2013

Application of Helmholtz/Hodge Decomposition to Finite Element Methods for Two-Dimensional Maxwell's Equations

Zhe Nan
Louisiana State University and Agricultural and Mechanical College

Follow this and additional works at: https://digitalcommons.lsu.edu/gradschool_dissertations

Part of the Applied Mathematics Commons

Recommended Citation
https://digitalcommons.lsu.edu/gradschool_dissertations/147

This Dissertation is brought to you for free and open access by the Graduate School at LSU Digital Commons. It has been accepted for inclusion in LSU Doctoral Dissertations by an authorized graduate school editor of LSU Digital Commons. For more information, please contact gradetd@lsu.edu.
APPLICATION OF HELMHOLTZ/HODGE DECOMPOSITION TO FINITE ELEMENT METHODS FOR TWO-DIMENSIONAL MAXWELL'S EQUATIONS

A Dissertation
Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
in
The Department of Mathematics

by
Zhe Nan
B.S. in Math., University of Science and Technology of China, 2005
M.S., Louisiana State University, 2009
May 2013
Acknowledgements

This dissertation would not be possible without several contributions.

First of all, it is a pleasure to thank my supervisors, Dr. Susanne C. Brenner and Dr. Li-Yeng Sung, for the successful completion of my dissertation. I am so honored to join their research group, and worked with them. Their untiring efforts, encouragement, guidance and support helped me greatly in my graduate studies as well as the successful completion of this dissertation. I will benefit from this valuable experience for my lifelong endeavors in academia.

I would like to thank all of my committee members: Dr. William A. Adkins, Dr. Hongchao Zhang and Dr. Guoping Zhang for their valuable suggestions and help in my academic research. My sincere thanks go to all the people who have contributed to my graduate studies in Louisiana State University.

I am greatly appreciative of all the people who were involved with my research, including other members from our research group: Jintao Cui, Shiyuan Gu and Yi Zhang for their generous help and discussions throughout these years. Many great thanks also go to all the support staff from the Department of Mathematics for their generous help.

I would also like to thank the National Science Foundation for the summer support under Grant No. DMS-07-13835.

Last but certainly not least, I cannot imagine going through the past few years in graduate study and completing this dissertation without endless support from my family and friends. I want to thank my loving wife Wanli Xu for her unconditional love, support and caring for me wholeheartedly throughout this special time. Special thanks also go to my dear friends Luo Xiao and Jianhua Chen.
# Table of Contents

Acknowledgements ................................................................. ii

List of Tables ................................................................. v

List of Figures ................................................................. vi

Abstract ................................................................................... vii

1 Introduction ........................................................................... 1
  1.1 Maxwell’s Equations and the Corresponding Interface Problems...... 1
     1.1.1 Maxwell’s Equations in Integral Form ................................. 1
     1.1.2 Maxwell’s Equations in Differential Form ......................... 2
     1.1.3 Constitutive Relations .................................................... 3
     1.1.4 Boundary Conditions and Interface Conditions ................. 4
     1.1.5 Time Harmonic Maxwell’s Equations ............................... 5
     1.1.6 Two-dimensional Maxwell’s Equations ............................ 6
     1.1.7 Weak Formulation of Maxwell’s Interface Problems ........... 7
  1.2 History of Finite Element Methods for Maxwell’s Equations and
     the Corresponding Interface Problems ..................................... 9

2 Fundamentals ......................................................................... 11
  2.1 Sobolev Spaces ............................................................... 11
  2.2 Regularity Results .......................................................... 14
     2.2.1 Regularity of Elliptic Problems ..................................... 14
     2.2.2 Regularity of Elliptic Interface Problems ......................... 16
  2.3 Extraction Formulas ........................................................ 21
     2.3.1 Extraction Formulas for Poisson Problems ....................... 22
     2.3.2 Extraction Formulas for Elliptic Interface Problems ........... 29
  2.4 Finite Element Methods ..................................................... 33
  2.5 Multigrid Methods ........................................................... 38

3 Vector Function Spaces and Helmholtz/Hodge Decompositions......... 42
  3.1 Definitions and Properties of the Vector Function Spaces $H(div; \Omega)$
     and $H(curl; \Omega)$ ............................................................ 42
  3.2 Two-dimensional Helmholtz/Hodge Decompositions .................. 45
     3.2.1 Characterization of Two-dimensional Divergence Free Vec-
         tor Fields and Curl Free Vector Fields ............................. 45
     3.2.2 Helmholtz/Hodge Decompositions ................................ 46

4 Maxwell Equations in Homogeneous Media .................................. 53
4.1 Introduction ................................................................. 53
4.2 Equation for $\xi = \nabla \times \mathbf{u}$ and $\phi$ ................................. 53
4.3 A $P_1$ Finite Element Method ............................................ 59
4.4 Convergence Analysis ..................................................... 61
4.5 Numerical Results .......................................................... 69

5 Multigrid Methods for Maxwell Equations in Heterogeneous Media .......... 74
5.1 Introduction ................................................................. 74
5.2 Equation for $\xi$ and $\phi$ ................................................ 76
5.3 Regularity, Stress Intensity Factors and Extraction Formulas ............. 78
5.4 The Algorithm .............................................................. 82
  5.4.1 The $k^{th}$ Level Iteration .......................................... 83
  5.4.2 The Full Multigrid Algorithms ...................................... 84
5.5 Convergence Analysis ..................................................... 86
5.6 Numerical Results .......................................................... 99

Appendix A: Proof for Theorem 2.4.7 ........................................ 109
Appendix B: Proof for (2.2.17) .............................................. 112
References ............................................................................. 116
Vita ...................................................................................... 121
List of Tables

4.1 Results for (4.1.1) on the L-shaped domain with exact solution given by (4.5.1) ............................................................. 70

4.2 Results for (4.1.1) on the doubly connected domain with exact solution given by (4.5.3) ............................................................. 72

4.3 Results for (4.1.1) on the doubly connected domain with right-hand side given by (4.5.4) ............................................................. 73

5.1 Contraction numbers for the symmetric W-cycle algorithm on the L-shaped domain with m smoothing steps for the case $\epsilon = [1/350, 1, 1/350]$ ............................................................. 100

5.2 Results of the full multigrid method with exaction formulas for Example 5.6.1 ............................................................. 103

5.3 Results of the full multigrid method without extraction formula for Example 5.6.1 ............................................................. 103

5.4 Pointwise errors for Example 5.6.1 using the full multigrid algorithms with extraction formulas, where $e^1_k, \eta^1_k, e^2_k, \eta^2_k, e^\infty_k$ and $\eta^\infty_k$ are defined by (5.6.3)-(5.6.8) ............................................................. 104

5.5 Pointwise errors for Example 5.6.1 using the full multigrid algorithms without extraction formulas, where $e^1_k, \eta^1_k, e^2_k, \eta^2_k, e^\infty_k$ and $\eta^\infty_k$ are defined by (5.6.3)-(5.6.8) ............................................................. 104

5.6 Results of the full multigrid method with extraction formulas for Example 5.6.2 ............................................................. 105

5.7 Results of the full multigrid method without extraction formulas for Example 5.6.2 ............................................................. 105

5.8 Results of the full multigrid method with extraction formulas for Example 5.6.3 ............................................................. 107

5.9 Results of the full multigrid method without extraction formulas for Example 5.6.3 ............................................................. 108
List of Figures

2.1 Examples of the domain $\Omega$ ................................................................. 17
2.2 Polar coordinates for the Sturm-Liouville problems ................................. 18
2.3 The domain $\Omega$ and its subdomains ......................................................... 20
4.1 A uniform mesh on the L-shaped domain .................................................... 69
4.2 A uniform mesh on the doubly connected domain ................................. 71
5.1 The domain $\Omega$ and its subdomains ......................................................... 100
5.2 The domain $\Omega$ and its subdomains ......................................................... 106
Abstract

In this work we apply the two-dimensional Helmholtz/Hodge decomposition to develop new finite element schemes for two-dimensional Maxwell’s equations. We begin with the introduction of Maxwell’s equations and a brief survey of finite element methods for Maxwell’s equations. Then we review the related fundamentals in Chapter 2. In Chapter 3, we discuss the related vector function spaces and the Helmholtz/Hodge decomposition which are used in Chapter 4 and 5. The new results in this dissertation are presented in Chapter 4 and Chapter 5. In Chapter 4, we propose a new numerical approach for two-dimensional Maxwell’s equations that is based on the Helmholtz/Hodge decomposition for divergence-free vector fields. In this approach an approximate solution for Maxwell’s equations can be obtained by solving standard second order scalar elliptic boundary value problems. This new approach is illustrated by a $P_1$ finite element method. In Chapter 5, we further extend the new approach described in Chapter 4 to the interface problem for Maxwell’s equations. We use the extraction formulas and multigrid method to overcome the low regularity of the solution for the Maxwell interface problem. The theoretical results obtained in this dissertation are confirmed by numerical experiments.
Chapter 1
Introduction

1.1 Maxwell’s Equations and the Corresponding Interface Problems

In this section we introduce several formulations of Maxwell equations, their boundary conditions and interface conditions. It is mainly based on the books [60, 7, 49, 40].

1.1.1 Maxwell’s Equations in Integral Form

Consider an open surface $S$ bounded by a closed contour $C$. The first two Maxwell’s equations are given in the following equations

$$\oint_C E(x, t) \cdot dl = -\frac{d}{dt} \iint_S B(x, t) \cdot dS$$  \hspace{1cm} (1.1.1)

and

$$\oint_C H(x, t) \cdot dl = \frac{d}{dt} \iint_S D(x, t) \cdot dS + \iint_S J \cdot dS,$$ \hspace{1cm} (1.1.2)

where

$E$ = electric field intensity,

$D$ = electric displacement,

$B$ = magnetic induction,

$H$ = magnetic field intensity,

and

$J$ = electric current density.
Next, consider a volume $V$ enclosed by a surface $S$. Two other Maxwell’s equations are given by the following equations

$$\int\int_{S} D(x, t) \cdot dS = \int\int\int_{V} \rho(x, t) dV \quad (1.1.3)$$

and

$$\int\int_{S} B(x, t) \cdot dS = 0, \quad (1.1.4)$$

where

$$\rho = \text{electric charge density in } V.$$

Remark 1.1.1. The integral form of Maxwell’s equations (1.1.1)-(1.1.4) is valid everywhere. We will use them to derive interface conditions.

1.1.2 Maxwell’s Equations in Differential Form

By applying Stokes’ theorem and Gauss’ theorem from calculus (cf. [57]), we can convert Maxwell’s equations in integral form into Maxwell’s equations in differential form.

By applying Stokes’ theorem to (1.1.1) and (1.1.2), we get

$$\int\int_{S} \nabla \times E(x, t) \cdot dS = -\frac{d}{dt} \int\int_{S} B \cdot dS \quad (1.1.5)$$

and

$$\int\int_{S} \nabla \times H(x, t) \cdot dS = \frac{d}{dt} \int\int_{S} E(x, t) \cdot dS + \int\int_{S} J \cdot dS. \quad (1.1.6)$$

Because of the arbitrariness of the surface $S$, equations (1.1.5) and (1.1.6) lead to the following differential equations

$$\nabla \times E(x, t) = -\frac{\partial}{\partial t} B(x, t) \quad (1.1.7)$$

and
\begin{equation}
\n\nabla \times \mathbf{H}(x,t) = \frac{\partial}{\partial t} \mathbf{D}(x,t) + \mathbf{J}.
\end{equation}

(1.1.8)

By applying Gauss’ theorem to equations (1.1.3) and (1.1.4), we obtain

\begin{equation}
\n\n\nabla \cdot \mathbf{D} = \rho
\end{equation}

(1.1.9)

and

\begin{equation}
\n\n\nabla \cdot \mathbf{B} = 0.
\end{equation}

(1.1.10)

By taking the divergence of (1.1.8), applying (1.1.9) and the vector identity (cf. [57])

\begin{equation}
\n\n\nabla \cdot (\nabla \times \mathbf{v}) = 0 \quad \text{for a smooth vector field } \mathbf{v},
\end{equation}

we obtain the conservation law

\begin{equation}
\n\n\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.
\end{equation}

(1.1.11)

1.1.3 Constitutive Relations

A medium affects electromagnetic fields through three phenomena: electric polarization, magnetic polarization, and electric conduction. Electric polarization leads to the constitutive relation for the electric field. In most cases it can be expressed as

\begin{equation}
\n\n\mathbf{D} = \epsilon \mathbf{E},
\end{equation}

(1.1.12)

where \( \mathbf{D} \) is called the electric flux density and \( \epsilon \) is called the permittivity of the dielectric medium. Magnetic polarization leads to the constitutive relation for the magnetic field. In most materials it can be expressed as

\begin{equation}
\n\n\mathbf{B} = \mu \mathbf{H},
\end{equation}

(1.1.13)
where $\mathbf{H}$ is called the magnetic field intensity and $\mu$ is called the permeability of the material. The electric conduction leads to the final constitutive relation

$$J_c = \sigma \mathbf{E},$$  \hspace{1cm} (1.1.14)

where $\sigma$ is called the conductivity and $J_c$ is called the conduction current, which can be regarded as a part of the total electric current.

1.1.4 Boundary Conditions and Interface Conditions

It is sufficient to consider interface conditions since the boundary of the domain is a special type of interface. Without loss of generality, we only consider an interface between two different mediums. Moreover, we assume a surface charge distribution over the interface. The surface charge density is defined as the amount of charge over a unit area on the surface. Applying (1.1.3) in a small cylinder with one of its faces in medium 1 and the other in medium 2 and letting its thickness $\Delta t \to 0$, we obtain

$$\mathbf{D}_1 \cdot \mathbf{n}_1 + \mathbf{D}_2 \cdot \mathbf{n}_2 = \rho_{e,s},$$  \hspace{1cm} (1.1.15)

where $\rho_{e,s}$ denotes the surface electric charge density, $\mathbf{n}_i \ (i = 1, 2)$ denotes the normal direction of the boundary of medium $i$, $\mathbf{D}_i \ (i = 1, 2)$ denotes the electric displacement in medium $i$.

Using a similar strategy, we obtain

$$\mathbf{B}_1 \cdot \mathbf{n}_1 + \mathbf{B}_2 \cdot \mathbf{n}_2 = 0,$$  \hspace{1cm} (1.1.16)

$$\mathbf{H}_1 \times \mathbf{n}_1 + \mathbf{H}_2 \times \mathbf{n}_2 = \mathbf{J}_s,$$  \hspace{1cm} (1.1.17)

and

$$\mathbf{E}_1 \times \mathbf{n}_1 + \mathbf{E}_2 \times \mathbf{n}_2 = 0,$$  \hspace{1cm} (1.1.18)
where \( J_s \) denotes the surface current density, \( E_i \) and \( H_i \) \((i = 1, 2)\) denote the electric and magnetic field intensity in medium \( i \). Here we define a notation to simplify the description of the interface conditions.

**Notation 1.1.2.** For a vector field \( u \) and \( \Gamma \) the interface between two mediums, let us denote

\[
[u \cdot n]_\Gamma := u_1 \cdot n_1 + u_2 \cdot n_2 \quad \text{or} \quad [u \cdot n] := u_1 \cdot n_1 + u_2 \cdot n_2 \quad \text{on the interface} \ \Gamma
\]

and

\[
[u \times n]_\Gamma := u_1 \times n_1 + u_2 \times n_2 \quad \text{or} \quad [u \times n] := u_1 \times n_1 + u_2 \times n_2 \quad \text{on the interface} \ \Gamma,
\]

where \( n_i \) denotes the normal direction of the interface with respect to medium \( i \).

Using this new notation, the above interface conditions can be written as

\[
[D \cdot n]_\Gamma = \rho_{e,s}, \quad (1.1.19)
\]
\[
[B \cdot n]_\Gamma = 0, \quad (1.1.20)
\]
\[
[E \times n]_\Gamma = 0, \quad (1.1.21)
\]
\[
[H \times n]_\Gamma = J_s. \quad (1.1.22)
\]

### 1.1.5 Time Harmonic Maxwell’s Equations

Now we derive time-harmonic Maxwell’s equations from the differential formulation of Maxwell’s equations \((1.1.7)-(1.1.10)\).

Assume that functions and vector fields in Maxwell’s equations have the form

\[
E = \Re \left( \vec{E}(x) \exp(-i\omega t) \right), \quad (1.1.23)
\]
\[
D = \Re \left( \vec{D}(x) \exp(-i\omega t) \right), \quad (1.1.24)
\]
\[
H = \Re \left( \vec{H}(x) \exp(-i\omega t) \right), \quad (1.1.25)
\]
\[
B = \Re \left( \vec{B}(x) \exp(-i\omega t) \right), \quad (1.1.26)
\]
\[ J = \Re(\hat{J}(x)\exp(-i\omega t)), \quad (1.1.27) \]
\[ \rho = \Re(\hat{\rho}(x)\exp(-i\omega t)), \quad (1.1.28) \]

where \( i = \sqrt{-1} \) and \( \Re(\cdot) \) denotes the real part of the expression in parentheses.

After substituting the relations (1.1.23)- (1.1.28) into (1.1.7)-(1.1.10), we get the time-harmonic Maxwell system:

\[ \nabla \times \hat{H} = -i\omega \hat{D} + \hat{J}, \quad (1.1.29) \]
\[ \nabla \times \hat{E} = i\omega \hat{B}, \quad (1.1.30) \]
\[ \nabla \cdot \hat{D} = \hat{\rho}, \quad (1.1.31) \]
\[ \nabla \cdot \hat{B} = 0. \quad (1.1.32) \]

Combining the constitutive relations (1.1.12)-(1.1.13), we can eliminate \( \hat{D}, \hat{B} \) and \( \hat{H} \) from (1.1.29)-(1.1.32) and obtain the following equation:

\[ \nabla \times (\mu^{-1}\nabla \times u) - \omega^2 \epsilon u = f, \quad (1.1.33) \]

where \( u = \hat{E} \) and \( f = i\omega \hat{J} \).

Correspondingly, the interface conditions (1.1.19)- (1.1.22) imply the following interface conditions for the solution \( u \) of (1.1.33):

\[ [(\epsilon \hat{u}) \cdot n]_\Gamma = \hat{\rho}_{e,s}, \quad (1.1.34) \]
\[ [\langle \nabla \times \hat{u} \rangle \cdot n]_\Gamma = 0, \quad (1.1.35) \]
\[ [\hat{u} \times n]_\Gamma = 0, \quad (1.1.36) \]
\[ [\langle \mu^{-1}\nabla \times \hat{u} \rangle \times n]_\Gamma = \hat{J}_s, \quad (1.1.37) \]

where \( \rho_{e,s} = \Re(\hat{\rho}_{e,s}\exp(-i\omega t)) \) and \( J_s = \Re(\hat{J}_s\exp(-i\omega t)) \).

### 1.1.6 Two-dimensional Maxwell’s Equations

In many cases, Maxwell’s equations can be reduced to a two dimensional problem.

