5-1-2012

Higher order weakly over-penalized symmetric interior penalty methods

Susanne C. Brenner  
*Louisiana State University*

Luke Owens  
*Automated Trading Desk*

Li Yeng Sung  
*Louisiana State University*

Follow this and additional works at: https://digitalcommons.lsu.edu/mathematics_pubs

**Recommended Citation**

This Article is brought to you for free and open access by the Department of Mathematics at LSU Digital Commons. It has been accepted for inclusion in Faculty Publications by an authorized administrator of LSU Digital Commons. For more information, please contact ir@lsu.edu.
Higher order weakly over-penalized symmetric interior penalty methods

Susanne C. Brenner\textsuperscript{a,\*}, Luke Owens\textsuperscript{b}, Li-Yeng Sung\textsuperscript{a}

\textsuperscript{a} Department of Mathematics and Center for Computation and Technology, Louisiana State University, Baton Rouge, LA 70803, United States
\textsuperscript{b} Automated Trading Desk, 11 Ewall Street, Mount Pleasant, SC 29464, United States

\begin{abstract}
In this paper we study higher order weakly over-penalized symmetric interior penalty methods for second-order elliptic boundary value problems in two dimensions. We derive $h$--$p$ error estimates in both the energy norm and the $L_2$ norm and present numerical results that corroborate the theoretical results.
\end{abstract}

\section{Introduction}

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and $f \in L_2(\Omega)$. We consider the Poisson problem with the homogeneous Dirichlet boundary condition as our model problem:

Find $u \in H_0^1(\Omega)$ such that

$$
\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \forall v \in H_0^1(\Omega).
$$

According to the elliptic regularity theory for nonsmooth domains [1--3],

$$
\|u\|_{H^{1+w}(\Omega)} \leq C_\alpha \|f\|_{L_2(\Omega)},
$$

where $\alpha = 1$ when $\Omega$ is convex and $\alpha$ is any number strictly less than $\pi/(\text{maximum reentrant angle})$ when $\Omega$ is nonconvex. We shall refer to $\alpha$ as the index of elliptic regularity.

Let $T_h$ be a quasi-uniform triangulation of the domain $\Omega$ where $h$ is the mesh parameter. We define $V_{h,r}$ to be the discontinuous $P_r$ ($r \geq 1$) finite element space with respect to the triangulation $T_h$, i.e.,

$$
V_{h,r} = \{ v \in L_2(\Omega) : v_T = v|_T \in P_r(T) \forall T \in T_h \},
$$

where $P_r(T)$ is the space of polynomials on $T$ with total degree less than or equal to $r$. 

\* Corresponding author.
\textit{E-mail addresses:} brenner@math.lsu.edu (S.C. Brenner), lowens@atdesk.com (L. Owens), sung@math.lsu.edu (L.-Y. Sung)

0377-0427/$-$ see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2012.01.025
The weakly over-penalized symmetric interior penalty (WOPSIP) method of order $r$ is:

Find $u_h \in V_{h,r}$ such that

$$a_{h,r}(u_h, v) = \int_{\Omega} f \cdot v \, dx \quad \forall \, v \in V_{h,r},$$

(1.4)

where the bilinear form $a_{h,r}(\cdot, \cdot)$ is defined by

$$a_{h,r}(w, v) = \sum_{T \in h} \int_T \nabla w \cdot \nabla v \, dx + \eta \sum_{e \in e_h} \frac{r}{|e|^{2r-1}} \int_e \Pi_e^{r-1} \mathbb{I} \cdot \Pi_e^{r-1} [v] \, ds.$$

(1.5)

Here $e_h$ is the set of the edges in $T_h$, $|e|$ is the length of the edge $e$, $[v]$ denotes the jump of $v$, and $\eta > 0$ is a penalty parameter that satisfies

$$\eta \leq \eta_0,$$

(1.6)

where $\eta_0 > 0$ is arbitrary but fixed. The operator $\Pi_e^{r-1}$ is the orthogonal projection from $L_2(e)$ onto $P_{r-1}(e)$, the space of polynomials of degree less than or equal to $r - 1$ on $e$.

The jumps are defined in the usual way [4,5]. Let $e$ be an interior edge shared by the triangles $T_1, T_2 \in T_h$. Then we define, on $e$,

$$[v] = v_1 n_1 + v_2 n_2,$$

(1.7)

where $v_1 = v|_{T_1}$, $v_2 = v|_{T_2}$ and $n_1$ (resp. $n_2$) is the unit normal of $e$ pointing towards the outside of $T_1$ (resp. $T_2$). We define, on an edge $e$ along $\partial \Omega$,

$$[v] = (v|_{e}) n,$$

(1.8)

where $n$ is the unit normal of $e$ pointing outside $\Omega$.

Note that the concept of jumps is well-defined on the space

$$H^1(\Omega, T_h) = \{ v \in L^2(\Omega) : v_T \in H^1(T) \ \forall T \in T_h \}$$

which of course contains the space $V_{h,r}$.

The WOPSIP method of order 1 was introduced in [6,7]. A comparison of WOPSIP with other interior penalty methods [8–11] can be found in [7]. The main advantage of the WOPSIP method is that it is stable for any positive penalty parameter (no need for parameter tuning) and at the same time it has optimal errors in both the energy norm and the $L^2$ norm. There is also a simple block diagonal preconditioner (with small blocks) that can offset the ill-conditioning due to over-penalization with the result that the preconditioned system behaves like a typical discrete system for second-order problems.

Moreover, it was shown in [12] that the WOPSIP method of order 1 is intrinsically parallel. In other words there exist two orderings of the degrees of freedom (dofs), the elementwise ordering and the edgewise ordering, such that the stiffness matrix can be written as the sum of two matrices, where one is block diagonal (with $3 \times 3$ blocks) in the elementwise ordering of the dofs and the other one is block diagonal (with $1 \times 1$ and $2 \times 2$ diagonal blocks) in the edgewise ordering of the dofs.

It is the goal of this paper to extend the results of [7,12] to WOPSIP methods of higher order. The rest of the paper is organized as follows. We derive the discretization errors of the general WOPSIP method in Section 2 and construct a simple preconditioner in Section 3 for WOPSIP methods of odd order, where we also discuss the intrinsic parallelism of these methods. Numerical results are presented in Section 4 and we end with some concluding remarks in Section 5. Appendix contains a proof that the nodal values at the nonstandard nodes introduced in Section 3 uniquely determine odd order polynomials.

