^{(+1)}$ (resp. $B_n^{(-1)}$ and $B_n^{(+1)}$) will be called **off–diagonal components** of $X_{n,k}$ (resp. of B_n) since they are conjugate each other; $p_n a_k^{(0)}$ and $(1-p_n) a_k^{(2)}$ (resp. $B_n^{(0)}$ and $B_n^{(2)}$) will be called **diagonal components** of $X_{n,k}$ (resp. of B_n) since they are self–adjoint.

As argued before, our particular choices $(\mathcal{A}, \phi)$ and $\{X_{n,k} : n \in \mathbb{N}^* \text{ and } k \leq n\}$ do **not** change the usual monotone Poisson CLT which says that the vacuum distribution of $\{B_n\}_{n=1}^\infty$ goes to $P_{mo}(\lambda)$ whenever $np_n \to \lambda$. Beyond this, one gets some new sight, e.g., one can study, for any $\{c_0, c_1, c_2\} \subset \mathbb{R}$,

$$\begin{aligned} &D_M(n, p_n; c_0, c_1, c_2) \\ :=&\text{the vacuum distribution of } c_1 \left(B_n^{(-1)} + B_n^{(+1)} \right) + c_0 B_n^{(0)} + c_2 B_n^{(2)} \end{aligned} \tag{1.5}$$

and the corresponding weak limit.

Versus the usual monotone Poisson CLT which is a consideration the weak limit of $D_M(n, p_n; 1, 1, 1)$ (i.e. the ϕ–distribution of $\{B_n\}_{n=1}^\infty$), the *quantized* monotone Poisson CLT is:

1) to examine, for any $m \in \mathbb{N}$ and $\varepsilon \in \{-1, 0, 1, 2\}^m$, the limit

$$\lim_{n\to\infty} \phi \left(B_n^{(\varepsilon(1))} \ldots B_n^{(\varepsilon(m))} \right) \tag{1.6}$$

i.e. to get the limits of the mixed–moments of $B_n^{(\varepsilon)}$'s, not only their sum;

2) to calculate, for any $\{c_0, c_1, c_2\} \subset \mathbb{R}$, the limit of

$$\lim_{N\to\infty} \phi \left(\exp \left(it \left(c_1 \left(B_n^{(-1)} + B_n^{(+1)} \right) + c_0 B_n^{(0)} + c_2 B_n^{(2)} \right) \right) \right) \tag{1.7}$$

i.e., to know the weak limit of $D_M(n, p_n; c_0, c_1, c_2)$, not only $D_M(n, p_n; 1, 1, 1)$, by means of the characteristic function;

3) to give a suitable representation to these limits.

As a consequence of the above 1) and 2), the quantized monotone Poisson CLT makes in evidence the **individual** contributions of $B_n^{(\varepsilon)}$'s in the CLT's procedure, not only their sum. In particular, under the condition $np_n \to \lambda$ (recall that this is a standard assumption for performing the Poisson type CLT), one has

$$B_n^{(\pm 1)} := \sqrt{p_n(1-p_n)} \sum_{k=1}^n a_k^{(\pm 1)} \approx \sqrt{\lambda} \frac{1}{\sqrt{n}} \sum_{k=1}^n a_k^{(\pm 1)}$$

and so by taking $c_0 = c_2 = 0$ and $c_1 = 1$ in above 2), one gets the monotone Laplace–de Moivre CLT. Therefore, the quantized monotone Poisson CLT tells us that *the monotone Laplace–de Moivre CLT is the off–diagonal part of the monotone Poisson CLT*. Moreover, by using the *representation* mentioned in above 3), one has also a view to understand the relationship between the monotone Poisson distribution and the monotone Gaussian distribution, namely, the arc–sine distribution.

Section 3 is devoted to set up a quantization of the monotone Poisson CLT. For formulating well our main result, the specific IFS $\Gamma\left(\mathbf{L}^2([0,1]), \{\chi_{\Delta_n}\}_{n=1}^\infty\right)$, namely the (continuous) monotone Fock space over $\mathbf{L}^2([0,1])$ introduced in [7], is required. Hereinafter, $\Delta_n := \left\{(t_n, \dots, t_1) : 0 \le t_1 < \dots < t_n \le 1\right\}$. Recall that this IFS is defined as $\mathbb{C} \oplus \bigoplus_{n=1}^\infty H_n$ and for any $n \in \mathbb{N}^*$, H_n is the pre–Hilbert space obtained by equipping the scalar product $\langle \cdot, \cdot \rangle_n$ on $\mathbf{L}^2([0,1]^n)$ as follows:

$$\langle F, G \rangle_n := \int_{[0,1]^n} \left(\overline{F} G \chi_{\Delta_n}\right)(t_n, \dots, t_1)\, dt_n \dots dt_1, \quad \forall F, G \in \mathbf{L}^2([0,1]^n) \tag{1.8}$$

Remark The two IFS $\Gamma_{mo}(\mathcal{H})$ (used to describe the objects *before* the CLT procedure) and $\Gamma\left(\mathbf{L}^2([0,1]), \{\chi_{\Delta_n}\}_{n=1}^\infty\right)$ (employed to represent the objects *after* the CLT procedure) are very different even if $\mathcal{H} = \mathbf{L}^2([0,1])$, the difference is easily to be seen just by definitions.

Our main task in Section 3 are

- to verify, for any $m \in \mathbb{N}$ and $\varepsilon \in \{-1, 0, 1, 2\}^m$, the existence of the limit (1.6) and prove that it has the form
$$\left\langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(n))} \Phi \right\rangle \lambda^{\sum_{k=1}^m (1 - |\varepsilon(k)|/2)} \tag{1.9}$$
where and throughout the paper, unless otherwise specified, $b^{(+1)}$ and $b^{(-1)}$ are the creation–annihilation operators on $\Gamma\left(\mathbf{L}^2([0,1]), \{\chi_{\Delta_n}\}_{n=1}^\infty\right)$ with the test function $\chi_{[0,1]}$; Φ is the vacuum vector; $b^{(0)} = b^{(-1)} b^{(+1)}$ and $b^{(2)} = \mathbf{1} - P_\Phi$; P_Φ :=the vacuum projector;
- to show that, for any $\{c_0, c_1, c_2\} \subset \mathbb{R}$, the limit (1.7) equals to
$$\left\langle \Phi, \exp\left(it\left(c_1\sqrt{\lambda}\left(b^{(-1)} + b^{(+1)}\right) + \lambda c_0 b^{(0)} + c_2 b^{(2)}\right)\right)\Phi \right\rangle \tag{1.10}$$
in other words, the vacuum distribution of the sequence $\Big\{c_1\left(B_n^{(-1)} + B_n^{(+1)}\right) + c_0 B_n^{(0)} + c_2 B_n^{(2)}\Big\}_{n=1}^\infty$ (i.e. $D_M(n, p_n; c_0, c_1, c_2)$) goes to the vacuum distribution of $c_1\sqrt{\lambda}\left(b^{(-1)} + b^{(+1)}\right) + \lambda c_0 b^{(0)} + c_2 b^{(2)}$ in the weak convergence.

Remark Recall from [7] that, on the IFS $\Gamma\left(\mathbf{L}^2([0,1]), \{\chi_{\Delta_n}\}_{n=1}^{\infty}\right)$, the vacuum distribution of $b^{(+1)}+b^{(-1)}$ is the standard (i.e. zero–mean and uni–variance) arc–sine distribution, i.e. the standard monotone–Gaussian distribution. In fact, it was proved in [7] that, on the IFS $\Gamma\left(\mathbf{L}^2([0,T]), \{\chi_{\Delta_n(T)}\}_{n=1}^{\infty}\right)$ (where, $0 \le T \le +\infty$ and $[0,+\infty]$ is understood as $[0,+\infty)$; $\Delta_n(T) := \{(t_n,\dots,t_1) : 0 \le t_1 < \dots < t_n < T\}$), the vacuum distribution of the field operator (i.e. sum of the creation and annihilation operators) with any test function $f \in \mathbf{L}^2([0,T])$ is the arc–sine distribution with mean zero and variance $\|f\|^2$.

Notice that, in the present paper, we use the creation–annihilation operators with the test function $\chi_{[0,1]}$ and the vacuum projector defined on the quite complicate IFS $\Gamma\left(\mathbf{L}^2([0,1]), \{\chi_{\Delta_n}\}_{n=1}^{\infty}\right)$ to represent our limits (1.6) and (1.7). This is considerable different from other (i.e., classical, Boolean, free) cases. In those cases (see [8], [9] and [10]), the analogies of the limits (1.6) and (1.7) can be represented on *one mode interacting Fock space* (1M–IFS in short) $\Gamma\left(\mathbb{C}, \{\omega_n\}_{n\ge1}\right)$ as shown in the following table:

	classical case	Boolean case	free case
1M-IFS $\Gamma\left(\mathbb{C}, \{\omega_n\}_{n\ge1}\right)$	$\omega_n = n\ \forall n \in \mathbb{N}$	$\omega_1 = 1,\ \omega_n = 0\ \forall n \ge 2$	$\omega_n = 1\ \forall n \ge 1$
$b^{(+1)}$	creator	creator	creator
$b^{(-1)}$	annihilator	annihilator	annihilator
$b^{(0)}$	$\mathbf{1}$	$b^{(-1)}b^{(+1)}$	$\mathbf{1}$
$b^{(2)}$	Λ	$b^{(+1)}b^{(-1)}$	$\mathbf{1} - P_\Phi$

where, Λ and P_Φ are, respectively, the number operator and the vacuum projector defined on the 1M-IFS $\Gamma(\mathbb{C}, \{\omega_n\}_{n\ge1})$. In other words, in the classical, Boolean and free cases, the canonical quantum decomposition (see [1] for the definition) of the corresponding Poisson distribution is $\left(\Gamma(\mathbb{C}, \{\omega_n\}_n),\ \{b^+, b, \alpha_\Lambda\}\right)$ with $b^+ := b^{(+1)}$, $b := b^{(-1)}$ and $\alpha_\Lambda := b^{(0)} + b^{(2)}$ given in the above table. But the monotone (as well as the anti–monotone) case is not like that—we give the representation mentioned in above 3) by employing $\Gamma\left(\mathbf{L}^2([0,1]), \{\chi_{\Delta_n}\}_{n=1}^{\infty}\right)$ and the creation–annihilation operators on it. In fact, $\Gamma\left(\mathbf{L}^2([0,1]), \{\chi_{\Delta_n}\}_{n=1}^{\infty}\right)$ is a *Type III* IFS (see [2] for the definition) and much more complicate than 1M–IFS. The reason of utilizing such an IFS, but not a simple 1M–IFS as in other cases, to give the representation mentioned in above 3) is the following **impossibility**: *let* $\left(\Gamma(\mathbb{C}, \{\omega_n\}_n),\ \{\tilde{b}^{(+1)}, \tilde{b}^{(-1)}, \tilde{\alpha}_\Lambda\}\right)$ *be a quantum decomposition of the monotone Poisson distribution with the parameter* λ*, where*

- $\Gamma(\mathbb{C}, \{\omega_n\}_n)$ *is a 1M-IFS,*
- $\tilde{b}^{(+1)}$, $\tilde{b}^{(-1)}$ *and* $\tilde{\alpha}_\Lambda$ *are the creation, annihilation and* **gradation preserving operators** *on* $\Gamma(\mathbb{C}, \{\omega_n\}_n)$,

then it is impossible to have neither

$$\begin{aligned}&\lim_{N\to\infty}\phi\left(\exp\left(it\left(c_1\left(B_N^{(-1)}+B_N^{(+1)}\right)+c\big(B_N^{(0)}+B_N^{(2)}\big)\right)\right)\right)\\&=\left\langle\Phi,\exp\left(it\left(c_1\left(\tilde{b}^{(-1)}+\tilde{b}^{(+1)}\right)+c\tilde{\alpha}_\Lambda\right)\right)\Phi\right\rangle,\quad\forall t,c,c_1\in\mathbb{R}\end{aligned}\tag{1.11}$$

nor its moment version:

$$\begin{aligned}&\lim_{N\to\infty}\phi\left(\left(c_1\left(B_N^{(-1)}+B_N^{(+1)}\right)+c\big(B_N^{(0)}+B_N^{(2)}\big)\right)^n\right)\\&=\left\langle\Phi,\left(c_1\left(\tilde{b}^{(-1)}+\tilde{b}^{(+1)}\right)+c\tilde{\alpha}_\Lambda\right)^n\Phi\right\rangle,\qquad\forall n\in\mathbb{N},\ c,c_1\in\mathbb{R}\end{aligned}\tag{1.12}$$

Clearly, (1.11) and (1.12) are equivalent since $\{B_n^\varepsilon : n\in\mathbb{N},\ \varepsilon\in\{-1,0,+1,2\}\}$ is a uniformly bounded family. We will show this impossibility in Section 4.

