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APPLICATIONS OF A SUPERPOSED ORNSTEIN-UHLENBECK TYPE PROCESSES

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Abstract. We discuss modeling of the evolution of risky assets using Fractal Activity Time Geometric Brownian Motion (FATGBM) and Ornstein-Uhlenbeck processes driven by Lévy process. The proposed strategy is tested using actual data and the model is calibrated for the chosen data.

1. Introduction

The Black-Scholes model is a critical foundation to modern quantitative finance. Moreover, stationary processes of the Ornstein-Uhlenbeck (OU)-type and their superpositions (sup-OU) provide convenient ways to construct weak or long range dependent (LRD) processes with given marginal distributions. Key ingredient that allows one to obtain OU-type processes with specified marginal distributions is the concept of *self-decomposability*. This opens the way to a variety of non-Gaussian distributions which display features such as semi-heavy tails and asymmetry, a point of considerable interest in fields of application such as finance and econometrics[13].

This expository article is organized as follows: In section 2, we study Ornstein-Uhlenbeck type processes, together with their finite and infinite sums. These processes are defined in a similar way to the OU-type via a Wiener process. Later the Wiener process is replaced by another Lévy process[3]. Such processes are known as superposed OU-type processes. This can be seen as a continuous time analogue of autoregressions. Again, the emphasis is on finding models with given marginal distributions.

After, we present several examples of processes with prescribed marginal distributions and long-range dependence which are constructed by means of sup-OU processes, such as Gamma (Γ), Inverse Gaussian (IG), Variance-Gamma (VG), Normal Inverse Gaussian (NIG), and discuss the covariance structure of sup-OU processes, together with examples.

In section 3, we then turn our attention to application to financial data. Indeed, since financial time series are often non-stationary processes, we will consider the log-return of the Fractal Activity Time Geometric Brownian Motion (FATGBM) model and also define a distribution theory to approach the model. We conclude

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by running hypotheses tests, and we investigate which distribution of the two models VG and NIG-sup-OU processes is the best fit for the selected data.

2. Ornstein-Uhlenbeck (OU) Processes

2.1. Primer on OU processes

The Ornstein-Uhlenbeck process is named after Leonard Ornstein and George Eugene Uhlenbeck. A real-valued ordinary Ornstein-Uhlenbeck (OU) process $Y = \{Y_t\}_{t\geq 0}$, induced by a Brownian motion B_t, is an a.s. and unique solution to the Langevin stochastic differential equation (SDE):

$$dY_t = -\lambda Y_t \, dt + dB_t.$$

We focus on OU-type processes which are analogues of the above mentioned *ordinary OU processes* where Brownian motion is replaced by a general Lévy process $Z = \{Z_t\}_{t\geq 0}$. So, the continuous time analogue of an autoregression of order 1 (AR(1)) is an OU-type *process* defined by (see [3])

$$dY_t = -\lambda Y_t \, dt + dZ_t \text{ or } Y_t = e^{-\lambda t} Y_0 + \int_0^t e^{-\lambda(t-s)} \, dZ_s, \tag{2.1}$$

where $\lambda > 0$ and Z is a homogeneous Lévy process independent of Y_0. By a Lévy process, we mean a process with independent increments which is continuous in probability. The process Z is termed the background driving Lévy process (BDLP) corresponding to the process Y (see [7, Page 4]). Our approach is, however, slightly different and simpler because, we discuss invariant marginal distributions rather than limit distributions.

Remark 2.1.

(i) The class of possible *marginal distributions* for the OU-type processes coincides with the class of *self-decomposable distributions*, which includes Gamma, Inverse Gaussian, Variance-Gamma, Normal Inverse Gaussian, and many other distributions useful for applications in finance. Self-decomposable distributions are specifically useful when modeling data that exhibits heavier tails and higher peaks than normal (frequently referred to as Leptokurtic distributions). Hence, the OU-type models are more flexible when fitting empirical data.

(ii) Frequently in the literature (see [2, page 6]), the background driving Lévy process (BDLP) of the given OU-type process is rescaled from Z_t to $Z_{\lambda t}$, so the above SDE (2.1) becomes

$$dY_t = -\lambda Y_t \, dt + dZ_{\lambda t} \quad \text{or} \quad dY_t = -\lambda Y_t \, dt + dZ(\lambda t),$$
$$\text{or} \quad Y_t = e^{-\lambda t} Y_0 + \int_0^t e^{-\lambda(t-s)} \, dZ(\lambda s).$$

The BDLP of $Z(t)$ is equivalent in law to $\{Z(\lambda t)\}_{\lambda \geq 0}$, (Lemma 2.1 in [1, page 6]). From [7, Page 4], the strong stationary solution of this equation exists if and only if $\mathbb{E}\big[\log\big(1 + |Z(1)|\big)\big] < \infty$.
Intuitively, the parameter λ controls how fast the process Y_t converges to its long term mean.

2.2. Existence of superposed OU (sup-OU) processes

Superpositions of OU-type processes, or sup-OU processes for short, were introduced in the papers of Barndorff-Nielsen [2], Barndorff-Nielsen and Shephard [5], Barndorff-Nielsen and Leonenko [4], among others. Here, we give an overview of the superposition type models, i.e. their existence. We start with the definition of finite and infinite superpositions and the necessary assumptions so that they are well defined. Later, we discuss the main properties including their covariance structure.

Assumptions [7]. We note the following:

(A1) Let $\{Y^{(k)}(t)\}_{k\geq 1}$ be the sequence of independent stationary OU-type processes, i.e. independent processes such that each $Y^{(k)}(t)$ is a stationary solution of the equation

$$dY^{(k)}(t) = -\lambda_k Y^{(k)}(t)dt + dZ^{(k)}(\lambda_k t), \quad t \geq 0,$$

in which the Lévy processes $\{Z^{(k)}\}_{k\geq 1}$ are independent, and $\lambda_k > 0$ for all $k \geq 1$. In addition, suppose the self-decomposable distribution of $Y^{(k)}$ has finite moments of order $p \geq 2$, and suppose it is closed under convolution with respect to at least one distributional parameter δ_k, and that cumulants of order $p \geq 2$ of the distribution of $Y^{(k)}$ are proportional to some parameter δ_k ;

(A2) Suppose

$$\sum_{k=1}^{\infty} \mathbb{E}\big[Y^{(k)}(t)\big] < \infty \text{ and } \sum_{k=1}^{\infty} \text{Var}\left[Y^{(k)}(t)\right] < \infty.$$

Armed with these assumptions, we have:

Definition 2.2.