2. Discretization errors

Let the mesh-dependent (energy) norm $\| \cdot \|_{h,r}$ be defined by

$$\|v\|^2_{h,r} = a_{h,r}(v, v) = \sum_{T \in T_h} \| \nabla v \|^2_{L^2(T)} + \eta \sum_{e \in e_h} \frac{r}{|e|^{2r-1}} \| \Pi_e^{r-1} [v] \|^2_{L^2(e)}.$$

(2.1)

for all $v \in H^1(\Omega, T_h)$.

The following estimate [cf. [13]] is standard:

$$\| u - u_h \|_{h,r} \leq \inf_{v \in V_h} \| u - v \|_{h,r} + \sup_{w \in V_{h,r}\setminus\{0\}} \frac{a_{h,r}(u - u_h, w)}{\| w \|_{h,r}}.$$

(2.2)

Let $w \in H^1(\Omega, T_h)$ be arbitrary. It follows from integration by parts that

$$\sum_{T \in T_h} \int_T \nabla u \cdot \nabla w \, dx = \sum_{e \in e_h} \int_e \nabla u \cdot \Pi_e^{r-1} [w] \, ds + \int_{\Omega} f \cdot v \, dx.$$

(2.3)
Remark 2.1. The integration by parts behind (2.3) is standard when the solution \( u \) of (1.1) belongs to \( H^2(\Omega) \), which is the case when \( \Omega \) is convex. The justification of (2.3) when \( \Omega \) is nonconvex can be found in [7].

Observe that \( u \in H^1_0(\Omega) \) implies \( \|u\|_r = 0 \) for all \( r \in \mathbb{R}_0 \). Therefore it follows from (1.5) and (2.3) that

\[
a_{h,r}(u, w) = \int_{\Omega} f w \, dx + \sum_{e \in \mathcal{E}_h} \int_e \nabla u \cdot \|w\| \, ds \quad \forall w \in H^1(\Omega, \mathcal{T}_h),
\]

which in view of (1.4) implies that

\[
a_{h,r}(u - u_h, w) = \sum_{e \in \mathcal{E}_h} \int_e \nabla u \cdot \|w\| \, ds \quad \forall w \in \mathcal{V}_{h,r}.
\]

We begin the analysis of the right-hand side of (2.5) with the following two preparatory lemmas. Throughout this section we will use \( C \), with or without subscripts, to denote a generic positive constant that is independent of both \( h \) and \( r \).

**Lemma 2.2.** We have

\[
\sum_{e \in \mathcal{E}_h} |e|^{-1} \|v\|^2_{L^2(e)} \leq C \left( \sum_{T \in \mathcal{T}_h} \|\nabla v\|^2_{L^2(T)} + \sum_{e \in \mathcal{E}_h} |e|^{-1} \|\Pi_{e}^{-1} v\|^2_{L^2(e)} \right) \leq C \|v\|^2_{H^1},
\]

for all \( v \in H^1(\Omega, \mathcal{T}_h) \).

**Proof.** Let \( v \in H^1(\Omega, \mathcal{T}_h) \) be arbitrary. We have, by the Cauchy–Schwarz inequality,

\[
\sum_{e \in \mathcal{E}_h} |e|^{-1} \|v\|^2_{L^2(e)} \leq 2 \sum_{e \in \mathcal{E}_h} |e|^{-1} (\|v - \Pi_e v\|_{L^2(e)} + \|\Pi_e^{-1} v\|_{L^2(e)}).
\]

Let \( \bar{v} \) be the piecewise constant function that equals \( \int_T v \, dx / |T| \) on each \( T \in \mathcal{T}_h \). It follows from (1.7), (1.8), the trace theorem (with scaling) and a standard interpolation error estimate [13] that

\[
\sum_{e \in \mathcal{E}_h} |e|^{-1} \|v - \Pi_e^{-1} v\|^2_{L^2(e)} \leq \sum_{e \in \mathcal{E}_h} |e|^{-1} \|v - \bar{v}\|^2_{L^2(e)} \leq C \sum_{T \in \mathcal{T}_h} \|\nabla v\|^2_{L^2(T)}.
\]

The estimate (2.6) follows from (2.1), (2.7) and (2.8). \( \square \)

**Lemma 2.3.** For any \( v, w \in \mathcal{V}_{h,r} \), we have

\[
\sum_{e \in \mathcal{E}_h} \int_e \|\nabla v\| \cdot \|w\| \, ds \leq Ch \left( \sum_{T \in \mathcal{T}_h} \|\nabla v\|^2_{L^2(T)} \right)^{1/2} \left( \eta \sum_{e \in \mathcal{E}_h} \frac{r}{|e|^{2r+1}} \|\Pi_e^{-1} w\|^2_{L^2(e)} \right)^{1/2}.
\]

Here the mean \( \|\nabla v\| \) on an interior edge shared by the triangles \( T_1 \) and \( T_2 \) is defined to be \( (\nabla v_1 + \nabla v_2)/2 \), and on a boundary edge \( e \) we take \( \|\nabla v\| = \nabla v \).

**Proof.** Since \( \Pi_e^{-1} : L^2(e) \to P_{r-1}(e) \) is the orthogonal projection operator and \( \|\nabla v\| \in P_{r-1}(e) \), we can write

\[
\sum_{e \in \mathcal{E}_h} \int_e \|\nabla v\| \cdot \|w\| \, ds = \sum_{e \in \mathcal{E}_h} \int_e \langle \nabla v \rangle \cdot \Pi_e^{-1} \|w\| \, ds.
\]

Hence, by the Cauchy–Schwarz inequality, we have

\[
\left| \sum_{e \in \mathcal{E}_h} \int_e \|\nabla v\| \cdot \|w\| \, ds \right| \leq \left( \eta^{-1} \sum_{e \in \mathcal{E}_h} \frac{|e|^{2r+1}}{r} \|\nabla v\|^2_{L^2(e)} \right)^{1/2} \left( \eta \sum_{e \in \mathcal{E}_h} \frac{r}{|e|^{2r+1}} \|\Pi_e^{-1} w\|^2_{L^2(e)} \right)^{1/2}.
\]

Furthermore, by a standard inverse estimate [14,11], we have

\[
\sum_{e \in \mathcal{E}_h} \|\nabla v\|^2_{L^2(e)} \leq C \sum_{e \in \mathcal{E}_h} \sum_{T \in \mathcal{T}_e} \|\nabla v\|^2_{L^2(T)} \leq C \sum_{T \in \mathcal{T}_h} \|\nabla v\|^2_{L^2(T)},
\]

where \( \mathcal{T}_e \) is the set of the triangles in \( \mathcal{T}_h \) that share \( e \) as a common edge.