2. $\Gamma_{mo}(\mathcal{H})$: Definition and a Brief Discussion

The following notations will be used frequently

$$\begin{aligned}\overline{\mathbb{F}_n}&:=\{\text{function from }\ \{1,\dots,n\}\ \text{ to }\mathbb{N}\}\\ \mathbb{F}_n&:=\left\{\mathbf{k}\in\overline{\mathbb{F}_n}:\mathbf{k}(i)\neq\mathbf{k}(i+1)\ \text{ for any }1\le i<n\right\}\\ \mathbb{F}_n^\vee&:=\big\{\mathbf{k}\in\mathbb{F}_n:\exists j\in\{1,\dots,n\},\mathbf{k}(1)>\dots>\mathbf{k}(j-1)>\mathbf{k}(j)\\&\qquad\qquad<\mathbf{k}(j+1)<\dots<\mathbf{k}(n)\big\}\\ \mathbb{F}_n^\uparrow&:=\{\mathbf{k}\in\mathbb{F}_n:\ \mathbf{k}(1)<\dots<\mathbf{k}(n)\}\subset\mathbb{F}_n^\vee\end{aligned}\tag{2.1}$$

Let $\mathcal{H}$ be a (pre–)Hilbert space with the onb $\{e_k\}_{k=1}^\infty$, one defines $\Gamma_{mo}(\mathcal{H})$, the *(discrete) monotone Fock space* over $\mathcal{H}$, as $\bigoplus_{n=0}^\infty\mathcal{H}_n$, where $\mathcal{H}_0:=\mathbb{C}$, $\mathcal{H}_1:=\mathcal{H}$ and for any $n\ge 2$,

$$\mathcal{H}_n=lin-sp.\left\{e_{\mathbf{k}(n)}\otimes\dots\otimes e_{\mathbf{k}(1)}:\mathbf{k}\in\mathbb{F}_n^\uparrow\right\}\tag{2.2}$$

equipped the usual tensor scalar product;

One calls $\Psi:=1\oplus0\oplus0\oplus\ldots\in\Gamma_{mo}(\mathcal{H})$ the vacuum vectors; for any $k\in\mathbb{N}^*$, calls the operator a_k^+ defined by the linearity and

$$\begin{aligned}&a_k^+\Psi:=e_k,\\&a_k^+e_{\mathbf{k}(n)}\otimes\dots\otimes e_{\mathbf{k}(1)}=\chi_{(\mathbf{k}(n)}(k)\,e_k\otimes e_{\mathbf{k}(n)}\otimes\dots\otimes e_{\mathbf{k}(1)},\quad\forall n\in\mathbb{N}^*,\ \mathbf{k}\in\mathbb{F}_n^\uparrow(n)\end{aligned}\tag{2.3}$$

the creation operator (with the test function e_k), hereinafter, for any $h\in\mathbb{N}$,

$$\chi_{(h}(k):=\begin{cases}1, & \text{if } k>h\\ 0, & \text{if } k\le h\end{cases}$$

in other words, $\chi_{(h}$ is the indicator function of the open interval $(h,+\infty)$. Obviously, $\left\|a_k^+\right\|=1$ and so its conjugate $a_k:=\left(a_k^+\right)^*$ is bounded linear operator and will be called the annihilation operator (with the test function e_k). Moreover, it is easy to check that, for any $k \in \mathbb{N}^*$,

$$a_k \Psi = 0,$$
$$a_k e_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)} = \delta_{k,\mathbf{k}(n)} e_{\mathbf{k}(n-1)} \otimes \ldots \otimes e_{\mathbf{k}(1)}, \quad \forall n \in \mathbb{N}^* \text{ and } \mathbf{k} \in \mathbb{F}_n^{\uparrow} \tag{2.4}$$

The above $\Gamma_{mo}(\mathcal{H})$ is in fact a particular IFS over $\mathcal{H}$: for any $n \geq 2$, one defines the operator λ_n on $\mathcal{H}^{\otimes n}$ by the linearity and

$$\lambda_n \left(e_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)} \right) = e_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)} \prod_{h=1}^{n-1} \chi_{(\mathbf{k}(h)} \left(\mathbf{k}(h+1) \right), \quad \forall \mathbf{k} \in \overline{\mathbb{F}_n} \tag{2.5}$$

then introduces

$$\langle x, y \rangle_n := (x, \lambda_n y)_{\otimes n} \ , \quad \forall x, y \in \mathcal{H}^{\otimes n}$$

where, $(\cdot, \cdot)_{\otimes n}$ is the usual tensor scalar product on $\mathcal{H}^{\otimes n}$. In fact, $\mathcal{H}_n$ is nothing else than $(\mathcal{H}^{\otimes n} / \ker \langle \cdot, \cdot \rangle_n \,, \ \langle \cdot, \cdot \rangle_n)$.

Proposition 2.1. *On $\Gamma_{mo}(\mathcal{H})$, the following affirmations hold:*

1) $\|a_k^+\| = \|a_k\| = 1$ and $\left(a_k^+\right)^2 = 0$ for any k; $a_k a_h^+ = 0$ and $a_k^+ a_h^+ = \chi_{(h}(k)\, a_k^+ a_h^+$ for any $h \neq k$;

2) For any k, let

$$\mathcal{H}_{n,k} := lin-sp. \left\{ e_{\mathbf{k}(n)} \otimes \ldots \otimes e_{\mathbf{k}(1)} : \mathbf{k} \in \mathbb{F}_n^{\uparrow} \text{ and } \mathbf{k}(n) = k \right\}, \quad \forall n \in \mathbb{N}^* \tag{2.6}$$

then

$$a_k^+ a_k = P_{[k]} := \text{the projector to } \bigoplus_{n=1}^{\infty} \mathcal{H}_{n,k} \tag{2.7}$$

and

$$a_k a_k^+ = P_{k)} := \text{the projector to } \mathbb{C} \oplus \bigoplus_{h<k} \bigoplus_{n=1}^{\infty} \mathcal{H}_{n,h} \tag{2.8}$$

3) For any k,

$$\mathcal{A}_k = lin-sp. \left\{ a_k, a_k^+, a_k P_{[k]}, a_k^+ P_{k)}, P_{[k]}, P_{k)} \right\} \tag{2.9}$$

*in other words, any word of $\mathcal{A}_k$ belongs to $\left\{ a_k, a_k^+, a_k P_{[k]}, a_k^+ P_{k)}, P_{[k]}, P_{k)} \right\}$; moreover $\left(a_k^+ P_{k)}\right)^*$ and $a_k P_{[k]}$ are conjugate each other and so $\mathcal{A}_k$ is a *–algebra;*

4) The family $\left\{ n^{-\left(1-\frac{|\varepsilon|}{2}\right)} \sum_{k=1}^n a_k^{(\varepsilon)} : \varepsilon \in \{-1,0,1,2\}, n \in \mathbb{N}^* \right\}$ is uniformly bounded. Consequently, for any such $\{p_n\}_{n=1}^{\infty} \subset [0,1]$ that $|np_n| \leq C$ for some C, the family $\left\{ B_n^{(\varepsilon)} : \varepsilon \in \{-1,0,1,2\}, n \in \mathbb{N}^* \right\}$ is uniformly bounded.

Proof. The affirmation 1) is a direct consequence of the definition of creation operator and (2.4).

For any $k \in \mathbb{N}^*$, one has

$$\Gamma_{mo}(\mathcal{H}) = \bigoplus_{n=1}^{\infty} \mathcal{H}_{n,k} \oplus \left(\mathbb{C} \oplus \bigoplus_{h \neq k} \bigoplus_{n=1}^{\infty} \mathcal{H}_{n,h} \right) \tag{2.10}$$

and the definition says that the restriction of $a_k^+ a_k$ to $\mathbb{C} \oplus \bigoplus_{h\neq k} \bigoplus_{n=1}^{\infty} \mathcal{H}_{n,h}$ (respectively, $\bigoplus_{n=1}^{\infty} \mathcal{H}_{n,k}$) equals to zero (respectively, the identity of $\bigoplus_{n=1}^{\infty} \mathcal{H}_{n,k}$). In other words, $a_k^+ a_k = P_{[k]}$. Similarly, one has

$$\Gamma_{mo}(\mathcal{H}) = \Big(\mathbb{C} \oplus \bigoplus_{h<k} \bigoplus_{n=1}^{\infty} \mathcal{H}_{n,h}\Big) \oplus \Big(\bigoplus_{h\geq k} \bigoplus_{n=1}^{\infty} \mathcal{H}_{n,h}\Big) \tag{2.11}$$

the definition says that the restriction of $a_k a_k^+$ to $\mathbb{C} \oplus \bigoplus_{h<k} \bigoplus_{n=1}^{\infty} \mathcal{H}_{n,h}$ and $\bigoplus_{h\geq k} \bigoplus_{n=1}^{\infty} \mathcal{H}_{n,h}$ equals to the identity (of $\mathbb{C} \oplus \bigoplus_{h<k} \bigoplus_{n=1}^{\infty} \mathcal{H}_{n,h}$) and zero respectively. In other words $a_k^+ a_k = P_{k)}$. So the affirmation 2) is obtained.

As a corollary of the affirmation 2) and the fact $a_k^{+2} = a_k^2 = 0$, one knows that $\mathcal{A}_k = lin - sp.\{a_k, a_k^+, a_k P_{[k]}, a_k^+ P_{k)}, P_{[k]}, P_{k)}\}$. Moreover, $\mathcal{A}_k$ is a *–algebra because of

$$\big(a_k^+ P_{k)}\big)^* = P_{k)}^* a_k = P_{k)} a_k = a_k a_k^+ a_k = a_k P_{[k]} \tag{2.12}$$

Finally, the affirmation 2) tells us that $a_k a_k^+$ is a projector (in particular, $\|a_k a_k^+\| = 1$) for any k. So

$$\Big\|\frac{1}{n}\sum_{k=1}^{n} a_k^{(0)}\Big\| = \Big\|\frac{1}{n}\sum_{k=1}^{n} a_k a_k^+\Big\| \leq 1 \tag{2.13}$$

The affirmation 2) tells us also that $\sum_{k=1}^{n} a_k^+ a_k$ is the projector to the subspace $\bigoplus_{k=1}^{n} \bigoplus_{m=1}^{\infty} \mathcal{H}_{m,k}$, so

$$\Big\|\sum_{k=1}^{n} a_k^{(2)}\Big\| = \Big\|\sum_{k=1}^{n} a_k^+ a_k\Big\| = 1 \tag{2.14}$$

Consequently, thanks to the fact $a_k a_h^+ = 0$ for any $h \neq k$,

$$\begin{aligned} &\Big\|\frac{1}{\sqrt{n}}\sum_{k=1}^{n} a_k^{(-1)}\Big\|^2 = \Big\|\frac{1}{\sqrt{n}}\sum_{k=1}^{n} a_k^{(+1)}\Big\|^2 \\ =&\Big\|\Big(\frac{1}{\sqrt{n}}\sum_{k=1}^{n} a_k^{(+1)}\Big)^*\Big(\frac{1}{\sqrt{n}}\sum_{k=1}^{n} a_k^{(+1)}\Big)\Big\| = \Big\|\frac{1}{n}\sum_{k=1}^{n} a_k a_k^+\Big\| \leq 1 \end{aligned} \tag{2.15}$$

□

For the simplicity, we will use subsequently

$$A_k^{(\varepsilon)} := \begin{cases} a_k, & \text{if } \varepsilon = -1 \\ a_k^+, & \text{if } \varepsilon = 1 \\ a_k P_{[k]}, & \text{if } \varepsilon = -2 \\ a_k^+ P_{k)}, & \text{if } \varepsilon = 2 \\ P_{[k]}, & \text{if } \varepsilon = -3 \\ P_{k)}, & \text{if } \varepsilon = 3 \end{cases} \tag{2.16}$$

Remark It is worth noticing that

• for any $\varepsilon < 0$ and $k \in \mathbb{N}^*$, $A_k^{(\varepsilon)}$ has the form $x\, a_k$, where x is (recall that $P_{[k]} = a_k^+ a_k$) either the identity (if $\varepsilon = -1$), or $a_k a_k^+$ (if $\varepsilon = -2$), or a_k^+ (if $\varepsilon = -3$);

• for any $\varepsilon \in \{-1, -2\}$ and $k \in \mathbb{N}^*$, $A_k^{(\varepsilon)}$ has also the form $a_k\, x$, where x is either the identity (if $\varepsilon = -1$) or $a_k^+ a_k$ (if $\varepsilon = -2$);

• for any $\varepsilon > 0$ and $k \in \mathbb{N}^*$, $A_k^{(\varepsilon)}$ has the form $x\, a_k^+$, where x is (recall that $P_{k)} = a_k a_k^+$) either the identity (if $\varepsilon = 1$), or $a_k^+ a_k$ (if $\varepsilon = 2$), or a_k (if $\varepsilon = 3$);

• for any $\varepsilon \in \{1, 2\}$ and $k \in \mathbb{N}^*$, $A_k^{(\varepsilon)}$ has also the form $a_k^+\, x$, where x is either the identity (if $\varepsilon = 1$) or $a_k a_k^+$ (if $\varepsilon = 2$).

Moreover, as shown in [4], for any $N \in \mathbb{N}^*$, $\mathbf{k} \in \overline{\mathbb{F}_N}$ and $\varepsilon \in \{-1, 1\}^N$, if the product $A_{\mathbf{k}(1)}^{(\varepsilon(1))} \ldots A_{\mathbf{k}(N)}^{(\varepsilon(N))}$ is non–zero, then it must have either the following *normally ordered* forms:

• $\lambda-$**form**, i.e., a product in the form of $a_{i_1}^+ \ldots a_{i_m}^+ a_{j_1} \ldots a_{j_n}$, where $m, n \in \mathbb{N}$, $i_1 < \ldots < i_m$ and $j_1 > \ldots > j_n$;

• $\pi-$**form**, i.e., in the form of $a_{i_1}^+ \ldots a_{i_m}^+ a_k a_k^+ a_{j_1} \ldots a_{j_n}$, where $m, n \in \mathbb{N}$ and $i_1 < \ldots < i_m < k > j_1 > \ldots > j_n$.

Proposition 2.2. *One has the following affirmations:*

1) For any $k \in \mathbb{N}^*$,

$$A_k^{(\varepsilon)} a_k^+ = 0, \quad \forall \varepsilon \in \{1, 2, 3\} \tag{2.17}$$

and

$$a_k \Psi = P_{[k]} \Psi = a_k P_{[k]} \Psi = 0; \qquad a_k^+ \Psi = a_k^+ P_{k)} \Psi = e_k; \qquad P_{k)} \Psi = \Psi \tag{2.18}$$

In particular,

$$\langle \Psi, x\Psi \rangle = \begin{cases} 1, & \text{if } x = P_{k)} \\ 0, & \text{if } x \in \left\{a_k, a_k^+, P_{[k]}, a_k^+ P_{k)}, a_k P_{[k]}\right\} \end{cases} \tag{2.19}$$

i.e., all words of $\mathcal{A}_k$*, except* $P_{k)}$*, has vacuum expectation zero.*

2) For any $k \neq h$

$$A_k^{(\varepsilon)} a_h^+ = \begin{cases} 0, & \text{if } \varepsilon < 0 \\ \chi_{(h}(k)\, a_k^+ a_h^+, & \text{if } \varepsilon = 1, 2 \\ \chi_{(h}(k)\, a_h^+, & \text{if } \varepsilon = 3 \end{cases} \tag{2.20}$$

3) For any $k < h$,

$$A_h^{(\varepsilon)} a_k = A_h^{(\varepsilon)} P_{[k]} = A_h^{(\varepsilon)} P_{k)} = A_h^{(\varepsilon)} a_k^+ = 0, \qquad \forall \varepsilon < 0 \tag{2.21}$$

$$a_k A_h^{(\varepsilon)} = P_{[k]} A_h^{(\varepsilon)} = P_{k)} A_h^{(\varepsilon)} = a_k^+ A_h^{(\varepsilon)} = 0, \qquad \forall \varepsilon \in \{1, 2\} \tag{2.22}$$

and

$$P_{h)} A_k^{(\varepsilon)} = A_k^{(\varepsilon)}, \qquad \forall \varepsilon \in \{\pm 1, \pm 2, \pm 3\} \tag{2.23}$$

In particular, for any $k < h$ *and* $\varepsilon' \in \{\pm 1, \pm 2, \pm 3\}$,

$$A_h^{(\varepsilon)} A_k^{(\varepsilon')} = 0, \quad \forall \varepsilon < 0; \qquad A_k^{(\varepsilon')} A_h^{(\varepsilon)} = 0, \quad \forall \varepsilon \in \{1, 2\} \tag{2.24}$$

More particular, for any $k_1 < h > k_2$ *and* $\varepsilon_1, \varepsilon_2 \in \{\pm 1, \pm 2, \pm 3\}$,

$$A_{k_1}^{(\varepsilon_1)} A_h^{(\varepsilon)} A_{k_2}^{(\varepsilon_2)} = 0, \qquad \forall \varepsilon \neq 3 \tag{2.25}$$

Proof. (2.17) is a simple consequence of the third affirmation of above Remark: $A_k^{(\varepsilon)} a_k^+ = x a_k^{+2} = 0$.