For example, in the case the region where an electromagnetic field exists is a
cylindrical body, the cross section of the cylinder is orthogonal to z-axis, and the electric field is orthogonal to z-axis and independent of the z variable, we could write electric field and magnetic field as

\[ \mathbf{E} = (E_x(x, y), E_y(x, y), 0), \]

and correspondingly,

\[ \mathbf{H} = (0, 0, H_z(x, y)). \]

In this situation, we will get a two-dimensional version of the equation (1.1.33):

\[ \nabla \times (\mu^{-1} \nabla \times \mathbf{u}) - \omega^2 \epsilon \mathbf{u} = \mathbf{f}, \tag{1.1.38} \]

where \( \mathbf{u} \) is a two-dimensional vector field.

### 1.1.7 Weak Formulation of Maxwell’s Interface Problems

In this section we derive the weak formulation for certain Maxwell’s interface problems. Let \( \Omega \subset \mathbb{R}^2 \) be a polygonal domain and \( \Omega_j, 1 \leq j \leq J \) be polygonal subdomains of \( \Omega \) that form a partition of \( \Omega \) (See Subsection 2.2.2 for more details). \( \Gamma \) denotes the interface between \( \Omega_j \)'s. Suppose that the vector function \( \mathbf{u} \) satisfying

\[ \mathbf{u}|_{\Omega_j} \in [C^2(\Omega_j)]^2 \cap [C^1(\overline{\Omega_j})]^2 \quad 1 \leq j \leq J \]

is the classical solution of Maxwell’s interface problem:

\[ \nabla \times (\mu_j^{-1} \nabla \times \mathbf{u}_j) - k^2 \epsilon_j \mathbf{u}_j = \mathbf{f}_j \quad \text{in the domain } \Omega_j, 1 \leq j \leq J, \tag{1.1.39a} \]

\[ \mathbf{n} \times \mathbf{u} = 0 \quad \text{on the boundary } \partial \Omega, \tag{1.1.39b} \]

\[ [\mathbf{n} \times \mathbf{u}] = 0 \quad \text{on the interface } \Gamma, \tag{1.1.39c} \]

\[ \nabla \cdot (\epsilon \mathbf{u}) = 0 \quad \text{in } \Omega_j, 1 \leq j \leq J, \tag{1.1.39d} \]

\[ [\mathbf{n} \cdot \epsilon \mathbf{u}] = 0 \quad \text{on the interface } \Gamma, \tag{1.1.39e} \]
\[ [\mu^{-1}\nabla \times \mathbf{u}] = 0 \] on the interface \( \Gamma \), \hspace{1cm} (1.1.39f)

where \( f_j \) is smooth in the closed subdomain \( \bar{\Omega}_j \), \( 1 \leq j \leq J \). We attempt to find a proper weak formulation of the above interface problem (1.1.39). Let \( \mathbf{v} \) be an arbitrary vector function in \( \mathbb{R}^2 \) satisfying

\[ \mathbf{v}|_{\Omega_j} \in \left[ C^1(\Omega_j) \right]^2 \cap \left[ C^0(\bar{\Omega}_j) \right]^2 \hspace{1cm} 1 \leq j \leq J. \]

Taking the dot product on both sides of (1.1.39a) by the vector function \( \mathbf{v}|_{\Omega_j} \), integrating over the subdomain \( \Omega_j \), we have

\[ \int_{\Omega_j} \nabla \times (\mu_j^{-1}\nabla \times \mathbf{u}_j) \cdot \mathbf{v} dx - \int_{\Omega_j} k^2\epsilon_j \mathbf{u}_j \cdot \mathbf{v} dx = \int_{\Omega_j} f_j \cdot \mathbf{v} dx. \hspace{1cm} (1.1.40) \]

Hence,

\[ \sum_{j=1}^{J} \left( \int_{\Omega_j} \nabla \times (\mu_j^{-1}\nabla \times \mathbf{u}_j) \cdot \mathbf{v} dx - \int_{\Omega_j} k^2\epsilon_j \mathbf{u}_j \cdot \mathbf{v} dx \right) = \sum_{j=1}^{J} \int_{\Omega_j} f_j \cdot \mathbf{v} dx. \hspace{1cm} (1.1.41) \]

Using integration by parts, we have that

\[ \int_{\Omega_j} \nabla \times (\mu_j^{-1}\nabla \times \mathbf{u}_j) \cdot \mathbf{v} dx = \int_{\Omega_j} \mu_j^{-1}(\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) dx + \int_{\Omega_j} \mu_j^{-1}(\nabla \times \mathbf{v}_j) \cdot (\mathbf{v} \times \mathbf{n}) ds. \hspace{1cm} (1.1.42) \]

Therefore, (1.1.41) and (1.1.42) lead to

\[ \sum_{j=1}^{J} \int_{\Omega_j} (\mu_j^{-1}\nabla \times \mathbf{u}_j \cdot \mathbf{v} - k^2\epsilon_j \mathbf{u}_j \cdot \mathbf{v}) dx + \sum_{j=1}^{J} \int_{\partial \Omega_j} \mu_j^{-1}(\nabla \times \mathbf{u}_j) \cdot (\mathbf{v} \times \mathbf{n}) ds = \sum_{j=1}^{J} \int_{\Omega_j} f_j \cdot \mathbf{v} dx, \hspace{1cm} (1.1.43) \]

or equivalently,

\[ \sum_{j=1}^{J} \int_{\Omega_j} (\mu_j^{-1}\nabla \times \mathbf{u}_j \cdot \mathbf{v} - k^2\epsilon_j \mathbf{u}_j \cdot \mathbf{v}) dx + \sum_{j=1}^{J} \left( \int_{\partial \Omega} \mu_j^{-1}(\nabla \times \mathbf{u}) \cdot (\mathbf{v} \times \mathbf{n}) ds \right) + \int_{\Gamma} \mu_j^{-1}(\nabla \times \mathbf{u}) \cdot [\mathbf{v} \times \mathbf{n}] ds = \sum_{j=1}^{J} \int_{\Omega_j} f_j \cdot \mathbf{v} dx. \hspace{1cm} (1.1.44) \]
By the homogeneous boundary and interface conditions (1.1.39b), (1.1.39c) and (1.1.39f), (1.1.44) implies that
\[ \sum_{j=1}^{J} \int_{\Omega_j} (\mu_j^{-1} \nabla \times u_j \cdot v - k_j^2 \epsilon_j u_j \cdot v) dx = \sum_{j=1}^{J} \int_{\Omega_j} f_j \cdot v dx, \]  
(1.1.45)
or,
\[ (\mu^{-1} \nabla \times u, \nabla \times v) - k^2 (\epsilon u, v) = (f, v), \]  
(1.1.46)
where \((\cdot, \cdot)\) denotes the summation of the \(L_2\) inner product on \(\Omega_j, 1 \leq j \leq J\). A natural choice of the variational space for the weak formulation (1.1.46) will be \(H_0(curl; \Omega) \cap H(div^0; \Omega; \epsilon)\).

So we are considering the weak formulation of Maxwell’s interface problem:
Find \(u \in H_0(curl; \Omega) \cap H(div^0; \Omega; \epsilon)\) such that
\[ (\mu^{-1} \nabla \times u, \nabla \times v) - k^2 (\epsilon u, v) = (f, v) \quad \forall v \in H_0(curl; \Omega) \cap H(div^0; \Omega; \epsilon). \]  
(1.1.47)

Remark 1.1.3. The formal definition of the space \(H_0(curl; \Omega) \cap H(div^0; \Omega; \epsilon)\) will be introduced in Chapter 3. For the moment, we just use it.

Remark 1.1.4. Here we consider Maxwell interface problem with homogeneous interface conditions, which means we assume there are no interface charge and interface current. The case where that there are interface charge and interface current can be reduced to the problem we consider here.

1.2 History of Finite Element Methods for Maxwell’s Equations and the Corresponding Interface Problems

The natural choice of variational space for the variational problem of Maxwell’s equations is \(H_0(curl; \Omega) \cap H(div^0; \Omega; \epsilon)\). However, every conforming finite element subspace in \(H_0(curl; \Omega) \cap H(div^0; \Omega; \epsilon)\) must be in \([H^1(\Omega)]^d\) \((d = 2\ or\ 3)\), since it consists of continuous piecewise polynomials, and the intersection of \([H^1(\Omega)]^d\) and
\[ H_0(\text{curl}; \Omega) \cap H(\text{div}^0; \Omega; \epsilon) \] is a proper closed subspace of \( H_0(\text{curl}; \Omega) \cap H(\text{div}^0; \Omega; \epsilon) \) when the domain \( \Omega \) has re-entrant corners [49]. Therefore, the resulting finite element space is not dense in \( H_0(\text{curl}; \Omega) \cap H(\text{div}^0; \Omega; \epsilon) \) as the mesh size goes to zero and hence the finite element solution may not converge to the exact \( H_0(\text{curl}; \Omega) \cap H(\text{div}^0; \Omega; \epsilon) \) solution [49]. Instead, some people use the larger space \( H_0(\text{curl}; \Omega) \) as the variational space and solve the curl-curl variational problem for Maxwell’s equations by \( H(\text{curl}) \)-conforming edge elements [39, 49, 51, 52]. More recently, successful algorithms have been discovered for this curl-curl problem that either solve a curl-curl and grad-div problem using nodal \( H^1 \) vector finite elements complemented by singular vector fields [31], or solve its regularized version using standard nodal \( H^1 \) vector finite elements [27]. Alternatively one can use nonconforming methods [15, 17, 18, 13, 19]. However, in the works we mentioned above, the dielectric and magnetic permeability were assumed to be constant. In this dissertation we will consider the case where the dielectric and magnetic permeability are piecewise constant. The main challenge in this situation is that the regularity of the solution could be much worse [28], and hence most of the existing methods fail. In order to develop a successful algorithm for Maxwell’s equations in heterogeneous media, a new algorithm for the homogeneous media case was first proposed in our work [14], which is based on Hodge decomposition. Following this new approach, an adaptive \( P_1 \) finite element method have been carried out in [16]. In this dissertation, we further extend this new approach to the heterogeneous media by exploiting extraction formulas and full multigrid methods [12, 21, 22, 23].
Chapter 2
Fundamentals

2.1 Sobolev Spaces

In this section we review some basic facts about Sobolev spaces. They are based on the references [1, 33, 34, 36]. First, let us define the notations for derivatives and related function spaces. Assume \( u : \Omega \to \mathbb{R}, \ x \in \Omega, \) where \( \Omega \) is a bounded open set in \( \mathbb{R}^d, \ d = 2 \) or 3.

**Notation 2.1.1.** A vector of the form \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \), where each component is a nonnegative integer, is called a multi-index of order \( |\alpha| = \sum_{i=1}^{n} \alpha_i. \) Then we denote

\[
D^\alpha u(x) := \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.
\]

**Notation 2.1.2.** The following function spaces are denoted by \( \mathcal{D}(\Omega), \mathcal{D}(\overline{\Omega}), \mathcal{C}^0(\overline{\Omega}), \)
respectively:

\[
\mathcal{D}(\Omega) := \{ v : v \text{ is smooth with compact support in the domain } \Omega \},
\]

\[
\mathcal{D}(\overline{\Omega}) := \{ v|_{\overline{\Omega}} : v|_{\Omega} \text{ is the restriction to } \Omega \text{ of a smooth function } v \text{ with compact support in } \mathbb{R}^d \},
\]

\[
\mathcal{C}^0(\overline{\Omega}) := \{ v : v \text{ is continuous in the domain } \overline{\Omega} \}.
\]

We also need the concept of weak derivatives.

**Definition 2.1.3.** Suppose \( u \) is locally integrable in \( \Omega \), and \( \alpha \) is a multi-index. If there exists a locally integrable function \( v \) such that

\[
\int_{\Omega} \phi v dx = (-1)^{|\alpha|} \int_{\Omega} u D^\alpha \phi dx \quad \forall \phi \in \mathcal{D}(\Omega),
\]

then \( v \) is called the \( \alpha^{th} \) weak derivative of \( u \), written as \( D^\alpha u = v. \)
Now let us define the Sobolev spaces.

**Definition 2.1.4.** Let $k$ be a nonnegative integer. The Sobolev space $H^k(\Omega)$ is defined as follows:

$$H^k(\Omega) = \{ u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega), \text{ for all } |\alpha| \leq k \}.$$  

We define the subspace $H^k_0(\Omega)$ of $H^k(\Omega)$ by

$$H^k_0(\Omega) = \text{the closure of } D(\Omega) \text{ in } H^k(\Omega).$$

**Definition 2.1.5.** Let $s = k + \sigma$, where $k$ is a nonnegative integer, and $0 < \sigma < 1$.

The fractional order Sobolev space $H^s(\Omega)$ is defined as follows:

$$H^s(\Omega) = \{ u \in L^2(\Omega) : u \in H^k(\Omega) \text{ and } \int_{\Omega \times \Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x - y|^{d + 2s}} dxdy < \infty, \forall |\alpha| = k \}.$$  

Next, we will discuss some properties of the Sobolev spaces $H^k(\Omega)$.

**Theorem 2.1.6.** For any nonnegative integer $k$, $H^k(\Omega)$ is a Hilbert space with the inner product

$$(u, v)_{H^k} = \sum_{|\alpha| \leq k} \int_{\Omega} D^\alpha u \cdot D^\alpha v dx \quad \forall u, v \in H^k(\Omega)$$

and the induced norm

$$\|u\|_{H^k(\Omega)}^2 = \left\{ \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^2(\Omega)}^2 \right\}^{1/2}.$$

For any positive number $s = k + \sigma$, where $k$ is a nonnegative integer and $0 < \sigma < 1$, $H^s(\Omega)$ is a Hilbert space with the inner product

$$(u, v)_{H^s(\Omega)} = (u, v)_{H^k(\Omega)} + \sum_{|\alpha| = k} \int_{\Omega \times \Omega} \frac{(D^\alpha u(x) - D^\alpha u(y))(D^\alpha v(x) - D^\alpha v(y))}{|x - y|^{d + 2s}} dxdy.$$
To better understand $H^k(\Omega)$, we need a density property of $H^k(\Omega)$. It turns out that the density property and many other properties of Sobolev spaces depend on the regularity of the domain $\Omega$. So let us first define a geometrical condition on the domain which will be sufficient for our subsequent purposes whenever the regularity of the boundary of the domain is needed.

**Definition 2.1.7.** Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with the boundary $\Gamma$. Then we say that the domain $\Omega$ has a Lipschitz continuous boundary if, for any point $x^0 \in \Gamma$, there exist $r > 0$ and a Lipschitz continuous function $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$, up to relabeling and reorienting the coordinates axes if necessary, such that

$$\Omega \cap B(x^0, r) = \{x \in B(x^0, r) : x_d > \gamma(x_1, x_2, \cdots, x_{d-1})\},$$

where $B(x^0, r) = \{x \in \mathbb{R}^d : \|x - x^0\| < r\}$ and $x = (x_1, x_2, \cdots, x_{d-1}, x_d)$. Similarly, we say that the domain $\Omega$ has a $C^1$ continuous boundary if, for any point $x^0 \in \Gamma$, there exist $r > 0$ and a $C^1$ continuous function $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$\Omega \cap B(x^0, r) = \{x \in B(x^0, r) : x_d > \gamma(x_1, x_2, \cdots, x_{d-1})\}.$$

**Theorem 2.1.8.** (Approximation by Smooth Functions on $\mathbb{R}^d$) Suppose that $\Omega$ has a Lipschitz continuous boundary, then $\mathcal{D}(\bar{\Omega})$ is dense in $H^k(\Omega)$.

To keep things simple, we will only state special cases of the Sobolev embedding theorems which will be needed in later chapters.

**Theorem 2.1.9.** (Embedding Theorem) Suppose that the domain $\Omega \subset \mathbb{R}^2$ has a Lipschitz continuous boundary. Then $H^1(\Omega)$ is compactly embedded in $L_q(\Omega)$ for $q \geq 1$, and $H^2(\Omega)$ is embedded in $C^{0,\eta}(\overline{\Omega})$ for $0 < \eta < 1$, where $C^{0,\eta}(\overline{\Omega})$ are the Hölder spaces defined by

$$C^{0,\eta}(\overline{\Omega}) = \{u \in C^0(\overline{\Omega}) : \sup_{\substack{x \neq y, \\ x, y \in \Omega}} \frac{|u(x) - u(y)|}{|x - y|^{\eta}} < \infty\} \quad (2.1.1)$$
and their corresponding norms are defined by
\[ \|u\|_{C^{0,\eta}(\bar{\Omega})} = \max_{x \in \Omega} |u(x)| + \sup_{x \neq y, x, y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\eta}. \]  

(2.1.2)

**Theorem 2.1.10.** (Trace Theorem) Suppose that \( \Omega \subset \mathbb{R}^2 \) has a \( C^1 \) continuous boundary \( \Gamma \). Then there exists a linear operator

\[ T : H^1(\Omega) \to L^2(\Gamma) \]

such that

\[ Tu = u|_\Gamma \quad \forall u \in D(\bar{\Omega}). \]

Moreover,

\[ \|Tu\|_{L^2(\Gamma)} \leq C\|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega), \]

where the constant \( C \) depends on \( \Omega \).

**Remark 2.1.11.** The trace theorem can be extended to polygonal domains. For details, see the reference [36].

### 2.2 Regularity Results

#### 2.2.1 Regularity of Elliptic Problems

In this subsection, we consider the regularity of elliptic problems with homogeneous Dirichlet boundary condition and Neumann boundary condition in nonconvex domains. The main results below are from the references [44, 36, 29, 50].

Suppose that \( \Omega \) is a polygonal domain in \( \mathbb{R}^2 \). Let \( c_1, c_2, \ldots, c_{N_\Omega} \) be the corners of \( \Omega \), \( \omega_1, \omega_2, \ldots, \omega_{N_\Omega} \) be the interior angles around those corners, and \( \omega = \max\{\omega_1, \omega_2, \ldots, \omega_{N_\Omega}\} \). For \( f \in L^2(\Omega) \), consider the Dirichlet problem:

Find \( u \in H^1_0(\Omega) \) such that

\[ -\Delta u = f \quad \text{in the domain } \Omega, \]  

(2.2.1a)
The regularity of the problem (2.2.1) is stated in the following theorem (cf. [29, 36]).

**Theorem 2.2.1.** (Regularity of the Dirichlet Problem of the Poisson Equation)

Suppose that $\Omega$ is nonconvex, i.e., $\omega > \pi$. Then the solution $u$ of (2.2.1) can be decomposed into a singular part $u_S$ and a regular part $u_R$, or equivalently, $u = u_S + u_R$, where $u_R \in H^2(\Omega)$. Moreover, there exist constants $\kappa_l$, for $\omega_l > \pi$, such that

$$u_S = \sum_{\omega_l > \pi} \kappa_l s_l,$$

and

$$s_l = r^{\frac{\omega_l}{\omega_l - \pi}} \sin \left(\frac{\pi}{\omega_l} \theta\right) \varrho_l(r)$$

is a function defined with respect to the polar coordinates $(r, \theta)$ around the corner $c_l$ and $\varrho_l(r)$ is a cut-off function which equals 1 near the corner and 0 away from the corner. We also have the elliptic regularity estimate

$$\|u_R\|_{H^2(\Omega)} + \sum_{\omega_l > \pi} |\kappa_l| \leq C\|f\|_{L^2(\Omega)}.$$

Under the same assumption for the domain $\Omega$ and $f$ as in the Dirichlet problem, we consider the Neumann problem:

Find $u \in H^1(\Omega)$ such that

$$-\Delta u = f \quad \text{in the domain } \Omega,$$  \hspace{1cm} (2.2.4a)

$$\frac{\partial u}{\partial n} = 0 \quad \text{on the boundary } \partial \Omega.$$  \hspace{1cm} (2.2.4b)

The regularity of the problem (2.2.4) is stated in the following theorem (cf. [29, 36]).