The estimate (2.9) follows from (1.6), (2.10) and (2.11). \( \square \)
We are now ready to establish an abstract discretization error estimate in the energy norm.

**Theorem 2.4.** We have

\[
\| u - u_h \|_{H,r} \leq C \left[ \inf_{v \in V_{h,r}} \left( \| u - v \|_{H,r}^2 + \sum_{e \in \mathcal{E}_h} |e| \| \nabla (u - v) \|_{L^2(e)}^2 \right)^{1/2} + h^r \|f\|_{L^2(\Omega)} \right].
\]  

(2.12)

**Proof.** Let \( v, w \in V_{h,r} \) be arbitrary. From (1.6), (2.5), (2.6), (2.9) and the Cauchy–Schwarz inequality, we have

\[
a_{h,r}(u - u_h, w) = \sum_{e \in \mathcal{E}_h} \int_e \|\nabla (u - v)\| \|w\| \, ds + \sum_{e \in \mathcal{E}_h} \int_e \|\nabla v\| \|w\| \, ds
\]

\[
\leq \left( \sum_{e \in \mathcal{E}_h} |e| \|\nabla (u - v)\|_{L^2(e)}^2 \right)^{1/2} \left( \sum_{e \in \mathcal{E}_h} |e|^{-1} \|w\|_{L^2(e)}^2 \right)^{1/2}
\]

\[
+ C h^r \left( \sum_{T \in \mathcal{T}_h} \|\nabla v\|_{L^2(T)}^2 \right)^{1/2} \left( \eta \sum_{e \in \mathcal{E}_h} \frac{r}{|e|^{2r+1}} \|f\|_{L^2(e)}^2 \right)^{1/2}
\]

\[
\leq C \left( \sum_{e \in \mathcal{E}_h} |e| \|\nabla (u - v)\|_{L^2(e)}^2 + h^{2r} \sum_{T \in \mathcal{T}_h} \|\nabla v\|_{L^2(T)}^2 \right)^{1/2} \|w\|_{H,r},
\]  

(2.13)

which together with (1.2) and (2.2) implies

\[
\| u - u_h \|_{H,r} \leq \| u - v \|_{H,r} + C \left( \sum_{e \in \mathcal{E}_h} |e| \|\nabla (u - v)\|_{L^2(e)}^2 + h^{2r} \sum_{T \in \mathcal{T}_h} \|\nabla (u - v)\|_{L^2(T)}^2 + h^{2r} \|u\|_{H^1(\Omega)}^2 \right)^{1/2}
\]

\[
\leq C \left( \| u - v \|_{H,r}^2 + \sum_{e \in \mathcal{E}_h} |e| \|\nabla (u - \tilde{u}_h)\|_{L^2(e)}^2 \right)^{1/2} + h^r \|f\|_{L^2(\Omega)}
\]

and hence (2.12). \( \square \)

Recall that the solution \( u \) of (1.1) belongs to \( H^s(\Omega) \) for some \( s > 3/2 \). According to the theory of the \( h-p \) version of finite elements \( [15,16,14] \), there exists \( \tilde{u}_h \in V_{h,r} \cap H^1(\Omega) \) such that

\[
|u - \tilde{u}_h|_{H^s(\Omega)} \leq C_s \left( \frac{h^{r-1}}{|r|} \right) \|u\|_{H^s(\Omega)}.
\]  

(2.14)

\[
\left( \sum_{e \in \mathcal{E}_h} |e| \|\nabla (u - \tilde{u}_h)\|_{L^2(e)}^2 \right)^{1/2} \leq C_s \left( \frac{h^{r-1}}{|r|} \right) \|u\|_{H^s(\Omega)},
\]  

(2.15)

where \( r = \min(r + 1, s) \), and the constant \( C_s \) depends on \( s \) but not on \( r \) or \( h \).

The estimates (2.12), (2.14) and (2.15) immediately imply the following concrete error estimate.

**Theorem 2.5.** Assuming the solution \( u \) of (1.1) belongs to \( H^s(\Omega) \) for \( s > 3/2 \), we have

\[
\| u - u_h \|_{H,r} \leq C_s \left( \frac{h^{r-1}}{|r|} \|u\|_{H^s(\Omega)} + h^r \|f\|_{L^2(\Omega)} \right),
\]  

(2.16)

where \( \mu = \min(r + 1, s) \) and the constant \( C_s \) depends on \( s \) but not on \( r \) or \( h \).

Finally we obtain an \( L^2(p) \) error estimate for the WOPSIP method by a duality argument.

**Theorem 2.6.** It holds that

\[
\| u - u_h \|_{L^2(\Omega)} \leq C_\alpha \left( h^{\alpha r - 1} + h^r \right) \|u - u_h\|_{H,r},
\]  

(2.17)

where \( \alpha > 1/2 \) is the index of elliptic regularity that appears in the estimate (1.2).

**Proof.** Let \( \phi \in H^1(\Omega) \) satisfy

\[
\int_\Omega \nabla \phi \cdot \nabla v \, dx = \int_\Omega (u - u_h) v \, dx \quad \forall v \in H^1(\Omega).
\]
Then we have the elliptic regularity estimate
\[ \| \phi \|_{H^{1+\alpha}(\Omega)} \leq C_{\alpha} \| u - u_{h} \|_{L_{2}(\Omega)}. \] (2.18)

According to (2.14) and (2.15) with \( \mu = s = 1 + \alpha \), there exists \( \phi_{h} \in V_{h, r} \cap H_{0}^{1}(\Omega) \) such that
\[ \| \phi - \phi_{h} \|_{H^{1}(\Omega)} \leq \frac{h^{\alpha}}{r^{\alpha}} \| \phi \|_{H^{1+\alpha}(\Omega)} \leq C_{\alpha} \frac{h^{\alpha}}{r^{\alpha-\frac{1}{2}}} \| u - u_{h} \|_{L_{2}(\Omega)}, \] (2.19)
\[ \left( \sum_{e \in \mathcal{E}_{h}} | e | \| \nabla (\phi - \phi_{h}) \|_{L_{2}(e)} \right)^{1/2} \leq C_{\alpha} \frac{h^{\alpha}}{r^{\alpha-1/2}} \| \phi \|_{H^{1+\alpha}(\Omega)} \leq C_{\alpha} \frac{h^{\alpha}}{r^{\alpha-2}} \| u - u_{h} \|_{L_{2}(\Omega)}, \] (2.20)

where we have also used the estimate (2.18).