(2.18) is trivially obtained just by the definitions of $P_{k)}$ and $P_{[k]}$. Consequently, one has (2.19) and the affirmation 1) is proved.

For any $k \neq h$, the definition says that

• $a_k a_h^+ = 0$ and so, as mentioned in above Remark, $A_k^{(\varepsilon)} a_h^+ = x a_k a_h^+ = 0$ for any $\varepsilon < 0$,

• the affirmation 1) of Proposition 2.1 gives

$$A_k^{(1)} a_h^+ = a_k^+ a_h^+ = \chi_{(h}(k)\, a_k^+ a_h^+ = \chi_{(h}(k)\, a_k^+ a_k a_k^+ a_h^+ = \chi_{(h}(k)\, a_k^+ P_{k)} a_h^+ = A_k^{(2)} a_h^+$$

• $A_k^{(3)} a_h^+ = P_{k)} a_h^+ = a_k a_k^+ a_h^+ = \chi_{(h}(k)\, a_h^+$.

Summing up, the affirmation 2) is proved. Now we turn to show the affirmation 3).

First of all, since $A_h^{(\varepsilon)}$ has the form $x\, a_h$ for any $\varepsilon < 0$ as mentioned in above Remark, one finds (2.21): for any $\varepsilon < 0$ and $k < h$ (so $a_h a_k = 0 = a_h a_k^+$),

$$\begin{aligned} A_h^{(\varepsilon)} a_k &= x\, a_h a_k = 0, & A_h^{(\varepsilon)} P_{[k]} &= x\, a_h a_k^+ a_k = 0 \\ A_h^{(\varepsilon)} P_{k)} &= x\, a_h a_k a_k^+ = 0, & A_h^{(\varepsilon)} a_k^+ &= x\, a_h a_k^+ = 0 \end{aligned}$$

Second of all, since $A_h^{(\varepsilon)}$ has the form $a_h\, x$ for any $\varepsilon \in \{1, 2\}$ as mentioned in above Remark, one gets (2.22): for any $\varepsilon \in \{1, 2\}$ and $k < h$ (so $a_k^+ a_h^+ = 0 = a_k a_h^+$),

$$\begin{aligned} a_k A_h^{(\varepsilon)} &= a_k a_h^+\, x = 0, & P_{[k]} A_h^{(\varepsilon)} &= a_k^+ a_k a_h^+\, x = 0 \\ P_{k)} A_h^{(\varepsilon)} &= a_k a_k^+ a_h^+\, x = 0, & a_k^+ A_h^{(\varepsilon)} &= a_k^+ a_h^+\, x = 0 \end{aligned}$$

Third of all, (2.23) is obtained just by the facts:

• $A_k^{(\varepsilon)} : \Gamma_{mo}(\mathcal{H}) \longmapsto \bigoplus_{0 \le j \le k} \mathcal{H}_j$;

• the restriction of $P_{h)}$ to $\bigoplus_{0 \le j \le k} \mathcal{H}_j$ is (because of $k < h$) identity.

Finally, (2.21) (resp. (2.22)) gives the first (resp. second) equality in (2.24). Moreover,

• in case $\varepsilon < 0$, the first equality in (2.24) and the fact $k_2 < h$ implies $A_h^{(\varepsilon)} A_{k_2}^{(\varepsilon_2)} = 0$;

• in case $\varepsilon \in \{1, 2\}$, the second equality in (2.24) and the fact $k_1 < h$ say $A_{k_1}^{(\varepsilon_1)} A_h^{(\varepsilon)} = 0$. □

Remark In terms of $A_k^{(\varepsilon)}$'s, the formulae (2.18) and (2.19) become to

$$A_k^{(\varepsilon)} \Psi = \begin{cases} 0, & \text{if } \varepsilon < 0 \\ e_k, & \text{if } \varepsilon \in \{1, 2\} \\ \Psi, & \text{if } \varepsilon = 3 \end{cases} ; \quad \left\langle \Psi, A_k^{(\varepsilon)} \Psi \right\rangle = \begin{cases} 1, & \text{if } \varepsilon = 3 \\ 0, & \text{otherwise} \end{cases} \tag{2.26}$$

Proposition 2.3. *For any* $n, h \in \mathbb{N}^*$ *and* $\varepsilon \in \{1,2,3\}^n$ *, for any* $\mathbf{k} \in \overline{\mathbb{F}_n}$*, there is* $C_h(\varepsilon, \mathbf{k}) \in \{0,1\}$ *such that*

$$A_{\mathbf{k}(1)}^{(\varepsilon(1))} \dots A_{\mathbf{k}(n)}^{(\varepsilon(n))} a_h^+ = C_h(\varepsilon, \mathbf{k}) \Big(\prod_{j \in \varepsilon^{-1}(\{1,2\})} a_{\mathbf{k}(j)}^+ \Big) a_h^+ \tag{2.27}$$

where, $\varepsilon^{-1}(\{1,2\}) := \{i : \varepsilon(i) \in \{1,2\}\}$ *. Moreover, one knows explicitly* $C_h(\varepsilon, \mathbf{k})$ *in two particular cases:*

- $C_h(\varepsilon, \mathbf{k}) = 0$ *if* $\mathbf{k}(n) \le h$;
- $C_h(\varepsilon, \mathbf{k}) = 1$ *if* $\mathbf{k}(1) > \dots > \mathbf{k}(n) > h$.

Proof. As a consequence of (2.20), one gets (2.27) for $n = 1$. In particular, the presence of $\chi_{(h}(\mathbf{k}(1))$ tells us that $C_h(\varepsilon, \mathbf{k}) = 0$ if $\mathbf{k}(1) \le h$, $C_h(\varepsilon, \mathbf{k}) = 1$ if $\mathbf{k}(1) > h$.

Suppose that the affirmation is already proved for $n \le m$, let's see it for $n = m+1$.

If $\varepsilon(m+1) = 3$, (2.20) gives $A_{\mathbf{k}(m+1)}^{(\varepsilon(m+1))} a_h^+ = \chi_{(h}(\mathbf{k}(m+1)) a_h^+$ and so

$$A_{\mathbf{k}(1)}^{(\varepsilon(1))} \dots A_{\mathbf{k}(m+1)}^{(\varepsilon(m+1))} a_h^+ = \chi_{(h}(\mathbf{k}(m+1)) A_{\mathbf{k}(1)}^{(\varepsilon(1))} \dots A_{\mathbf{k}(m)}^{(\varepsilon(m))} a_h^+$$

The assumption of induction says that there exists $C_h(\varepsilon_m, \mathbf{k}_m) \in \{0,1\}$, where $\varepsilon_m := \varepsilon\big|_{\{1,\dots,m\}}$ and $\mathbf{k}_m := \mathbf{k}\big|_{\{1,\dots,m\}}$, such that

$$A_{\mathbf{k}(1)}^{(\varepsilon(1))} \dots A_{\mathbf{k}(m)}^{(\varepsilon(m))} a_h^+ = C_h(\varepsilon_m, \mathbf{k}_m) \Big(\prod_{j \in \varepsilon_m^{-1}(\{1,2\})} a_{\mathbf{k}(j)}^+ \Big) a_h^+ \tag{2.28}$$

In our case (i.e. $\varepsilon(m+1) = 3$), one has $\varepsilon_m^{-1}(\{1,2\}) = \varepsilon^{-1}(\{1,2\})$ and so (2.27) holds by taking $C_h(\varepsilon, \mathbf{k}) := \chi_{(h}(\mathbf{k}(m+1)) C_h(\varepsilon_m, \mathbf{k}_m)$, which belongs to $\{0,1\}$ since the induction's assumption says $C_h(\varepsilon_m, \mathbf{k}_m) \in \{0,1\}$.

If $\varepsilon(m+1) \in \{1,2\}$, (2.20) gives $A_{\mathbf{k}(m+1)}^{(\varepsilon(m+1))} a_h^+ = \chi_{(h}(\mathbf{k}(m+1)) a_{\mathbf{k}(m+1)}^+ a_h^+$ and so

$$A_{\mathbf{k}(1)}^{(\varepsilon(1))} \dots A_{\mathbf{k}(m+1)}^{(\varepsilon(m+1))} a_h^+ = \chi_{(h}(\mathbf{k}(m+1)) A_{\mathbf{k}(1)}^{(\varepsilon(1))} \dots A_{\mathbf{k}(m)}^{(\varepsilon(m))} a_{\mathbf{k}(m+1)}^+ a_h^+$$

The assumption of induction says that there exists $C_{\mathbf{k}(m+1)}(\varepsilon_m, \mathbf{k}_m) \in \{0,1\}$ with $\varepsilon_m := \varepsilon\big|_{\{1,\dots,m\}}$ and $\mathbf{k}_m := \mathbf{k}\big|_{\{1,\dots,m\}}$, such that

$$A_{\mathbf{k}(1)}^{(\varepsilon(1))} \dots A_{\mathbf{k}(m)}^{(\varepsilon(m))} a_{\mathbf{k}(m+1)}^+ = C_{\mathbf{k}(m+1)}(\varepsilon_m, \mathbf{k}_m) \Big(\prod_{j \in \varepsilon_m^{-1}(\{1,2\})} a_{\mathbf{k}(j)}^+ \Big) a_{\mathbf{k}(m+1)}^+ \tag{2.29}$$

In our case (i.e., $\varepsilon(m+1) \in \{1,2\}$), one has $\varepsilon_m^{-1}(\{1,2\}) \cup \{m+1\} = \varepsilon^{-1}(\{1,2\})$ and so (2.27) holds by taking $C_h(\varepsilon, \mathbf{k}) := \chi_{(h}(\mathbf{k}(m+1)) C_{\mathbf{k}(m+1)}(\varepsilon_m, \mathbf{k}_m)$, which belongs to $\{0,1\}$ since the induction's assumption says $C_{\mathbf{k}(m+1)}(\varepsilon_m, \mathbf{k}_m) \in \{0,1\}$.

In addition,

- if $\mathbf{k}(m+1) \le h$, $C_h(\varepsilon, \mathbf{k})$ given in above equals to zero because of the appearance of the factor $\chi_{(h}(\mathbf{k}(m+1))$;
- if $\mathbf{k}(1) > \dots > \mathbf{k}(m) > \mathbf{k}(m+1) > h$, one obtains that $\chi_{(h}(\mathbf{k}(m+1)) = 1$ and the induction's assumption says $C_{\mathbf{k}(m+1)}(\varepsilon_m, \mathbf{k}_m) = 1 = C_h(\varepsilon_m, \mathbf{k}_m)$. So $C_h(\varepsilon, \mathbf{k}) = 1$. □

Remark Just for convenience, for any $\mathbf{k} : \{1, \dots, n\} \mapsto \mathbb{N}^*$, we have labelled and will label the tensor product of vectors $\{e_{\mathbf{k}(j)} : j \in \{1, \dots, n\}\}$ (respectively, the product of operators $\{x_{\mathbf{k}(j)} : j \in \{1, \dots, n\}\}$) by $e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)}$ (respectively, $x_{\mathbf{k}(1)} \dots x_{\mathbf{k}(n)}$).

Theorem 2.4. *$\{\mathcal{A}_k\}_{k=1}^{\infty}$ is monotone independent with respect to the vacuum state.*

Proof. By the definition, one needs to check the "$V-$*form factorization*" principle and the "*local maximum*" principle.

First of all, we show the $V-$form factorization principle: *for any $n \geq 2$ and $\mathbf{k} \in \mathbb{F}_n^{\vee}$, for any $x_j \in \mathcal{A}_{\mathbf{k}(j)}$ with $j \in \{1, \dots, n\}$, the following equality holds:*

$$\langle \Psi, x_1 \dots x_n \Psi \rangle = \prod_{j=1}^{n} \langle \Psi, x_j \Psi \rangle \tag{2.30}$$

By the multi–linearity, it is sufficient to prove (2.30) in case each x_j being a word of the algebra $\mathcal{A}_{\mathbf{k}(j)}$, i.e. each x_j belongs to $\{a_{\mathbf{k}(j)}, a^+_{\mathbf{k}(j)}, P_{\mathbf{k}(j))}, P_{[\mathbf{k}(j)]}, a_{\mathbf{k}(j)} P_{\mathbf{k}(j))}, a^+_{\mathbf{k}(j)} P_{[\mathbf{k}(j)]}\}$, in other words, to prove, with the notations introduced in (2.16),

$$\left\langle \Psi, A_{\mathbf{k}(1)}^{(\varepsilon(1))} \dots A_{\mathbf{k}(n)}^{(\varepsilon(n))} \Psi \right\rangle = \prod_{j=1}^{n} \left\langle \Psi, A_{\mathbf{k}(j)}^{(\varepsilon(j))} \Psi \right\rangle \tag{2.31}$$

It is clear (see (2.26)) that

$$\left\langle \Psi, A_k^{(\varepsilon)} \Psi \right\rangle = \begin{cases} 1, & \text{if } \varepsilon = 3 \\ 0, & \text{otherwise} \end{cases}$$

and so

$$\prod_{j=1}^{n} \left\langle \Psi, A_{\mathbf{k}(j)}^{(\varepsilon(j))} \Psi \right\rangle = \begin{cases} 1, & \text{if } \varepsilon(j) = 3 \text{ for all } j \in \{1, \dots, n\} \\ 0, & \text{otherwise} \end{cases}$$

(2.31) is trivial for $n = 1$. Suppose that (2.31) holds for $n = m$ and let's see it for $n = m + 1$.

In case $\varepsilon(m+1) < 0$, (2.26) gives $A_{\mathbf{k}(m+1)}^{(\varepsilon(m+1))} \Psi = 0$ and so the both sides of (2.31) are zero.