**Theorem 2.2.2.** (Regularity of the Neumann Problem of the Poisson Equation)

Suppose that $\Omega$ is nonconvex, i.e., $\omega > \pi$. Then the solution $u$ of (2.2.1) can be
decomposed into a singular part $u_S$ and a regular part $u_R$, or equivalently, $u = u_S + u_R$, where $u_R \in H^2(\Omega)$. Moreover, there exist constants $\kappa_l$, for $\omega_l \geq \pi$, such that

$$u_S = \sum_{\omega_l > \pi} \kappa_l s_l, \quad (2.2.5)$$

where

$$s_l = r \frac{\pi}{\omega_l} \cos\left(\frac{\pi}{\omega_l} \theta l\right) \rho_l(r)$$

is a function defined with respect to the polar coordinates $(r, \theta)$ around the corner $c_l$ and $\rho_l(r)$ is a cut-off function which equals 1 near the corner and 0 away from the corner. We also have the elliptic regularity estimate

$$\|u_R\|_{H^2(\Omega)} + \sum_{\omega_l > \pi} |\kappa_l| \leq C\|f\|_{L^2(\Omega)}. \quad (2.2.6)$$

Remark 2.2.3. The functions $s_l$ in (2.2.2) and (2.2.5) are called singular functions. The constants $\kappa_l$ in (2.2.2) and (2.2.5) are called stress intensity factors.

2.2.2 Regularity of Elliptic Interface Problems

In this subsection we discuss the regularity of elliptic interface problems. The main references are [41, 42, 43, 46, 6, 32, 53, 54, 55, 56].

Suppose that $\Omega$ is a polygonal domain in $\mathbb{R}^2$, and $\Omega_j, 1 \leq j \leq J$, are polygonal subdomains of $\Omega$ that form a partition of $\Omega$ (See Figure 2.1), i.e.,

$$\Omega_{j_1} \cap \Omega_{j_2} = \emptyset \quad \text{for } j_1 \neq j_2 \text{ and } \bigcup_{j=1}^{J} \Omega_j = \bar{\Omega}.$$

Let $f \in L^2(\Omega)$, $\rho_j$ be positive constants and $\rho : \Omega \to \mathbb{R}$ be a function defined by

$$\rho(x) = \rho_j \quad \forall x \in \Omega_j, 1 \leq j \leq J.$$

Denote the interface between the subdomains $\Omega_j$ by $\Gamma$. 

16
Consider the following elliptic interface problem with Neumann boundary conditions:

Find \( u \) such that

\[
-\rho_j \Delta u = f \quad \text{in} \quad \Omega_j, \quad 1 \leq j \leq J,
\]

(2.2.7a)

\[
\frac{\partial u}{\partial n} = 0 \quad \text{on the boundary} \quad \partial \Omega,
\]

(2.2.7b)

\[
[u] = 0 \quad \text{on the interface} \quad \Gamma,
\]

(2.2.7c)

\[
\left[ \frac{\rho \partial u}{\partial n} \right] = 0 \quad \text{on the interface} \quad \Gamma.
\]

(2.2.7d)

Here \([u]\) denotes the jump of \( u \) and \([u] = 0\) on the interface \( \Gamma \) means that \( u \) is continuous across \( \Gamma \) and \([\rho \frac{\partial u}{\partial n}]\) denotes

\[
\left[ \frac{\partial u}{\partial n} \right] = \rho_- \frac{\partial u}{\partial n_-} + \rho_+ \frac{\partial u}{\partial n_+},
\]

where \( \rho_- \) (resp. \( \rho_+ \)) denotes the weight \( \rho \) in the subdomain \( \Omega_- \) (resp. \( \Omega_+ \)) and \( n_- \) (resp. \( n_+ \)) denotes the unit normal along the interface \( \Gamma \) when we view \( \Gamma \) as the boundary of the subdomain \( \Omega_- \) (resp. \( \Omega_+ \)).

Define the weak bilinear form \( a_\rho(\cdot, \cdot) \) by

\[
a_\rho(u, v) = \int_\Omega \rho \nabla u \cdot \nabla vdx \quad \forall u, v \in H^1(\Omega).
\]
FIGURE 2.2. Polar coordinates for the Sturm-Liouville problems.

Then the weak form of the interface problem is:

Find $u \in H^1(\Omega)$ such that

$$a_\rho(u, v) = (f, v) \quad \forall v \in H^1(\Omega).$$

(2.2.8)

Away from the vertices of $\Omega_1, \cdots, \Omega_J$, the solution $u$ of (2.2.8) has the standard regularity. In other words, $u \in H^2(\Omega_{j,\delta})$ for $1 \leq j \leq J$, where $\Omega_{j,\delta}$ is obtained from $\Omega_j$ by excising the closure of a disc $D(p, \delta)$ ($\delta > 0$ is arbitrary) around the vertex $p$. At a vertex $p$ common to more than one subdomain (i.e., an interface vertex), the solution $u$ of (2.2.8) is in general singular in the sense that it does not belong to $H^2(D(p, \delta) \cap \Omega_j)$ for those subdomains $\Omega_j$ that have nonempty intersection with $D(p, \delta)$. Below we will discuss the details of the interface singularities.

The discussion of the interface singularities for (2.2.8) are divided into two cases depending on whether the interface vertex $p$ belongs to the boundary of $\Omega$ or the interior of $\Omega$ (See Figure 2.2).

Case 1. The interface vertex $p$ belongs to the boundary of $\Omega$. Let $(\lambda_k, \Theta_k)$, $\lambda_k = \sigma_k^2 > 0$, $k = 1, 2, 3, \cdots$ be the eigenvalues and eigenfunctions of the Sturm-Liouville problem around the interface vertex $p$,

$$\Theta''(\theta) + \lambda \Theta(\theta) = 0 \quad \text{for } \omega_{j-1} \leq \theta \leq \omega_j \text{ and } 1 \leq j \leq J,$$

(2.2.9a)

$$\Theta'(0+) = \Theta'(-) = 0,$$

(2.2.9b)
\[ \Theta(\omega_j-) = \Theta(\omega_j+) \quad \text{for } 1 \leq j \leq J - 1, \]  
(2.2.9c)

\[ \rho_j \Theta'(\omega_j-) = \rho_{j+1} \Theta'(\omega_j+) \quad \text{for } 1 \leq j \leq J - 1, \]  
(2.2.9d)

where the \( \Theta_k \)'s satisfy

\[ \sum_{j=1}^{J} \int_{\omega_{j-1}}^{\omega_j} \Theta_i(\theta)\Theta_k(\theta) \rho_j d\theta = \delta_{ik}. \]  
(2.2.10)

Moreover,

\[ u - \sum_{\sigma_k < 1} \kappa_k \rho \Theta_k(\theta) \in H^2(\Omega_j \cap D(p, \delta)), \quad 1 \leq j \leq J, \]  
(2.2.11)

where the \( \kappa_k \)'s are called stress intensity factors, which can be computed by an extraction formula.

**Case 2.** The interface vertex \( p \) belongs to the interior of \( \Omega \). Let \( (\lambda_k, \Theta_k) \), \( \lambda_k = \sigma_k^2 > 0 \), \( k = 1, 2, 3, \cdots \) be the eigenvalues and eigenfunctions of the Sturm-Liouville problem around the interior vertex \( p \):

\[ \Theta''(\theta) + \lambda \Theta(\theta) = 0 \quad \text{for } \omega_{j-1} \leq \theta \leq \omega_j \text{ and } 1 \leq j \leq J, \]  
(2.2.12a)

\[ \Theta(\omega_j-) = \Theta(\omega_j+) \quad \text{for } 1 \leq j \leq J - 1, \]  
(2.2.12b)

\[ \rho_j \Theta'(\omega_j-) = \rho_{j+1} \Theta'(\omega_j+) \quad \text{for } 1 \leq j \leq J - 1, \]  
(2.2.12c)

\[ \Theta(0+) = \Theta(2\pi-) = 0, \]  
(2.2.12d)

\[ \rho_1 \Theta'(0+) = \rho_j \Theta'(2\pi-), \]  
(2.2.12e)

where the \( \Theta_k \)'s satisfy

\[ \sum_{j=1}^{J} \int_{\omega_{j-1}}^{\omega_j} \Theta_i(\theta)\Theta_k(\theta) \rho_j d\theta = \delta_{ik}. \]  
(2.2.13)

Moreover,

\[ u - \sum_{\sigma_k < 1} \kappa_k \rho \Theta_k(\theta) \in H^2(\Omega_j \cap D(p, \delta)), \quad 1 \leq j \leq J. \]  
(2.2.14)
For the simplicity of presentation we will assume from here on there is only one interface vertex $p$ of the subdomains near which $u$ is singular. In this case, let $\varrho_{\text{cut}}$ be a cut-off function which equals 1 in a neighborhood of the interface vertex $p$. Let $s_l$ be defined by

$$s_l = r^{\sigma_l}(\Theta_l(\theta))\varrho_{\text{cut}},$$

where $\lambda_l = \sigma_l^2$, $l \geq 1$, are the eigenvalues of Sturm-Liouville problem at $p$ and $\Theta_l$’s are the corresponding eigenfunctions. Let

$$u_S = \sum_{0<\sigma_l<1} \kappa_l s_l,$$

and $u_R = u - u_S$, then we have

$$u = u_S + u_R,$$

where $u_R|_{\Omega_j} \in H^2(\Omega_j)$. Moreover, we have the elliptic regularity estimate

$$\sum_{j=1}^{J} \|u_R\|_{H^2(\Omega_j)} + \sum_{0<\sigma_l<1} |\kappa_l| \leq C\|f\|_{L_2(\Omega)}.$$  \hfill (2.2.16)

**Example 2.2.4.** Let us consider the Sturm-Liouville problem on an $L$-shape domain with vertices $(0, 0)$, $(0, 1)$, $(-1, 1)$, $(-1, -1)$, $(1, -1)$, and $(0, 1)$, which is partitioned into three squares $\Omega_1$, $\Omega_2$ and $\Omega_3$ (See Figure 2.3). So the interface vertex is $(0, 0)$ and $\omega_0 = \frac{\pi}{2}$, $\omega_1 = \pi$, $\omega_2 = \frac{3\pi}{2}$ and $\omega_3 = 2\pi$. 

![Figure 2.3](image-url)
Suppose that $\rho_1 = \rho_3 = 50$, $\rho_2 = 1$. Then $\sigma_1 = 0.126276410744819...$ is the positive square root of the first or smallest eigenvalue of this Sturm-Liouville problem, and it is the only one which is less than one.

Actually, in this example, we can compute the first positive eigenvalue of the corresponding Sturm-Liouville problem which is between 0 and 1 by the following formula:

$$\sin\left(\frac{\sigma_1 \pi}{2}\right) = \sqrt{\frac{\rho_2 (\rho_1 + \rho_2 + \rho_3)}{\rho_2 (\rho_1 + \rho_2 + \rho_3) + \rho_1 \rho_3}}, \quad (2.2.17)$$

where $\lambda_1 = \sigma_1^2$.

Using the equation (2.2.17) with $\rho_2 = 1$ and $\rho_1 = \rho_3$, we can construct a Sturm-Liouville problem whose first positive eigenvalue is as small as we want. If $\rho_1 = \rho_3 = 50$ and $\rho_2 = 1$, then $\sigma_1 = 0.126276410744819...$ If $\rho_1 = \rho_3 = 350$ and $\rho_2 = 1$, then $\sigma_1 = 0.048066746316346...$ If $\rho_1 = \rho_3 = 1300$ and $\rho_2 = 1$, then $\sigma_1 = 0.024962282535010...$

**Remark 2.2.5.** Note that if $\rho_1 = \rho_2 = \rho_3 = 1$, then $\sigma_1 = \frac{2}{3}$.

### 2.3 Extraction Formulas

From Subsection 2.2.1, we know that the solutions of Poisson problems with Dirichlet or Neumann boundary conditions on nonconvex domains have singular representations $u = \sum_{0 < \sigma_l < 1} \kappa_l s_l + u_R$, where $u_R \in H^2(\Omega)$ and $s_l$ is determined by the interior angle of the corner. The same is true for the elliptic interface problem. The goal of this section is to develop formulas for computing the stress intensity factors $\kappa_l$ in different cases. These formulas are called extraction formulas [5, 2, 30, 45, 61, 37, 50] and [41, 42, 32, 53, 54, 55].
2.3.1 Extraction Formulas for Poisson Problems

First let us consider the Poisson problem with the Dirichlet boundary condition on nonconvex domain. Suppose that $\Omega$ is a nonconvex polygonal domain and $f \in L_2(\Omega)$. Consider the following Dirichlet problem:

Find $u \in H^1_0(\Omega)$ such that

\[ -\Delta u = f \quad \text{in the domain } \Omega, \quad (2.3.1a) \]
\[ u = 0 \quad \text{on the boundary } \partial \Omega. \quad (2.3.1b) \]

From Subsection 2.2.1, we know that

\[ u = \sum_{\omega_l > \pi} \kappa_l s_l + u_R, \quad (2.3.2) \]

where $u_R \in H^2(\Omega)$ and $s_l$ has the form

\[ s_l^* = \frac{r^{-\omega_l}}{\pi} \sin \left( \frac{\pi}{\omega_l} \theta \right) \varrho_l(r) \]

around the corner. Here $\omega_l$ is the interior angle of the corner where $\omega_l > \pi$, $(r, \theta)$ are the local polar coordinates around the corner, and $\varrho_l$ is a cut-off function which equals 1 around the corner and 0 away from the corner.

In the following lemma, we derive the extraction formula for computing the stress intensity factor $\kappa_l$ in the equation (2.3.2). First, let us define a related function.

**Definition 2.3.1.** (Dual Singular Function) Given the singular function $s = r^{\sigma} \sin(\sigma \theta) \varrho(r)$ around a corner of a polygonal domain, we call the function $s^* = r^{-\sigma} \sin(\sigma \theta) \varrho(r)$ the dual singular function.

**Lemma 2.3.2.** (Extraction Formula for $\kappa_l$) Let $u \in H^1_0(\Omega)$ be the weak solution of the Poisson problem of (2.3.1). Then the stress intensity factors $\kappa_l$ in the singular representation (2.3.2) can be computed by the extraction formula

\[ \kappa_l = \frac{1}{\pi} \int_{\Omega} (fs_l^* + u \Delta s_l^*) dx, \quad (2.3.3) \]
where

\[ s_l = r^{\frac{\pi}{\omega_l}} \sin\left(\frac{\pi}{\omega_l} \theta\right) \varrho_l(r) \]

and

\[ s_l^* = r^{-\frac{\pi}{\omega_l}} \sin\left(\frac{\pi}{\omega_l} \theta\right) \varrho_l(r) \]

**Proof.** Without loss of generality, we assume there is only one corner \( p_l \) with the interior angle \( \omega_l > \pi \). Given any small \( \delta > 0 \), denote \( D_\delta = B(p_l, \delta) \cap \Omega \), where \( B(p_l, \delta) = \{ x \in \mathbb{R}^2 : \| x - p_l \| < \delta \} \). Denote \( \Omega_\delta = \Omega \setminus D_\delta \).

For a small \( \delta \), consider the integral

\[ I_\delta = \int_{\Omega_\delta} (f s_l^* + u \Delta s_l^*) dx. \quad (2.3.4) \]

Using Lebesgue's dominated convergence theorem, we can show that

\[ \lim_{\delta \to 0} I_\delta = \int_{\Omega} (f s_l^* + u \Delta s_l^*) dx. \quad (2.3.5) \]

We rewrite (2.3.4) as

\[ I_\delta = I_1^\delta + \kappa_1 I_2^\delta, \quad (2.3.6) \]

where

\[ I_1^\delta = \int_{\Omega_\delta} (- (\Delta u R) s_l^* + u R \Delta s_l^*) dx \]

and

\[ I_2^\delta = \int_{\Omega_\delta} (- (\Delta s_l) s_l^* + s_l \Delta s_l^*) dx. \]

After applying Green's formula and a direct computation, we have

\[ \lim_{\delta \to 0} I_2^\delta = \pi. \quad (2.3.7) \]

So it remains to show that \( \lim_{\delta \to 0} I_1^\delta = 0 \). Applying Green's formula again, we have

\[ I_1^\delta = \int_{\partial \Omega_\delta} \left( - \frac{\partial u R}{\partial n} s_l^* + u R \frac{\partial s_l^*}{\partial n} \right) ds = I_3^\delta + I_4^\delta. \quad (2.3.8) \]

Here

\[ I_3^\delta = \int_{\Gamma_\delta} \left( - \frac{\partial u R}{\partial n} s_l^* + u R \frac{\partial s_l^*}{\partial n} \right) ds \quad (2.3.9) \]
and

\[ I_4^\delta = \int_{\partial \Omega_\delta \setminus \Gamma_\delta} (\frac{\partial u_R}{\partial \nu} s^*_l + u_R \frac{\partial s^*_l}{\partial \nu}) ds, \]  

(2.3.10)

where \( \Gamma_\delta = \partial B(p_l, \delta) \cap \bar{\Omega} \).

Note that \( u_R = 0 \) and \( s^*_l = 0 \) on the boundary \( \partial \Omega \) and \( \partial \Omega_\delta \setminus \Gamma_\delta \subset \partial \Omega \), so \( I_4^\delta = 0 \).