(1) In view of (A1) with $p = 2$, a **finite superposition** for $m \in \mathbb{Z}$ is given by

$$Y_m(t) = \sum_{k=1}^{m} Y^{(k)}(t), \quad t \in \mathbb{R}.$$

(2) In view of (A1) with $p = 2$, and (A2), an **infinite superposition** is given by

$$Y_\infty(t) = \sum_{k=1}^{\infty} Y^{(k)}(t), \quad t \in \mathbb{R}.$$

Remark 2.3.

(i) Under the stated assumptions, (especially (A2)), infinite superposition is well-defined in the sense of mean-square or almost-sure convergence.

(ii) Although assumption (A1) may seem restrictive, it is satisfied for many examples with traceable distributions of superpositions. In subsection 2.4, we see a number of examples where both assumptions (A1) and (A2) are satisfied. These examples include Gamma, Inverse Gaussian, Variance-Gamma, Normal Inverse Gaussian and other well known distributions. Their superpositions have the marginal distributions that belong to the same class as the marginal distributions of the components of superposition.

2.3. The covariance structure of sup-OU processes

Finite sup-OU processes([7]).

In this case, the covariance $R_{Y_m}(t)$ function of the resulting process is given by the formula:

$$R_{Y_m}(t) = \mathrm{Cov}(Y_m(0),\, Y_m(t)) = \sum_{k=1}^{m} \mathrm{Var}\left[Y^{(k)}(t)\right] e^{-\lambda_k t},$$

We recall that a stochastic process is short-range dependent (SRD) if its covariance function is integrable, i.e. $\int_{\mathbb{R}} |R_{Y_m}(t)|\, dt < \infty$.

Since for all $t \in \mathbb{R}$, $\mathrm{Var}\left[Y^{(k)}(t)\right] e^{-\lambda_k t} > 0$, the condition of integrability above becomes

$$\int_{\mathbb{R}} |R_{Y_m}(t)|\, dt = \int_{\mathbb{R}} \sum_{k=1}^{m} \mathrm{Var}\left[Y^{(k)}(t)\right] e^{-\lambda_k t}\, dt < \infty,$$

We can conclude that the *finite superposition* is a *short-range dependent* (SRD) process.

Infinite sup-OU processes([7]).

Here, the *covariance* $R_{Y_\infty}(t)$ function is of the form,

$$R_{Y_\infty}(t) = \mathrm{Cov}(Y_\infty(0),\, Y_\infty(t)) = \sum_{k=1}^{\infty} \mathrm{Var}\left[Y^{(k)}(t)\right] e^{-\lambda^{(k)} t}, \tag{2.2}$$

And under the condition (A1), the variance of $Y^{(k)}(t)$ is proportional to δ_k, that is $\mathrm{Var}\left[Y^{(k)}(t)\right] = C\delta_k$, where the constant C does not depend on k and reflects parameters of the marginal distribution of $Y^{(k)}$. If one chooses δ_k,

$$\delta_k = k^{-(1+2(1-H))}, \quad \frac{1}{2} < H < 1, \quad \lambda^{(k)} = \frac{\lambda}{k}$$

for some $\lambda > 0$, then we get that the covariance function of an infinite superposition (2.2) is

$$R_{Y_\infty}(t) = C \sum_{k=1}^{\infty} \delta_k e^{-\lambda_k t} = C \sum_{k=1}^{\infty} \frac{1}{k^{(1+2(1-H))}} e^{-\lambda^{(k)} t}. \tag{2.3}$$

The lemma below shows that the covariance function (2.3) is not integrable for the chosen parameters δ_k and $\lambda^{(k)}$, thus, the process obtained via infinite superposition exhibits *long-range dependence* (LRD).

Lemma 2.4. [7, Lemma 1]. *For the infinite superposition (2.2) of* OU-type *processes that satisfy condition (A1) with $p = 2$ and condition (A2), the covariance function of $Y_\infty(t)$ given by (2.3) with $\lambda^{(k)} = \lambda/k$ and $\delta_k = k^{-(1+2(1-H))}$, $\frac{1}{2} < H < 1$, can be written as*

$$R_{Y_\infty}(t) = \frac{L(t)}{t^{2(1-H)}}, \quad t > 0$$

where L is a slowly varying function at infinity.

Remark 2.5. As a consequence of this lemma, $R_{Y_\infty}(t) \sim t^{-2d}L(t),$ as $t \to \infty$ with $d = (1 - H) \in (0, \frac{1}{2})$. So, the process $Y_\infty(t)$ obtained via infinite superposition exhibits *long-range dependence* (LRD).

2.4. Examples of OU-type superpositions (sup-OU)

The examples in this section have been discussed in [2, 13]. We present them to illustrate that conditions (A1) and (A2) are satisfied for a number of OU-type processes.