Note that (2.11), (2.18) and (2.19) imply
\[ \sum_{e \in \mathcal{E}_{h}} \frac{| e |}{r} \| \nabla \phi_{h} \|_{L_{2}(e)} \leq C \| \phi_{h} \|_{H^{1}(\Omega)}^{2} \leq C \| \phi_{h} \|_{H^{1}(\Omega)}^{2} + \| \phi \|_{H^{1}(\Omega)}^{2} \leq C \| u - u_{h} \|_{L_{2}(\Omega)}^{2}. \] (2.21)

Applying (2.4) to \( \phi \) and (2.5), we find
\[ \| u - u_{h} \|_{L_{2}(\Omega)}^{2} = a_{h, r}(\phi, u - u_{h}) - \sum_{e \in \mathcal{E}_{h}} \int_{e} \nabla \phi \cdot \| u - u_{h} \| \, ds \]
\[ = a_{h, r}(\phi - \phi_{h}, u - u_{h}) - \sum_{e \in \mathcal{E}_{h}} \int_{e} \| \nabla (\phi - \phi_{h}) \| \cdot \| u - u_{h} \| \, ds \]
\[ - \sum_{e \in \mathcal{E}_{h}} \int_{e} \| \nabla \phi_{h} \| \cdot \| u - u_{h} \| \, ds. \] (2.22)

We now bound each of the three terms on the right-hand side of (2.22) separately.

The first one can be bounded using (2.19):
\[ a_{h, r}(\phi - \phi_{h}, u - u_{h}) \leq \| \phi - \phi_{h} \|_{H_{0}^{1}(\Omega)} \| u - u_{h} \|_{H_{0}^{1}(\Omega)} \leq C_{\alpha} \frac{h^{\alpha}}{r^{\alpha}} \| u - u_{h} \|_{L_{2}(\Omega)} \| u - u_{h} \|_{H_{0}^{1}(\Omega)}. \] (2.23)

We bound the second term using (2.6) and (2.20) and the Cauchy–Schwarz inequality:
\[ \left| \sum_{e \in \mathcal{E}_{h}} \int_{e} \| \nabla (\phi - \phi_{h}) \| \cdot \| u - u_{h} \| \, ds \right| \leq \left( \sum_{e \in \mathcal{E}_{h}} | e | \| \nabla (\phi - \phi_{h}) \|_{L_{2}(e)}^{2} \right)^{1/2} \left( \sum_{e \in \mathcal{E}_{h}} | e |^{-1} \| u - u_{h} \|_{L_{2}(e)}^{2} \right)^{1/2} \leq C_{\alpha} \frac{h^{\alpha}}{r^{\alpha-2}} \| u - u_{h} \|_{L_{2}(\Omega)} \| u - u_{h} \|_{H_{0}^{1}(\Omega)}. \] (2.24)

Finally, we use (1.6), (2.21) and the fact that \( \nabla \phi_{h} |_{e} \in P_{r-1}(e) \) for all \( e \in \mathcal{E}_{h} \) to bound the third term:
\[ \left| \sum_{e \in \mathcal{E}_{h}} \int_{e} \| \nabla \phi_{h} \| \cdot \| u - u_{h} \| \, ds \right| = \left| \sum_{e \in \mathcal{E}_{h}} \int_{e} \| \nabla \phi_{h} \| \cdot P_{r-1}^{\leq 1}(u - u_{h}) \, ds \right| \]
\[ \leq \left( \eta^{-1} \sum_{e \in \mathcal{E}_{h}} | e |^{2r+1} \| \nabla \phi_{h} \|_{L_{2}(e)}^{2} \right)^{1/2} \left( \eta \sum_{e \in \mathcal{E}_{h}} | e |^{2r+1} \| P_{r-1}^{\leq 1}(u - u_{h}) \|_{L_{2}(e)}^{2} \right)^{1/2} \leq \mathcal{C} \| u - u_{h} \|_{L_{2}(\Omega)} \| u - u_{h} \|_{H_{0}^{1}(\Omega)}. \] (2.25)

Combining (2.22)–(2.25) we have the estimate (2.17). \( \square \)

**Remark 2.7.** Note that the error estimates in Theorems 2.4–2.6 are derived under the assumption (1.6). If we keep track of the dependence of the penalty parameter \( \eta \) as it approaches 0, then the factor \( \eta^{-1/2} \) will appear on the right-hand sides of (2.12), (2.16) and (2.17). This adverse effect of a small \( \eta \) is observed in the numerical results (cf. Fig. 4.1).

### 3. The Preconditioner

In this section we construct a simple preconditioner for odd order WOPSIP methods that reduces the condition number of the discrete problem from \( \Theta(h^{-2r-1}) \) to \( \Theta(h^{-2}) \). We will use \( C \) (with or without subscripts) to denote a positive constant that is independent of \( h \), but which can depend on \( r \) and the minimum angle of \( T_{h} \).
Let \( A_{h,r} : V_{h,r} \rightarrow V_{h,r}^\prime \) be defined by
\[
\langle A_{h,r} w, v \rangle = a_{h,r}(w, v) \quad \forall v, w \in V_{h,r},
\]
where \( \langle \cdot, \cdot \rangle \) is the canonical bilinear form on \( V_{h,r} \times V_{h,r} \). In terms of \( A_{h,r} \), the discrete problem can be written as
\[
A_{h,r} u_{h,r} = \phi_{h,r},
\]
where \( \phi_{h,r} \in V_{h,r}^\prime \) is defined by
\[
\langle \phi_{h,r}, v \rangle = \int_\Omega f v \, dx \quad \forall v \in V_{h,r}.
\]

Let the operator \( B_{h,r} : V_{h,r} \rightarrow V_{h,r}^\prime \) be defined by
\[
\langle B_{h,r} w, v \rangle = \sum_{t \in \mathcal{T}_h} \sum_{p \in \mathcal{N}_T} w(p) v(p) + \eta \sum_{e \in \mathcal{E}_h} \frac{r}{e|e|^{1/2}} \int_e \Pi_{e}^{r-1} \| w \| \cdot \Pi_{e}^{r-1} \| v \| \, ds \quad \forall v, w \in V_{h,r},
\]
where \( \mathcal{N}_T \) is the set of nodal points in the triangle \( T \) (see Fig. 3.1). These nodal points consist of \( r \) Gauss nodes along each edge of \( T \) and \( (r-1)(r-2)/2 \) nodes interior to \( T \). These interior nodes are chosen so that polynomials of degree less than or equal to \( r-3 \) are uniquely determined by their values at these nodes. A proof that a polynomial in \( P_r(T) \) is uniquely determined by its values at these nonstandard nodes is given in the Appendix.