For sufficiently small \( \delta \), by a direct computation, we further simplify \( I_3^\delta \) in the following way:

\[ I_3^\delta = - \int_{\Gamma_\delta} (\frac{\partial u_R}{\partial r} s^*_l + u_R \frac{\partial s^*_l}{\partial r}) ds = L_1 + L_2, \]  

(2.3.11)

where

\[ L_1 = \int_0^{\omega_l} \frac{\partial u_R}{\partial r} \sin\left(\frac{\pi}{\omega_l} \theta\right) \delta^{1 - \frac{\pi}{\omega_l}} d\theta \]  

(2.3.12)

and

\[ L_2 = \frac{\pi}{\omega_l} \int_0^{\omega_l} u_R \sin\left(\frac{\pi}{\omega_l} \theta\right) \delta^{-\frac{\pi}{\omega_l}} d\theta. \]  

(2.3.13)

Since \( u_R \in H^2(\Omega) \), it follows from the Sobolev embedding theorem (cf. Theorem 2.1.9) that

\[ u_R \in C^{0,\eta}(\bar{\Omega}) \quad \text{for any} \quad 0 < \eta < 1 \]  

(2.3.14)

and hence there exists a positive constant \( C_\eta \) depending only on \( \eta \) and \( \Omega \) such that

\[ |u_R(x) - u_R(y)| \leq C_\eta |x - y|^\eta \quad \text{for} \quad x, y \in \Omega, \]  

(2.3.15)

which together with the fact that \( u_R(p_l) = 0 \) implies

\[ |u_R(\delta, \theta)| \leq C_\eta \delta^\eta. \]  

(2.3.16)

It follows from (2.3.13) and (2.3.16) that, for some positive constant \( C_\eta' \) depending only on \( \eta \) and \( \Omega \),

\[ |L_2| \leq C_\eta' \delta^{\eta - \frac{\pi}{\omega_l}}, \]  

(2.3.17)

which implies that, if we choose an \( \eta \) so that \( \eta - \frac{\pi}{\omega_l} > 0 \), then

\[ \lim_{\delta \to 0} L_2 = 0. \]  

(2.3.18)
Let \( z = \frac{\partial u_R}{\partial r} \). Since \( u_R \in H^2(\Omega) \), we have \( z \in H^1(\Omega) \) and \( z \in H^1(D_\delta) \). Let \( \hat{z}(r, \theta) = z(\delta r, \theta) \), then \( \hat{z} \in H^1(D_1) \).

By the Cauchy-Schwartz inequality, there exists a constant \( C \) depending only on \( \omega_l \) such that
\[
\left| \int_0^{\omega_l} \frac{\partial u_R}{\partial r} \sin(\frac{\pi}{\omega_l} \theta) d\theta \right| = \left| \int_0^{\omega_l} z(\delta, \theta) \sin(\frac{\pi}{\omega_l} \theta) d\theta \right| = \left| \int_0^{\omega_l} \hat{z}(1, \theta) \sin(\frac{\pi}{\omega_l} \theta) d\theta \right| \leq C \| \hat{z} \|_{L^2(\Gamma_1)},
\]
which together with Theorem 2.1.10 implies
\[
\left| \int_0^{\omega_l} \frac{\partial u_R}{\partial r} \sin(\frac{\pi}{\omega_l} \theta) d\theta \right| \leq C \| \hat{z} \|_{H^1(D_1)}. \tag{2.3.19}
\]

Since \( \hat{z} \in H^1(D_1) \), by Theorem 2.1.9, we have \( \hat{z} \in L^q(D_1) \) for any \( q \geq 1 \).

Using Hölder’s inequality, we obtain the estimate
\[
\| \hat{z} \|_{L^2(D_1)} \leq C_q \| \hat{z} \|_{L^q(D_1)} \quad \text{for} \quad q > 2, \tag{2.3.20}
\]
where \( C_q \) is a positive constant depending only on \( q \) and \( \omega_l \).

A direct computation implies
\[
\| \hat{z} \|_{L^q(D_1)}^q = \int_0^{\omega_l} \int_0^\delta |\hat{z}(r, \theta)|^q r d\theta dr = \frac{1}{\delta^2} \int_0^\delta \int_0^{\omega_l} |\hat{z}|^q r d\theta dr = \frac{1}{\delta^2} \| \hat{z} \|_{L^q(D_3)}^q
\]
and hence
\[
\| \hat{z} \|_{L^q(D_1)} = \delta^{-2/q} \| \hat{z} \|_{L^q(D_3)}. \tag{2.3.21}
\]

Similarly, we derive
\[
| \hat{z} |_{H^1(D_1)} = | \hat{z} |_{H^1(D_3)}. \tag{2.3.22}
\]

Combining (2.3.12), (2.3.19)–(2.3.22), we obtain that, for any given \( q > 2 \),
\[
|L_1| \leq C'_q \left( \delta^{1 - \frac{\omega_l}{\pi}} | \hat{z} |_{L^q(D_3)} + \delta^{1 - \frac{\omega_l}{\pi}} | \hat{z} |_{H^1(D_3)} \right), \tag{2.3.23}
\]
where $C'_q$ is a positive constant depending only on $q$ and $\omega_l$. It follows from Theorem 2.1.9 and (2.3.23) that, for a small $\delta$,

$$|L_1| \leq C'_q\delta^{1-\frac{\omega_l}{\omega_{l'}}-\frac{q}{q}} \|z\|_{H^1(\Omega)}.$$  

(2.3.24)

Choose $q$ such that $\frac{2}{q} < 1 - \frac{\pi}{\omega_{l'}}$, then (2.3.24) implies

$$\lim_{\delta \to 0} L_1 = 0,$$

(2.3.25)

which completes the proof.

Next we consider the following Poisson problem with Neumann boundary condition:

Find $u \in H^1(\Omega)$ such that

$$-\Delta u = f \quad \text{in the domain } \Omega,$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on the boundary } \partial \Omega.$$  

(2.3.26a)

(2.3.26b)

From Subsection 2.2.1, we know that

$$u = \sum_{\omega_l > \pi} \kappa_l s_l + u_R,$$

(2.3.27)

where $u_R \in H^2(\Omega)$ and $s_l$ has the form

$$r^{\frac{\pi}{\omega_l}} \cos\left(\frac{\pi}{\omega_l} \theta\right) q_l(r)$$

around the corner. Here $\omega_l$ is the interior angle of the corner where $\omega_l > \pi$, $(r, \theta)$ are the local polar coordinates around the corner, and $q_l$ is the cut-off function which equals 1 around the corner and 0 away from the corner.

In this case, the corresponding dual singular functions are of the form

$$s^* = r^{-\sigma} \cos(\sigma \theta) q(r)$$

around the particular corners of the polygonal domain. Then the extraction formulas for this problem are formulated in the following lemma.
Lemma 2.3.3. (Extraction Formula for $\kappa_l$) Let $u \in H^1(\Omega)$ be the weak solution of the Poisson problem (2.3.26). Then the stress intensity factors $\kappa_l$ in the singular representation (2.3.27) can be computed by the extraction formula

$$
\kappa_l = \frac{1}{\pi} \int_{\Omega} (f s_l^* + u \Delta s_l^*) dx,
$$

(2.3.28)

where

$$
s_l = r^{-\omega_l} \cos(\frac{\pi}{\omega_l} \theta) \varrho_l(r),
$$

$$
s_l^* = r^{-\omega_l} \cos(\frac{\pi}{\omega_l} \theta) \varrho_l(r).
$$

Proof. Without loss of generality, we assume there is only one corner $p_l$ with the interior angle $\omega_l > \pi$. Given any small $\delta > 0$, denote $D_\delta = B(p_l, \delta) \cap \Omega$, where $B(p_l, \delta) = \{ x \in \mathbb{R}^2 : \| x - p_l \| < \delta \}$. Denote $\Omega_\delta = \Omega \setminus D_\delta$.

For a small $\delta$, consider the integral

$$
I_\delta = \int_{\Omega_\delta} (f s_l^* + u \Delta s_l^*) dx.
$$

(2.3.29)

Using Lebesgue’s dominated convergence theorem, we can show that

$$
\lim_{\delta \to 0} I_\delta = \int_{\Omega} (f s_l^* + u \Delta s_l^*) dx.
$$

(2.3.30)

We rewrite (2.3.29) as

$$
I_\delta = I_1^\delta + \kappa_l I_2^\delta,
$$

(2.3.31)

where $I_1^\delta = \int_{\Omega_\delta} (-(\Delta u_R) s_l^* + u_R \Delta s_l^*) dx$ and $I_2^\delta = \int_{\Omega_\delta} (-(\Delta s_l) s_l^* + s_l \Delta s_l^*) dx$.

After applying Green’s formula and a direct computation, we have

$$
\lim_{\delta \to 0} I_2^\delta = \pi.
$$

(2.3.32)

So it remains to show that $\lim_{\delta \to 0} I_1^\delta = 0$.

Applying Green’s formula again, we have

$$
I_1^\delta = \int_{\partial \Omega_\delta} \left( -\frac{\partial u_R}{\partial n} s_l^* + u_R \frac{\partial s_l^*}{\partial n} \right) ds = I_3^\delta + I_4^\delta.
$$

(2.3.33)
Here

\[ I_3^\delta = \int_{\Gamma_3} \left( -\frac{\partial u_R}{\partial n} s_l^* + u_R \frac{\partial s_l^*}{\partial n} \right) ds \tag{2.3.34} \]

and

\[ I_4^\delta = \int_{\partial \Omega \setminus \Gamma_3} \left( -\frac{\partial u_R}{\partial n} s_l^* + u_R \frac{\partial s_l^*}{\partial n} \right) ds, \tag{2.3.35} \]

where \( \Gamma_\delta = \partial B(p_l, \delta) \cap \bar{\Omega} \).

Note that \( \frac{\partial u_R}{\partial n} = 0 \) and \( \frac{\partial s_l^*}{\partial n} = 0 \) on the boundary \( \partial \Omega \) and \( \partial \Omega_\delta \setminus \Gamma_\delta \subset \partial \Omega \), so \( I_4^\delta = 0 \).

For sufficiently small \( \delta \), by a direct computation, we further simplify \( I_3^\delta \) in the following way:

\[ I_3^\delta = -\int_{\Gamma_3} \left( -\frac{\partial u_R}{\partial r} s_l^* + u_R \frac{\partial s_l^*}{\partial r} \right) ds = L_1 + L_2, \tag{2.3.36} \]

where

\[ L_1 = \int_0^{\omega_l} \frac{\partial u_R}{\partial r} \cos(\frac{\pi}{\omega_l} \theta) \delta^\frac{\pi}{\omega_l} d\theta \tag{2.3.37} \]

and

\[ L_2 = \frac{\pi}{\omega_l} \int_0^{\omega_l} u_R \cos(\frac{\pi}{\omega_l} \theta) \delta^\frac{\pi}{\omega_l} d\theta. \tag{2.3.38} \]

The regular part \( u_R \) at the vertex \( p_l \) may not be zero for the Neumann problem.

But we can rewrite (2.3.38) as

\[ L_2 = \frac{\pi}{\omega_l} \delta^\frac{\pi}{\omega_l} \int_0^{\omega_l} \left[ (u_R - u_R(p_l)) \cos(\frac{\pi}{\omega_l} \theta) + u_R(p_l) \cos(\frac{\pi}{\omega_l} \theta) \right] d\theta. \tag{2.3.39} \]

Since \( \int_0^{\omega_l} \cos(\frac{\pi}{\omega_l} \theta) d\theta = 0 \), we have

\[ L_2 = \frac{\pi}{\omega_l} \delta^\frac{\pi}{\omega_l} \int_0^{\omega_l} \left( \tilde{u}_R \right) \cos(\frac{\pi}{\omega_l} \theta) d\theta, \tag{2.3.40} \]

where \( \tilde{u}_R = u_R - u_R(p_l) \).

Now we can repeat the argument in the proof of Lemma 2.3.2 to prove \( \lim_{\delta \to 0} L_1 = 0 \) and \( \lim_{\delta \to 0} L_2 = 0 \).
2.3.2 Extraction Formulas for Elliptic Interface Problems

In addition to the assumptions for the domain $\Omega$ in Subsection 2.2.2 we further assume for simplicity the domain $\Omega$ has only one interface vertex $p$. From Subsection 2.2.2, we have, for the solution of the interface problem (2.2.8), the following singular function representation

$$u = \sum_{0<\sigma_l<1} \kappa_l s_l + u_R,$$  \hspace{1cm} (2.3.41)

where $u_R \in H^2(\Omega_j), 1 \leq j \leq J$ and $s_l$ has the form $r^{\sigma_l} \Theta_l(\theta) \varrho_l(r)$ around the interface vertex $p$. Here $\{\sigma_l^2, \Theta_l\}, 0 < \sigma_l < 1$ are the first few eigenvalues and eigenfunctions of the corresponding Sturm-Liouville problem (2.2.9) or (2.2.12), $(r, \theta)$ are the local polar coordinates around the interface vertex $p$, and $\varrho_l$ is the cut-off function which equals 1 around the interface vertex $p$ and 0 away from $p$.

In this case, the dual singular function of $s_l = r^{\sigma_l} \Theta_l(\theta) \varrho_l(r)$ is defined by $s_l^* = r^{-\sigma_l} \Theta_l(\theta) \varrho_l(r)$. We have the following lemma on the extraction formula for the stress intensity factors of the interface problem (2.2.8).

**Lemma 2.3.4.** (Extraction Formula for $\kappa_l$) Let $u \in H^1(\Omega)$ be the weak solution of the elliptic interface problem (2.2.8). Then the stress intensity factors $\kappa_l$ in (2.3.41) can be computed by the extraction formula

$$\kappa_l = \frac{1}{2\sigma_l} \int_{\Omega} (f s_l^* + \rho u \Delta s_l^*) dx.$$  \hspace{1cm} (2.3.42)

**Proof.** Given any small $\delta > 0$, denote $D_\delta = B(p, \delta) \cap \Omega$, where $B(p, \delta) = \{x \in \mathbb{R}^2 : \|x-p\| < \delta\}$. Denote $\Omega_\delta = \Omega \setminus D_\delta$ and $\Omega_{\delta,j} = \Omega_\delta \cap \Omega_j$.

For a small $\delta$, consider the integral

$$I_\delta = \int_{\Omega_\delta} (f s_l^* + \rho u \Delta s_l^*) dx.$$  \hspace{1cm} (2.3.43)

Using Lebesgue’s dominated convergence theorem, we can show that

$$\lim_{\delta \to 0} I_\delta = \int_{\Omega} (f s_l^* + \rho u \Delta s_l^*) dx.$$  \hspace{1cm} (2.3.44)
We rewrite (2.3.43) as

\[ I_\delta = I_1^\delta + I_2^\delta, \]  

(2.3.45)

where \( I_1^\delta = \int_{\Omega_\delta} (-\rho(\Delta u_R)s^*_l + \rho u_R \Delta s^*_l)dx \) and \( I_2^\delta = \sum_{0<\sigma_l<1} \kappa_l \int_{\Omega_\delta} (-\rho(\Delta s^*_l)s^*_l + \rho s^*_l \Delta s^*_l)dx. \)

Applying Green’s formula on each subdomain of \( \Omega_\delta \) and using the fact that \( \frac{\partial s_l}{\partial n} = 0 \) and \( \frac{\partial s^*_l}{\partial n} = 0 \) on the boundary \( \partial \Omega \), we have

\[ I_2^\delta = \sum_{0<\sigma_l<1} -\kappa_l \int_{\partial D_\delta} (-\rho \frac{\partial s^*_l}{\partial r} s^*_l + \rho s^*_l \frac{\partial s^*_l}{\partial r})ds. \]  

(2.3.46)

When \( \delta \) is small, (2.2.9)-(2.2.10) (or (2.2.12)-(2.2.13)) and the definition of \( s_l \) and \( s^*_l \) imply

\[
\int_{\partial D_\delta} (-\rho \frac{\partial s^*_l}{\partial r} s^*_l + \rho s^*_l \frac{\partial s^*_l}{\partial r})ds = \sum_{j=1}^{J} \int_{\theta_{j-1}}^{\theta_j} (-\rho_j \sigma_l \delta^{\sigma_l-\sigma_l} \Theta_l \Theta_l)
+ \rho_j (-\sigma_l) \delta^{\sigma_l-\sigma_l} \Theta_l \Theta_l) d\theta
= -\delta^{\sigma_l-\sigma_l} (\sigma_l + \sigma_l') \delta_l \]  

(2.3.47)

and hence \( I_2^\delta = 2\kappa_l \sigma_l \). So it remains to show that \( \lim_{\delta \to 0} I_1^\delta = 0. \)

Applying Green’s formula on \( \Omega_{\delta,j} \) for \( 1 \leq j \leq J \), we have

\[
I_1^\delta = \sum_{j=1}^{J} \int_{\partial \Omega_{\delta,j}} (-\rho_j \frac{\partial u_R}{\partial n} s^*_l + \rho_j u_R \frac{\partial s^*_l}{\partial n})ds = I_3^\delta + I_4^\delta. \]  

(2.3.48)

Here

\[
I_3^\delta = \int_{\Gamma_\delta} (-\rho \frac{\partial u_R}{\partial n} s^*_l + \rho u_R \frac{\partial s^*_l}{\partial n})ds \]  

(2.3.49)

and

\[
I_4^\delta = \sum_{j=1}^{J} \int_{\partial \Omega_{\delta,j} \setminus \Gamma_\delta} (-\rho \frac{\partial u_R}{\partial n} s^*_l + \rho u_R \frac{\partial s^*_l}{\partial n})ds, \]  

(2.3.50)

where \( \Gamma_\delta = \partial B(p, \delta) \cap \bar{\Omega} \).

Since \( \frac{\partial u_R}{\partial n} \) and \( \frac{\partial s^*_l}{\partial n} \) are zero on the boundary \( \partial \Omega \), \[ \rho \frac{\partial u_R}{\partial n} \] and \[ \rho \frac{\partial s^*_l}{\partial n} \] are zero on the interface \( \Gamma \), it follows that \( I_4^\delta = 0. \)
For sufficiently small $\delta$, we further simplify $I_3^\delta$ in the following way:

$$I_3^\delta = -\int_{\Gamma_3} (-\rho \frac{\partial u_R}{\partial r} s^*_i + \rho u_R \frac{\partial s^*_i}{\partial r}) ds = L_1 + L_2, \quad (2.3.51)$$

where

$$L_1 = \sum_{j=1}^{J} \int^{\theta_j}_{\theta_{j-1}} \rho_j \frac{\partial u_R}{\partial r} \Theta_i \delta^{1-\sigma_i} d\theta \quad (2.3.52)$$

and

$$L_2 = \sigma_l \sum_{j=1}^{J} \int^{\theta_j}_{\theta_{j-1}} \rho_j u_R \Theta_i \delta^{\sigma_i} d\theta. \quad (2.3.53)$$

The regular part $u_R$ at the interface vertex $p$ may not be zero, but we can rewrite (2.3.53) as

$$L_2 = \sigma_l \sum_{j=1}^{J} \int^{\theta_j}_{\theta_{j-1}} \left[ \rho_j (u_R - u_R(p)) \Theta_i \delta^{\sigma_i} + \rho_j u_R(p) \Theta_i \delta^{-\sigma_i} \right] d\theta. \quad (2.3.54)$$

Since $\Theta_i''(\theta) = -\sigma_i^2 \Theta_i(\theta)$ for $\theta_{j-1} < \theta < \theta_j$ and $1 \leq j \leq J$, we have

$$\sigma_l \sum_{j=1}^{J} \int^{\theta_j}_{\theta_{j-1}} \rho_j u_R(p) \Theta_i \delta^{\sigma_i} d\theta = -\frac{1}{\sigma_l} \delta^{-\sigma_i} u_R(p) \sum_{j=1}^{J} \int^{\theta_j}_{\theta_{j-1}} \rho_j \Theta_i'' d\theta \quad (2.3.55)$$

which, together with the conditions (2.2.9b) and (2.2.9d) (or (2.2.12c) and (2.2.12e)), implies

$$\sigma_l \sum_{j=1}^{J} \int^{\theta_j}_{\theta_{j-1}} \rho_j u_R(p) \Theta_i \delta^{\sigma_i} d\theta = 0. \quad (2.3.56)$$

Therefore,

$$L_2 = \sigma_l \sum_{j=1}^{J} \int^{\theta_j}_{\theta_{j-1}} \rho_j \tilde{u}_R \Theta_i \delta^{\sigma_i} d\theta, \quad (2.3.57)$$

where $\tilde{u}_R = u_R - u_R(p)$.