2.4.1. Gamma sup-OU process

Let us find the superposition of the stationary Gamma OU-type processes. Suppose $\{Y^{(k)}(t),\ t \geq 0\}$, $k \geq 1$ are independent stationary OU-type Gamma processes with marginals $\Gamma(\alpha_k, \beta)$, $k \geq 1$ where $\alpha_k = \alpha k^{-(1+2(1-H))},\ H \in (\frac{1}{2}, 1)$. In addition, the stationary Gamma OU-type process $\{Y(t), t \geq 0\}$ with gamma marginal distribution has the cumulant generating function [7, page 16],

$$\kappa_X(z) = \log \mathbb{E}\big[e^{izY(t)}\big] = -\alpha \log\left(1 - \frac{iz}{\beta}\right) = \sum_{m=1}^{\infty} \frac{\alpha(m-1)!}{\beta^m}\frac{(iz)^m}{m!}, \quad z < \beta.$$

Therefore, we get $\mathbb{E}(Y(t)) = \frac{\alpha}{\beta}$ and $\mathrm{Var}(Y(t)) = \frac{\alpha}{\beta^2}$. The processes satisfy condition (A1) with the choice $\delta_k = \alpha_k$. Furthermore, since $\sum_{k=1}^{\infty} \alpha_k < \infty$, the condition (A2) is satisfied as well, hence *infinite superposition* of the stationary Gamma OU-type processes is well defined. Thus, the sup-OU process given by, $Y_\infty(t) = \sum_{k=1}^{\infty} Y^{(k)}(t),\ t \geq 0$, has a marginal distribution $\Gamma\left(\sum_{k=1}^{\infty}\alpha_k, \beta\right)$ and the covariance function is of the form, $R_{Y_\infty}(t) = \frac{1}{\beta^2}\sum_{k=1}^{\infty}\alpha_k e^{-\lambda^{(k)}t}$.

2.4.2. Inverse Gaussian sup-OU process

The inverse Gaussian law $\mathrm{IG}(\delta, \gamma)$ is the distribution on the positive half-line having density (see [2])

$$\frac{\delta}{\sqrt{2\pi}} e^{\delta\gamma} x^{-3/2} \exp\left\{ -\frac{1}{2}\left(\frac{\delta^2}{x} + \gamma^2 x\right)\right\}\mathbf{1}_{\{x>0\}},$$

where $\delta > 0$ and $\gamma \geq 0$. The distribution is self-decomposable for $\gamma > 0$, so there exists a stationary OU-type process $\{Y(t), t \geq 0\}$ with $\mathrm{IG}(\delta, \gamma)$ marginal distribution and correlation function $\exp\{-\lambda|t|\}$.
The stationary inverse Gaussian OU-type process has the cumulant generating function ([7])

$$\kappa_X(z) = \log \mathbb{E}\big[e^{izY(t)}\big] = \delta\left(\gamma - \sqrt{\gamma^2 - 2iz}\right) = \sum_{m=1}^{\infty} \frac{\delta(2m)!}{\gamma^{2m-1}(2m-1)m!\,2^m}\frac{(iz)^m}{m!},$$

with $|z| < \frac{\gamma^2}{2}$.
We can obtain easily the expectation and the variance of the stationary IG-OU-type process $\{Y(t), t \geq 0\}$, i.e. we have $\mathbb{E}(Y(t)) = \frac{\delta}{\gamma}$ and $\mathrm{Var}(Y(t)) = \frac{\delta}{\gamma^3}$. It follows that, independent stationary OU-type processes $\{Y^{(k)}(t),\ t \geq 0\}$, $k \geq 1$

with marginals $\mathrm{IG}(\delta_k,\, \gamma)$, $k \geq 1$ where $\delta_k = \delta k^{-(1+2(1-H))}$, $H \in (\frac{1}{2},\, 1)$ satisfy conditions (A1) and (A2). Thus, the Inverse Gaussian sup-OU process $Y_\infty(t) = \sum_{k=1}^{\infty} Y^{(k)}(t),\ t \geq 0$, has a marginal distribution $\mathrm{IG}\left(\sum_{k=1}^{\infty} \delta_k,\, \gamma\right)$ and their covariance function (see (2.2)) is of the form, $R_{Y_\infty}(t) = \frac{1}{\gamma^3} \sum_{k=1}^{\infty} \delta_k e^{-\lambda^{(k)} t}$.

2.4.3. Variance-Gamma sup-OU process

We now consider another self-decomposable distribution known as Variance-Gamma distribution denoted by $\mathrm{VG}\,(\kappa,\, \alpha,\, \beta,\, \mu\,)$ with parameters $\kappa > 0,\ 0 < |\beta| < \alpha,\ \mu \in \mathbb{R},\ \gamma^2 = \alpha^2 - \beta^2$, and pdf given by (see [13])

$$\mathrm{vg}(x) = \frac{\gamma^{2\kappa}}{\sqrt{\pi}\Gamma(\kappa)(2\alpha)^{\kappa-1/2}} |x-\kappa|^{\kappa-1/2} K_{\kappa-1/2}(\alpha|x-\mu|) e^{\beta(x-\mu)}, \quad x \in \mathbb{R},$$

where $K_\nu(z)$ denotes modified Bessel function of the second kind. Since the Variance-Gamma distribution is self-decomposable, the corresponding stationary OU-type process, denoted as $\{Y(t),\ t \geq 0\}$, exists and its cumulant generating function (cgf) is given by (see [7]):

$$\kappa_X(z) = \log \mathbb{E}\big[e^{izY(t)}\big] = i\mu z + 2\kappa \log\left(\frac{\gamma}{\sqrt{\alpha^2 - (\beta + 2iz)^2}}\right), \quad |\beta + z| < \alpha.$$

It follows that $\mathrm{VG}\,(\kappa,\, \alpha,\, \beta,\, \mu\,)$ distribution is closed under convolution with respect to the parameters κ and μ.

We have $\mathbb{E}(Y(t)) = \mu + \dfrac{2\kappa\beta}{\gamma^2}$ and $\mathrm{Var}(Y(t)) = \frac{2\kappa}{\gamma^2}\left(1 + 2\left(\frac{\beta}{\gamma}\right)^2\right)$ with $\gamma^2 = \alpha^2 - \beta^2$.

So the independent stationary OU-type processes $\{Y^{(k)}(t),\, t \geq 0\}$, $k \geq 1$ with marginals $\mathrm{VG}\,(\kappa_k,\, \alpha,\, \beta,\, \mu_k\,)$ where $\sum_{k=1}^{\infty} \mu_k$ converges and a specific choice for the parameter κ_k, i.e. $\kappa_k = \kappa k^{-(1+2(1-H))}$, $\mu_k = \dfrac{\kappa_k}{c}$, $k = 1, 2, \ldots$, $H \in \left(\dfrac{1}{2},\, 1\right)$, $c \in \mathbb{R}^*$, satisfy (A1) and (A2).