The following lemma establishes that the condition number of the preconditioned system \( B_{h,r}^{-1} A_{h,r} \) is \( O(h^{-2}) \). Thus \( B_{h,r}^{-1} A_{h,r} \) behaves like a second-order differential operator.

**Theorem 3.1.** All of the eigenvalues of \( B_{h,r}^{-1} A_{h,r} \) are positive, and the following estimate holds:
\[
\kappa(B_{h,r}^{-1} A_{h,r}) = \frac{\lambda_{\text{max}}(B_{h,r}^{-1} A_{h,r})}{\lambda_{\text{min}}(B_{h,r}^{-1} A_{h,r})} \leq C_r h^{-2},
\]
where the positive constant \( C_r \) depends only on \( r \) and the minimum angle of \( \mathcal{T}_h \).

**Proof.** Since the operator \( B_{h,r}^{-1} A_{h,r} \) is symmetric positive definite with respect to the inner product \( \langle B_{h,r}^{-1}, \cdot \rangle \) on \( V_{h,r} \), all of the eigenvalues of \( B_{h,r}^{-1} A_{h,r} \) are positive, and it follows from the Raleigh quotient formula [17] that
\[
\lambda_{\text{max}}(B_{h,r}^{-1} A_{h,r}) = \max_{v \in V_{h,r} \setminus \{0\}} \frac{\langle A_{h,r} v, v \rangle}{\langle B_{h,r} v, v \rangle},
\]
\[
\lambda_{\text{min}}(B_{h,r}^{-1} A_{h,r}) = \min_{v \in V_{h,r} \setminus \{0\}} \frac{\langle A_{h,r} v, v \rangle}{\langle B_{h,r} v, v \rangle}.
\]

By a simple scaling argument we find
\[
\| \nabla v \|^2_{L^2(T)} \leq C_z \sum_{p \in \mathcal{N}_T} v^2(p) \quad \forall v \in V_{h,r}
\]
for some constant \( C_z \geq 1 \), which yields
\[
\langle A_{h,r} v, v \rangle \leq C_z \langle B_{h,r} v, v \rangle \quad \forall v \in V_{h,r},
\]
and so by (3.5)
\[
\lambda_{\text{max}}(B_{h,r}^{-1} A_{h,r}) \leq C_z.
\]

In the other direction we first combine Lemma 2.2 and the Poincaré–Friedrichs inequality (cf. [18])
\[
\| v \|^2_{L^2(\Omega)} \leq C_z \left( \sum_{T \in \mathcal{T}_h} \| \nabla v \|^2_{L^2(T)} + \sum_{e \in \mathcal{E}_h} \frac{1}{|e|} \| v \|^2_{L^2(\partial e)} \right) \quad \forall v \in H^1(\Omega, \mathcal{T}_h)
\]
to obtain the estimate

$$
\|v\|_{L^2(\Omega)}^2 \leq C_s \left( \sum_{T \in \mathcal{T}_h} \|\nabla v\|_{L^2(T)}^2 + \sum_{e \in \partial h} |e|^{-1} \|I_{e}^{r-1} v\|_{L^2(e)}^2 \right) \quad \forall v \in H^1(\Omega, \mathcal{T}_h).
$$

(3.7)

Then using the facts that

$$
h \approx |e| \quad \forall e \in \partial h \quad \text{and} \quad \|v\|_{L^2(\Omega)}^2 \approx h^2 \sum_{T \in \mathcal{T}_h, p \in N_T} v^2(p) \quad \forall v \in V_{h,r},
$$

we find, by (1.6) and (3.7),

$$
h^2 \langle B_{h,r} v, v \rangle \leq C_1 \left( \|v\|_{L^2(\Omega)}^2 + \eta \sum_{e \in \partial h} \frac{r}{|e|^{2r-1}} \|I_{e}^{r-1} v\|_{L^2(e)}^2 \right)
$$

$$
\leq C_1 \left( C_s \left[ \sum_{T \in \mathcal{T}_h} \|\nabla v\|_{L^2(T)}^2 + \sum_{e \in \partial h} |e|^{-1} \|I_{e}^{r-1} v\|_{L^2(e)}^2 \right] + \eta \sum_{e \in \partial h} \frac{r}{|e|^{2r-1}} \|I_{e}^{r-1} v\|_{L^2(e)}^2 \right)
$$

$$
\leq C_s \langle A_{h,r} v, v \rangle \quad \forall v \in V_{h,r},
$$

which by (3.6) yields

$$
\lambda_{\text{min}}(B_{h,r}^{-1} A_{h,r}) \geq h^2/C_s. \quad \square
$$

Remark 3.2. Theorem 3.1 also holds for WOPSIP methods of arbitrary order if we take $\mathcal{M}^T$ in (3.3) to be the set of the standard nodes for the Lagrange finite element. However the matrix representing $B_{h,r}$ with respect to the standard nodes does not have a nice block diagonal structure.

Let $B_{h,r}$ be the matrix representing the operator $B_{h,r}$ with respect to the nodal basis of $V_{h,r}$ and the dual basis of $V_{h,r}'$, i.e.,

$$
\mathbf{w}^T B_{h,r} \mathbf{v} = \langle B_{h,r} w, v \rangle \quad \forall w, v \in V_{h,r},
$$

(3.8)

where $\mathbf{w}$ (resp. $\mathbf{v}$) is the coordinate vector for $w$ (resp. $v$) that stores the values of $w$ (resp. $v$) at the nodes. In view of (3.3), we can write

$$
B_{h,r} = I_{h,r} + (\eta r)J_{h,r},
$$

(3.9)
where \( \text{Id}_{h,r} \) is the identity matrix and

\[
\mathbf{w^T J}_{h,r} \mathbf{v} = \sum_{e \in \mathcal{E}_h} \frac{1}{|e|^{2r+1}} \int_e \Pi_e^{-1} \| \mathbf{w} \| \cdot \Pi_e^{-1} \| \mathbf{v} \| \, ds \quad \forall \mathbf{v}, \, \mathbf{w} \in \mathbf{V}_{h,r}.
\] (3.10)