Denote $L_{1,j} = \int^{\theta_j}_{\theta_{j-1}} \rho_j \frac{\partial u_R}{\partial r} \Theta_i \delta^{1-\sigma_i} d\theta$ and $L_{2,j} = \sigma_l \int^{\theta_j}_{\theta_{j-1}} \rho_j \tilde{u}_R \Theta_i \delta^{\sigma_i} d\theta$.

Since $\tilde{u}_R \in H^2(\Omega_j)$ (cf. Subsection 2.2.2) and $\frac{\partial \tilde{u}_R}{\partial r} = \frac{\partial u_R}{\partial r} \in H^1(\Omega_j)$, we can repeat the argument in the proof of Lemma 2.3.2 to prove $\lim_{\delta \to 0} L_{1,j} = 0$ and $\lim_{\delta \to 0} L_{2,j} = 0$ for $1 \leq j \leq J$. Hence $\lim_{\delta \to 0} L_1 = 0$ and $\lim_{\delta \to 0} L_2 = 0$. \hfill $\Box$
It is not difficult to extend the extraction formula in Lemma 2.3.4 to the non-homogeneous interface problem with the Neumann boundary condition.

In addition to the assumptions for the problem (2.2.7), we further assume there are functions \( g_\Gamma \) and \( h_\Gamma \) on \( \Gamma \) such that there exists a function \( G \) on \( \bar{\Omega} \) which satisfies

\[
G|_{\Omega_j} \in H^2(\Omega_j) \quad \text{for} \quad 1 \leq j \leq J,
\]

\[
\rho \frac{\partial G}{\partial n} = g_\Gamma \quad \text{on the boundary } \partial \Omega,
\]

\[
[G] = 0 \quad \text{on the interface } \Gamma,
\]

and

\[
\left[ \rho \frac{\partial G}{\partial n} \right] = h_\Gamma \quad \text{on the interface } \Gamma.
\]

Now we consider the nonhomogeneous interface problem with the Neumann boundary condition:

Find \( u \in H^1(\Omega) \) such that

\[
-\rho_j \Delta u = f \quad \text{in } \Omega_j, 1 \leq j \leq J, \tag{2.3.58a}
\]

\[
\rho \frac{\partial u}{\partial n} = g_\Gamma \quad \text{on the boundary } \partial \Omega, \tag{2.3.58b}
\]

\[
[u] = 0 \quad \text{on the interface } \Gamma, \tag{2.3.58c}
\]

\[
\left[ \rho \frac{\partial u}{\partial n} \right] = h_\Gamma \quad \text{on the interface } \Gamma. \tag{2.3.58d}
\]

By our assumptions, \( \tilde{u} = u - G \) is the solution of the following homogeneous interface problem with Neumann boundary condition:

\[
-\rho_j \Delta \tilde{u} = f + \rho_j \Delta G \quad \text{in } \Omega_j, 1 \leq j \leq J, \tag{2.3.59a}
\]

\[
\rho \frac{\partial \tilde{u}}{\partial n} = 0 \quad \text{on the boundary } \partial \Omega, \tag{2.3.59b}
\]

\[
[\tilde{u}] = 0 \quad \text{on the interface } \Gamma, \tag{2.3.59c}
\]

\[
\left[ \rho \frac{\partial \tilde{u}}{\partial n} \right] = 0 \quad \text{on the interface } \Gamma. \tag{2.3.59d}
\]
So Lemma 2.3.4 implies that the stress intensity factors $\kappa_l$ for the problem (2.3.58) can be computed by

$$\kappa_l = \frac{1}{2\sigma_l} \int_{\Omega} ((f + \rho\Delta G)s_i^* + \rho(u - G)\Delta s_i^*)dx. \quad (2.3.60)$$

By Green’s formula and a similar argument in the proof of Lemma 2.3.3, (2.3.60) implies that

$$\kappa_l = \frac{1}{2\sigma_l} \left[ \int_{\Omega} (f s_i^* + \rho u\Delta s_i^*)dx + \int_{\partial\Omega} \rho h s_i^* ds + \int_{\Gamma} \rho g s_i^* ds \right]. \quad (2.3.61)$$

In summary, we have the following lemma.

**Lemma 2.3.5.** (Extraction Formula for $\kappa_l$ of the Nonhomogeneous Interface Problem with the Neumann Boundary Condition) Let $u \in H^1(\Omega)$ be the weak solution of the elliptic interface problem (2.3.58). Then $\tilde{u} = u - G$ has the singular function representation (2.3.41) and the stress intensity factors $\kappa_l$ in (2.3.41) can be computed by the extraction formula (2.3.61).

### 2.4 Finite Element Methods

In this section we discuss some basic facts about finite element methods. The basic references are [20, 25].

We will use the Poisson problem with the Dirichlet boundary condition as a model problem.

Suppose that $\Omega$ is a polygonal domain and $f \in L_2(\Omega)$. Consider the following Dirichlet problem:

Find $u \in H^1(\Omega)$ such that

$$-\Delta u = f \quad \text{in the domain } \Omega, \quad (2.4.1a)$$

$$u = 0 \quad \text{on the boundary } \partial\Omega. \quad (2.4.1b)$$
Its weak formulation is to find $u \in H^1_0(\Omega)$ such that

$$a(u, v) = (f, v) \quad \forall v \in H^1_0(\Omega), \quad (2.4.2)$$

where

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx.$$

The well-posedness of the problem (2.4.2) is guaranteed by the following Lax-Milgram theorem (cf. [33]).

**Theorem 2.4.1.** Let $H$ be a Hilbert space with inner product $(\cdot, \cdot)$, and let $B(u, v)$ be a bilinear form on $H \times H$, $u \in H$, $v \in H$ such that

$$|B(u, v)| \leq C_1 \|u\|_H \|v\|_H, \quad (2.4.3a)$$

$$|B(u, u)| \geq C_2 \|u\|_H^2, \quad (2.4.3b)$$

with $C_1 > 0$, $C_2 > 0$.

Let $f \in H'$, i.e., $f$ is a bounded linear functional on $H$. Then there exists a unique $u_0 \in H$ such that

$$B(u_0, v) = f(v) \quad \forall v \in H. \quad (2.4.4)$$

We want to construct a finite dimensional subspace of $H^1_0(\Omega)$ and solve the equation (2.4.2) on that subspace. This can be carried out by a conforming finite element method. Now we introduce the basic terminology of this method.

**Definition 2.4.2.** (Triangulation) Let $\Omega$ be a polygonal domain. A triangulation $\mathcal{T}_h$ of $\Omega$ is a subdivision consisting of triangles with the property that no vertex of any triangle lies in the interior of an edge of another triangle. Denote $h = \max_{T \in \mathcal{T}_h} \text{diam} T.$
**Definition 2.4.3. (Quasi-uniform)** If a family \( \mathcal{T}_h \) of triangulations of \( \Omega \) satisfies the following condition: there exists a positive constant \( C \) such that

\[
\min \{ \operatorname{diam} B_T : T \in \mathcal{T}_h \} \geq C h
\] (2.4.5)

for all \( h \), where \( B_T \) is the largest ball inscribed in \( T \) and \( h = \max_{T \in \mathcal{T}_h} \operatorname{diam} T \), then we say this family is quasi-uniform.

**Definition 2.4.4. (\( P_1 \) Finite Element Space on \( \mathcal{T}_h \))** Let \( V_h \) be the space of continuous piecewise \( P_1 \) polynomials on the triangulation \( \mathcal{T}_h \), or,

\[
V_h = \{ v \in C^0(\overline{\Omega}) : v|_T \text{ is a first order polynomial for any } T \in \mathcal{T}_h \}.
\]

Let \( \hat{V}_h \) be the subspace of \( V_h \) defined by

\[
\hat{V}_h = \{ v \in V_h : v|_{\partial \Omega} = 0 \}
\]

or

\[
\hat{V}_h = V_h \cap H^1_0(\Omega).
\]

**Remark 2.4.5.** \( \hat{V}_h \) is a subspace of the space \( H^1_0(\Omega) \), so we refer to the corresponding finite element method as a conforming finite element method.

Once the finite element space is chosen (\( \hat{V}_h \) in our case), the discrete version of the weak problem (2.4.2) is:

Find \( u_h \in \hat{V}_h \) such that

\[
a(u_h, v) = (f, v) \quad \text{for all } v \in \hat{V}_h.
\] (2.4.6)

Because of the Lax-Milgram Theorem 2.4.1, the well-posedness of the equation (2.4.6) can be easily verified.

From (2.4.2) and (2.4.6), we have the following Galerkin orthogonality ([20, Proposition 2.5.9]) for \( u - u_h \):

\[
a(u - u_h, v) = 0 \quad \text{for } v \in \hat{V}_h.
\] (2.4.7)
Next, we consider the error between the two functions $u$ and $u_h$. First, we consider Céa’s lemma ([20, Theorem 2.8.1]), which is a consequence of (2.4.7). It shows that the approximation $u_h$ to $u$ is quasi-optimal.

**Lemma 2.4.6.** (Céa’s Lemma) Suppose $u$ and $u_h$ are the solutions of (2.4.2) and (2.4.6). Then there exists a positive constant $C$ independent of the subspace $\hat{V}_h$, such that

$$
\|u - u_h\|_{H^1(\Omega)} \leq C \inf_{v \in \hat{V}_h} \|u - v\|_{H^1(\Omega)}.
$$

Because of Céa’s lemma, we can focus on finding a specific function $v \in V_h$ where $\|u - v\|_{H^1(\Omega)}$ can be estimated in terms of $h$. That specific function is the interpolant of $u$ on $V_h$. Let $\Pi_h : C^0(\bar{\Omega}) \to V_h$ be the nodal interpolant operator defined by

$$
\Pi_h v = v \quad \text{at all the vertices of } \mathcal{T}_h.
$$

Then we have the following interpolation error estimate. In the case of a convex polygon, this is a standard result. A proof of the general case is given in Appendix A.

**Theorem 2.4.7.** (Interpolation Error Estimate) Let $u$ be the solution of (2.4.2). Let $c_1, c_2, \ldots, c_{N_\Omega}$ be the corners of $\Omega$ and $\omega_1, \omega_2, \ldots, \omega_{N_\Omega}$ be the interior angle of the corners. Let $\omega = \max\{\omega_1, \omega_2, \ldots, \omega_{N_\Omega}\}$ and

$$
\beta = \max\{1, \frac{\pi}{\omega}\}.
$$

If $\beta = 1$, we have

$$
\|u - \Pi_h u\|_{L^2(\Omega)} + h|u - \Pi_h u|_{H^1(\Omega)} \leq C h^2 \|u\|_{H^2(\Omega)}, \quad (2.4.8)
$$

where the constant $C$ is independent of the mesh size $h$. If $\beta < 1$, we have

$$
\|u - \Pi_h u\|_{L^2(\Omega)} + h|u - \Pi_h u|_{H^1(\Omega)} \leq C h^{1+\beta}(\|u_R\|_{H^2(\Omega)} + \sum_{\omega_i > \pi} |\kappa_i|), \quad (2.4.9)
$$
where the constant $C$ is independent of the mesh size $h$.

Remark 2.4.8. Based on the regularity result for $u$ (Theorem 2.2.1), we know that $u \in C^0(\Omega)$ and hence $\Pi_h u$ is well-defined.

Applying Theorem 2.2.1, Lemma 2.4.6 and Theorem 2.4.7, we have the following error estimate.

**Theorem 2.4.9.** $(H^1$ Error Estimate for $u_h)$ Let $u$ be the solution of (2.4.2) and $u_h$ be the solution of (2.4.6). Let $c_1, c_2, \cdots, c_{N_\Omega}$ be the corners of $\Omega$ and $\omega_1, \omega_2, \cdots, \omega_{N_\Omega}$ be the interior angle of the corners. Let $\omega = \max\{\omega_1, \omega_2, \cdots, \omega_{N_\Omega}\}$ and

$$\beta = \max\{1, \frac{\pi}{\omega}\}.$$  

Then we have

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch^\beta \|f\|_{L^2(\Omega)}, \quad (2.4.10)$$

where the constant $C$ is independent of the mesh size $h$.

Remark 2.4.10. The above discussion can be generalized to the elliptic interface problem.

We now consider the error estimate for $u - u_h$ in the $L_2$ norm. To estimate $\|u - u_h\|_{L^2(\Omega)}$, we use a duality argument. Let $w$ be the solution of

$$-\Delta w = e \quad \text{in} \quad \Omega, \quad (2.4.11a)$$

$$w = 0 \quad \text{on the boundary} \quad \partial \Omega, \quad (2.4.11b)$$

where $e = u - u_h$. The variational formulation of this problem is: find $w \in H^1_0(\Omega)$ such that

$$a(w, v) = (e, v) \quad \forall v \in H^1_0(\Omega). \quad (2.4.12)$$

Therefore

$$\|u - u_h\|_{L^2(\Omega)}^2 = (u - u_h, u - u_h)$$
\[ = a(w, u - u_h) \]
\[ = a(w - \Pi_h w, u - u_h) \quad \text{(by Galerkin orthogonality (2.4.7))} \]
\[ \leq C\|w - \Pi_h w\|_{H^1(\Omega)}\|u - u_h\|_{H^1(\Omega)} \]
\[ \leq Ch^\beta\|u - u_h\|_{H^1(\Omega)}\|e\|_{L^2(\Omega)} \quad \text{(by (2.2.3) and (2.4.8)).} \]

By Theorem 2.4.9, we have \(\|u - u_h\|_{H^1(\Omega)} \leq Ch^\beta\|f\|_{L^2(\Omega)}.\) Therefore,
\[ \|u - u_h\|_{L^2(\Omega)} \leq Ch^{2\beta}\|f\|_{L^2(\Omega)}. \]

Thus we have proved the following theorem.

**Theorem 2.4.11.** (L^2 Error Estimate for \(u_h\)) Let \(u\) be the solution of (2.4.2) and \(u_h\) be the solution of (2.4.6). Let \(c_1, c_2, \ldots, c_{N_\Omega}\) be the corners of \(\Omega\) and \(\omega_1, \omega_2, \ldots, \omega_{N_\Omega}\) be the interior angle of the corners. Let \(\omega = \max\{\omega_1, \omega_2, \ldots, \omega_{N_\Omega}\}\) and
\[ \beta = \max\{1, \frac{\pi}{\omega}\}. \]

Then we have
\[ \|u - u_h\|_{L^2(\Omega)} \leq Ch^{2\beta}\|f\|_{L^2(\Omega)}, \quad (2.4.13) \]
where the constant \(C\) is independent of the mesh size \(h\).

### 2.5 Multigrid Methods

In this section we discuss multigrid methods. The main references are [38, 10, 64, 11, 24, 20].

We consider the model problem (2.4.2). To approximate the solution \(u\), we construct a nested sequence of triangulations \(\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_k, \ldots\) over the polygonal domain \(\Omega\) by the following procedure. Suppose that \(\mathcal{T}_1\) is given, then the triangulation \(\mathcal{T}_k\) is obtained by connecting the midpoints of the edges of the triangles of
the coarser triangulation \( \mathcal{T}_{k-1} \) for \( k > 1 \). On the triangulation \( \mathcal{T}_k \) we define the finite element space \( V_k \subset V = H^1_0(\Omega) \):

\[
V_k = \{ v \in C^0(\Omega) : v|_T \text{ is a first order polynomial} \quad \forall T \in \mathcal{T}_k \} \cap H^1_0(\Omega).
\]

It is easy to see that \( V_{k-1} \subset V_k \).

First we introduce the basic terminology for multigrid methods. We define a mesh-dependent inner product \((\cdot, \cdot)_k\) on \( V_k \) by

\[
(v, w)_k = h_k^2 \sum_{i=1}^{n_k} v(p_i) w(p_i),
\]

where \( \{p_i\}_{i=1}^{n_k} \) is the set of internal vertices of \( \mathcal{T}_k \).

The linear operators \( A_k : V_k \to V_k \) are defined by

\[
(A_k v, w)_k = a(v, w) \quad \forall v, w \in V_k.
\]

The operators \( Q_k : L^2(\Omega) \to V_k \) are defined by

\[
(Q_k u, v)_k = (u, v) \quad \forall u \in L^2(\Omega), v \in V_k.
\]

The discrete weak problem (2.4.6) is then equivalent to

\[
A_k u_k = F_k, \quad \tag{2.5.1}
\]

where \( F_k = Q_k f \).

The coarse-to-fine operator \( I^k_{k-1} : V_{k-1} \to V_k \) is defined to be the natural injection, or equivalently,

\[
I^k_{k-1} v = v \quad \forall v \in V_{k-1}.
\]

The fine-to-coarse operator \( I^{k-1}_k : V_k \to V_{k-1} \) is defined to be the transpose of \( I^k_{k-1} \), or equivalently,

\[
(I^{k-1}_k v, w)_{k-1} = (v, I^k_{k-1} w)_k \quad \forall v \in V_k, v \in V_{k-1}.
\]
Algorithm 2.5.1. (The $k^{\text{th}}$ Level Iteration) $MG(k, z_0, F_k)$ is the approximate solution of the equation

$$A_k z = F_k$$

obtained by the $k^{\text{th}}$ level iteration with initial guess $z_0$. Let $R_k : V_k \to V_k$ be an approximation of $A_k^{-1}$ and $R_1 = A_1^{-1}$.

For $k = 1$, $MG(1, z_0, F_1)$ is the solution obtained from a direct method. In other words,

$$MG(1, z_0, g) = R_1 F_1.$$ 

For $k > 1$, $MG(k, z_0, F_k)$ is obtained recursively in three steps.

Presmoothing Step. For $1 \leq l \leq m_1$, let

$$z_l = z_{l-1} + R_k(F_k - A_k z_{l-1}).$$

Error Correction Step. Let $\bar{F}_{k-1} := I_{k-1}^{k-1}(F_k - A_k z_{m_1})$ and $q_0 = 0$. For $1 \leq i \leq p$, let

$$q_i = MG(k-1, q_{i-1}, \bar{F}_{k-1}).$$

Then we define

$$z_{m_1+1} = z_{m_1} + I_{k-1}^k q_p.$$ 

Postsmoothing Step. For $m_1 + 2 \leq l \leq m_1 + m_2 + 1$, let

$$z_l = z_{l-1} + R_k(F_k - A_k z_{l-1}).$$

Then the final output of the $k^{\text{th}}$ level iteration is

$$MG(k, z_0, F_k) = z_{m_1+m_2+1}.$$ 

Here $p = 1$ or $p = 2$. When $p = 1$ it is called a $V$-cycle method. When $p = 2$ it is called a $W$-cycle method.
When applying the $k^{th}$ level iteration to (2.5.1), we use the following approach. We take the initial guess to be $I^k_{k-1} \hat{u}_{k-1}$, where $\hat{u}_{k-1}$ is the approximate solution already obtained for the equation $A_{k-1}u_{k-1} = F_{k-1}$. Then we apply the $k^{th}$ level iteration $r$ times.