In case $\varepsilon(m+1) = 3$, (2.26) tells us $A_{\mathbf{k}(m+1)}^{(\varepsilon(m+1))} \Psi = \Psi$ and $\left\langle \Psi, A_{\mathbf{k}(m+1)}^{(\varepsilon(m+1))} \Psi \right\rangle = 1$. So the induction's assumption gives

$$\begin{aligned} \left\langle \Psi, A_{\mathbf{k}(1)}^{(\varepsilon(1))} \dots A_{\mathbf{k}(m+1)}^{(\varepsilon(m+1))} \Psi \right\rangle &= \left\langle \Psi, A_{\mathbf{k}(1)}^{(\varepsilon(1))} \dots A_{\mathbf{k}(m)}^{(\varepsilon(m))} \Psi \right\rangle \\ &= \prod_{j=1}^{m} \left\langle \Psi, A_{\mathbf{k}(j)}^{(\varepsilon(j))} \Psi \right\rangle = \prod_{j=1}^{m+1} \left\langle \Psi, A_{\mathbf{k}(j)}^{(\varepsilon(j))} \Psi \right\rangle \end{aligned}$$

Now we see the case $\varepsilon(m+1) \in \{1,2\}$. In this case, one has $A_{\mathbf{k}(m+1)}^{(\varepsilon(m+1))}\Psi = a_{\mathbf{k}(m+1)}^{+}\Psi$ and consequently,

$$A_{\mathbf{k}(1)}^{(\varepsilon(1))}\dots A_{\mathbf{k}(m+1)}^{(\varepsilon(m+1))}\Psi = A_{\mathbf{k}(1)}^{(\varepsilon(1))}\dots A_{\mathbf{k}(m)}^{(\varepsilon(m))}a_{\mathbf{k}(m+1)}^{+}\Psi$$
$$\left\langle \Psi, A_{\mathbf{k}(m+1)}^{(\varepsilon(m+1))}\Psi\right\rangle = \left\langle \Psi, a_{\mathbf{k}(j)}^{+}\Psi\right\rangle = 0$$

Therefore, what needed to show is

$$\left\langle \Psi, A_{\mathbf{k}(1)}^{(\varepsilon(1))}\dots A_{\mathbf{k}(m)}^{(\varepsilon(m))}a_{\mathbf{k}(m+1)}^{+}\Psi\right\rangle = 0 \tag{2.32}$$

If $\mathbf{k}(m) < \mathbf{k}(m+1)$, (2.20) says that $A_{\mathbf{k}(m)}^{(\varepsilon(m))}a_{\mathbf{k}(m+1)}^{+} = 0$ and therefore (2.32) holds. So we need only to consider the case $\mathbf{k}(m) > \mathbf{k}(m+1)$. Moreover, in this case, the fact $\mathbf{k} \in \mathbb{F}_{m+1}^{\vee}$ makes sure that $\mathbf{k}(1) > \ldots > \mathbf{k}(m) > \mathbf{k}(m+1)$. In order to get (2.32), one must examine ε.

Firstly we see the case of $\varepsilon(j) > 0$ for any j. In this case, the Proposition 2.3 says that $A_{\mathbf{k}(1)}^{(\varepsilon(1))}\dots A_{\mathbf{k}(m)}^{(\varepsilon(m))}a_{\mathbf{k}(m+1)}^{+}$ has the form $\left(\prod_{1\le j\le m:\varepsilon(j)\in\{1,2\}} a_{\mathbf{k}(j)}^{+}\right)a_{\mathbf{k}(m+1)}^{+}$ and so (2.32) is obtained.

Secondly we see the case of $\varepsilon(m) < 0$. In this case, the formula (2.21) and the fact $\mathbf{k}(m) > \mathbf{k}(m+1)$ say that $A_{\mathbf{k}(m)}^{(\varepsilon(m))}a_{\mathbf{k}(m+1)}^{+} = 0$ and so (2.32) holds.

Finally we see the case of $\varepsilon(m) > 0$ and $\varepsilon(j) < 0$ for some j. In this case, $m_0 := \max\{j : \varepsilon(j) < 0\}$ belongs surely to $\{1,\ldots,m-1\}$, $\varepsilon(m_0) < 0$ and $\varepsilon(m_0+1) > 0$. The Proposition 2.3 says that the product $A_{\mathbf{k}(m_0+1)}^{(\varepsilon(m_0+1))}\dots A_{\mathbf{k}(m)}^{(\varepsilon(m))}a_{\mathbf{k}(m+1)}^{+}$ must have the form $C\left(\prod_{m_0+1\le j\le m:\varepsilon(j)\in\{1,2\}} a_{\mathbf{k}(j)}^{+}\right)a_{\mathbf{k}(m+1)}^{+}$. So the formula (2.21) guarantees, in virtue of the facts $\varepsilon(m_0) < 0$ and $\mathbf{k}(m_0) \neq \mathbf{k}(j)$ for all $j > m_0$, that

$$\begin{aligned}&A_{\mathbf{k}(m_0)}^{(\varepsilon(m_0))}A_{\mathbf{k}(m_0+1)}^{(\varepsilon(m_0+1))}\dots A_{\mathbf{k}(m)}^{(\varepsilon(m))}a_{\mathbf{k}(m+1)}^{+}\\ =&CA_{\mathbf{k}(m_0)}^{(\varepsilon(m_0))}\Big(\prod_{m_0+1\le j\le m:\varepsilon(j)\in\{1,2\}} a_{\mathbf{k}(j)}^{+}\Big)a_{\mathbf{k}(m+1)}^{+} = 0\end{aligned}$$

Now we prove the “local maximum”principle:

$$x_{k_1}x_hx_{k_2} = \left\langle \Phi, x_h\Phi\right\rangle x_{k_1}x_{k_2} \tag{2.33}$$

for any $x_{k_1} \in \mathcal{A}_{k_1}, x_{k_2} \in \mathcal{A}_{k_2}$ *and* $x_h \in \mathcal{A}_h$, *whenever* $k_1 < h > k_2$.

The multi–linearity permits us to prove the thesis by assuming that $x_h = A_h^{(\varepsilon)}$ and $x_{k_j} = A_{k_j}^{(\varepsilon_j)}$ for any $j \in \{1,2\}$ and $\{\varepsilon,\varepsilon_1,\varepsilon_2\} \subset \{\pm1,\pm2,\pm3\}$. That is to prove

$$\begin{aligned}&A_{k_1}^{(\varepsilon_1)}A_h^{(\varepsilon)}A_{k_2}^{(\varepsilon_2)} = \left\langle \Phi, A_h^{(\varepsilon)}\Phi\right\rangle A_{k_1}^{(\varepsilon_1)}A_{k_2}^{(\varepsilon_2)},\\ &\qquad\forall k_1 < h > k_2 \text{ and } \{\varepsilon,\varepsilon_1,\varepsilon_2\} \subset \{\pm1,\pm2,\pm3\}\end{aligned} \tag{2.34}$$

The two equalities in (2.24) tell us that $A_{k_1}^{(\varepsilon_1)}A_h^{(\varepsilon)}A_{k_2}^{(\varepsilon_2)} = 0$ whenever $k_1 < h > k_2$ and $\varepsilon \neq 3$. On the other hand, $\left\langle \Phi, A_h^{(\varepsilon)}\Phi\right\rangle = 0$ if $\varepsilon \neq 3$. So both sides of the equality in (2.34) are zero if $\varepsilon \neq 3$.

In case $\varepsilon = 3$, one has $A_h^{(3)} = P_{h)}$, $\left\langle \Phi, A_h^{(3)}\Phi\right\rangle = \left\langle \Phi, P_{h)}\Phi\right\rangle = 1$ and moreover the formula (2.23) gives $A_h^{(3)}A_{k_2}^{(\varepsilon_2)} = P_{h)}A_{k_2}^{(\varepsilon_2)} = A_{k_2}^{(\varepsilon_2)}$ whenever $h > k_2$. Therefore, for any $k_1 < h > k_2$

$$A_{k_1}^{(\varepsilon_1)} A_h^{(3)} A_{k_2}^{(\varepsilon_2)} = A_{k_1}^{(\varepsilon_1)} A_{k_2}^{(\varepsilon_2)} = \langle \Phi, A_h^{(3)} \Phi \rangle A_{k_1}^{(\varepsilon_1)} A_{k_2}^{(\varepsilon_2)} \qquad \square$$

3. Quantization of Monotone Poisson CLT

Let's introduce, for any $n \in \mathbb{N}^*$, $p \in [0,1]$ and $\varepsilon \in \{-1,1\}$, for any $g : [0,1] \longmapsto \mathbb{C}$ Riemannian integrable function,

$$B_n^{(\varepsilon)}(g) := \sqrt{p(1-p)} \sum_{k=1}^{n} g^{(\varepsilon)} \Big(\frac{k}{n}\Big) a_k^{(\varepsilon)}; \quad g^{(\varepsilon)} := \begin{cases} g, & \text{if } \varepsilon = 1 \\ \overline{g}, & \text{if } \varepsilon = -1 \end{cases} \tag{3.1}$$

Obviously, by taking g as constant 1, $B_n^{(\varepsilon)}(1)$ is just that introduced in (1.4) with $p_n = p$. Moreover, one denotes, for any $m \in \mathbb{N}^*$,

- $\mathbf{L}_R([0,1]^m)$:=the totality of the Riemannian integrable functions defined on $[0,1]^m$;
- $\mathcal{L}([0,1]) := \mathbf{L}_R([0,1])$ and for any $m \geq 2$, $\mathcal{L}([0,1]^m)$:=the set of all such function $f \in \mathbf{L}_R([0,1]^m)$ that for any $1 \leq k < m$, for any $(t_1, \dots, t_{k-1}) \in [0,1]^k$, $f(t_1, \dots, t_{k-1}, \cdot) : [0,1]^{m-k} \longmapsto \mathbb{C}$ is Riemannian integrable.

Clearly, $\mathcal{L}([0,1]^m)$ is a $*$–algebra and furthermore, it is actually bigger than $\mathbf{C}([0,1]^m)$:=the set of all continuous functions on $[0,1]^m$; actually smaller than $\mathbf{L}_R\big([0,1]^m\big)$ when $m \geq 2$—the 2–dimensional Pringsheim's function belongs to $\mathbf{L}_R\big([0,1]^2\big) \setminus \mathcal{L}\big([0,1]^2\big)$. Moreover, for any $m \geq 2$,

- for any $1 \leq h \neq k \leq m$ and for any $a, b \in \mathbb{R}$, the function $(t_1, \dots, t_m) \longmapsto a\chi_{t_h)}(t_k) + b\chi_{t_h]}(t_k)$ belongs to $\mathcal{L}([0,1]^m)$;
- for any $\{g_1, \dots, g_m\} \subset \mathbf{L}_R([0,1])$, the function $(t_1, \dots, t_m) \longmapsto \prod_{k=1}^{m} g_k(t_k)$ belongs to $\mathcal{L}([0,1]^m)$.

Proposition 3.1. *$B_n^{(\pm)}(g)$'s introduced in* (3.1) *and $B_n^{(\varepsilon)}$'s introduced in (1.4) with $p_n = p$ possess the following properties:*

1) for any $n, N \in \mathbb{N}^*$ and $\{g, g_1, \dots, g_n\} \subset \mathbf{L}_R([0,1])$

$$B_N^{(-1)}(g)\Psi = B_N^{(2)}\Psi = 0; \quad B_N^{(0)}\Psi = pN\Psi \tag{3.2}$$

$$B_N^{(+1)}(g_n) \dots B_N^{(+1)}(g_1) = (p(1-p))^{n/2} \sum_{\mathbf{k} \in \mathbb{F}_n^{\uparrow}} \Big(\prod_{j=1}^{n} g_j \Big(\frac{\mathbf{k}(j)}{N}\Big)\Big) a_{\mathbf{k}(n)}^{+} \dots a_{\mathbf{k}(1)}^{+} \tag{3.3}$$

and in particular

$$\Big\langle \Psi, B_N^{(+1)}(g_n) \dots B_N^{(+1)}(g_1)\Psi \Big\rangle = 0 \tag{3.4}$$

2) for any $n \in \mathbb{N}^*$ and $\{g, g_1, \dots, g_n\} \subset \mathbf{L}_R([0,1])$

$$B_N^{(2)} B_N^{(+1)}(g_n) \dots B_N^{(+1)}(g_1)\Psi = (1-p) B_N^{(+1)}(g_n) \dots B_N^{(+1)}(g_1)\Psi \tag{3.5}$$

$$\begin{aligned}&B_N^{(0)}B_N^{(+1)}(g_n)\dots B_N^{(+1)}(g_1)\Psi\\=&p\sum_{h=1}^{N}B_N^{(+1)}\big(g_n\chi_{\frac{h}{N})}\big)B_N^{(+1)}(g_{n-1})\dots B_N^{(+1)}(g_1)\Psi\\=&pN\ B_N^{(+1)}(g_n(1-M))B_N^{(+1)}(g_{n-1})\dots B_N^{(+1)}(g_1)\Psi\end{aligned}\tag{3.6}$$

and

$$\begin{aligned}&B_N^{(-1)}(g)B_N^{(+1)}(g_n)\dots B_N^{(+1)}(g_1)\Psi\\=&p(1-p)\sum_{h=1}^{N}(\overline{g}g_n)\Big(\frac{h}{N}\Big)B_N^{(+1)}\big(g_{n-1}\chi_{\frac{h}{N})}\big)B_N^{(+1)}(g_{n-2})\dots B_N^{(+1)}(g_1)\Psi\end{aligned}\tag{3.7}$$

where, M is the following multiplication operator on $\mathbf{L}^2([0,1])$:

$$(Mg)(t):=tg(t)\,,\quad \forall g\in\mathbf{L}^2([0,1])\ \text{ and } t\in[0,1]\tag{3.8}$$

Proof. The affirmation 1) is just a direct consequence of the definition of $B_N^{(\varepsilon)}$'s and $B_N^{(\pm)}(g)$ given in (1.4) and (3.1) respectively.