Let \( e \in \mathcal{E}_h, x_1, \ldots, x_r \) be the Gauss nodes on \( e \), and \( \omega_1 > 0, \ldots, \omega_r > 0 \) be the Gauss weights associated with the Gauss nodes in the unit interval. Then for any polynomial \( p \) of degree less than or equal to \( r \) and any polynomial \( q \) of degree less than or equal to \( r-1 \), we have (cf. [19])

\[
\sum_{i=1}^r \omega_i p(x_i) q(x_i) = \frac{1}{|e|} \int_e pq \, ds = \frac{1}{|e|} \int_e (\Pi_e^{-1} p) q \, ds = \sum_{i=1}^r \omega_i (\Pi_e^{-1} p)(x_i) q(x_i).
\]

Therefore \( \Pi_e^{-1} p \) is the polynomial of degree less than or equal to \( r-1 \) determined by

\[
(\Pi_e^{-1} p)(x_i) = p(x_i) \quad \text{for} \ 1 \leq i \leq r.
\] (3.11)

It follows from (1.7) and (3.11) that, for any interior edge \( e \) shared by the triangles \( T_1 \) and \( T_2 \),

\[
\frac{1}{|e|^{2r+1}} \int_e \Pi_e^{-1} \| \mathbf{w} \| \cdot \Pi_e^{-1} \| \mathbf{v} \| \, ds = \frac{1}{|e|^{2r}} \sum_{i=1}^r \omega_i (\Pi_e^{-1} \| \mathbf{w} \|)(x_i)(\Pi_e^{-1} \| \mathbf{v} \|)(x_i)
\]

\[
= \frac{1}{|e|^{2r}} \sum_{i=1}^r \omega_i \left[ w_1 (x_i) v_1 (x_i) + w_2 (x_i) v_2 (x_i)
\right.
\]

\[
- w_1 (x_i) v_2 (x_i) - w_2 (x_i) v_1 (x_i)\] (3.12)

where \( w_j = \mathbf{w}|_{T_j} \) and \( v_j = \mathbf{v}|_{T_j} \) for \( j = 1, 2 \).

Similarly, for a boundary edge \( e \), it follows from (1.8) and (3.11) that

\[
\frac{1}{|e|^{2r+1}} \int_e \Pi_e^{-1} \| \mathbf{w} \| \cdot \Pi_e^{-1} \| \mathbf{v} \| \, ds = \frac{1}{|e|^{2r}} \sum_{i=1}^r \omega_i w(x_i) v(x_i).
\] (3.13)

In view of (3.9), (3.10), (3.12) and (3.13), the matrix \( \mathbf{J}_{h,r} \) (and hence also the matrix \( \mathbf{B}_{h,r} \)) is block diagonal with \( 1 \times 1 \) and \( 2 \times 2 \) blocks provided that the dofs are ordered so that the two dofs associated with the same Gauss node on an interior edge are always consecutive. The \( 2 \times 2 \) blocks are associated with the dofs at the Gauss nodes of the interior edges and the \( 1 \times 1 \) blocks are associated with all the other dofs. We shall refer to this ordering of the dofs as an edgewise ordering. With respect to such an ordering the evaluation of \( \mathbf{B}_{h,r}^{-1} \) becomes trivial.

**Remark 3.3.** We can also order the dofs so that the ones associated with the nodes on the same triangle are always consecutive and we will refer to this as an elementwise ordering. Let \( \mathbf{D}_{h,r} \) be the matrix defined by

\[
\mathbf{w^T D}_{h,r} \mathbf{v} = \sum_{r \in \mathcal{R}_h} \int_r \nabla \mathbf{w} \cdot \nabla \mathbf{v} \, dx \quad \forall \mathbf{v}, \, \mathbf{w} \in \mathbf{V}_{h,r},
\]

where \( \mathbf{w} \) (resp. \( \mathbf{v} \)) is the coordinate vector for \( \mathbf{w} \) (resp. \( \mathbf{v} \)) in an elementwise ordering. Then \( \mathbf{D}_{h,r} \) is a block diagonal matrix with \( m \times m \) blocks, where \( m = (r+1)(r+2)/2 \) is the dimension of \( P_r \).

**Remark 3.4.** Let \( \mathbf{P}_{h,r} \) be the permutation matrix that transforms the coordinate vector of a finite element function in an edgewise ordering of the dofs to the coordinate vector in an elementwise ordering. Then the stiffness matrix \( \mathbf{A}_{h,r} \) that represents the operator \( \mathbf{A}_{h,r} \) in the edgewise ordering can be written as

\[
\mathbf{A}_{h,r} = \mathbf{P}_{h,r}^T \mathbf{D}_{h,r} \mathbf{P}_{h,r} + (\eta r) \mathbf{J}_{h,r}.
\]

Since both \( \mathbf{D}_{h,r} \) and \( \mathbf{J}_{h,r} \) are block diagonal matrices with small blocks whose sizes are independent of \( h \), the evaluation of \( \mathbf{A}_{h,r} \mathbf{v} \) can be easily parallelized. Thus we can say that the odd order WOPSIP methods are intrinsically parallel.

**Remark 3.5.** The multiplication of a vector by the permutation matrix \( \mathbf{P}_{h,r} \) (or \( \mathbf{P}_{h,r}^T \)) is equivalent to a relabeling of the coordinates of the vector. Such an operation is not intrinsically parallel, but it also does not involve floating point arithmetic.

### 4. Numerical results

First we solve the model problem (1.1) on the L-shaped domain

\[
\Omega = (-1, 1) \times (-1, 1) \setminus [0, 1] \times [0, 1]
\]
using the MATLAB backslash solve. In order to avoid round-off errors due to the over-penalization, the matrix $B_{h,r}^{-1}$ is computed by using

\[ B_{h,r}^{-1}A_{h,r} = (B_{h,r}^{-1}G_{h,r}) + (\eta r)(B_{h,r}^{-1}J_{h,r}), \]

with the exact solution

\[ u(x_1, x_2) = x_1(1 - x_1^2)x_2(1 - x_2^2). \]

The results for $r = 1, 3, 5$ are presented in Tables 4.1–4.3, where the relative errors in the piecewise $H^1$ semi-norm, the $L_2$ norm, and the energy norm $\| \cdot \|_{h,r}$ are defined by

\[
\alpha_k^1 = \left( \frac{\sum_{T \in T_k} \| \nabla (u - u_k^1) \|^2_{L_2(T)}}{\| \nabla u \|^2_{L_2(\Omega)}} \right)^{1/2}, \quad \beta_k^1 = \frac{\| u - u_k^1 \|^2_{L_2(\Omega)}}{\| u \|^2_{L_2(\Omega)}}, \quad \gamma_k^1 = \frac{\| u - u_k^1 \|_{h,r}}{\| u \|_{h,r}}.
\]

We use uniform grids in our computation where the length of a horizontal/vertical edge in grid $T_k$ is $h_k = 2^{1-k}$, and we set the penalty parameter $\eta$ to be 1.