**Algorithm 2.5.2.** (The Full Multigrid Algorithm) For $k = 1$, $\hat{u}_1 = R_1 F_1$.

For $k \geq 2$, the approximate solution $\hat{u}_k$ is obtained recursively from

\[
\begin{align*}
    \hat{u}_0 &= I^k_{k-1} \hat{u}_{k-1}, \\
    u_l^k &= MG(k, u_{l-1}^k, F_k), \quad 1 \leq l \leq r, \\
    \hat{u}_k &= u_r^k.
\end{align*}
\]

For simplicity, we consider the convergence of the one-sided $W$-cycle method, i.e., $p = 2, m_1 = m$ and $m_2 = 0$ in the algorithm 2.5.1. Then we have the following convergence result (cf. [20, Theorem 6.5.9]).

**Theorem 2.5.3.** (Convergence of the $k^{th}$ Level Iteration for the One-sided $W$-Cycle) For any $0 < \gamma < 1$, $m$ can be chosen large enough such that

\[
\|z - MG(k, z_0, F_k)\|_{H^1(\Omega)} \leq \gamma \|z - z_0\|_{H^1(\Omega)}, \quad \text{for } k = 1, 2, \ldots.
\]

**Remark 2.5.4.** Similar convergence results also hold for other $W$-cycle methods.

The convergence of the full multigrid method is a consequence of the convergence of $k^{th}$ level iteration (cf. [20, Theorem 6.7.1])

**Theorem 2.5.5.** (Full Multigrid Convergence) If the $k^{th}$ level iteration is a contraction with a contraction number $\gamma$ independent of $k$ and if $r$ is large enough, then there exists a constant $C > 0$ such that

\[
\|u_k - \hat{u}_k\|_{H^1(\Omega)} \leq Ch_k^\beta |u|_{H^{1+\beta}(\Omega)},
\]

where $\beta = \max\{1, \frac{\pi}{\omega}\}$.
Chapter 3

Vector Function Spaces and Helmholtz/Hodge Decompositions

The numerical methods we develop in this dissertation are based on the Helmholtz/Hodge decomposition for vector fields. Since Maxwell’s equations involve the divergence operator and curl operator, it is easy to see that Helmholtz/Hodge decomposition can play a role in the study of Maxwell’s equation. In this chapter, we review the Helmholtz/Hodge decomposition for two-dimensional vector fields, since we will focus on the two-dimensional Maxwell’s equations. The main references are [14, 35, 49].

3.1 Definitions and Properties of the Vector Function Spaces $H(div; \Omega)$ and $H(curl; \Omega)$

Let $\Omega \subset \mathbb{R}^2$ be an open set with a Lipschitz boundary $\Gamma$. The vector function spaces naturally related to the variational formulation of Maxwell’s equations are $H(div, \Omega)$ and $H(curl, \Omega)$.

**Definition 3.1.1.** Let $\mathbf{u} = (u_1(x_1,x_2), u_2(x_1,x_2))$ belong to $[L_2(\Omega)]^2$. We say that $\nabla \cdot \mathbf{u} \in L_2(\Omega)$ if there exists a function $v \in L_2(\Omega)$ such that

$$(v, \phi) = -(\mathbf{u}, \nabla \phi) \quad \forall \phi \in \mathcal{D}(\Omega).$$

We will then take $\nabla \cdot \mathbf{u}$ top be $v$. The vector function space $H(div; \Omega)$ is defined as follows:

$$H(div; \Omega) = \{\mathbf{u} \in [L_2(\Omega)]^2 : \nabla \cdot \mathbf{u} \in L_2(\Omega)\},$$

with the norm

$$\|\mathbf{u}\|_{H(div; \Omega)} = \left\{\|\mathbf{u}\|^2_{L_2(\Omega)} + \|\nabla \cdot \mathbf{u}\|^2_{L_2(\Omega)}\right\}^{1/2}.$$
Definition 3.1.2. Let \( u = (u_1(x_1, x_2), u_2(x_1, x_2)) \) belong to \([L_2(\Omega)]^2\). We say that \( \nabla \times u \in L_2(\Omega) \) if there exists a function \( v \in L_2(\Omega) \) such that

\[
(v, \phi) = (u, \nabla \times \phi) \quad \forall \phi \in \mathcal{D}(\Omega),
\]

where \( \nabla \times \phi = \left( \frac{\partial \phi}{\partial x_2}, -\frac{\partial \phi}{\partial x_1} \right) \). We then define \( \nabla \times u \) to be \( v \). The vector function space \( H(curl; \Omega) \) is defined as follows:

\[
H(curl; \Omega) = \{ u \in [L_2(\Omega)]^2 : \nabla \times u \in L_2(\Omega) \},
\]

with the norm

\[
\|u\|_{H(curl; \Omega)} = \left\{ \|u\|_{L_2(\Omega)}^2 + \|\nabla \times u\|_{L_2(\Omega)}^2 \right\}^{1/2}.
\]

First we discuss the properties of the spaces \( H(div; \Omega) \).

Theorem 3.1.3. (cf. [35, Theorem 2.4.]) The space \( \mathcal{D}(\bar{\Omega})^2 \) is dense in \( H(div; \Omega) \).

Because of the following theorem (cf. [35, Theorem 2.5]), the normal trace of \( H(div; \Omega) \) can be defined.

Theorem 3.1.4. The mapping \( \gamma_n : v \to v \cdot n|_{\Gamma} \) defined on \( \mathcal{D}(\bar{\Omega})^2 \) can be extended by continuity to a continuous linear mapping, still denoted by \( \gamma_n \), from \( H(div; \Omega) \) into \( H^{-1/2}(\Gamma) \), the dual space of \( H^{1/2}(\Gamma) \).

We call \( \gamma_n v \) the normal trace of \( v \) on \( \Gamma \) and it is denoted by \( v \cdot n \). We also denote \( (\gamma_n v)(\phi) \) for any \( \phi \in H^{1/2}(\Gamma) \) and \( \gamma_n v \in H^{-1/2}(\Gamma) \) by \( <v \cdot n, \phi>_{\Gamma} \).

Because of Theorem 3.1.4, we have a generalized versions of Green’s formulas ([35, Corollary 2.6]).

Corollary 3.1.5. Let \( v \in H(div; \Omega) \) and \( \phi \in H^{1}(\Omega) \), then

\[
(v, \nabla \phi) + (\nabla \cdot v, \phi) = <v \cdot n, \phi>_{\Gamma}.
\]
If \( u \in H^1(\Omega) \) and \( \Delta u \in L_2(\Omega) \), then \( \nabla u \) is in \( H(\text{div}; \Omega) \). Therefore Corollary 3.1.5 implies the following corollary.

**Corollary 3.1.6.** Let \( u \in H^1(\Omega) \) and \( \Delta u \in L_2(\Omega) \). Then \( \frac{\partial u}{\partial n} \in H^{-1/2}(\Gamma) \) and

\[
(\nabla u, \nabla v) + (\Delta u, v) = \langle \frac{\partial u}{\partial n}, v \rangle_{\Gamma} \quad \forall v \in H^1(\Omega).
\]

**Definition 3.1.7.** Let \( H_0(\text{div}; \Omega) \) denote

\[
\text{Ker}(\gamma_n) = \{ u \in H(\text{div}; \Omega) : u \cdot n|_{\Gamma} = 0 \}.
\]

Next, we will discuss the properties of \( H(\text{curl}; \Omega) \). Note that, in the two-dimensional case, the function \( v = (v_1, v_2) \in H(\text{curl}; \Omega) \) if and only if the function \( w = (-v_2, v_1) \in H(\text{div}; \Omega) \). Therefore, all the properties of \( H(\text{div}; \Omega) \) have similar versions for \( H(\text{curl}; \Omega) \) (cf. [35]).

**Theorem 3.1.8.** The space \( D(\bar{\Omega})^2 \) is dense in \( H(\text{curl}; \Omega) \).

Let us denote the tangential vector of \( \Gamma \) by \( \tau \) such that \( n \) and \( \tau \) obey the right-hand rule. Then we have the following extension theorem ([35, Theorem 2.11]).

**Theorem 3.1.9.** The mapping \( \gamma_\tau : v \to v \cdot \tau|_{\Gamma} \) defined on \( D(\bar{\Omega})^2 \) can be extended by continuity to a continuous linear mapping, still denoted by \( \gamma_\tau \), from \( H(\text{curl}; \Omega) \) into \( H^{-1/2}(\Gamma) \). Moreover, the following Green's formula holds:

\[
(\nabla \times v, \phi) - (v, \nabla \times \phi) = \langle \gamma_\tau v, \phi \rangle_{\Gamma} \quad \forall v \in H(\text{curl}; \Omega), \phi \in H^1(\Omega).
\]

**Remark 3.1.10.** We call \( \gamma_\tau v \) the tangential trace of \( v \) on \( \Gamma \) and it is denoted by \( v \cdot \tau \).

We also denote \( (\gamma_\tau v)(\phi) \), for any \( \phi \in H^{1/2}(\Gamma) \) and \( \gamma_\tau v \in H^{-1/2}(\Gamma) \), by \( < v \cdot \tau, \phi >_{\Gamma} \).

**Definition 3.1.11.** Let \( H_0(\text{curl}; \Omega) \) denote

\[
\text{Ker}(\gamma_\tau) = \{ u \in H(\text{curl}; \Omega) : u \cdot \tau|_{\Gamma} = 0 \}.
\]
The following lemma ([35, Lemma 2.4]) gives us a criterion for $H_0(\text{curl}; \Omega)$.

**Lemma 3.1.12.** A vector function $f$ of $H(\text{curl}; \Omega)$ belongs to $H_0(\text{curl}; \Omega)$ if and only if

$$(f, \nabla \times \phi) - (\nabla \times f, \phi) = 0 \quad \forall \phi \in H^1(\Omega).$$

### 3.2 Two-dimensional Helmholtz/Hodge Decompositions

In this section we extend the following classical Stokes’ Theorem.

**Theorem 3.2.1.** If a $C^1$ vector field has a vanishing curl in a simply-connected region of $\mathbb{R}^2$, then this vector field is the gradient of a function.

If a $C^1$ vector field has a vanishing divergence in a simply-connected region of $\mathbb{R}^2$, then this vector field is the curl of a function.

Let us first state a characterization for two-dimensional divergence free vector fields.

### 3.2.1 Characterization of Two-dimensional Divergence Free Vector Fields and Curl Free Vector Fields

We will not restrict ourselves to a simply-connected domain here. Instead, the domain can be multiply-connected.

Denote by $\Gamma_0$ the exterior boundary of $\Omega$ and by $\Gamma_i, 1 \leq i \leq p$, the other components of the boundary $\Gamma$. Then we have the following characterization for two-dimensional divergence free vector fields.

**Theorem 3.2.2.** (cf. [35, Theorem 3.1.]) A vector field $v \in [L_2(\Omega)]^2$ satisfies

$$\nabla \cdot v = 0 \quad \text{and} \quad <v \cdot n, 1>_{\Gamma_i} = 0 \quad \text{for} \quad 0 \leq i \leq p$$

if and only if there exists a stream function $\phi \in H^1(\Omega)$ such that:

$$v = \nabla \times \phi,$$
where $\nabla \times \phi = (\frac{\partial \phi}{\partial x_2}, -\frac{\partial \phi}{\partial x_1})$. Moreover $\phi$ is unique up to a constant in $H^1(\Omega)$.

Remark 3.2.3. If the domain $\Omega$ is simply-connected, then $v = \nabla \times \phi$ if and only if $\nabla \cdot v = 0$.

Similarly, we have the following characterization for two-dimensional curl-free vector fields.

**Theorem 3.2.4.** A vector field $v \in [L_2(\Omega)]^2$ satisfies:

$$\nabla \times v = 0 \quad \text{and} \quad <v \cdot \tau, 1 >_{\Gamma_i} = 0 \quad \text{for} \quad 0 \leq i \leq p$$

if and only if there exists a potential function $\phi \in H^1(\Omega)$ such that

$$v = \nabla \phi.$$

Moreover $\phi$ is unique up to a constant in $H^1(\Omega)$.

### 3.2.2 Helmholtz/Hodge Decompositions

There are many different ways to decompose a vector field into a divergence free field and a curl free field. Here, we introduce several Helmholtz/Hodge decompositions for $[L_2(\Omega)]^2$, $H_0(\text{curl}; \Omega)$, $H(\text{div}^0; \Omega)$ and $H(\text{div}^0; \Omega) \cap H_0(\text{curl}; \Omega)$, respectively.

We first introduce an orthogonal decomposition for the space $[L_2(\Omega)]^2$ with respect to weighted inner product that will be used in Chapter 4. As in Subsection 3.2.1, the domain $\Omega$ is multiply-connected. First we introduce a weighted inner product on the space $[L_2(\Omega)]^2$.

Let $\epsilon$ be a bounded positive function in $\Omega$. Define the weighted $L_2(\Omega; \epsilon)$ inner product $(\cdot, \cdot)_{L_2(\Omega; \epsilon)}$ on $[L_2(\Omega)]^2$ by

$$(v, w)_{L_2(\Omega; \epsilon)} = \int_{\Omega} \epsilon (v \cdot w) dx \quad \forall v, w \in [L_2(\Omega)]^2.$$

For distinction we use the notation $[L_2(\Omega; \epsilon)]^2$ for the space $[L_2(\Omega)]^2$ equipped with the weighted inner product $(\cdot, \cdot)_{L_2(\Omega; \epsilon)}$. 

46
**Definition 3.2.5.** The subspace $H(div^0; \Omega; \epsilon)$ is defined by

$$H(div^0; \Omega; \epsilon) = \{ v \in [L_2(\Omega)]^2 : \nabla \cdot (\epsilon v) = 0 \}.$$

**Definition 3.2.6.** The space $\mathcal{H}(\Omega; \epsilon)$ is defined by

$$\mathcal{H}(\Omega; \epsilon) = \{ \phi \in H^1(\Omega) : (\epsilon \nabla \phi, \nabla v) = 0 \quad \forall v \in H^1_0(\Omega);$$

$$\phi|_{\Gamma_0} = 0; \quad \phi|_{\Gamma_i} = \text{a constant} \quad \text{for} \quad 1 \leq i \leq p \}.$$

We have the following decomposition theorem for $[L_2(\Omega; \epsilon)]^2$.

**Theorem 3.2.7.** With respect to the weighted $L_2(\Omega; \epsilon)$ inner product, we have the decomposition:

$$[L_2(\Omega; \epsilon)]^2 = K \oplus H \oplus G,$$

where

$$K = \epsilon^{-1} \nabla \times H^1(\Omega) = \{ \epsilon^{-1} \nabla \times \phi : \phi \in H^1(\Omega) \},$$

$$G = \nabla H^1_0(\Omega) = \{ \nabla \phi : \phi \in H^1_0(\Omega) \},$$

and

$$H = \nabla \mathcal{H}(\Omega; \epsilon) = \{ \nabla \phi : \phi \in \mathcal{H}(\Omega; \epsilon) \}.$$

Moreover, with respect to the same weighted inner product, we have the decomposition:

$$H(div^0; \Omega; \epsilon) = K \oplus H.$$  \hspace{1cm} (3.2.2)

To prepare for the proof of Theorem 3.2.7, we first prove a few lemmas.

**Lemma 3.2.8.** We have an orthogonal decomposition

$$[L_2(\Omega; \epsilon)]^2 = H(div^0; \Omega; \epsilon) \oplus G$$

with respect to the weighted $L_2(\Omega; \epsilon)$ inner product.
Proof. Let $\mathbf{v} \in [L_2(\Omega)]^2$. Then

$$\mathbf{v} \in H(\text{div}^0; \Omega; \epsilon)$$

iff

$$(\epsilon \mathbf{v}, \nabla \phi) = 0 \quad \forall \phi \in \mathcal{D}(\Omega)$$

iff

$$(\epsilon \mathbf{v}, \nabla \phi) = 0 \quad \forall \phi \in H_0^1(\Omega).$$

In other words, $H(\text{div}^0; \Omega; \epsilon)$ is the orthogonal complement of $G = \nabla H_0^1(\Omega)$ with respect to the weighted $L_2(\Omega; \epsilon)$ inner product.

Next we will show that $G = \nabla H_0^1(\Omega)$ is closed in $[L_2(\Omega; \epsilon)]^2$. First we note that it is equivalent to show $G = \nabla H_0^1(\Omega)$ is closed in $[L_2(\Omega)]^2$, since the norms induced by the $L_2(\Omega)$ inner product and the weighted $L_2(\Omega; \epsilon)$ inner product are equivalent on the space $[L_2(\Omega)]^2$.

Let $\phi_n$ be a sequence in $H_0^1(\Omega)$ such that the sequence $\nabla \phi_n$ converges to a function $\mathbf{v}$ in $[L_2(\Omega)]^2$ and hence $\nabla \phi_n$ is a Cauchy sequence in $[L_2(\Omega)]^2$. Because of Poincare’s inequality (cf. Proposition (5.3.5), [20]), $\phi_n$ is a Cauchy sequence in $H_0^1(\Omega)$ and hence the sequence $\phi_n$ converges to a function $\phi$ in $H_0^1(\Omega)$ and hence $\nabla \phi_n$ converges to a function $\nabla \phi$ in $[L_2(\Omega)]^2$. So $\mathbf{v} = \nabla \phi$, where $\phi \in H_0^1(\Omega)$. Therefore $G = \nabla H_0^1(\Omega)$ is closed in $[L_2(\Omega)]^2$.

Since $G = \nabla H_0^1(\Omega)$ is closed in $[L_2(\Omega; \epsilon)]^2$ and $H(\text{div}^0; \Omega; \Omega)$ is the orthogonal complement of $G = \nabla H_0^1(\Omega)$ in $[L_2(\Omega; \epsilon)]^2$, we have the orthogonal decomposition

$$[L_2(\Omega; \epsilon)]^2 = H(\text{div}^0; \Omega; \epsilon) \oplus G$$

with respect to the weighted $L_2(\Omega; \epsilon)$ inner product. 

□
Lemma 3.2.9. Let $\zeta \in H^1(\Omega)$ such that the trace of $\zeta$ on $\Gamma_i$ is a constant $\gamma_i$ for $0 \leq i \leq p$. Then we have

$$(\nabla \times \psi, \nabla \zeta) = 0 \quad \forall \psi \in H^1(\Omega). \quad (3.2.3)$$

Proof. Let $v = \nabla \times \psi$. By Theorem 3.2.2, we have

$$< v \cdot n, 1 >_{\Gamma_i} = 0 \quad \forall 0 \leq i \leq p. \quad (3.2.4)$$

So Corollary 3.1.5 and (3.2.4) imply that

$$(\nabla \times \psi, \nabla \zeta) = (v, \nabla \zeta) = \sum_{i=0}^{p} < v \cdot n, \zeta >_{\Gamma_i}$$

$$= \sum_{i=0}^{p} \zeta |_{\Gamma_i} < v \cdot n, 1 >_{\Gamma_i} = 0.$$

This lemma leads to the following corollary.