We obtain, the Variance-Gamma sup-OU process as $Y_\infty(t) = \sum_{k=1}^{\infty} Y^{(k)}(t),\ t \geq 0$, with marginal distribution $\mathrm{VG}\left(\sum_{k=1}^{\infty} \kappa_k,\, \alpha,\, \beta,\, \sum_{k=1}^{\infty} \mu_k\right)$ and covariance function $R_{Y_\infty}(t) = \dfrac{2}{\gamma^2}\left(1 + 2\left(\dfrac{\beta}{\gamma}\right)^2\right) \sum_{k=1}^{\infty} \kappa_k e^{-\lambda^{(k)} t}$.

2.4.4. Normal inverse Gaussian sup-OU process

The inverse Gaussian law $\mathrm{NIG}(\alpha,\, \beta,\, \delta, \mu)$ with parameters $\delta > 0,\ 0 < |\beta| < \alpha,\ \mu \in \mathbb{R}$, is the symmetric distribution on the real line having density (see [13])

$$\mathrm{nig}(x) = \frac{\delta\alpha}{\pi} \exp\left\{\delta\sqrt{\alpha^2 - \beta^2} + \beta x - \beta\mu\right\} \frac{K_1\big(\alpha g(x-\mu)\big)}{g(x-\mu)}, \quad x \in \mathbb{R},$$

where $K_\nu(z)$ denotes the modified Bessel function of the second kind of order ν and $g(x) = \sqrt{\delta^2 + x^2}$. Since it is self-decomposable, the corresponding stationary OU-type process $\{Y(t),\ t \geq 0\}$ exists and the stationary normal inverse Gaussian OU-type process has cumulant generating function given by ([7]),

$$\kappa_X(z) = \log \mathbb{E}\big[e^{izY(t)}\big] = i\mu z + \delta\bigg(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iz)^2}\bigg), \quad |\beta + z| < \alpha.$$

It follows that NIG$(\alpha,\ \beta,\ \delta,\ \mu)$ distribution is closed under convolution with respect to parameters δ and μ.
We have $\mathbb{E}(Y(t)) = \mu + \dfrac{\delta\beta}{\sqrt{\alpha^2 - \beta^2}}$ and $\mathrm{Var}(Y(t)) = \dfrac{\delta\alpha^2}{\left(\alpha^2 - \beta^2\right)^{3/2}}$. The independent stationary OU-type processes $\{Y^{(k)}(t),\ t \geq 0\},\ k \geq 1$ with marginals NIG$(\alpha,\ \beta,\ \delta_k, \mu_k),\ k \geq 1$ where $\sum_{k=1}^{\infty} \mu_k < \infty$. With a specific choice for the parameters δ_k, given as $\delta_k = \delta k^{-(1+2(1-H))},\ \mu_k = \dfrac{\delta_k}{c},\ k = 1, 2, \ldots, H \in \left(\dfrac{1}{2}, 1\right), c \in \mathbb{R}^*$, conditions (A1) and (A2) hold, and we obtain, the Normal Inverse Gaussian sup-OU process as $Y_\infty(t) = \sum_{k=1}^{\infty} Y^{(k)}(t),\ t \geq 0$, with marginal distribution NIG $\left(\alpha,\ \beta,\ \sum_{k=1}^{\infty} \delta_k,\ \sum_{k=1}^{\infty} \mu_k\right)$. Their covariance function is of the form,

$$R_{Y_\infty}(t) = \frac{\alpha^2}{\left(\alpha^2 - \beta^2\right)^{3/2}} \sum_{k=1}^{\infty} \delta_k e^{-\lambda^{(k)} t}.$$

3. Fractal Activity Time Geometric Brownian Motion Process

Fractal Activity Time Geometric Brownian Motion (FATGBM) models for risky assets are due to [8], see also [10, 9, 6]. This model generalizes classical GBM by using a random subordinator, as opposed to time, to evaluate the standard Brownian motion. The model describes stock price process $S(t)$ of a risky asset at time t.

Definition 3.1. A *fractal activity time* $\{T_t,\ t \geq 0\}$ is a $\mathcal{F}_t$-adapted, right continuous, positive, non-decreasing random process with long range dependence, that is having stationary but not necessarily independent increments, starting at $T_0 = 0$ almost surely (a.s.).

Definition 3.2 (FATGBM)**.** A *fractal activity time geometric Brownian motion* process $\{S(t),\ t \geq 0\}$ is defined by $S_t = S_0 \exp\{\mu t + \theta T_t + \sigma B(T_t)\}$, where $B(t)$ is an $\mathcal{F}_t$-adapted standard Brownian motion (Wiener process) independent of activity time $\{T_t\}$. The constants $\mu \in \mathbb{R}, \theta \in \mathbb{R}$, and $\sigma > 0$, respectively the location (drift), skew (asymmetry) and scale (volatility) parameters.
The increments over unit time are denoted by $\tau_t = T_t - T_{t-1},\ t = 1, 2, \ldots$

Assumption 1. The activity time process $\{T_t,\, t \geq 0\}$ has continuous sample paths, i.e. the map $t \longmapsto T_t(\omega)$ is a continuous function of t for all paths ω.

In the next theorem, we review some concepts of Stochastic Calculus. It is assumed that the reader has some background on stochastic integration w.r.t. Wiener process (Itó integral).

Theorem 3.3 (**Itó's formula**, [15]). *Let $X = \{X_t,\, t \geq 0\}$ be an Itó process in the form $dX_t = \mu_t\, dt + \sigma_t\, dB_t$, and $F : [0,\, T] \times \mathbb{R} \to \mathbb{R},\ (t,\, x) \to F(t,\, x)$ with $F \in \mathbb{C}^{1,\,2},\ F = F(t,\, X_t)$ then it holds*

$$F(t,\, X_t) = F(0,\, X_0) + \int_0^t F_t(s,\, X_s) ds + \int_0^t F_x(s,\, X_s) dX_s + \frac{1}{2}\int_0^t F_{xx}(s,\, X_s) d\langle X\rangle_s. \tag{3.1}$$

For the quadratic variation of $\langle X\rangle$ of X it holds $\langle X\rangle_t = \int_0^t \sigma_s^2\, ds$ and using dX_t the above equation can be rewritten as

$$F(t,\, X_t) = F(0,\, X_0) + \int_0^t \left[F_t(s,\, X_s) + \mu_s F_x(s,\, X_s) + \frac{1}{2}\sigma_s^2 F_{xx}(s,\, X_s)\right] ds + \int_0^t \sigma_s F_x(s,\, X_s) dB_s.$$

or in differential form $dF = \left[F_t + \mu_t F_x + \frac{1}{2}\sigma_t^2 F_{xx}\right] dt + \sigma_t F_x\, dB_t$.