Thanks to the definition of $B_N^{(2)}$, the formula (3.3) and the fact $a_h^+a_ha_k^+ = \delta_{h,k}a_k^+$, one gets first of all (3.5) as follows

$$\begin{aligned}&B_N^{(2)}B_N^{(+1)}(g_n)\dots B_N^{(+1)}(g_1)\Psi\\=&(1-p)(p(1-p))^{n/2}\sum_{\mathbf{k}\in\mathbb{F}_n^{\uparrow}}\sum_{h=1}^{N}\Big(\prod_{j=1}^{n}g_j\Big(\frac{\mathbf{k}(j)}{N}\Big)\Big)a_h^+a_ha_{\mathbf{k}(n)}^+\dots a_{\mathbf{k}(1)}^+\Psi\\=&(1-p)(p(1-p))^{n/2}\sum_{\mathbf{k}\in\mathbb{F}_n^{\uparrow}}\Big(\prod_{j=1}^{n}g_j\Big(\frac{\mathbf{k}(j)}{N}\Big)\Big)a_{\mathbf{k}(n)}^+\dots a_{\mathbf{k}(1)}^+\Psi\\=&(1-p)B_N^{(+1)}(g_n)B_N^{(+1)}(g_{n-1})\dots B_N^{(+1)}(g_1)\Psi\end{aligned}$$

It follows from the definition of $B_N^{(0)}$, the formula (3.3) and the fact $a_ha_h^+a_k^+ = \chi_{h)}(k)\,a_k^+$ that

$$\begin{aligned}&B_N^{(0)}B_N^{(+1)}(g_n)\dots B_N^{(+1)}(g_1)\Psi\\=&p(p(1-p))^{n/2}\sum_{\mathbf{k}\in\mathbb{F}_n^{\uparrow}}\sum_{h=1}^{N}\Big(\prod_{j=1}^{n}g_j\Big(\frac{\mathbf{k}(j)}{N}\Big)\Big)a_ha_h^+a_{\mathbf{k}(n)}^+\dots a_{\mathbf{k}(1)}^+\Psi\\=&p(p(1-p))^{n/2}\sum_{\mathbf{k}\in\mathbb{F}_n^{\uparrow}}\sum_{h=1}^{N}\chi_{h)}(\mathbf{k}(n))\Big(\prod_{j=1}^{n}g_j\Big(\frac{\mathbf{k}(j)}{N}\Big)\Big)a_{\mathbf{k}(n)}^+\dots a_{\mathbf{k}(1)}^+\Psi\end{aligned}\tag{3.9}$$

Since $\chi_{s)}(t)=\chi_{cs)}(ct)$ for any $c>0$ and $\{s,t\}\subset\mathbb{R}$, one gets

$$\chi_{h)}(\mathbf{k}(n))\,g_n\Big(\frac{\mathbf{k}(n)}{N}\Big)=\chi_{\frac{h}{N})}\Big(\frac{\mathbf{k}(n)}{N}\Big)g_n\Big(\frac{\mathbf{k}(n)}{N}\Big)=\big(g_n\chi_{\frac{h}{N})}\big)\Big(\frac{\mathbf{k}(n)}{N}\Big)$$

and therefore (3.9) becomes to

$$\begin{aligned} & B_N^{(0)} B_N^{(+1)}(g_n) \ldots B_N^{(+1)}(g_1) \Psi \\ = & p \sum_{h=1}^{N} B_N^{(+1)}\big(g_n \chi_{\frac{h}{N})}\big) B_N^{(+1)}(g_{n-1}) \ldots B_N^{(+1)}(g_1) \Psi \end{aligned} \tag{3.10}$$

On the other hand, since $\sum_{h=1}^{N} \chi_{h)}(\mathbf{k}(n)) = N - \mathbf{k}(n) = N\Big(1 - \frac{\mathbf{k}(n)}{N}\Big)$ for any $\mathbf{k}(n) \in \{1, \ldots, N\}$, one obtains

$$\sum_{h=1}^{N} \chi_{h)}(\mathbf{k}(n)) g_n\Big(\frac{\mathbf{k}(n)}{N}\Big) = N\Big(1 - \frac{\mathbf{k}(n)}{N}\Big) g_n\Big(\frac{\mathbf{k}(n)}{N}\Big) = N(g_n(1-M))\Big(\frac{\mathbf{k}(n)}{N}\Big)$$

and so

$$\begin{aligned} & B_N^{(0)} B_N^{(+1)}(g_n) \ldots B_N^{(+1)}(g_1) \Psi \\ = & pN\, B_N^{(+1)}(g_n(1-M)) B_N^{(+1)}(g_{n-1}) \ldots B_N^{(+1)}(g_1) \Psi \end{aligned} \tag{3.11}$$

Thus, one gets (3.6) by combining (3.10) together with (3.11).

Finally, by using the definition of $B_N^{(-1)}(g)$, the formulae (3.3) and the fact $\chi_{s)}(t) = \chi_{cs)}(ct)$ for any $c > 0$ and $\{s, t\} \subset \mathbb{R}$, one obtains (3.7) as follows:

$$\begin{aligned} & B_N^{(-1)}(g) B_N^{(+1)}(g_n) \ldots B_N^{(+1)}(g_1) \Psi \\ = & (p(1-p))^{\frac{n+1}{2}} \sum_{\mathbf{k} \in \mathbb{F}_n^{\uparrow}} \sum_{h=1}^{N} \overline{g}\Big(\frac{h}{N}\Big)\Big(\prod_{j=1}^{n} g_j\Big(\frac{\mathbf{k}(j)}{N}\Big)\Big) a_h a_{\mathbf{k}(n)}^+ \ldots a_{\mathbf{k}(1)}^+ \Psi \\ = & (p(1-p))^{\frac{n+1}{2}} \sum_{\mathbf{k} \in \mathbb{F}_n^{\uparrow}} \sum_{h=1}^{N} \delta_{h,\mathbf{k}(n)} \overline{g}\Big(\frac{h}{N}\Big)\Big(\prod_{j=1}^{n} g_j\Big(\frac{\mathbf{k}(j)}{N}\Big)\Big) a_{\mathbf{k}(n-1)}^+ \ldots a_{\mathbf{k}(1)}^+ \Psi \\ = & (p(1-p))^{\frac{n+1}{2}} \sum_{\mathbf{k} \in \mathbb{F}_{n-1}^{\uparrow}} \sum_{h=1}^{N} (\overline{g} g_n)\Big(\frac{h}{N}\Big) \\ & \qquad \chi_{h)}(\mathbf{k}(n-1))\Big(\prod_{j=1}^{n-1} g_j\Big(\frac{\mathbf{k}(j)}{N}\Big)\Big) a_{\mathbf{k}(n-1)}^+ \ldots a_{\mathbf{k}(1)}^+ \Psi \\ = & p(1-p) \sum_{h=1}^{N} (\overline{g} g_n)\Big(\frac{h}{N}\Big) B_N^{(+1)}\big(g_{n-1}\chi_{\frac{h}{N})}\big) B_N^{(+1)}(g_{n-2}) \ldots B_N^{(+1)}(g_1) \Psi \qquad \square \end{aligned}$$

Remark 1) This Proposition explains the motivation of introducing *test function* into $B_N^{(\varepsilon)}$ and defining $B_N^{(\varepsilon)}(g)$ for $\varepsilon \in \{-1, 1\}$. In fact $B_N^{(\varepsilon)}$ is nothing but $B_N^{(\varepsilon)}(g)$ with $g := \chi_{[0,1]}$. In the classical, Boolean and free cases, for any $n \in \mathbb{N}^*$, $\Big\{c\Big(B_N^{(+1)}\Big)^n \Psi : c \in \mathbb{C}\Big\}$ is $B_N^{(0)}-$invariant:

$$B_N^{(0)}\big(B_N^{(+1)}\big)^n \Psi = C_{N,n}^{0,1}\big(B_N^{(+1)}\big)^n \Psi$$

with

$$C_{N,n}^{0,1} := \begin{cases} 0, & \text{in classical and Boolean cases,} \\ p\left(N-n\right), & \text{in free case,} \end{cases}$$

But in monotone case, $\left\{c\big(B_N^{(+1)}\big)^n\Psi : c\in\mathbb{C}\right\}$ is not $B_N^{(0)}$−invariant, in fact

$$B_N^{(0)}\big(B_N^{(+1)}\big)^n\Psi = p\sum_{h=1}^{N} B_N^{(+1)}\big(\chi_{\frac{h}{N}})\big)\,\big(B_N^{(+1)}\big)^{n-1}\Psi$$

the test functions are no more always $\chi_{[0,1]}$. Similar situation happens as well as for $B_N^{(-1)}\big(B_N^{(+1)}\big)^n\Psi$.

2) As a multiplication operator, $\chi_{[0,1]}$ is the identity of $\mathbf{L}^2\left([0,1]\right)$. So we say sometimes the test function is constant 1 when it is $\chi_{[0,1]}$. In fact, it is more natural to interpret $B_N^{(\varepsilon)}$ as $B_N^{(\varepsilon)}\left(\mathbf{1}\right)$ for $\varepsilon\in\{0,2\}$ and where $\mathbf{1}$ is the identity operator of $\mathbf{L}^2\left([0,1]\right)$.

Recall that, on the IFS $\Gamma\left(\mathbf{L}^2\left([0,1]\right),\{\chi_{\Delta_n}\}_{n=1}^{\infty}\right)$, by denoting $b^{(+1)}(f)$ and $b^{(-1)}(f)$ the *creation–annihilation operators with the test function* $f\in\mathbf{L}^2\left([0,1]\right)$, one of main result in [7] is the determination of annihilation operator, which is in fact a continuous version of (3.7): for any $n\in\mathbb{N}^*$, $\{f,g,g_1,\ldots,g_n\}\subset\mathbf{L}_2\left([0,1]\right)$

$$\begin{aligned} & b^{(-1)}\left(f\right)b^{(+1)}\left(g\right)b^{(+1)}\left(g_n\right)\ldots b^{(+1)}\left(g_1\right)\Phi \\ &= \int_0^1 dt\ \left(\overline{f}g\right)(t)\,b^{(+1)}\left(g_n\chi_{t)}\right)b^{(+1)}\left(g_{n-1}\right)\ldots b^{(+1)}\left(g_1\right)\Phi \\ &= \int_0^1 dt\ \left(\overline{f}g\right)b^{(+1)}\left(g_n\chi_{t)}\right)b^{(+1)}\left(g_{n-1}\chi_{t)}\right)\ldots b^{(+1)}\left(g_1\chi_{t)}\right)\Phi \end{aligned} \tag{3.12}$$

where, the second equality (although it looks strange) is a consequence of the following fact: for any $c\in[0,1]$, $n\in\mathbb{N}^*$ and $\{g_1,\ldots,g_n\}\subset\mathbf{L}_2\left([0,1]\right)$, the two formally different vectors

$$b^{(+1)}\left(g_n\chi_{c)}\right)b^{(+1)}\left(g_{n-1}\right)\ldots b^{(+1)}\left(g_1\right)\Phi$$

and

$$b^{(+1)}\left(g_n\chi_{c)}\right)b^{(+1)}\left(g_{n-1}\chi_{c)}\right)\ldots b^{(+1)}\left(g_1\chi_{c)}\right)\Phi$$

are the same element of the (pre–)Hilbert space H_n equipped the scalar product (1.8).

Furthermore, (3.5) has naturally a trivial continuous version: for any $n\in\mathbb{N}^*$ and $\{g_1,\ldots,g_n\}\subset\mathbf{L}_2\left([0,1]\right)$,

$$\big(1-P_\Phi\big)b^{(+1)}\left(g_n\right)\ldots b^{(+1)}\left(g_1\right)\Phi = b^{(+1)}\left(g_n\right)\ldots b^{(+1)}\left(g_1\right)\Phi \tag{3.13}$$

where, recall from Section 1 that P_Φ is the vacuum projector defined on the IFS $\Gamma\left(\mathbf{L}^2\left([0,1]\right),\{\chi_{\Delta_n}\}_{n=1}^{\infty}\right)$.

The following result provide a continuous version of (3.6).

Proposition 3.2. *On the IFS* $\Gamma\left(\mathbf{L}^2([0,1]),\{\chi_{\Delta_n}\}_{n=1}^{\infty}\right)$, *one defines* $b^{(0)}$ *by the linearity and*

$$b^{(0)}\Phi := \Phi \tag{3.14}$$

$$b^{(0)}\left(b^{(+1)}(g_n)\,b^{(+1)}(g_{n-1})\ldots b^{(+1)}(g_1)\,\Phi\right)$$
$$:=b^{(+1)}((\mathbf{1}-M)\,g_n)\,b^{(+1)}(g_{n-1})\ldots b^{(+1)}(g_1)\,\Phi,\quad \forall n\in\mathbb{N}^*,\ \{g_k\}_{k=1}^{n}\subset\mathbf{L}^2([0,1])$$

then

$$b^{(0)} = b\left(\chi_{[0,1]}\right)b^{+}\left(\chi_{[0,1]}\right) \tag{3.15}$$

Proof. For any $n\in\mathbb{N}^*$, $F\in\mathbf{L}_2([0,1]^n)$ and $\{f,g,g_1,\ldots,g_n\}\subset\mathbf{L}_2([0,1])$, (3.12) gives

$$\left\langle F, b^{(-1)}(f)\,b^{(+1)}(g)\,b^{(+1)}(g_n)\ldots b^{(+1)}(g_1)\,\Phi\right\rangle$$
$$=\int_0^1 dt\ \left(\overline{f}g\right)(t)\left\langle F, b^{(+1)}\left(g_n\chi_{t)}\right)\ldots b^{(+1)}\left(g_1\chi_{t)}\right)\Phi\right\rangle$$
$$=\int_0^1 dt\ \left(\overline{f}g\right)(t)\int_0^t dt_n\int_0^{t_n}dt_{n-1}\ldots\int_0^{t_2}dt_1\overline{F}(t_n,\ldots,t_1)\prod_{k=1}^{n}g_k(t_k) \tag{3.16}$$

In particular, by applying the formula

$$\int_0^1 dt\int_0^t ds\,G(s)=\int_0^1 ds\,G(s)\int_s^1 dt=\int_0^1(1-s)G(s)ds,\quad \forall G\in\mathbf{L}^1([0,1])$$

by taking $f=g=\chi_{[0,1]}$ and

$$G(t_n):=\int_0^{t_n}dt_{n-1}\ldots\int_0^{t_2}dt_1\overline{F}(t_n,\ldots,t_1)\prod_{k=1}^{n}g_k(t_k)$$

the expression in (3.16) equals to

$$\int_0^1 dt_n\,(1-t_n)\int_0^{t_n}dt_{n-1}\ldots\int_0^{t_2}dt_1\overline{F}(t_n,\ldots,t_1)\prod_{k=1}^{n}g_k(t_k)$$

i.e

$$\int_0^1 dt_n\int_0^{t_n}dt_{n-1}\ldots\int_0^{t_2}dt_1\overline{F}(t_n,\ldots,t_1)\left((1-M)g_n\right)(t_n)\prod_{k=1}^{n-1}g_k(t_k)$$

Therefore, (3.16) becomes, for $f=g=\chi_{[0,1]}$, to

$$\left\langle F, b^{(-1)}\left(\chi_{[0,1]}\right)b^{(+1)}\left(\chi_{[0,1]}\right)b^{(+1)}(g_n)\ldots b^{(+1)}(g_1)\,\Phi\right\rangle$$
$$=\left\langle F, b^{(+1)}(g_n(1-M))\,b^{(+1)}(g_{n-1})\ldots b^{(+1)}(g_1)\,\Phi\right\rangle$$

and the arbitrariness of F gives the equality

$$b^{(-1)}\left(\chi_{[0,1]}\right)b^{(+1)}\left(\chi_{[0,1]}\right)b^{(+1)}(g_n)\ldots b^{(+1)}(g_1)\,\Phi$$
$$=b^{(+1)}(g_n(1-M))\,b^{(+1)}(g_{n-1})\ldots b^{(+1)}(g_1)\,\Phi \tag{3.17}$$

By combining (3.14) and (3.17), the arbitrarily of n and $\{g_k\}_{k=1}^n \subset \mathbf{L}^2([0,1])$, one obtains the thesis. $\square$

Remark By recalling from Section 1 that $b^{(2)} := 1 - P_\Phi$, one knows that (3.12), (3.13) and (3.14) are actually continuous analogies of (3.7), (3.5) and (3.6) respectively.