Corollary 3.2.10. We have $\nabla H^1_0(\Omega) \subset H_0(\text{curl}; \Omega)$.

Proof. Obviously, $\nabla H^1_0(\Omega) \subset H(\text{curl}; \Omega)$. Lemma 3.2.9 implies that for any $\zeta \in H^1_0(\Omega)$,

$$(\nabla \times \psi, \nabla \zeta) = 0 = (\psi, \nabla \times \nabla \zeta) \quad \forall \psi \in H^1(\Omega). \quad (3.2.5)$$

So $\nabla \zeta \in H_0(\text{curl}; \Omega)$ by Lemma 3.1.12 and hence $\nabla H^1_0(\Omega) \subset H_0(\text{curl}; \Omega)$. \qed

Lemma 3.2.11. Let $\varphi \in \mathcal{H}(\Omega; \epsilon)$. Then

$$\int_{\Gamma_i} \epsilon \frac{\partial \varphi}{\partial n} ds = 0 \quad \text{for} \quad 1 \leq i \leq p \quad (3.2.6)$$

if and only if $\varphi = 0$. 

49
Proof. It is sufficient to show that if \( \varphi \in H(\Omega; \epsilon) \) and \( \varphi \) satisfies (3.2.6), then \( \varphi = 0 \).

By Corollary 3.1.5 and Definition 3.2.6, we have

\[
(\epsilon \nabla \varphi, \nabla \varphi) = \sum_{i=0}^{p} \epsilon \frac{\partial \varphi}{\partial n} \cdot \varphi >_{\Gamma_i} \sum_{i=0}^{p} \varphi|_{\Gamma_i} \epsilon \frac{\partial \varphi}{\partial n} , 1 >_{\Gamma_i} = 0.
\]

So \( \varphi = 0 \), since the domain \( \Omega \) is connected and \( \varphi = 0 \) on \( \Gamma_0 \).

Lemma 3.2.12. We have an orthogonal decomposition

\[ H(div^0; \Omega; \epsilon) = K \oplus H \]

with respect to the weighted \( L_2(\Omega; \epsilon) \) inner product.

Proof. Lemma 3.2.9 implies that \( K \) and \( H \) are orthogonal to each other under the weighted \( L_2(\Omega; \epsilon) \) inner product. \( H \) is also closed, since it is a finite dimensional space. Next we will show that \( K \) is closed in \([L_2(\Omega; \epsilon)]^2\).

First we note that it is equivalent to show \( K \) is closed in \([L_2(\Omega)]^2\), since the norms induced by the \( L_2(\Omega) \) inner product and the weighted \( L_2(\Omega; \epsilon) \) inner product are equivalent on the space \([L_2(\Omega)]^2\). Furthermore, it is equivalent to show that \( \nabla \times H^1(\Omega) \) is closed \([L_2(\Omega)]^2\).

Now let \( \phi_n \) be a sequence in \( H^1(\Omega) \) such that the sequence \( \nabla \times \phi_n \) converges to a function \( \nu \) in \([L_2(\Omega)]^2\) and hence \( \nabla \times \phi_n \) is a Cauchy sequence in \([L_2(\Omega)]^2\). Because of Friedrichs’ inequality (cf. [20, Lemma 4.3.14]), it follows that the sequence \( \phi_n - \bar{\phi}_n \) is a Cauchy sequence in \( H^1(\Omega) \) and hence converges to a function \( \phi \) in \( H^1(\Omega) \), where \( \bar{\phi}_n = \frac{1}{|\Omega|} \int_{\Omega} \phi_n dx \). Hence \( \nabla \times \phi_n = \nabla \times (\phi_n - \bar{\phi}_n) \) converges to the function \( \nabla \times \phi \) in \([L_2(\Omega)]^2\). So \( \nu = \nabla \times \phi \), where \( \phi \in H^1(\Omega) \). Therefore \( \nabla \times H^1(\Omega) \) is closed in \([L_2(\Omega)]^2\).

The remaining task of the proof is to show that \( H(div^0; \Omega; \epsilon) = K + H \).
Let \( \mathbf{v} \) be a vector function in \( H(\text{div}^0; \Omega; \epsilon) \), there exists a function \( \varphi \in H \) such that
\[
< \epsilon \frac{\partial \varphi}{\partial n}, 1 >_{\Gamma_i} = < \epsilon \mathbf{v} \cdot \mathbf{n}, 1 >_{\Gamma_i}, \quad \forall 1 \leq i \leq p,
\]
since by Lemma 3.2.11, the mapping defined by
\[
\varphi \rightarrow \begin{bmatrix}
< \epsilon \nabla \varphi \cdot \mathbf{n}, 1 >_{\Gamma_1} \\
< \epsilon \nabla \varphi \cdot \mathbf{n}, 1 >_{\Gamma_2} \\
\vdots \\
< \epsilon \nabla \varphi \cdot \mathbf{n}, 1 >_{\Gamma_p}
\end{bmatrix}
\]
is an isomorphism from \( \mathcal{H}(\Omega; \epsilon) \) to \( \mathbb{R}^p \).

Now \( \nabla \cdot (\epsilon \mathbf{v} - \epsilon \nabla \varphi) = 0 \) and
\[
< (\epsilon \mathbf{v} - \epsilon \nabla \varphi) \cdot \mathbf{n}, 1 >_{\Gamma_i} = 0 \quad \text{for} \quad 1 \leq i \leq p.
\]
Therefore, by Theorem 3.2.2, there exists \( \phi \in H^1(\Omega) \) such that \( \epsilon \mathbf{v} - \epsilon \nabla \varphi = \nabla \times \phi \).

Let \( \mathbf{v}_1 = \epsilon^{-1} \nabla \times \phi \), then we have
\[
\mathbf{v} = \mathbf{v}_1 + \nabla \varphi,
\]
where \( \mathbf{v}_1 \in K \) and \( \varphi \in HG \).

Proof. (Proof of Theorem 3.2.7) Using Lemma 3.2.8 and Lemma 3.2.12, we have Theorem 3.2.7.

Theorem 3.2.7 leads to the decomposition for \( H_0(\text{curl}; \Omega) \) and \( H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega; \epsilon) \) in the following corollaries.

**Corollary 3.2.13.** (Decomposition for \( H_0(\text{curl}; \Omega) \)) Suppose that \( \mathbf{v} \in H_0(\text{curl}; \Omega) \), then there exist a unique \( \mathbf{\hat{v}} \) and \( \varphi \) such that
\[
\mathbf{v} = \mathbf{\hat{v}} + \nabla \varphi,
\]
where \( \mathbf{\hat{v}} \in H_0(\text{curl}; \Omega) \cap H(\text{div}^0; \Omega; \epsilon) \) and \( \varphi \in H^1_0(\Omega) \).
Proof. Since \( v \in H_0(\text{curl}; \Omega) \subset [L_2(\Omega)]^2 \), by Theorem 3.2.7, there exists a unique \( \hat{v} \in H(\text{div}^0; \Omega; \varepsilon) \) and \( \varphi \in H^1_0(\Omega) \) such that

\[
v = \hat{v} + \nabla \varphi.
\]

Note that \( \nabla \varphi \in H_0(\text{curl}; \Omega) \) by Lemma 3.2.9 and \( v \in H_0(\text{curl}; \Omega) \), which imply that \( \hat{v} \in H_0(\text{curl}; \Omega) \). \( \square \)

A similar argument as in the proof of Corollary 3.2.13 leads to the following decomposition.

**Corollary 3.2.14.** (Decomposition for \( H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega; \varepsilon) \)) Suppose that \( v \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega; \varepsilon) \), then there exist a unique \( \hat{v} \) and \( \varphi \) such that

\[
v = \hat{v} + \nabla \varphi,
\]

where \( \hat{v} \in H_0(\text{curl}; \Omega) \cap H(\text{div}^0; \Omega; \varepsilon) \) and \( \varphi \in H^1_0(\Omega) \). Moreover \( \varphi \) satisfies

\[
(\varepsilon \nabla \varphi, \nabla \phi) = (\nabla \cdot (\varepsilon v), \phi) \quad \forall \phi \in H^1_0(\Omega).
\]
Chapter 4
Maxwell Equations in Homogeneous Media

4.1 Introduction

For simplicity, we assume that $\epsilon = 1$ and $\mu = 1$. We will follow the notation introduced in Subsection 3.2.2 but with $\epsilon$ suppressed. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain, $\Gamma_0$ the exterior boundary, $\Gamma_1, \cdots \Gamma_p$ the components of the interior boundary, and $\alpha$ be a constant. Then the weak formulation of the Maxwell's equations is:

For $f \in [L_2(\Omega)]^2$, find $\mathbf{u} \in H_0(curl; \Omega) \cap H(div^0; \Omega)$ such that

$$\left(\nabla \times \mathbf{u}, \nabla \times \mathbf{v}\right) + \alpha (\mathbf{u}, \mathbf{v}) = (f, \mathbf{v}) \quad \forall \mathbf{v} \in H_0(curl; \Omega) \cap H(div^0; \Omega).$$

We assume that $-\alpha$ is not a Maxwell eigenvalue so that (4.1.1) is uniquely solvable.

Remark 4.1.1. For $\alpha \leq 0$, (4.1.1) is exactly the weak form for (1.1.38). For $\alpha > 0$, (4.1.1) is related to the time-domain Maxwell's equations.

We will use the Helmholtz/Hodge decomposition from Chapter 3 to reduce (4.1.1) to standard second order scalar elliptic boundary value problems. The presentation in this chapter follows [14] closely.

4.2 Equation for $\xi = \nabla \times \mathbf{u}$ and $\phi$

Because of the Helmholtz/Hodge decomposition for $H(div^0; \Omega)$ (See Theorem 3.2.7), we can reformulate the problem (4.1.1) as coupled elliptic problems.

Theorem 4.2.1. Suppose that the solution $\mathbf{u} \in H_0(curl; \Omega) \cap H(div^0; \Omega)$ is decomposed as in Theorem 3.2.7,

$$\mathbf{u} = \nabla \phi + \nabla \varphi,$$

(4.2.1)
where $\varphi \in H(\Omega)$ and $\phi \in H^1(\Omega)$ satisfies $(\phi, 1) = 0$. Then $\phi$ is determined by

$$
(\nabla \times \phi, \nabla \times \psi) = (\xi, \psi) \quad \forall \psi \in H^1(\Omega)
$$

(4.2.2)

and the constraint

$$
(\phi, 1) = 0,
$$

(4.2.3)

where the function $\xi = \nabla \times \hat{u} \in H^1(\Omega)$ is determined by

$$
(\nabla \times \xi, \nabla \times \psi) + \alpha(\xi, \psi) = (f, \nabla \times \psi) \quad \forall \psi \in H^1(\Omega)
$$

(4.2.4)

when $\alpha \neq 0$, and by (4.2.4) together with the constraint

$$
(\xi, 1) = 0
$$

(4.2.5)

when $\Omega$ is simply connected and $\alpha = 0$. Moreover, when $p \geq 1$ and $\alpha \neq 0$, $\varphi$ can be determined by

$$
(\nabla \varphi, \nabla \psi) = \frac{1}{\alpha} (f, \nabla \psi) \quad \forall \psi \in H(\Omega).
$$

(4.2.6)

To prove Theorem 4.2.1, we need the following lemma concerning with the strong form of (4.1.1).

**Lemma 4.2.2.** The solution $\hat{u}$ of (4.1.1) satisfies

$$
\nabla \times (\nabla \times \hat{u}) + \alpha \hat{u} = Qf
$$

in the sense of distribution, where $Q : [L_2(\Omega)]^2 \to H(div^0; \Omega)$ is the orthogonal projection.

**Proof.** Let $\zeta \in [\mathcal{D}(\Omega)]^2$ be a $C^\infty$ vector field with compact support in $\Omega$. So $\zeta \in H_0(curl; \Omega)$, $Q\zeta \in H(div^0; \Omega)$ and $\zeta - Q\zeta \in \nabla H_0^1(\Omega)$ by Lemma 3.2.8.

Because of Corollary 3.2.10, we have $\zeta - Q\zeta \in H_0(curl; \Omega)$ and hence $Q\zeta \in H_0(curl; \Omega)$. Therefore,

$$
Q\zeta \in H_0(curl; \Omega) \cap H(div^0; \Omega).
$$

(4.2.7)
Furthermore, we have
\[ \nabla \times (\zeta - Q\zeta) = 0, \quad (4.2.8) \]
since \( \nabla \times (\nabla H^1_0(\Omega)) = \{0\} \), and for \( \hat{u} \in H(\text{div}^0, \Omega) \), we have
\[ (\hat{u}, \zeta - Q\zeta) = 0, \quad (4.2.9) \]
by Lemma 3.2.8 and the fact that \( \zeta - Q\zeta \in \nabla H^1_0(\Omega) \).

Using (4.1.1), (4.2.7), (4.2.8) and (4.2.9), we have
\[
(\nabla \times \hat{u}, \nabla \times \zeta) + \alpha(\hat{u}, \zeta) = (\nabla \times \hat{u}, \nabla \times (Q\zeta + (\zeta - Q\zeta))) + \alpha(\hat{u}, Q\zeta + (\zeta - Q\zeta)) = (\nabla \times \hat{u}, \nabla \times Q\zeta) + \alpha(\hat{u}, Q\zeta) = (f, Q\zeta) = (Qf, \zeta),
\]
which completes the proof.

\[ \Box \]

**Remark 4.2.3.** Lemma 4.2.2 implies that \( \xi = \nabla \times \hat{u} \in H^1(\Omega) \) and
\[
\nabla \times \xi + \alpha \hat{u} = Qf. \quad (4.2.10)
\]

Now we are ready to prove Theorem 4.2.1.

**Proof. (Proof of Theorem 4.2.1)** First, let us justify (4.2.2) by using (4.2.1), Lemma 3.1.12 and Lemma 3.2.9. Let \( \psi \in H^1(\Omega) \) be arbitrary. We have
\[ (\nabla \times \phi, \nabla \times \psi) = (\nabla \times \phi + \nabla \varphi, \nabla \times \psi) = (\hat{u}, \nabla \times \varphi) = (\nabla \times \hat{u}, \varphi) = (\xi, \varphi). \]

To justify (4.2.6) when \( p \geq 1 \), we take \( \mathbf{v} = \nabla \psi \) in (4.1.1) where \( \psi \in \mathcal{H}(\Omega) \), and replace \( \hat{u} \) by the Hodge/Helmholtz decomposition (4.2.1). Then we obtain the equation
\[ \alpha(\nabla \times \phi + \nabla \varphi, \nabla \psi) = (f, \nabla \psi), \]
which implies (4.2.6) by Lemma 3.2.9.

Now let us justify (4.2.4). Let \( \psi \in H^1(\Omega) \) be arbitrary. Since \( \nabla \times \psi \in H(div; \Omega) \) by Lemma 3.2.2, we have

\[
(f, \nabla \times \psi) = (Qf, \nabla \times \psi) = (\nabla \times \xi + \alpha \hat{u}, \nabla \times \psi) \quad \text{(by the equation (4.2.10))}
\]

\[
= (\nabla \times \xi, \nabla \times \psi) + \alpha(\nabla \times \hat{u}, \psi) \quad \text{(by Lemma 3.1.12)}
\]

which gives (4.2.4). The constraint \( (\xi, 1) = 0 \) follows immediately from Lemma 3.1.12.

\[\square\]

Note that \( \mathcal{H}(\Omega) \) is a finite dimensional space with the basis \( \{\varphi_1, \varphi_2, \cdots, \varphi_p\} \), where \( \varphi_i, 1 \leq i \leq p \) satisfies that

\[
\Delta \varphi_i = 0, \quad \text{(4.2.11a)}
\]

\[
\varphi_i|_{r_0} = 0, \quad \text{(4.2.11b)}
\]

\[
\varphi_i|_{r_j} = \delta_{ij} \quad \text{for } 1 \leq j \leq p, \quad \text{(4.2.11c)}
\]

i.e.,

\[
(\nabla \varphi_i, \nabla v) = 0 \quad \text{for } v \in H^1_0(\Omega), \quad \text{(4.2.12a)}
\]

\[
\varphi_i|_{r_0} = 0, \quad \text{(4.2.12b)}
\]

\[
\varphi_i|_{r_j} = \delta_{ij} \quad \text{for } 1 \leq j \leq p, \quad \text{(4.2.12c)}
\]

therefore \( \varphi \) in (4.2.6) can be written as

\[
\sum_{i=1}^{p} c_i \varphi_i, \quad \text{(4.2.13)}
\]
where the coefficients $c_i$'s are determined by the symmetric positive-definite system

$$\sum_{i=1}^{p} (\nabla \varphi_i, \nabla \varphi_k)c_i = \frac{1}{\alpha}(f, \nabla \varphi_k) \quad \text{for} \quad 1 \leq k \leq p. \quad (4.2.14)$$

Next we discuss the relation between the solvability of (4.1.1) and the solvability of (4.2.4) under the condition that $-\alpha(\neq 0)$ is not a Maxwell eigenvalue.

**Lemma 4.2.4.** For $\alpha \neq 0$, the problem (4.1.1) is uniquely solvable if and only if the problem (4.2.4) is uniquely solvable.

**Proof.** Let $\alpha$ be nonzero. Since $H^1(\Omega)$ is compactly embedded in $L_2(\Omega)$ by the Rellich-Kondrachov theorem [1] and $H_0(curl; \Omega) \cap H(div^0; \Omega)$ is compactly embedded in $[L_2(\Omega)]^2$ by a result of Weber [62], we can apply the Fredholm alternative [33] to consider only the homogeneous equation corresponding to (4.1.1)

$$(\nabla \times w, \nabla \times v) + \alpha(w, v) = 0 \quad \forall v \in H_0(curl; \Omega) \cap H(div^0; \Omega), \quad (4.2.15)$$

and the homogeneous equation corresponding to (4.2.4)

$$(\nabla \times \eta, \nabla \times \psi) + \alpha(\eta, \psi) = 0 \quad \forall \psi \in H^1(\Omega). \quad (4.2.16)$$

By the Fredholm alternative (cf. [34, Theorem 5.11]), it suffices to show that (4.2.15) has a nontrivial solution $w \in H_0(curl; \Omega) \cap H(div^0; \Omega)$ if and only if (4.2.16) has a nontrivial solution $\eta \in H^1(\Omega)$.

Suppose there exists a nontrivial $w \in H_0(curl; \Omega) \cap H(div^0; \Omega)$ that satisfies (4.2.15). Let $\eta = \nabla \times w$, then $\eta \in H^1(\Omega)$ and (4.2.16) holds as a special case of (4.2.4) where $f = 0$.