We now present corollary from the Itó formula given in Theorem 3.3, by changing the representation of the Itó process X. It is useful to show that the paradigm FATGBM model S_t in mathematical finance is a solution of the given SDE, so it also follows the next lemma.

Corollary 3.4. *Suppose $X = \{X_t,\, t \geq 0\}$ an Itó process of the form $dX_t = \mu_t\, dt + \theta\, dT_t + \sigma_t\, dB_{T_t}$, T_t random activity time as definition 3.1 and $F : [0,\, T] \times \mathbb{R} \to \mathbb{R},\ (t,\, x) \to F(t,\, x)$ with $F \in \mathbb{C}^{1,\,2}$ then, it holds [in differential form with $F = F(t,\, X_t)$], $dF = \left[F_t + \mu_t F_x\right] dt + \left[\theta F_x + \frac{1}{2}\sigma_t^2 F_{xx}\right] dT_t + \sigma_t F_x\, dB_{T_t}$.*

Proof. Since dX_t can be written $dX_t = \left(\mu_t + \theta T_t'\right) dt + \sigma_t\, dB_{T_t}$, with $dT_t = T_t' dt$, the quadratic variation is given by $\langle X\rangle_s = \int_0^t \sigma_s\, dB_{T_s} = \int_0^t \sigma_s^2\, dT_s$ and $d\,\langle X\rangle_s = \sigma_s^2\, dT_s$. By the usual Itó's formula (3.1), we get

$$F(t,\, X_t) = F(0,\, X_0) + \int_0^t \left[F_t(s,\, X_s) + \mu_s F_x(s,\, X_s)\right] ds + \int_0^t \left[\theta F_x + \frac{1}{2}\sigma_s^2 F_{xx}(s,\, X_s)\right] dT_s + \int_0^t \sigma_s F_x(s,\, X_s) dB_{T_s}.$$

So, with $F = F(t,\, X_t)$, in differential form we have $dF = \left[F_t + \mu_t F_x\right] dt + \left[\theta F_x + \frac{1}{2}\sigma_t^2 F_{xx}\right] dT_t + \sigma_t F_x\, dB_{T_t}$. □

Lemma 3.5. *Let $B(t)$ be an $\mathcal{F}_t$-adapted standard Brownian motion and T_t a fractal activity time. Let $\mu \in \mathbb{R},\ \theta \in \mathbb{R}$ and $\sigma > 0$ be constants. Assume that, assumption 1 holds, then the unique strong solution to the SDE*

$$dS_t = S_t\left[\mu\, dt + (\theta + \frac{1}{2}\sigma^2)\, dT_t + \sigma\, dB_{T_t}\right],$$

with initial condition $S(0) = s(0)$, *is given by* $S_t = S_0 \exp\{\mu t + \theta T_t + \sigma B_{T_t}\}$.

Proof. As in definition 3.2, FATGBM model is given by $S_t = S_0 e^{\mu t + \theta T_t + \sigma B_{T_t}}$. We now show that FATGBM is a unique solution to the time-change SDE below

$$\frac{dS_t}{S_t} = \mu\, dt + (\theta + \frac{1}{2}\sigma^2)\, dT_t + \sigma\, dB_{T_t}.$$

By corollary 3.4, set $S_t = F(X_t) = S_0 e^{X_t}$ with $X_t = \mu t + \theta T_t + \sigma B_{T_t}$ $\mu_t = \mu$ and $\sigma_t = \sigma$. We have $dX_t = \mu\, dt + \theta\, dT_t + \sigma\, dB_{T_t}$. Since $F_t = 0$, $F_x = F_{xx} = F$, we get

$$dS_t = S_t\big[\mu\, dt + (\theta + \frac{1}{2}\sigma^2)\, dT_t + \sigma\, dB_{T_t}\big].$$

□

3.1. Log-returns in the FATGBM model.

As with the Black-Scholes model, we are more interested in modeling log-returns instead of the stock price process, hence the FATGBM log-returns are of the form,

$$\begin{aligned} X_t &= \log(S_t) - \log(S_{t-1}) = \mu + \theta(T_t - T_{t-1}) + \sigma\big(B_{T_t} - B_{T_{t-1}}\big) \\ X_t &\overset{\mathcal{D}}{=} \mu + \theta\tau_t + \sigma B_{T_t - T_{t-1}} \overset{\mathcal{D}}{=} \mu + \theta\tau_t + \sigma B_{\tau_t} \overset{\mathcal{D}}{=} \mu + \theta\tau_t + \sigma\sqrt{\tau_t} B_1, \end{aligned} \tag{3.2}$$

provided that $\tau_t = T_t - T_{t-1}$.
In distribution theory, the conditional distributions of the log-returns X_t are *mixed normal* that is $X_t|\tau_t \overset{\mathcal{D}}{=} \mathcal{N}(\mu + \theta\tau_t,\, \sigma^2\tau_t)$.