Theorem 3.3. *For any $m \in \mathbb{N}$, $n \in \mathbb{N}^*$ and $\varepsilon \in \{-1,0,1,2\}^n$, for any $f \in \mathbf{L}_R([0,1]^m)$ and such $\{g_1,\ldots,g_n\} \subset \mathcal{L}\left([0,1]^{m+1}\right)$ that g_k is constant 1 (more precisely, $\chi_{[0,1]}$), whenever $\varepsilon(k) \in \{0,2\}$, if $p = p_N$ verifies $Np_N \to \lambda$, one has*

$$
\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N^m} \sum_{1\le h_1,\ldots,h_m\le N} f\Big(\frac{h_1}{N},\ldots,\frac{h_m}{N}\Big) \\
&\Big\langle \Psi, B_N^{(\varepsilon(n))}\Big(g_n\Big(\frac{h_1}{N},\ldots,\frac{h_m}{N},\cdot\Big)\Big)\ldots B_N^{(\varepsilon(1))}\Big(g_1\Big(\frac{h_1}{N},\ldots,\frac{h_m}{N},\cdot\Big)\Big)\Psi\Big\rangle \\
=&\lambda^{\sum_{j=1}^n (1-|\varepsilon(j)|/2)} \int_{[0,1]^m} f(t_1,\ldots,t_m) \\
&\big\langle \Phi, b^{(\varepsilon(n))}\big(g_n(t_1,\ldots,t_m,\cdot)\big)\ldots b^{(\varepsilon(1))}\big(g_1(t_1,\ldots,t_m,\cdot)\big)\Phi\big\rangle dt_1\ldots dt_m \qquad (3.18)
\end{aligned}
$$

Remark 1) Notice that, if one takes $g_j \in \mathbf{L}_R\left([0,1]^{m+1}\right)$, then, as mentioned in the begin of this section, we can not be sure that $g_j\left(\frac{h_1}{N},\ldots,\frac{h_m}{N},\cdot\right)$ is Riemannian integrable for all N and $1 \le h_1,\ldots,h_m \le N$.

On the other hand, if one takes $g_j \in \mathbf{C}\left([0,1]^{m+1}\right)$ for all $j \in \{1,\ldots,n\}$, then (3.7) says that the function $B_N^{(-1)}(g)\, B_N^{(+1)}(g_n)\ldots B_N^{(+1)}(g_1)\,\Psi$ is no more continuous even if $B_N^{(+1)}(g_n)\ldots B_N^{(+1)}(g_1)\,\Psi$ is continuous.

Just by these considerations, we use *strange* $\mathcal{L}([0,1]^m)$ but neither $\mathbf{L}_R([0,1]^m)$ nor $\mathbf{C}([0,1]^m)$.

2) We have written formally *test function* for $B_N^{(\varepsilon)}$ and $b^{(\varepsilon)}$ when $\varepsilon \in \{0,2\}$. This gives a convenience to express all $B_N^{(\varepsilon)}$'s (respectively, $b^{(\varepsilon)}$'s) in the same way and what we should remember is just the test function is the constant 1 if $\varepsilon \in \{0,2\}$.

Proof of Theorem 3.3: Notice that the both sides of (3.18) are zero if either $\varepsilon(1) \in \{-1,2\}$ or $\varepsilon(j) = 1$ for all $j \in \{1,\ldots,n\}$. Therefore we need to prove (3.18) only for such ε that

$$\varepsilon(1) \in \{0,1\} \text{ and } \varepsilon^{-1}(\{-1,0,2\}) \neq \emptyset \qquad (3.19)$$

The proof will be performed by applying the induction principle.

If $n = 1$, the facts of $\varepsilon(1) \in \{0,1\}$ and $\langle \Psi, B_N^{(1)}\Psi\rangle = 0 = \langle \Phi, b^{(1)}\Phi\rangle$ permit us to examine only the case $\varepsilon(1) = 0$. In this case, the definition gives

$$\begin{aligned}
&\frac{1}{N^m}\sum_{1\le h_1,\dots,h_m\le N} f\Big(\frac{h_1}{N},\dots,\frac{h_m}{N}\Big)\big\langle\Psi, B_N^{(0)}\Psi\big\rangle\\
=&\frac{p_N}{N^m}\sum_{1\le h_1,\dots,h_m\le N} f\Big(\frac{h_1}{N},\dots,\frac{h_m}{N}\Big)\sum_{k=1}^N\big\langle\Psi, a_k a_k^+\Psi\big\rangle\\
=&\frac{Np_N}{N^m}\sum_{1\le h_1,\dots,h_m\le N} f\Big(\frac{h_1}{N},\dots,\frac{h_m}{N}\Big)\longrightarrow\lambda\int_{[0,1]^m} f(t_1,\dots,t_m)\,dt_1\dots dt_m\\
=&\lambda^{1-|0|/2}\int_{[0,1]^m} f\big(t_1,\dots,t_m\big)\big\langle\Phi, b^{(0)}\Phi\big\rangle dt_1\dots dt_m
\end{aligned}$$

For $n=2$, Proposition 3.1 and the definition of $b^{(\varepsilon)}$'s say that the both sides of (3.18) are zero whenever $\varepsilon=(\varepsilon(2),\varepsilon(1))\notin\{(0,0),(-1,1)\}$.

If $\varepsilon(1)=\varepsilon(2)=0$, one has

$$\begin{aligned}
&\frac{1}{N^m}\sum_{1\le h_1,\dots,h_m\le N} f\Big(\frac{h_1}{N},\dots,\frac{h_m}{N}\Big)\big\langle\Psi, B_N^{(0)}B_N^{(0)}\Psi\big\rangle\\
=&\frac{p_N^2}{N^m}\sum_{1\le h_1,\dots,h_m\le N} f\Big(\frac{h_1}{N},\dots,\frac{h_m}{N}\Big)\sum_{k_1,k_2=1}^N\big\langle\Psi, a_{k_2}a_{k_2}^+a_{k_1}a_{k_1}^+\Psi\big\rangle\\
=&\frac{N^2p_N^2}{N^m}\sum_{1\le h_1,\dots,h_m\le N} f\Big(\frac{h_1}{N},\dots,\frac{h_m}{N}\Big)
\end{aligned}$$

which tends, as $N\to\infty$, to $\lambda^2\int_{[0,1]^m} f(t_1,\dots,t_m)\,dt_1\dots dt_m$, i.e.,

$$\lambda^{1-|\varepsilon(1)|/2+1-|\varepsilon(2)|/2}\int_{[0,1]^m} f(t_1,\dots,t_m)\Big\langle\Phi, b^{(\varepsilon(1))}b^{(\varepsilon(2))}\Phi\Big\rangle dt_1\dots dt_m\Big|_{\varepsilon(1)=\varepsilon(0)=0}$$

since (3.14) gives $\big\langle\Phi, b^{(\varepsilon(1))}b^{(\varepsilon(2))}\Phi\big\rangle\Big|_{\varepsilon(1)=\varepsilon(0)=0}=\big\langle\Phi, b^{(0)}b^{(0)}\Phi\big\rangle=1$.

If $\varepsilon(1)=1$ and $\varepsilon(2)=-1$, Proposition 3.1 tells us that

$$\begin{aligned}
&\frac{1}{N^m}\sum_{1\le h_1,\dots,h_m\le N} f\Big(\frac{h_1}{N},\dots,\frac{h_m}{N}\Big)\\
&\qquad\Big\langle\Psi, B_N^{(-1)}\Big(g_2\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)B_N^{(1)}\Big(g_1\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\Psi\Big\rangle\\
=&\frac{1}{N^m}\sum_{1\le h_1,\dots,h_m\le N} f\Big(\frac{h_1}{N},\dots,\frac{h_m}{N}\Big)p_N(1-p_N)\sum_{k=1}^N\big(\overline{g}_2g_1\big)\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\frac{k}{N}\Big)\\
=&\frac{Np_N(1-p_N)}{N^{m+1}}\sum_{1\le h_1,\dots,h_m,k\le N} f\Big(\frac{h_1}{N},\dots,\frac{h_m}{N}\Big)\big(\overline{g}_2g_1\big)\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\frac{k}{N}\Big)
\end{aligned}$$

which goes, as $N\to\infty$, to

$$\lambda\int_{[0,1]^{m+1}} f(t_1,\dots,t_m)\big(\overline{g}_2g_1\big)(t_1,\dots,t_m,t)\,dt_1\dots dt_m dt$$

i.e.

$$\lambda \int_{[0,1]^m} f(t_1, \ldots, t_m) \Big\langle \Phi, b^{(-1)}(g_2(t_1, \ldots, t_m, \cdot)) \\ b^{(1)}(g_1(t_1, \ldots, t_m, \cdot))\,\Phi \Big\rangle dt_1 \ldots dt_m$$

For any $n, m \in \mathbb{N}^*$ and for any $\varepsilon \in \{-1, 0, 1, 2\}^{n+1}$, we can assume, as mentioned before, $\varepsilon(1) \in \{0, 1\}$.

The 1^{st} case: $\varepsilon(1) = 0$. In this case, the formula (3.3) gives

$$\begin{aligned}&\frac{1}{N^m} \sum_{1 \le h_1, \ldots, h_m \le N} f\Big(\frac{h_1}{N}, \ldots, \frac{h_m}{N}\Big) \\ &\Big\langle \Psi, B_N^{(\varepsilon(n+1))}\Big(g_{n+1}\Big(\frac{h_1}{N}, \ldots, \frac{h_m}{N}, \cdot\Big)\Big) \ldots B_N^{(\varepsilon(2))}\Big(g_2\Big(\frac{h_1}{N}, \ldots, \frac{h_m}{N}, \cdot\Big)\Big) B_N^{(0)} \Psi \Big\rangle \\ =&\frac{N p_N}{N^m} \sum_{1 \le h_1, \ldots, h_m \le N} f\Big(\frac{h_1}{N}, \ldots, \frac{h_m}{N}\Big) \\ &\Big\langle \Phi, B_N^{(\varepsilon(n+1))}\Big(g_{n+1}\Big(\frac{h_1}{N}, \ldots, \frac{h_m}{N}, \cdot\Big)\Big) \ldots B_N^{(\varepsilon(2))}\Big(g_2\Big(\frac{h_1}{N}, \ldots, \frac{h_m}{N}, \cdot\Big)\Big) \Phi \Big\rangle\end{aligned}$$

which goes, in virtue of the induction's assumption, to

$$\begin{aligned}&\lambda \lambda^{\sum_{j=2}^{n+1}(1-|\varepsilon(j)|/2)} \int_{[0,1]^m} f(t_1, \ldots, t_m) \\ &\Big\langle \Phi, b^{(\varepsilon(n+1))}(g_{n+1}(t_1, \ldots, t_m, \cdot)) \ldots b^{(\varepsilon(2))}(g_2(t_1, \ldots, t_m, \cdot))\,\Phi \Big\rangle dt_1 \ldots dt_m\end{aligned}$$

i.e., thanks to (3.14) and the fact $\lambda^{1-|\varepsilon(1)|/2}\big|_{\varepsilon(1)=0} = \lambda$,

$$\begin{aligned}&\lambda^{\sum_{j=1}^{n+1}(1-|\varepsilon(j)|/2)} \int_{[0,1]^m} f(t_1, \ldots, t_m) \\ &\Big\langle \Phi, b^{(\varepsilon(n+1))}(g_{n+1}(t_1, \ldots, t_m, \cdot)) \ldots b^{(\varepsilon(2))}(g_2(t_1, \ldots, t_m, \cdot))\, b^{(0)}\Phi \Big\rangle dt_1 \ldots dt_m\end{aligned}$$

The 2^{nd} case: $\varepsilon(1) = 1$. (3.19) and the fact $\varepsilon(1) = 1$ tell us that there must be a unique $r \in \{1, \ldots, n\}$ such that $\varepsilon(j) = 1$ for ant $j \le r$ and $\varepsilon(r+1) \in \{-1, 0, 2\}$.

If $\varepsilon(r+1) = 2$, (3.5) gives

$$\begin{aligned}&B_N^{(2)} B_N^{(+1)}\Big(g_r\Big(\frac{h_1}{N}, \ldots, \frac{h_m}{N}, \cdot\Big)\Big) \ldots B_N^{(+1)}\Big(g_1\Big(\frac{h_1}{N}, \ldots, \frac{h_m}{N}, \cdot\Big)\Big)\Psi \\ =&(1 - p_N)\, B_N^{(+1)}\Big(g_r\Big(\frac{h_1}{N}, \ldots, \frac{h_m}{N}, \cdot\Big)\Big) \ldots B_N^{(+1)}\Big(g_1\Big(\frac{h_1}{N}, \ldots, \frac{h_m}{N}, \cdot\Big)\Big)\Psi\end{aligned}$$

So, the induction's assumption gives

$$\begin{aligned}
&\frac{1}{N^m}\sum_{1\le h_1,\dots,h_m\le N} f\Big(\frac{h_1}{N},\dots,\frac{h_m}{N}\Big)\\
&\Big\langle \Psi, B_N^{(\varepsilon(n+1))}\Big(g_{n+1}\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\dots B_N^{(\varepsilon(1))}\Big(g_1\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\Psi\Big\rangle\\
=&\frac{(1-p_N)}{N^m}\sum_{1\le h_1,\dots,h_m\le N} f\Big(\frac{h_1}{N},\dots,\frac{h_m}{N}\Big)\\
&\Big\langle \Psi, B_N^{(\varepsilon(n+1))}\Big(g_{n+1}\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\dots B_N^{(\varepsilon(r+2))}\Big(g_{r+2}\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\\
&\qquad B_N^{(\varepsilon(r))}\Big(g_r\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\dots B_N^{(\varepsilon(1))}\Big(g_1\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\Psi\Big\rangle\\
\longrightarrow&\lambda^{\sum_{j\in\{1,\dots,n+1\}\setminus\{r+1\}}(1-|\varepsilon(j)|/2)}\int_{[0,1]^m} f(t_1,\dots,t_m)\\
&\Big\langle \Phi, b^{(\varepsilon(n+1))}(g_{n+1}(t_1,\dots,t_m,\cdot))\dots b^{(\varepsilon(r+2))}(g_{r+2}(t_1,\dots,t_m,\cdot))\\
&\quad b^{(\varepsilon(r))}(g_r(t_1,\dots,t_m,\cdot))\dots b^{(\varepsilon(1))}(g_1(t_1,\dots,t_m,\cdot))\,\Phi\Big\rangle dt_1\dots dt_m \qquad (3.20)
\end{aligned}$$

Since $\varepsilon(j)=1$ for any $j\le r$ and $\varepsilon(r+1)=2$, since $b^{(2)}=1-P_\Phi$, one has

$$\begin{aligned}
&b^{(\varepsilon(r))}(g_r(t_1,\dots,t_m,\cdot))\dots b^{(\varepsilon(1))}(g_1(t_1,\dots,t_m,\cdot))\,\Phi\\
=&b^{(2)}b^{(\varepsilon(r))}(g_r(t_1,\dots,t_m,\cdot))\dots b^{(\varepsilon(1))}(g_1(t_1,\dots,t_m,\cdot))\,\Phi\\
=&b^{(\varepsilon(r+1))}(g_r(t_1,\dots,t_m,\cdot))\,b^{(\varepsilon(r))}(g_r(t_1,\dots,t_m,\cdot))\dots b^{(\varepsilon(1))}(g_1(t_1,\dots,t_m,\cdot))\,\Phi
\end{aligned}$$

and

$$\sum_{j\in\{1,\dots,n+1\}\setminus\{r+1\}}(1-|\varepsilon(j)|/2)=\sum_{j=1}^{n+1}(1-|\varepsilon(j)|/2)$$

the expression in the right hand side of (3.20) equals to nothing else than

$$\begin{aligned}
&\lambda^{\sum_{j=1}^{n+1}(1-|\varepsilon(j)|/2)}\int_{[0,1]^m} f(t_1,\dots,t_m)\\
&\Big\langle \Phi, b^{(\varepsilon(n+1))}(g_{n+1}(t_1,\dots,t_m,\cdot))\dots b^{(\varepsilon(1))}(g_1(t_1,\dots,t_m,\cdot))\,\Phi\Big\rangle dt_1\dots dt_m
\end{aligned}$$