The results in Tables 4.1–4.3 clearly demonstrate the estimate (2.16), where $\mu = r + 1$. The order of convergence for the $L_2$ error is better than the one predicted by (2.17). This is likely due to the superconvergence effect that comes from the smoothness of $u$ and the uniformity of the computational grids.

In our computation we use the standard nodes for Lagrange finite elements, and we solve the discrete system (cf. (3.2))

\[ A_{h,r}u_{h,r} = \phi_{h,r}, \]

by solving the preconditioned system (cf. (3.3))

\[ B_{h,r}^{-1}A_{h,r}u_{h,r} = B_{h,r}^{-1}\phi_{h,r} \]

using the MATLAB backslash solve. In order to avoid round-off errors due to the over-penalization, the matrix $B_{h,r}^{-1}A_{h,r}$ is computed by using

\[ B_{h,r}^{-1}A_{h,r} = (B_{h,r}^{-1}G_{h,r}) + (\eta r)(B_{h,r}^{-1}J_{h,r}). \]
Table 4.4
A comparison of the energy norm errors between elements of order 1, 3, and 5 for the singular solution ($\eta = 1$ except $\eta = 0$).

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\gamma^1$</th>
<th>$\gamma^3$</th>
<th>$\gamma^5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.619E+00</td>
<td>1.785E+00</td>
<td>1.660E+00</td>
</tr>
<tr>
<td>2</td>
<td>1.122E+00</td>
<td>2.640E+01</td>
<td>1.031E+01</td>
</tr>
<tr>
<td>3</td>
<td>5.222E-01</td>
<td>4.211E+02</td>
<td>2.420E+02</td>
</tr>
<tr>
<td>4</td>
<td>2.536E-01</td>
<td>1.817E-02</td>
<td>1.516E-02</td>
</tr>
<tr>
<td>5</td>
<td>1.259E-01</td>
<td>1.119E-02</td>
<td>9.555E-03</td>
</tr>
<tr>
<td>6</td>
<td>6.360E-02</td>
<td>7.039E-03</td>
<td>6.029E-03</td>
</tr>
<tr>
<td>7</td>
<td>3.266E-02</td>
<td>4.425E-03</td>
<td>3.791E-03</td>
</tr>
<tr>
<td>8</td>
<td>1.707E-02</td>
<td>3.612E-03</td>
<td>–</td>
</tr>
<tr>
<td>9</td>
<td>9.116E-03</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

Table 4.5
A comparison of the $L^2$ norm errors between elements of order 1, 3, and 5 for the singular solution ($\eta = 1$ except $\eta = 0$).

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\beta^1$</th>
<th>$\beta^3$</th>
<th>$\beta^5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.475E+00</td>
<td>3.337E+00</td>
<td>3.098E+00</td>
</tr>
<tr>
<td>2</td>
<td>1.199E+00</td>
<td>7.348E-02</td>
<td>1.073E-02</td>
</tr>
<tr>
<td>3</td>
<td>2.911E-01</td>
<td>2.903E-03</td>
<td>1.114E-03</td>
</tr>
<tr>
<td>4</td>
<td>7.468E-02</td>
<td>8.252E-04</td>
<td>3.537E-04</td>
</tr>
<tr>
<td>5</td>
<td>2.028E-02</td>
<td>2.741E-04</td>
<td>1.121E-04</td>
</tr>
<tr>
<td>6</td>
<td>5.885E-03</td>
<td>9.344E-05</td>
<td>3.554E-05</td>
</tr>
<tr>
<td>7</td>
<td>1.835E-03</td>
<td>3.266E-05</td>
<td>1.130E-05</td>
</tr>
<tr>
<td>8</td>
<td>6.127E-04</td>
<td>1.172E-05</td>
<td>–</td>
</tr>
<tr>
<td>9</td>
<td>2.164E-04</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

where $J_{h,r}$ is the matrix representing the jump term (cf. (3.10)) and $G_{h,r}$ is the matrix representing the Dirichlet form, i.e.,

$$w^T G_{h,r} v = \sum_{T \in \mathcal{H}_h} \int_T \nabla w \cdot \nabla v \, dx \quad \forall v, w \in V_{h,r}.$$  

Moreover taking a smaller $\eta$ can also reduce the round-off errors.

Next we examine the performance of the WOPSIP methods when the exact solution of (1.1) on the L-shaped domain is the singular solution

$$u(x, y) = (1 - x^2)(1 - y^2)r^{2/3} \sin \left( \frac{2}{3} \left( \theta - \frac{\pi}{2} \right) \right),$$

where $(r, \theta)$ are the polar coordinates at the origin. We observe from the results in Tables 4.4 and 4.5 that both the energy error and the $L^2$ error decrease as $h$ decreases or $r$ increases. This is consistent with (2.16) and (2.17).

As mentioned above, taking $\eta$ small can reduce the round-off error. The drawback is that the constant in (2.16) is inversely proportional to $\eta^2$ (see Remark 2.7). A graph of the energy norm error (for the smooth solution) versus $k$ with $r = 3$ and varying penalty parameters is given in Fig. 4.1. The graph demonstrates that the rate of convergence in the energy norm is independent of $\eta$, but the constant in (2.12) is affected by the penalty parameter.

In our final set of numerical experiments we analyze the condition number $\kappa(h_k, r)$ of the preconditioned system. The values of $\kappa(h_k, r) \times h_k^2$ for various $k$ and $r = 1, 3, 5$ are given in Table 4.6. The table shows that $\kappa(h_k, r) \approx C_i h_k^{-2}$, which is consistent with the estimate (3.4). Also note that $C_1 < C_3 < C_5$. 

Table 4.6
$\kappa(h_k, r) \times h_k^2$ for $r = 1, 3, 5$ ($\eta = 1$).