3.1.1. Moments

Assuming the finiteness of at least fourth order moment of τ_t, the log returns have the moments up to the fourth order;

$$\mathbb{E}(X_t) = \mu + \theta\mathbb{E}(\tau_t) \text{ and } \mathrm{Var}(X_t) = \mathbb{E}\big(X_t - \mathbb{E}(X_t)\big)^2 = \sigma^2\mathbb{E}(\tau_t) + \theta^2 M_2 \tag{3.3}$$

$$\mathbb{E}\big(X_t - \mathbb{E}(X_t)\big)^3 = 3\theta\sigma^2 M_2 + \theta^3 M_3 \tag{3.4}$$

$$\mathbb{E}\big(X_t - \mathbb{E}(X_t)\big)^4 = 3\sigma^4\big(M_2 + (\mathbb{E}(\tau_t))^2\big) + 6\sigma^2\theta^2\big(\mathbb{E}(\tau_t)M_2 + M_3\big) + \theta^4 M_4 \tag{3.5}$$

where $M_i = \mathbb{E}\big(\tau_t - \mathbb{E}(\tau_t)\big)^i$, $i = 2, 3, 4$. Even when $\theta = 0$, the variance is time dependent, and the model is *heteroskedastic*.

3.1.2. Asymmetry and Leptokurtic

The distribution of X_t has the coefficients of skewness and of kurtosis (fatness of tails) respectively given by,

$$\gamma_1 = \frac{3\theta\sigma^2 M_2 + \theta^3 M_3}{(\sigma^2\mathbb{E}(\tau_t) + \theta^2 M_2)^{3/2}} \tag{3.6}$$

and

$$\gamma_2 = \frac{3\sigma^4\big(M_2 + (\mathbb{E}(\tau_t)^2)\big) + 6\sigma^2\theta^2\big(\mathbb{E}(\tau_t)M_2 + M_3\big) + \theta^4 M_4}{(\sigma^2\mathbb{E}(\tau_t) + \theta^2 M_2)^2}. \tag{3.7}$$

3.2. Estimation of parameters using method of moments (MoM)

1) **The** VG **model.** We assume that increments of the activity time follow a marginal gamma distribution $\Gamma(\alpha, \beta)$. In addition, since $\mathbb{E}(\tau_t) < \infty$, without loss of generality, we can assume that $\mathbb{E}(\tau_t) = 1$. The scaling constant gets absorbed into θ and σ. Since $\mathbb{E}(\tau_t) = 1$, the moments of the Gamma distribution are $M_2 = \frac{1}{\alpha}$, $M_3 = \frac{2}{\alpha^2}$, $M_4 = \frac{3(\alpha+2)}{\alpha^3}$. We can consider $X_t|\tau_t \sim \mathcal{N}(\mu, \sigma^2\tau_t)$ where τ_t has a gamma distribution. Under the above assumptions, when increments of activity time follow a marginal gamma distribution $\Gamma(\alpha, \alpha)$, the returns X_t will have a marginal VG distribution (see [14]), denoted as $\mathrm{VG}(\mu, \theta, \sigma^2, \alpha)$ with μ (location), θ (asymmetry), σ (scale), α (shape). Its density is given by

$$\mathrm{vg}(x) = \sqrt{\frac{2}{\pi}}\frac{\alpha^\alpha e^{-\frac{(x-\mu)\theta}{\sigma^2}}}{\sigma\Gamma(\alpha)}\left(\frac{|x-\mu|}{\sqrt{\theta^2+2\alpha\sigma^2}}\right)^{\alpha-\frac{1}{2}} K_{\alpha-\frac{1}{2}}\left(\frac{|x-\mu|\sqrt{\theta^2+2\alpha\sigma^2}}{\sigma^2}\right), \ x \in \mathbb{R}.$$

Consequently, from [(3.3), (3.4), (3.5), (3.6), (3.7)] we may successfully obtain approximations to $\hat{\sigma}$, $\hat{\alpha}$, $\hat{\theta}$ and $\hat{\mu}$ following

$$\hat{\sigma} = s, \ \hat{\alpha} = \frac{3}{\hat{\gamma}_2}, \ \hat{\theta} = \frac{\hat{\gamma}_1\hat{\alpha}\hat{\sigma}}{3}, \ \hat{\mu} = \bar{X} - \hat{\theta}$$

where $s^2 = \hat{\mu}_2, \ \hat{\mu}_1 = \frac{1}{N}\sum_{t=1}^{N} X_t = \bar{X}, \ \hat{\mu}_k = \frac{1}{N}\sum_{t=1}^{N}(X_t - \bar{X})^k, \ k = 2, 3, 4.$

2) **The** NIG **model.** We assume that increments of the activity time follow a marginal Inverse Gaussian distribution $\mathrm{IG}(\delta, \gamma)$. In addition, suppose that the conditional distribution of X_t given τ_t, denoted by $X_t|\tau_t \sim \mathcal{N}(\mu + \kappa\tau_t, \tau_t)$ where τ_t has a Inverse Gaussian distribution. If τ_t itself follows an $IG(\delta, \gamma)$ distribution with parameters δ and γ, then the resulting mixed distribution is $\mathrm{NIG}(\mu, \kappa, \alpha, \delta)$ where $\alpha = \sqrt{\kappa^2 + \gamma^2}$. Moreover, by looking at the distribution theory in subsection 3.1, we take $\sigma^2 = 1$, $\theta = \kappa$ for our NIG-distribution, and the density of X_t in equation (3.2) becomes

$$\mathrm{nig}(x) = \frac{\alpha}{\pi} e^{\delta\sqrt{\alpha^2-\kappa^2}+\kappa(x-\mu)} k(x)^{-\frac{1}{2}} K_1\big(\alpha\delta k(x)^{\frac{1}{2}}\big), \ \text{for all} \ \ x \in \mathbb{R},$$

where $k(x) = 1 + \left(\frac{x-\mu}{\delta}\right)^2$ and $K_1(x)$ denotes the modified Bessel function of the third kind of order 1 evaluated at x, with real model parameters μ (location), δ (scale), κ (skewness), and α (shape). Hence, after some computation [(3.3), (3.4), (3.5), (3.6), (3.7)], one obtains that,

$$\hat{\gamma} = \frac{3}{s\sqrt{3\hat{\gamma}_2 - 5\gamma_1^2}}, \ \hat{\kappa} = \frac{\hat{\gamma}_1 s\hat{\gamma}^2}{3}, \ \hat{\delta} = \frac{s^2\hat{\gamma}^3}{\hat{\kappa}^2 + \hat{\gamma}^2}, \ \hat{\mu} = \bar{X} - \hat{\kappa}\frac{\hat{\delta}}{\hat{\gamma}} \quad \text{with} \ \ s^2 = \hat{\mu}_2.$$

3.3. Goodness-of-Fit tests [(VG, NIG)-models]

The data used in the study consists of the daily returns of the prices, between April 18, 2008 and May 17, 2019 for seven selected stocks market indexes from USA, United Kingdom, and Germany. Moreover, we explore the data specified as a Major World Indexes in Yahoo finance. It is called S&P 500 (Standard & Poor's 500), DOW 30 (Dow Jones Industrial Average), NASDAQ 100 (National

Association of Securities Dealers Automated Quotations), SmallCap 2000 (the Russell 2000 Index), CBOE (Chicago Board Options Exchange), FTSE 100 (Financial Times Stock Exchange 100 Index) and also DAX (Deutscher Aktienindex (German stock index)).