If $\varepsilon(r+1)=0$, (3.6) gives

$$
\begin{aligned}
&B_N^{(\varepsilon(r+1))}\Big(g_{r+1}\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\dots B_N^{(\varepsilon(1))}\Big(g_1\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\Psi\\
=&B_N^{(0)}B_N^{(+1)}\Big(g_r\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\dots B_N^{(+1)}\Big(g_1\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\Psi\\
=&Np_N\ B_N^{(+1)}\Big(g_r\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\left(1-M\left(\cdot\right)\right)\Big)\\
&\qquad B_N^{(+1)}\Big(g_{r-1}\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\dots B_N^{(+1)}\Big(g_1\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\Psi\\
=&Np_N\ B_N^{(\varepsilon(r))}\Big(g_r\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\left(1-M\left(\cdot\right)\right)\Big)\\
&\qquad B_N^{(\varepsilon(r-1))}\Big(g_{r-1}\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\dots B_N^{(\varepsilon(1))}\Big(g_1\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\Psi
\end{aligned}
$$

and so

$$
\begin{aligned}
&\frac{1}{N^m}\sum_{1\le h_1,\dots,h_m\le N}f\Big(\frac{h_1}{N},\dots,\frac{h_m}{N}\Big)\\
&\Big\langle\Phi,B_N^{(\varepsilon(n+1))}\Big(g_{n+1}\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\dots B_N^{(\varepsilon(1))}\Big(g_1\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\Phi\Big\rangle\\
=&\frac{Np_N}{N^m}\sum_{1\le h_1,\dots,h_m\le N}f\Big(\frac{h_1}{N},\dots,\frac{h_m}{N}\Big)\Big\langle\Phi,B_N^{(\varepsilon(n+1))}\Big(g_{n+1}\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\\
&\dots B_N^{(\varepsilon(r+2))}\Big(g_{r+2}\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)B_N^{(\varepsilon(r))}\Big(g_r\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\left(1-M\left(\cdot\right)\right)\Big)\\
&B_N^{(\varepsilon(r-1))}\Big(g_{r-1}\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\dots B_N^{(\varepsilon(1))}\Big(g_1\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\Phi\Big\rangle \qquad (3.21)
\end{aligned}
$$

which goes, thanks to the induction's assumption, to

$$
\begin{aligned}
&\lambda\lambda^{\sum_{j\in\{1,\dots,n+1\}\setminus\{r+1\}}(1-|\varepsilon(j)|/2)}\int_{[0,1]^m}dt_1\dots dt_m\ f\left(t_1,\dots,t_m\right)\\
&\quad\big\langle\Phi,b^{(\varepsilon(n+1))}\left(g_{n+1}\left(t_1,\dots,t_m,\cdot\right)\right)\dots b^{(\varepsilon(r+2))}\left(g_{r+2}\left(t_1,\dots,t_m,\cdot\right)\right)\\
&\quad b^{(\varepsilon(r))}\left(g_r\left(t_1,\dots,t_m,\cdot\right)\left(1-M\right)\left(\cdot\right)\right)b^{(\varepsilon(r-1))}\left(g_{r-1}\left(t_1,\dots,t_m,\cdot\right)\right)\\
&\qquad\qquad\dots b^{(\varepsilon(1))}\left(g_1\left(t_1,\dots,t_m,\cdot\right)\right)\Phi\big\rangle \qquad (3.22)
\end{aligned}
$$

Thanks to the facts $\varepsilon\left(j\right)=1$ for any $j\le r$ and $\varepsilon\left(r+1\right)=0$, thanks to the definition of $b^{(0)}$ (i.e., (3.14)), one finds

$$
\begin{aligned}
&b^{(\varepsilon(r))}\left(g_r\left(t_1,\dots,t_m,\cdot\right)\left(1-M\right)\left(\cdot\right)\right)b^{(\varepsilon(r-1))}\left(g_{r-1}\left(t_1,\dots,t_m,\cdot\right)\right)\\
&\qquad\qquad\dots b^{(\varepsilon(1))}\left(g_1\left(t_1,\dots,t_m,\cdot\right)\right)\Phi\\
=&b^{(0)}b^{(\varepsilon(r))}\left(g_r\left(t_1,\dots,t_m,\cdot\right)\right)\dots b^{(\varepsilon(1))}\left(g_1\left(t_1,\dots,t_m,\cdot\right)\right)\Phi\\
=&b^{(\varepsilon(r+1))}\left(g_{r+1}\left(t_1,\dots,t_m,\cdot\right)\right)b^{(\varepsilon(r))}\left(g_r\left(t_1,\dots,t_m,\cdot\right)\right)\\
&\qquad\qquad\dots b^{(\varepsilon(1))}\left(g_1\left(t_1,\dots,t_m,\cdot\right)\right)\Phi
\end{aligned}
$$

and

$$
\lambda\lambda^{\sum_{j\in\{1,\dots,n+1\}\setminus\{r+1\}}(1-|\varepsilon(j)|/2)}=\lambda^{\sum_{j=1}^{n+1}(1-|\varepsilon(j)|/2)}
$$

Consequently, the expression in (3.21) goes to

$$\lambda^{\sum_{j=1}^{n+1}(1-|\varepsilon(j)|/2)}\int_{[0,1]^m} dt_1\dots dt_m\ f(t_1,\dots,t_m)$$
$$\Big\langle \Phi, b^{(\varepsilon(n+1))}(g_{n+1}(t_1,\dots,t_m,\cdot))\dots b^{(\varepsilon(1))}(g_1(t_1,\dots,t_m,\cdot))\,\Phi\Big\rangle$$

The last case to be seen is $\varepsilon(r+1)=-1$. In this case, (3.7) gives

$$\begin{aligned}
&B_N^{(\varepsilon(r+1))}\Big(g_{r+1}\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\dots B_N^{(\varepsilon(1))}\Big(g_1\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\Psi\\
=&p_N(1-p_N)\sum_{h=1}^{N}(\overline{g}_{r+1}g_r)\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\frac{h}{N}\Big)\\
&\quad B_N^{(+1)}\Big(g_{r-1}\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\chi_{\frac{h}{N})}(\cdot)\Big)B_N^{(+1)}\Big(g_{r-2}\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\\
&\quad\dots B_N^{(+1)}\Big(g_1\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\Psi
\end{aligned}\tag{3.23}$$

So, one has

$$\begin{aligned}
&\frac{1}{N^m}\sum_{1\le h_1,\dots,h_m\le N} f\Big(\frac{h_1}{N},\dots,\frac{h_m}{N}\Big)\\
&\quad\Big\langle\Psi, B_N^{(\varepsilon(n+1))}\Big(g_{n+1}\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\dots B_N^{(\varepsilon(1))}\Big(g_1\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\Psi\Big\rangle\\
=&\frac{p_N(1-p_N)}{N^m}\sum_{1\le h_1,\dots,h_m\le N} f\Big(\frac{h_1}{N},\dots,\frac{h_m}{N}\Big)\sum_{h=1}^{N}(\overline{g}_{r+1}g_r)\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\frac{h}{N}\Big)\\
&\quad\Big\langle\Psi, B_N^{(\varepsilon(n+1))}\Big(g_{n+1}\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\dots B_N^{(\varepsilon(r+2))}\Big(g_{r+2}\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\\
&\quad B_N^{(\varepsilon(r-1))}\Big(g_{r-1}\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\chi_{\frac{h}{N})}(\cdot)\Big)B_N^{(\varepsilon(r-2))}\Big(g_{r-2}\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\\
&\quad\dots B_N^{(\varepsilon(1))}\Big(g_1\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\cdot\Big)\Big)\Psi\Big\rangle
\end{aligned}\tag{3.24}$$

By denoting, for any $\forall\,(t_1,\dots,t_m,t)\in[0,1]^{m+1}$ and $s\in[0,1]$,

$$\begin{aligned}
\widetilde{f}(t_1,\dots,t_m,t)&:=f(t_1,\dots,t_m)\,(\overline{g}_{r+1}g_r)(t_1,\dots,t_m,t)\\
\widetilde{g}_{r-1}(t_1,\dots,t_m,t)(s)&:=g_{r-1}(t_1,\dots,t_m,s)\,\chi_{t)}(s)\\
\widetilde{g}_j(t_1,\dots,t_m,t)(s)&:=g_j(t_1,\dots,t_m,s)\,,\qquad\forall j\ne r-1
\end{aligned}\tag{3.25}$$

the expression in (3.24) becomes to

$$\begin{aligned}
&\frac{Np_N(1-p_N)}{N^{m+1}}\sum_{1\le h_1,\dots,h_m,h\le N}\widetilde{f}\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\frac{h}{N}\Big)\\
&\Big\langle\Psi, B_N^{(\varepsilon(n+1))}\Big(\widetilde{g}_{n+1}\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\frac{h}{N},\cdot\Big)\Big)\dots B_N^{(\varepsilon(r+2))}\Big(\widetilde{g}_{r+2}\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\frac{h}{N},\cdot\Big)\Big)\\
&\quad B_N^{(\varepsilon(r-1))}\Big(\widetilde{g}_{r-1}\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\frac{h}{N},\cdot\Big)\Big)\dots B_N^{(\varepsilon(1))}\Big(\widetilde{g}_1\Big(\frac{h_1}{N},\dots,\frac{h_m}{N},\frac{h}{N},\cdot\Big)\Big)\Psi\Big\rangle
\end{aligned}$$

which goes, thanks to the induction assumption, to

$$\lambda\lambda^{\sum_{j\in\{1,\ldots,n+1\}\setminus\{r,r+1\}}(1-|\varepsilon(j)|/2)}\int_{[0,1]^{m+1}}dt_1\ldots dt_m dt\ \widetilde{f}(t_1,\ldots,t_m,t)$$
$$\Big\langle\Phi, b^{(\varepsilon(n+1))}(\widetilde{g}_{n+1}(t_1,\ldots,t_m,t,\cdot))\ldots b^{(\varepsilon(r+2))}(\widetilde{g}_{r+2}(t_1,\ldots,t_m,t,\cdot))$$
$$b^{(\varepsilon(r-1))}(\widetilde{g}_{r-1}(t_1,\ldots,t_m,t,\cdot))\ldots b^{(\varepsilon(1))}(\widetilde{g}_1(t_1,\ldots,t_m,t,\cdot))\Phi\Big\rangle \tag{3.26}$$

On the IFS $\Gamma\left(\mathbf{L}^2([0,1]),\{\chi_{\Delta_n}\}_{n=1}^{\infty}\right)$, as shown in [7], for $\varepsilon(r+1)=-1$

$$\begin{aligned}
&\int_{[0,1]^m}dt_1\ldots dt_m f(t_1,\ldots,t_m)\\
&\qquad\langle\Phi, b^{(\varepsilon(n+1))}(g_{n+1}(t_1,\ldots,t_m,\cdot))\ldots b^{(\varepsilon(1))}(g_1(t_1,\ldots,t_m,\cdot))\Phi\rangle\\
&=\int_{[0,1]^m}dt_1\ldots dt_m f(t_1,\ldots,t_m)\ \langle\Phi, b^{(\varepsilon(n+1))}(g_{n+1}(t_1,\ldots,t_m,\cdot))\\
&\qquad\ldots b^{(\varepsilon(r+2))}(g_{r+2}(t_1,\ldots,t_m,\cdot))\, b^{(-1)}(g_{r+1}(t_1,\ldots,t_m,\cdot))\\
&\qquad\quad b^{(+1)}(g_r(t_1,\ldots,t_m,\cdot))\ldots b^{(+1)}(g_1(t_1,\ldots,t_m,\cdot))\Phi\rangle\\
&=\int_{[0,1]^m}dt_1\ldots dt_m f(t_1,\ldots,t_m)\int_0^1 dt\ \left(\overline{g}_{r+1}g_r\right)(t_1,\ldots,t_m,t)\\
&\qquad\langle\Phi, b^{(\varepsilon(n+1))}(g_{n+1}(t_1,\ldots,t_m,\cdot))\ldots b^{(\varepsilon(r+2))}(g_{r+2}(t_1,\ldots,t_m,\cdot))\\
&\qquad b^{(+1)}\left(g_{r-1}(t_1,\ldots,t_m,\cdot)\,\chi_{t)}(\cdot)\right) b^{(+1)}(g_{r-2}(t_1,\ldots,t_m,\cdot))\\
&\qquad\qquad\qquad\qquad\ldots b^{(\varepsilon(1))}(g_1(t_1,\ldots,t_m,\cdot))\Phi\rangle\\
&=\int_{[0,1]^{m+1}}dt_1\ldots dt_m dt\ \widetilde{f}(t_1,\ldots,t_m,t)\\
&\qquad\langle\Phi, b^{(\varepsilon(n+1))}(\widetilde{g}_{n+1}(t_1,\ldots,t_m,t,\cdot))\ldots b^{(\varepsilon(r+2))}(\widetilde{g}_{r+2}(t_1,\ldots,t_m,t,\cdot))\\
&\qquad\quad b^{(\varepsilon(r-1))}(\widetilde{g}_{r-1}(t_1,\ldots,t_m,t,\cdot))\ldots b^{(\varepsilon(1))}(\widetilde{g}_1(t_1,\ldots,t_m,t,\cdot))\Phi\rangle
\end{aligned}$$

Moreover, in case $\varepsilon(r+1)=-1$ and $\varepsilon(r)=1$, one has

$$\lambda\lambda^{\sum_{j\in\{1,\ldots,n+1\}\setminus\{r,r+1\}}(1-|\varepsilon(j)|/2)}=\lambda^{\sum_{j=1}^{n+1}(1-|\varepsilon(j)|/2)}$$

and so the expression (3.26) is in fact

$$\lambda^{\sum_{j=1}^{n+1}(1-|\varepsilon(j)|/2)}\int_{[0,1]^m}dt_1\ldots dt_m f(t_1,\ldots,t_m)$$
$$\Big\langle\Phi, b^{(\varepsilon(n+1))}(g_{n+1}(t_1,\ldots,t_m,\cdot))\ldots b^{(\varepsilon(1))}(g_1(t_1,\ldots,t_m,\cdot))\Phi\Big\rangle$$

Summing up, we have prove the thesis. □

As a consequence of Propositions 2.1 and Theorem 3.3, one gets the following

Theorem 3.4. *If $p = p_N$ verifies $Np_N \to \lambda$, then for any $\{c_0, c_1, c_2\} \subset \mathbb{R}$,*

$$\lim_{N\to\infty}\left\langle \Psi, \left(c_1\left(B_N^{(-1)}+B_N^{(+1)}\right)+c_0B_N^{(0)}+c_2B_N^{(2)}\right)^n\Psi\right\rangle$$
$$=\left\langle \Phi, \left(c_1\sqrt{\lambda}\left(b^{(-1)}+b^{(+1)}\right)+c_0\lambda b^{(0)}+c_2b^{(2)}\right)^n\Phi\right\rangle, \qquad \forall n\in\mathbb{N} \tag{3.27}$$

and

$$\lim_{N\to\infty}\left\langle \Psi, \exp\left(it\left(c_1\left(B_N^{(-1)}+B_N^{(+1)}\right)+c_0B_N^{(0)}+c_2B_N^{(2)}\right)\right)\Psi\right\rangle$$
$$=\left\langle \Phi, \exp\left(it\left(c_1\sqrt{\lambda}\left(b^{(-1)}+b^{(+1)}\right)+c_0\lambda b^{(0)}+c_2b^{(2)}\right)\right)\Phi\right\rangle, \quad \forall t\in\mathbb{R} \tag{3.28}$$

4. Impossibility to Have Quantum Decomposition of $P_{mo}(\lambda)$ by Using 1M-IFS

The main goal of this section is to show the impossibility sketched in the end of Section 1. Namely, we will show that it is impossible to have (1.12) (or equivalently, (1.11)) in which $\left(\Gamma(\mathbb{C},\{\omega_n\}_n), \left\{\tilde{b}^{(+1)}, \tilde{b}^{(-1)}, \tilde{\alpha}_\Lambda\right\}\right)$ is a quantum decomposition of the monotone Poisson distribution $P_{mo}(\lambda)$.