<table>
<thead>
<tr>
<th>$k$</th>
<th>$r = 1$</th>
<th>$r = 3$</th>
<th>$r = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.8715</td>
<td>26.0387</td>
<td>276.7830</td>
</tr>
<tr>
<td>2</td>
<td>2.2262</td>
<td>24.9683</td>
<td>261.1012</td>
</tr>
<tr>
<td>3</td>
<td>1.9108</td>
<td>24.6917</td>
<td>257.9308</td>
</tr>
<tr>
<td>4</td>
<td>1.8442</td>
<td>24.6468</td>
<td>256.4157</td>
</tr>
<tr>
<td>5</td>
<td>1.8288</td>
<td>24.5553</td>
<td>256.3327</td>
</tr>
<tr>
<td>6</td>
<td>1.8183</td>
<td>24.5521</td>
<td>256.2193</td>
</tr>
<tr>
<td>7</td>
<td>1.8179</td>
<td>24.5487</td>
<td>–</td>
</tr>
<tr>
<td>8</td>
<td>1.8179</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

where $J_{h,r}$ is the matrix representing the jump term (cf. (3.10)) and $G_{h,r}$ is the matrix representing the Dirichlet form, i.e.,

$$w^T G_{h,r} v = \sum_{T \in \mathcal{H}_h} \int_T \nabla w \cdot \nabla v \, dx \quad \forall v, w \in V_{h,r}.$$  

Moreover taking a smaller $\eta$ can also reduce the round-off errors.

Next we examine the performance of the WOPSIP methods when the exact solution of (1.1) on the L-shaped domain is the singular solution

$$u(x, y) = (1 - x^2)(1 - y^2)r^{2/3} \sin \left( \frac{2}{3} \left( \theta - \frac{\pi}{2} \right) \right),$$

where $(r, \theta)$ are the polar coordinates at the origin. We observe from the results in Tables 4.4 and 4.5 that both the energy error and the $L^2$ error decrease as $h$ decreases or $r$ increases. This is consistent with (2.16) and (2.17).

As mentioned above, taking $\eta$ small can reduce the round-off error. The drawback is that the constant in (2.16) is inversely proportional to $\eta^2$ (see Remark 2.7). A graph of the energy norm error (for the smooth solution) versus $k$ with $r = 3$ and varying penalty parameters is given in Fig. 4.1. The graph demonstrates that the rate of convergence in the energy norm is independent of $\eta$, but the constant in (2.12) is affected by the penalty parameter.

In our final set of numerical experiments we analyze the condition number $\kappa(h_k, r)$ of the preconditioned system. The values of $\kappa(h_k, r) \times h_k^2$ for various $k$ and $r = 1, 3, 5$ are given in Table 4.6. The table shows that $\kappa(h_k, r) \approx C_i h_k^{-2}$, which is consistent with the estimate (3.4). Also note that $C_1 < C_3 < C_5$. 

Table 4.6
$\kappa(h_k, r) \times h_k^2$ for $r = 1, 3, 5$ ($\eta = 1$).
5. Concluding remarks

We have extended the convergence results for the weakly over-penalized symmetric interior penalty (WOPSIP) method to higher order elements. More precisely, we have established $h-p$ error estimates in the energy norm and the $L_2$ norm for arbitrary order elements. In addition we were able to construct a simple block diagonal preconditioner for the resulting discrete system for the odd order elements. This preconditioner reduces the condition number from $h^{-2r-1}$ to $h^{-2}$. We have also provided numerical results that verify both our theoretical convergence estimates and our condition number estimates.

For simplicity we have focused on a simple model problem, but the results can be extended to problems with variable coefficients and nonconforming meshes as in [7].

We note that an $h$-adaptive algorithm using the WOPSIP method with the $P_1$ finite element has been analyzed in [20]. The results in this paper provide the foundation for exploring $h-p$ adaptivity within the WOPSIP framework.

Acknowledgment

This work was supported in part by the National Science Foundation under Grant No. DMS-10-16332.

Appendix. Nonstandard DOFs for odd order polynomial spaces

Let $r$ be an odd positive integer. In this appendix we will show that any polynomial of degree less than or equal to $r$ on a triangle $T$ is uniquely determined by its values at the $r$ Gauss nodes on each edge of $T$ and at $(r-1)(r-2)/2$ interior nodes that uniquely determined $P_{r-3}(T)$.

Observe first that by affine invariance we may assume that $T$ is an equilateral triangle. Moreover, since $3r + [(r-1)(r-2)/2] = (r+1)(r+2)/2$ is the dimension of $P_r(T)$, it suffices to show that given any node $x$ we can find a polynomial $p$ in $P_r(T)$ such that $p(x) = 1$ and $p$ vanishes at all the other nodes.

Case 1. $x$ is one of the interior nodes.

Since the interior nodes uniquely determine $P_{r-3}(T)$, there exists a polynomial $p$ of the form $q\lambda_1\lambda_2\lambda_3$ with the desired property, where $q \in P_{r-3}$ and $\lambda_1, \lambda_2, \lambda_3$ are the barycentric coordinates on $T$.

Case 2. $x$ is a midpoint of an edge of $T$.

Since $T$ is an equilateral triangle, there exist $(r-1)/2$ circles that pass through all the Gauss nodes on the edges except the three midpoints. Furthermore there is a line that passes through the two midpoints different from $x$. Therefore there exists a polynomial $\tilde{p} \in P_r$ such that $\tilde{p}(x) = 1$ and $\tilde{p}$ vanishes at all the other boundary nodes. We can then apply the result in Case 1 to find a polynomial $q \in P_r$ such that $p = \tilde{p} + q$ has the desired property.

Case 3. $x$ is a Gauss node on an edge $e$ of $T$ and $x$ is not the midpoint.

In this case there are $r-2$ lines that pass through the vertex of $T$ opposite $e$ and the $r-2$ Gauss nodes on $e$ different from $x$ and the midpoint. On the other hand the two edges different from $e$ pass through all the Gauss nodes on those edges. Therefore there exists a polynomial $\tilde{p} \in P_r$ such that $\tilde{p}(x) = 1$ and $\tilde{p}$ vanishes at all the other boundary nodes except the midpoint of $e$. We can then apply the results from Cases 1 and 2 to find a polynomial $q \in P_r$ such that $p = \tilde{p} + q$ has the desired property.

The geometric constructions associated with Cases 2 and 3 are illustrated in Fig. A.1 for $r = 3$.

References