To apply, we also calculate the daily log-returns stock price indexes as $X_t := \log(S_t/S_{t-1})$, where S_t is the closing value of each stock price index.
Furthermore, as we say on the introduction 1, we investigate the distribution(s) between VG and NIG models sup-OU processes, which conclude the best fit of the stock price indexes log-returns by using the hypotheses testing. So, let us formulate a hypotheses for each model.

✓ **Variance-Gamma.**

$$\begin{cases} H_0: & \text{Stock price indexes log-returns follow the VG distribution} \\ H_1: & \text{Stock price indexes log-returns do not follow the VG distribution.} \end{cases}$$

✓ **Normal Inverse Gaussian.**

$$\begin{cases} H_0: & \text{Stock price indexes log-returns follow the NIG distribution} \\ H_1: & \text{Stock price indexes log-returns do not follow the NIG distribution.} \end{cases}$$

In order to compare our models successfully, we measure the distance between the empirical asset returns and the theoretical distributions under different models employing the [Kolmogorov-Smirnov (K-S) and Anderson-Darling (A-D)] goodness-of-fit tests. Moreover, the *Kolmogorov-Smirnov statistic*[11] consists to find the difference between $F_n(x, \mathbb{X})$ and $F_0(x)$. This difference D_n is defined to be the supremum over the absolute differences between $F_n(x, \mathbb{X})$ and $F_0(x)$, which is given by

$$D_n = \sup_{x\in\mathbb{R}} |F_n(x, \mathbb{X}) - F_0(x)|, \tag{3.8}$$

where $F_n(x, \mathbb{X})$ and $F_0(x)$ are a cumulative distribution function (c.d.f) of the empirical distribution function and the theoretical distribution respectively (which can be c.d.f of either VG or NIG).
Also, knowing the cumulative density function (c.d.f) of the proposed distribution $F_X(x_i)$, $i = 1, 2, \cdots, N$, the *Anderson-Darling statistic*[17] is computed by

$$A^2 = -N - \frac{1}{N}\sum_{i=1}^{N}(2i-1)\big[\log(F_X(x_i)) + \log\big(1 - F_X(x_{N+1-i})\big)\big]. \tag{3.9}$$

Remark 3.6.

(i) As [17], D_α refers to the upper-tail critical value of the K-S statistic at level α, giving a $(1-\alpha)$ percent confidence interval. In this case, Proschan[16] uses the approximation value of K-S critical values to be $D_\alpha = \dfrac{1.63}{\sqrt{N}}$ and $D_\alpha = \dfrac{1.358}{\sqrt{N}}$ for sample size $N \geq 50$, this corresponds respectively to the significance level 1% and 5%.

(ii) Furthermore, using M.A. Stephens [17], we suggests that, the Anderson-Darling (A-S) critical values (A_α) for a known cumulative density function, with large sample size N, are given respectively 3.878 and 2.492 for 1% and 5% significance level.

(iii) If the K-S value D_α [as in (i)] for significant level $\alpha = 1\%$ or 5%, exceed the corresponding test statistic D_n, [as in Equation (3.8)] determined from the given data set, then we decide to accept the null hypotheses using this test. The same also for A-D test [the statistic test A^2 is in Equation (3.9)].

Based on the used packages in the statistical software R, we produced a code and therefore, we can easily see the fitting of VG, NIG distribution of all chosen indexes returns in Figure 1 and 2. Given in Table 1 is also a summary of the test results.