This goal will be achieved as follows: on the one hand, by taking $c_0 = c_2 = 0$ and $c_1 = 1$, Theorem 3.4 tells us that, with the assumption $Np_N \to \lambda$,

$$\lim_{N\to\infty}\left\langle \Psi, \left(B_N^{(-1)}+B_N^{(+1)}\right)^n\Psi\right\rangle = \left\langle \Phi, \lambda^{\frac{n}{2}}\left(b^{(-1)}+b^{(+1)}\right)^n\Phi\right\rangle, \quad \forall n\in\mathbb{N} \tag{4.1}$$

and

$$\lim_{N\to\infty}\left\langle \Psi, \exp\left(it\left(B_N^{(-1)}+B_N^{(+1)}\right)\right)\Psi\right\rangle$$
$$=\left\langle \Phi, \exp\left(it\sqrt{\lambda}\left(b^{(-1)}+b^{(+1)}\right)\right)\Phi\right\rangle, \qquad \forall t\in\mathbb{R} \tag{4.2}$$

where, $b^{(+1)}$ and $b^{(-1)}$ are the creation–annihilation operators with the test function $\chi_{[0,1]}$ defined on the IFS $\Gamma\left(\mathbf{L}^2([0,1]), \{\chi_{\Delta_n}\}_{n=1}^\infty\right)$. Therefore, as shown in [7], the distribution of $b^{(-1)}+b^{(+1)}$ is the uni–variance arc–sine distribution, consequently

$$\sqrt{\lambda}\big(b^{(-1)}+b^{(+1)}\big) \text{ has the arc–sine distribution with the variance } \lambda \tag{4.3}$$

On the other hand, if (1.12) holds (equivalently, (1.11) holds), by taking $c_1 = 1$, one gets (recall that $\phi := \langle\Psi, \cdot\Psi\rangle$)

$$\lim_{N\to\infty}\left\langle \Psi, \left(B_N^{(-1)}+B_N^{(+1)}+c\big(B_N^{(0)}+B_N^{(2)}\big)\right)^n\Psi\right\rangle$$
$$=\left\langle \Phi, \big(\tilde{b}^{(-1)}+\tilde{b}^{(+1)}+c\tilde{\alpha}_\Lambda\big)^n\Phi\right\rangle, \qquad \forall n\in\mathbb{N},\ c\in\mathbb{R} \tag{4.4}$$

in particular, by taking $c = 1$ and 0 respectively,

$$\lim_{N\to\infty}\left\langle \Psi, \left(B_N^{(-1)}+B_N^{(+1)}+B_N^{(0)}+B_N^{(2)}\right)^n\Psi\right\rangle$$
$$=\lim_{N\to\infty}\left\langle \Psi, B_N^n\Psi\right\rangle = \left\langle \Phi, \big(\tilde{b}^{(-1)}+\tilde{b}^{(+1)}+\tilde{\alpha}_\Lambda\big)^n\Phi\right\rangle, \qquad \forall n\in\mathbb{N} \tag{4.5}$$

and

$$\lim_{N\to\infty}\left\langle\Psi,\left(B_N^{(-1)}+B_N^{(+1)}\right)^n\Psi\right\rangle=\left\langle\Phi,\left(\tilde b^{(-1)}+\tilde b^{(+1)}\right)^n\Phi\right\rangle,\qquad\forall n\in\mathbb{N}\tag{4.6}$$

So by combining (4.2), (4.3) and (4.6), one concludes that **if** (1.12) **holds, then** $\tilde b^{(-1)}+\tilde b^{(+1)}$ **and** $b^{(-1)}+b^{(+1)}$ **have the same arc–sine distribution** as the weak limit of the vacuum distribution if the sequence $\left\{B_N^{(-1)}+B_N^{(+1)}\right\}_{N=1}^{\infty}$. We prove that this is not the case.

Since $\left(\Gamma\left(\mathbb{C},\{\omega_n\}_n\right),\left\{\tilde b^{(+1)},\tilde b^{(-1)},\tilde\alpha_\Lambda\right\}\right)$ is a quantum decomposition of the distribution $P_{mo}(\lambda)$, we know that,

• with respect to the state $\langle\Phi,\cdot\,\Phi\rangle$, the distribution of $\tilde b^{(-1)}+\tilde b^{(+1)}+\tilde\alpha_\Lambda$ is $P_{mo}(\lambda)$,

• by denoting $\{\omega_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=0}^{\infty}$ the Jacobi coefficients of the distribution $P_{mo}(\lambda)$,

$$\left\langle\left(\tilde b^{(+1)}\right)^n\Phi,\left(\tilde b^{(+1)}\right)^n\Phi\right\rangle=\omega_1\cdot\ldots\cdot\omega_n,\qquad\forall n\in\mathbb{N}^*\tag{4.7}$$

and

$$\begin{aligned}&\tilde b^{(-1)}\left(\tilde b^{(+1)}\right)^n\Phi=\omega_n\left(\tilde b^{(+1)}\right)^{n-1}\Phi,\quad\forall n\in\mathbb{N}^*;\qquad\tilde b^{(-1)}\Phi=0\\&\tilde\alpha_\Lambda\left(\tilde b^{(+1)}\right)^n\Phi=\alpha_n\left(\tilde b^{(+1)}\right)^n\Phi,\quad\forall n\in\mathbb{N}\end{aligned}\tag{4.8}$$

Moreover, by denoting simply $a^+:=\tilde b^{(+1)}$, $a:=\tilde b^{(-1)}$ and $\alpha_\Lambda:=\tilde\alpha_\Lambda$, one has

$$\begin{aligned}u_n&:=\text{the } n\text{–th moment of } P_{mo}(\lambda)=\left\langle\Phi,\left(a+a^++\alpha_\Lambda\right)^n\Phi\right\rangle\\&\overset{(4.5)}{=}\lim_{N\to\infty}\left\langle\Psi,B_N^n\Psi\right\rangle,\qquad\forall n\in\mathbb{N}\end{aligned}\tag{4.9}$$

A simple calculation (see [13]) gives u_n for $n=1,2,3,4$ as follows:

$$u_1=\lambda;\quad u_2=\lambda^2+\lambda;\quad u_3=\lambda^3+\frac{5}{2}\lambda^2+\lambda;\quad u_4=\lambda^4+\frac{13}{3}\lambda^3+\frac{9}{2}\lambda^2+\lambda,\quad\ldots\tag{4.10}$$

Now, if $\left(\Gamma\left(\mathbb{C},\{\omega_n\}_n\right),\left\{\tilde b^{(+1)},\tilde b^{(-1)},\tilde\alpha_\Lambda\right\}\right)$ is a quantum decomposition of the distribution $P_{mo}(\lambda)$, we are able to calculate $\{u_1,u_2,u_3,u_4\}$ by using (4.7), (4.8) and (4.9).

First of all,

$$\lambda\overset{(4.10)}{=}u_1=\left\langle\Phi,\left(a+a^++\alpha_\Lambda\right)\Phi\right\rangle\overset{(4.8)}{=}\alpha_0\tag{4.11}$$

In order to determine u_n for $n\ge 2$, we introduce

$$v_n:=\left\langle\Phi,\left(a+a^++\alpha_\Lambda\right)^n a^+\Phi\right\rangle,\qquad\forall n\in\mathbb{N}^*$$

Then

$$v_1=\left\langle\Phi,\left(a+a^++\alpha_\Lambda\right)a^+\Phi\right\rangle=\left\langle\Phi,aa^+\Phi\right\rangle\overset{(4.7)}{=}\omega_1\tag{4.12}$$

and for any $n\ge 2$

$$\begin{aligned}u_n&=\left\langle\Phi,\left(a+a^++\alpha_\Lambda\right)^n\Phi\right\rangle\\&\overset{(4.8)}{=}\alpha_0\left\langle\Phi,\left(a+a^++\alpha_\Lambda\right)^{n-1}\Phi\right\rangle+\left\langle\Phi,\left(a+a^++\alpha_\Lambda\right)^{n-1}a^+\Phi\right\rangle\\&=\alpha_0u_{n-1}+v_{n-1}\end{aligned}\tag{4.13}$$

By using the formulae (4.11) and (4.12), i.e., $\alpha_0 = \lambda$ and $v_1 = \omega_1$, one has

$$\lambda^2 + \lambda \overset{(4.10)}{=} u_2 \overset{(4.13)}{=} \alpha_0 u_1 + v_1 = \lambda^2 + \omega_1$$

and so

$$\omega_1 = v_1 = \lambda \tag{4.14}$$

By using the fact

$$\left\langle \Phi, aa^{+2}\Phi \right\rangle = \left\langle \Phi, a^+ a^{+2}\Phi \right\rangle = \left\langle \Phi, \alpha_\Lambda a^{+2}\Phi \right\rangle = 0$$

one obtains,

$$\begin{aligned} v_2 &= \left\langle \Phi, \left(a + a^+ + \alpha_\Lambda\right)^2 a^+\Phi \right\rangle \\ &= \left\langle \Phi, \left(a + a^+ + \alpha_\Lambda\right) \alpha_\Lambda a^+\Phi \right\rangle + \left\langle \Phi, \left(a + a^+ + \alpha_\Lambda\right) aa^+\Phi \right\rangle \\ &= \alpha_1 v_1 + \omega_1 u_1 = \lambda^2 + \lambda\alpha_1 \end{aligned}$$

and therefore

$$\lambda^3 + \frac{5}{2}\lambda^2 + \lambda \overset{(4.10)}{=} u_3 \overset{(4.13)}{=} \alpha_0 u_2 + v_2 = \lambda\left(\lambda^2 + \lambda\right) + \lambda^2 + \lambda\alpha_1 \tag{4.15}$$

This formula gives first of all

$$\alpha_1 = \frac{\lambda}{2} + 1 \tag{4.16}$$

and by combining (4.15) with (4.10) and (4.11), one obtains furthermore

$$v_2 = u_3 - \alpha_0 u_2 = \lambda^3 + \frac{5}{2}\lambda^2 + \lambda - \lambda\left(\lambda^2 + \lambda\right) = \lambda\left(\frac{3}{2}\lambda + 1\right) \tag{4.17}$$

By using the facts

- $\left\langle \Phi, (a + a^+ + \alpha_\Lambda)^2 a^{+2}\Phi \right\rangle = \left\langle \Phi, a^2 a^{+2}\Phi \right\rangle = \omega_1\omega_2$,
- $\alpha_\Lambda a^+\Phi = \alpha_1 a^+\Phi$,
- $aa^+\Phi = \omega_1\Phi$,

one gets, with the help of (4.14,4.16,4.17),

$$\begin{aligned} v_3 &= \left\langle \Phi, \left(a + a^+ + \alpha_\Lambda\right)^3 a^+\Phi \right\rangle \\ &= \left\langle \Phi, \left(a + a^+ + \alpha_\Lambda\right)^2 \alpha_\Lambda a^+\Phi \right\rangle + \left\langle \Phi, \left(a + a^+ + \alpha_\Lambda\right)^2 aa^+\Phi \right\rangle \\ &\qquad + \left\langle \Phi, \left(a + a^+ + \alpha_\Lambda\right)^2 a^{+2}\Phi \right\rangle \\ &= \alpha_1 v_2 + \omega_1 u_2 + \omega_1\omega_2 = \left(\frac{\lambda}{2} + 1\right)\lambda\left(\frac{3}{2}\lambda + 1\right) + \lambda\left(\lambda^2 + \lambda\right) + \lambda\omega_2 \\ &= \lambda\Big(\lambda^2 + \lambda + \Big(\frac{\lambda}{2} + 1\Big)\Big(\frac{3\lambda}{2} + 1\Big) + \omega_2\Big) = \lambda\Big(\frac{7}{4}\lambda^2 + 3\lambda + 1 + \omega_2\Big) \end{aligned}$$

Therefore,

$$\begin{aligned} &\lambda^4 + \frac{13}{3}\lambda^3 + \frac{9}{2}\lambda^2 + \lambda \overset{(4.10)}{=} u_4 \overset{(4.13)}{=} \alpha_0 u_3 + v_3 \\ =&\lambda\Big(\lambda^3 + \frac{5}{2}\lambda^2 + \lambda + \frac{7}{4}\lambda^2 + 3\lambda + 1 + \omega_2\Big) = \lambda\Big(\lambda^3 + \frac{17}{4}\lambda^2 + 4\lambda + 1 + \omega_2\Big) \end{aligned}$$

and consequently

$$\omega_2 = \frac{1}{12}\lambda^2 + \frac{1}{2}\lambda \tag{4.18}$$

The facts $\omega_1 = \lambda$ and $\omega_2 = \frac{1}{12}\lambda^2 + \frac{1}{2}\lambda$ tell us that the **symmetric part** of the quantum decomposition of the monotone Poisson distribution, i.e. the distribution of $a+a^+$, can **not** be the arc–sine distribution since the Jacobi coefficient $\{\omega_n\}_{n\geq 1}$ of the arc–sine distribution on any interval $(-c, c)$ with $c > 0$ must verify

$$\omega_n = \frac{\omega_1}{2}, \quad \forall n \geq 2$$

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