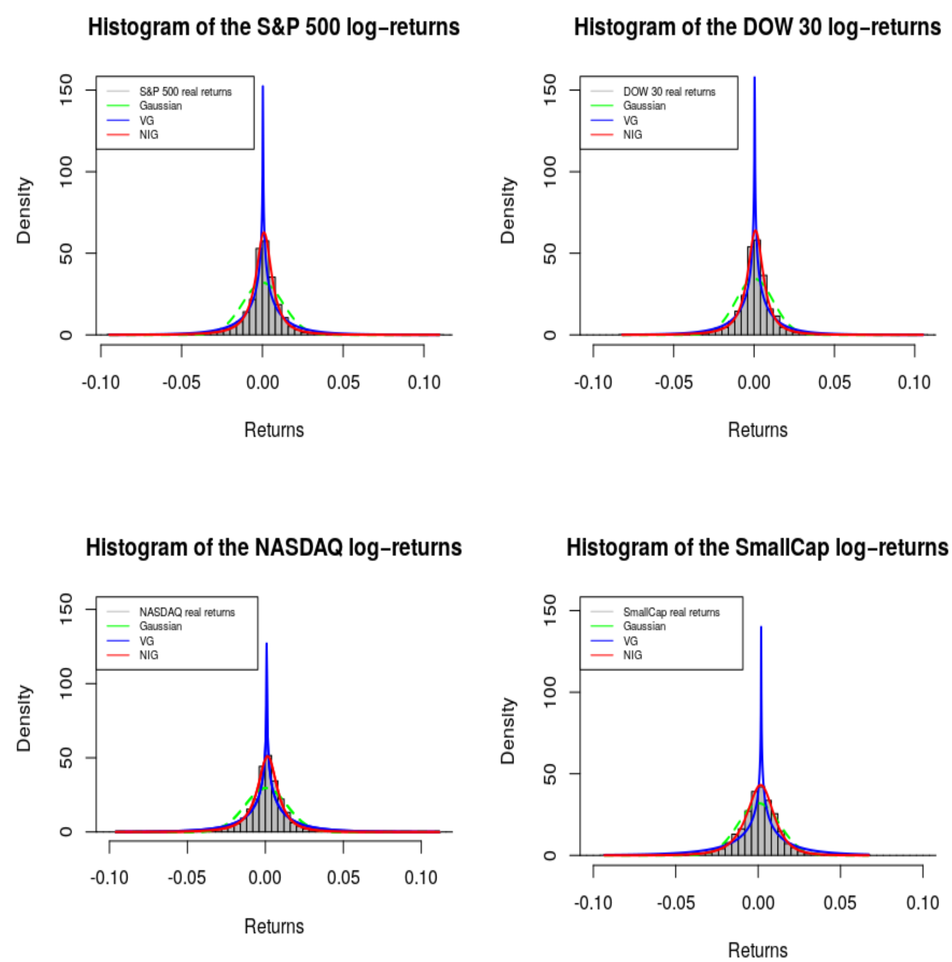

FIGURE 1. Fitting of VG, NIG distribution parameters to the S&P 500, DOW 30, NASDAQ, and SmallCap indexes returns.

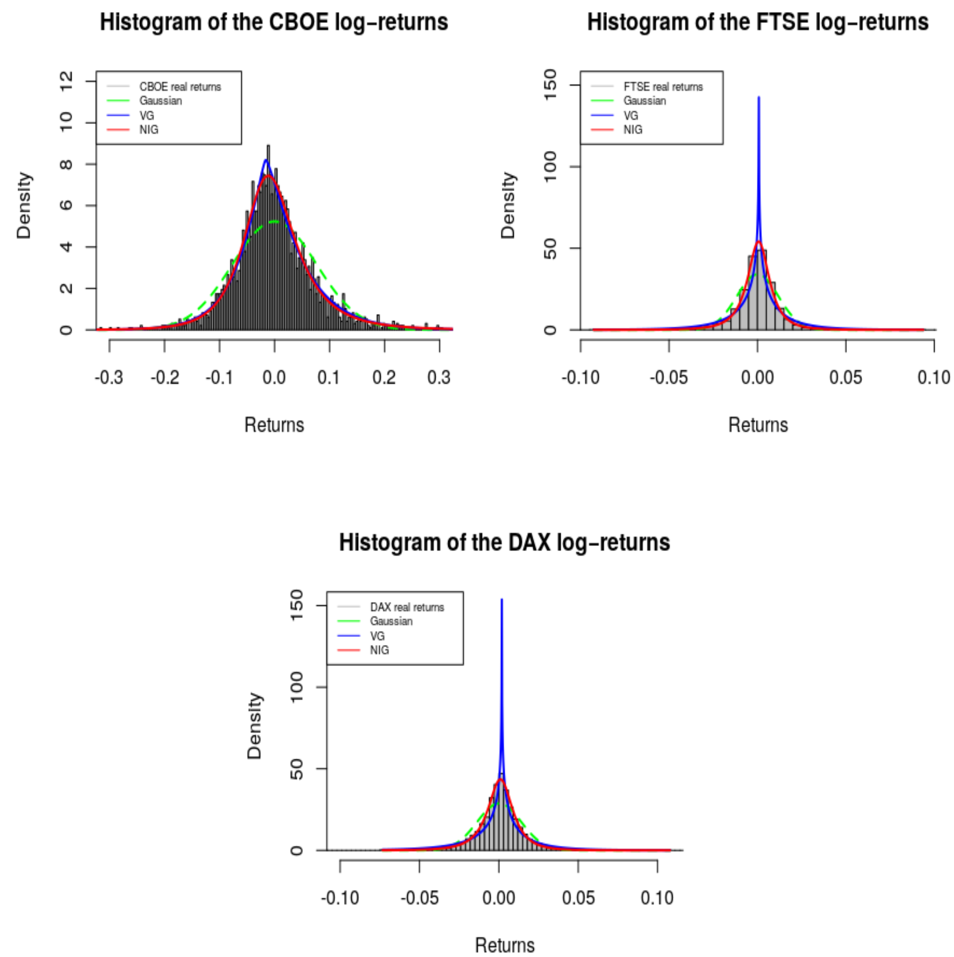

FIGURE 2. Fitting of VG, NIG distribution parameters to the remaining indexes returns.

Decision Made	K-S test		A-D test	
	VG	NIG	VG	NIG
Number of statistics which decline/fail to reject H_0	1	7	1	7
Number of statistics which reject H_0	6	0	6	0

TABLE 1. A summary of the test results for the Goodness-of-Fit.

4. Conclusion

In this article, we have shown that FATGBM models with a specific time-change, namely, the models use superpositions of OU-type processes (supOU) to construct the increments of the activity time. Moreover, we focus on the application of finite superposition in modeling financial time series and the superiority of the model in question over the Black-Scholes model when modeling log-returns, such as a risky asset returns.

To summarize our results, under the VG model sup-OU process, also based on the critical values of K-S and A-D goodness-of-fit tests, at level 1% and 5%, we found that, only one out of the seven cases of stock price market indexes log-returns decline to reject the null hypotheses that means, one of the stock price market indexes log-returns follows the VG distribution. On the other hand, under the NIG model sup-OU process, we have seen that seven out of seven cases fail to reject the null hypotheses that the stock price market indexes log-returns follows the NIG distribution.

In addition, we also observed, the probability distributions of the Lévy processes improves upon the fit of the probability distribution of the FATGBM model for developed market stock indexes.

Our results suggest that returns of empirical daily indexes from the period 2008 to 2019, fit better to the NIG distribution for all sub-samples/full samples in developed markets, instead of VG distribution which has fitted only one market index in our case. Additionally, the VG and NIG models sup-OU processes have more parameters, and then have more flexibility in fitting to the empirical distributions compared with the Gaussian model. Therefore, more precisely, based on all provided results, we conclude that NIG sup-OU process gives the best fit than VG sup-OU process, with respect to the financial data (stock price indexes of a risky asset).

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