Suppose there exists a nontrivial $\eta \in H^1(\Omega)$ that satisfies (4.2.16). Since $\alpha \neq 0$, we deduce from (4.2.16) that $(\eta, 1) = 0$. Let $w = \nabla \times \rho$, where $\rho \in H^1(\Omega)$ is defined by the Neumann problem

$$(\nabla \times \rho, \nabla \times \psi) = (\eta, \psi) \quad \psi \in H^1(\Omega), \quad (4.2.17a)$$

57
\((\rho, 1) = 0. \) \hspace{1cm} (4.2.17b)

Then Theorem 3.2.2 and (4.2.17a) imply

\[ \eta = \nabla \times w \] 

(4.2.18)

and \( w \in H(\text{div}^0; \Omega) \). Note that \( w \in H(\text{curl}; \Omega) \) by (4.2.16). Since (4.2.17a) can also be written as

\[ (w, \nabla \times \psi) = (\nabla \times w, \psi) \quad \forall \psi \in H^1(\Omega), \]

we have \( w \in H_0(\text{curl}; \Omega) \) by Lemma 3.1.12. Therefore, \( w \in H_0(\text{curl}; \Omega) \cap H(\text{div}^0; \Omega) \).

It is easy to see that \( w \) is nontrivial. To check that it satisfies (4.2.15), we take an arbitrary \( v \in H_0(\text{curl}; \Omega) \cap H(\text{div}^0; \Omega) \) and write its Hodge decomposition (cf. Theorem 3.2.7) as

\[ v = \nabla \times \phi + \nabla \varphi, \] 

(4.2.19)

where \( \phi \in H^1(\Omega) \) and \( \varphi \in H(\Omega) \). Note that, by Lemma 3.2.9, we have

\[ (\nabla \times \eta, \nabla \varphi) = 0 \quad \text{and} \quad (w, \nabla \varphi) = (\nabla \times \rho, \nabla \varphi) = 0. \] 

(4.2.20)

It follows from Lemma 3.1.12, (4.2.16), (4.2.18), (4.2.19) and (4.2.20) that

\[ (\nabla \times w, \nabla \times v) = (\eta, \nabla \times v) = (\nabla \times \eta, v) \]

\[ = (\nabla \times \eta, \nabla \times \phi + \nabla \varphi) \]

\[ = (\nabla \times \eta, \nabla \times \phi) \]

\[ = -\alpha(\eta, \phi) \]

\[ = -\alpha(\nabla \times w, \phi) \]

\[ = -\alpha(w, \nabla \times \phi) \]

\[ = -\alpha(w, \nabla \times \phi + \nabla \varphi) = -\alpha(w, v), \]

58
i.e., \( w \) satisfies (4.2.15).

It follows from Theorem 4.2.1 and Lemma 4.2.4 that we can solve (4.1.1) by the following numerical procedure under the assumption that \(-\alpha\) is not a Maxwell eigenvalue.

Step 1. Compute a numerical approximation \( \tilde{\xi} \) of \( \xi \) by solving (4.2.4) when \( \alpha \neq 0 \), and by solving (4.2.4) with the constraint \( (\xi, 1) = 0 \) when \( \Omega \) is simply connected and \( \alpha = 0 \).

Step 2. Compute a numerical approximation \( \tilde{\phi} \) of \( \phi \) by solving (4.2.2) under the constraint \( (\phi, 1) = 0 \), where \( \xi \) is replaced by \( \tilde{\xi} \).

Step 3. Compute numerical approximations \( \tilde{\varphi}_1, \cdots, \tilde{\varphi}_p \) of \( \varphi_1, \cdots, \varphi_p \) by solving the boundary value problems in (4.2.11).

Step 4. Compute numerical approximations \( \tilde{c}_1, \cdots, \tilde{c}_p \) by solving (4.2.14), where \( \varphi_1, \cdots, \varphi_p \) are replace by \( \tilde{\varphi}_1, \cdots, \tilde{\varphi}_p \).

Step 5. The numerical approximation \( \tilde{u} \) for \( u \) is given by

\[
\tilde{u} = \nabla \times \tilde{\phi} + \sum_{i=1}^{p} \tilde{c}_i \nabla \tilde{\varphi}_i.
\]

### 4.3 A \( P_1 \) Finite Element Method

In this section, we use a \( P_1 \) finite element method to demonstrate our approach.

Let \( \mathcal{T}_h \) be a quasi-uniform simplicial triangulation of \( \Omega \) with mesh size \( h \) and \( V_h \subset H^1(\Omega) \) be the \( P_1 \) finite element space associated with \( \mathcal{T}_h \) (See Section 2.4).

For \( \alpha \neq 0 \), the \( P_1 \) finite element method for (4.2.4) is to find \( \xi_h \in V_h \) such that

\[
(\nabla \times \xi_h, \nabla \times v) + \alpha(\xi_h, v) = (f, \nabla \times v) \quad \forall v \in V_h.
\]
For $\alpha > 0$, the problem (4.3.1) is symmetric positive-definite and hence well-posed. It is also well-posed for $\alpha < 0$ provided $-\alpha$ is not a Maxwell eigenvalue and $h$ is sufficiently small (cf. Lemma 4.4.2).

Note that when $\alpha \neq 0$ (4.3.1) implies

$$(\xi_h, 1) = 0. \quad (4.3.2)$$

When $\Omega$ is simply connected and $\alpha = 0$, $\xi_h \in V_h$ is determined by (4.3.1) together with the constraint (4.3.2). It is a well-posed problem because of the Poincare-Friedrichs inequality (cf. [20])

$$\|v\|_{L^2(\Omega)} \leq C(|(v, 1)| + \|\nabla \times v\|_{L^2(\Omega)}) \quad \forall v \in H^1(\Omega). \quad (4.3.3)$$

The $P_1$ finite element approximation $\phi_h$ of $\phi$ is then determined by

$$(\nabla \times \phi_h, \nabla \times v) = (\xi_h, v) \quad \forall v \in V_h, \quad (4.3.4a)$$

$$(\phi_h, 1) = 0. \quad (4.3.4b)$$

The problem (4.3.4) is well-posed because of (4.3.2) and (4.3.3).

For the multiply connected domain $\Omega$ (i.e., $p \neq 0$), we have the approximation $\varphi_h$ of $\varphi$ as follows:

$$\varphi_h = \sum_{i=1}^{p} c_{i,h} \nabla \varphi_{i,h},$$

where $\varphi_{i,h}$ is determined by

$$(\nabla \varphi_{i,h}, \nabla v) = 0 \quad \forall v \in \tilde{V}_h = V_h \cap H^1_0(\Omega), \quad (4.3.5a)$$

$$\varphi_{i,h}|_{\Gamma_0} = 0, \quad (4.3.5b)$$

$$\varphi_{i,h}|_{\Gamma_k} = \delta_{jk} \quad \text{for} \quad 1 \leq k \leq p, \quad (4.3.5c)$$

and the $c_{i,h}$’s are determined by

$$\sum_{i=1}^{p} (\nabla \varphi_{i,h}, \nabla \varphi_{k,h}) c_{i,h} = \frac{1}{\alpha} (f, \nabla \varphi_{k,h}) \quad \text{for} \quad 1 \leq k \leq p. \quad (4.3.6)$$
Finally, we approximate $\hat{u}$ by the piecewise constant vector field $\hat{u}_h$ defined by

$$
\hat{u}_h = \nabla \times \phi_h + \sum_{i=1}^{p} c_{i,h} \nabla \varphi_{i,h}.
$$

\[4.3.7\]

### 4.4 Convergence Analysis

In this section, we use standard techniques to analyze the $P_1$ finite element method in Section 4.3, since (4.2.2), (4.2.1) and (4.2.6) only involve standard second order scalar elliptic problems. Before doing this, let us introduce the related interpolation error estimates which are similar to the one introduced in Section 2.4.

Let the index $\beta$ be defined by

$$
\beta = \min(1, \min_{1 \leq l \leq N0} \pi/\omega_l),
$$

\[4.4.1\]

where $\omega_1, \omega_2, \ldots, \omega_{N0}$ are the interior angles at the corners of $\Omega$.

We have the following estimate for the solution of (4.3.5):

$$
\|\varphi_i - \Pi_h \varphi_i\|_{L^2(\Omega)} + h|\varphi_i - \Pi_h \varphi_i|_{H^1(\Omega)} \leq Ch^{1+\beta},
$$

\[4.4.2\]

where $\Pi_h$ is the nodal interpolation operator for the $P_1$ finite element.

Similarly, for the solution $\zeta$ of the Laplace equation with homogeneous Neumann boundary condition, we have

$$
\|\zeta - \Pi_h \zeta\|_{L^2(\Omega)} + h|\zeta - \Pi_h \zeta|_{H^1(\Omega)} \leq Ch^{1+\beta}\|g\|_{L^2(\Omega)},
$$

\[4.4.3\]

where $g$ is the right-hand side function.

We begin by comparing $\xi_h$ and $\xi = \nabla \times \hat{u}$. The following result is obtained by using (4.2.4), (4.3.1), (4.4.3) and a standard duality argument.

**Lemma 4.4.1.** For $\alpha > 0$ (general $\Omega$) and $\alpha = 0$ (simply connected $\Omega$), we have

$$
\|\xi - \xi_h\|_{L^2(\Omega)} \leq Ch^{\beta} \inf_{v \in \mathcal{V}_h} \|\nabla \times (\xi - v)\|_{L^2(\Omega)}.
$$

\[4.4.4\]
Proof. Combining (4.2.4) and (4.3.1), we have the Galerkin orthogonality

\[(\nabla \times (\xi - \xi_h), \nabla \times v) + \alpha((\xi - \xi_h), v) = 0 \quad \text{for} \quad v \in V_h. \quad (4.4.5)\]

From (4.4.5) we conclude that

\[((\xi - \xi_h), 1) = 0 \quad (4.4.6)\]

when \(\alpha \neq 0\). If \(\alpha = 0\) (simply connected \(\Omega\)), then, from (4.2.5) and (4.3.2), we have the equation (4.4.6).

Since \(\xi, \xi_h \in H^1(\Omega)\), it follows from (4.4.6) and Poincaré-Friedrichs inequality (cf. [20, (10.6.1)]) that

\[\|\xi - \xi_h\|_{L^2(\Omega)} \leq C|\xi - \xi_h|_{H^1(\Omega)}, \quad (4.4.7)\]

where the positive constant \(C\) depends only on the domain \(\Omega\).

Now we estimate \(|\xi - \xi_h|_{H^1(\Omega)}\). Let \(v \in V_h\). It follows from (4.4.5) and (4.4.7) that

\[|\xi - \xi_h|_{H^1(\Omega)}^2 + \alpha\|\xi - \xi_h\|_{L^2(\Omega)}^2 = (\nabla \times (\xi - \xi_h), \nabla \times (\xi - \xi_h)) + \alpha(\xi - \xi_h, \xi - \xi_h)\]
\[= (\nabla \times (\xi - \xi_h), \nabla \times (\xi - v)) + \alpha(\xi - \xi_h, \xi - v)\]
\[\leq C|\xi - \xi_h|_{H^1(\Omega)}|\xi - v|_{H^1(\Omega)},\]

which implies

\[|\xi - \xi_h|_{H^1(\Omega)} \leq C|\xi - v|_{H^1(\Omega)} \quad \forall v \in V_h. \quad (4.4.8)\]

We prove an error estimate for \(\xi - \xi_h\) in the \(L^2\) norm by a duality argument.

Let \(\zeta\) be the solution of

\[(\nabla \times \zeta, \nabla \times v) + \alpha(\zeta, v) = (e, v) \quad \forall v \in H^1(\Omega), \quad (4.4.9)\]

where \(e = \xi - \xi_h\). When \(\alpha = 0\), (4.4.9) is uniquely solvable up to an additive constant (cf. [20, Section 5.2]) and we assume its solution \(\zeta\) satisfying the constant
\((\zeta,1) = 0\). It follows from (4.4.3), (4.4.5), (4.4.7), (4.4.8), and (4.4.9) that

\[
\|\xi - \xi_h\|^2_{L^2(\Omega)} = (e, \xi - \xi_h)
\]

\[
= (\nabla \times \zeta, \nabla \times (\xi - \xi_h)) + \alpha(\zeta, \xi - \xi_h)
\]

\[
= (\nabla \times (\zeta - \Pi_h \zeta), \nabla \times (\xi - \xi_h)) + \alpha(\zeta - \Pi_h \zeta, \xi - \xi_h)
\]

\[
\leq C(\|\zeta - \Pi_h \zeta\|_{L^2(\Omega)} + \|\zeta - \Pi_h \zeta\|_{L^2(\Omega)})
\]

\[
\times (|\xi - \xi_h|_{H^1(\Omega)} + \|\xi - \xi_h\|_{L^2(\Omega)})
\]

\[
\leq C h^\beta \|\xi - \xi_h\|_{L^2(\Omega)} |\xi - \xi_h|_{H^1(\Omega)}
\]

\[
\leq C h^\beta \|\xi - \xi_h\|_{L^2(\Omega)} \inf_{v \in V_h} |\xi - v|_{H^1(\Omega)},
\]

which implies (4.4.4).

In the case \(\alpha < 0\), we have the following result by using the approach of Schatz [58], where the required well-posedness of the continuous problem (4.2.6) is guaranteed by Lemma 4.2.4.

**Lemma 4.4.2.** The discrete problem (4.3.1) is well-posed for \(\alpha < 0\), provided \(-\alpha\) is not a Maxwell eigenvalue and \(h\) is sufficiently small. Under these conditions the estimate (4.4.4) remains valid.

**Proof.** First we establish an a priori estimate. Assume that the solution \(\xi_h\) of the discrete problem (4.3.1) exists. Then we apply the same duality argument as in the proof of Lemma 4.4.1 to obtain the estimate (cf. (4.4.10))

\[
\|\xi - \xi_h\|_{L^2(\Omega)} \leq C h^\beta |\xi - \xi_h|_{H^1(\Omega)}.
\]

Let \(v \in V_h\). It follows from (4.4.5) and (4.4.7) that

\[
|\xi - \xi_h|^2_{H^1(\Omega)} + \alpha \|\xi - \xi_h\|^2_{L^2(\Omega)} = (\nabla \times (\xi - \xi_h), \nabla \times (\xi - v)) + \alpha(\xi - \xi_h, \xi - v)
\]

\[
\leq C|\xi - \xi_h|_{H^1(\Omega)} |\xi - v|_{H^1(\Omega)},
\]
which together with the estimate (4.4.11) implies that

\[ |\xi - \xi_h|^2_{H^1(\Omega)} \leq C(|\xi - \xi_h|_{H^1(\Omega)} |\xi - v|_{H^1(\Omega)} + h^{2\beta} |\xi - \xi_h|^2_{H^1(\Omega)}). \]  

(4.4.12)

Hence, for \( h \) sufficiently small, we have

\[ |\xi - \xi_h|_{H^1(\Omega)} \leq C|\xi - v|_{H^1(\Omega)} \quad \forall v \in V_h, \]  

(4.4.13)

which together with (4.4.11) implies the estimate (4.4.4).

If \( \xi_h \) is the solution of (4.3.1) corresponding to \( f = 0 \), then \( \xi = 0 \) is a solution of (4.3.1) and it follows from (4.4.4) that \( \xi_h = 0 \) (let \( v = 0 \) in (4.4.4)). Hence the homogeneous discrete problem has a unique solution and, since \( V_h \) is finite-dimensional, this implies that the discrete problem (4.3.1) is well-posed for \( h \) sufficiently small. \( \Box \)

Under the assumption \( f \in [L^2(\Omega)]^2 \), we have the following stability estimate from the well-posedness of the continuous problem:

\[ \|\xi\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}, \]  

(4.4.14)

which together with (4.4.4) immediately implies the following corollary.

**Corollary 4.4.3.** Under the assumptions in Lemmas 4.4.1 and 4.4.2, we have

\[ \|\xi - \xi_h\|_{L^2(\Omega)} \leq C h^{\beta} \|f\|_{L^2(\Omega)}. \]  

Next we compare \( \phi_h \) and \( \phi \).

**Lemma 4.4.4.** For \( h \) sufficiently small, we have

\[ \|\nabla \times (\phi - \phi_h)\|_{L^2(\Omega)} \leq C (h^{\beta} \inf_{v \in V_h} \|\nabla \times (\xi - v)\|_{L^2(\Omega)} + \inf_{v \in V_h} \|\nabla \times (\phi - v)\|_{L^2(\Omega)}). \]  

(4.4.15)
Proof. Since \((\xi, 1) = 0\), we can define \(\tilde{\phi}_h \in V_h\) to be the unique solution of

\[
(\nabla \times \tilde{\phi}_h, \nabla \times v) = (\xi, v) \quad \forall v \in V_h, \quad (4.4.16a)
\]

\[
(\tilde{\phi}_h, 1) = 0. \quad (4.4.16b)
\]

It follows from (4.3.4) and (4.4.16) that

\[
(\nabla \times (\tilde{\phi}_h - \phi_h), \nabla \times v) = (\xi - \xi_h, v) \quad \forall v \in V_h, \quad (4.4.17)
\]

and \((\phi_h - \tilde{\phi}_h, 1) = 0\). We then obtain, by (4.3.3), (4.4.4) and (4.4.17),

\[
\| \nabla \times (\tilde{\phi}_h - \phi_h) \|_{L^2(\Omega)}^2 = (\xi - \xi_h, \tilde{\phi}_h - \phi_h)
\]

\[
\leq C \| \xi - \xi_h \|_{L^2(\Omega)} \| \tilde{\phi}_h - \phi_h \|_{L^2(\Omega)}
\]

\[
\leq C h^\beta \inf_{v \in V_h} \| \nabla \times (\xi - v) \|_{L^2(\Omega)} \| \nabla \times (\tilde{\phi}_h - \phi_h) \|_{L^2(\Omega)},
\]

which implies

\[
\| \nabla \times (\tilde{\phi}_h - \phi_h) \|_{L^2(\Omega)} \leq C h^\beta \inf_{v \in V_h} \| \nabla \times (\xi - v) \|_{L^2(\Omega)}. \quad (4.4.18)
\]

Comparing (4.2.2) and (4.4.16a), we have the Galerkin orthogonality

\[
(\nabla \times (\phi - \tilde{\phi}_h), \nabla \times v) = 0 \quad \forall v \in V_h,
\]

which implies that, for \(v \in V_h\),

\[
\| \nabla \times (\phi - \tilde{\phi}_h) \|_{L^2(\Omega)}^2 = (\nabla \times (\phi - \tilde{\phi}_h), \nabla \times (\phi - v))
\]

\[
\leq \| \nabla \times (\phi - \tilde{\phi}_h) \|_{L^2(\Omega)} \| \nabla \times (\phi - v) \|_{L^2(\Omega)}
\]

and hence

\[
\| \nabla \times (\phi - \tilde{\phi}_h) \|_{L^2(\Omega)} \leq \inf_{v \in V_h} \| \nabla \times (\phi - v) \|_{L^2(\Omega)}. \quad (4.4.19)
\]

Since \(\tilde{\phi}_h \in V_h\), the estimate (4.4.19) implies

\[
\| \nabla \times (\phi - \tilde{\phi}_h) \|_{L^2(\Omega)} = \inf_{v \in V_h} \| \nabla \times (\phi - v) \|_{L^2(\Omega)}. \quad (4.4.20)
\]

The estimate (4.4.15) follows from (4.4.18) and (4.4.20).
Note that (4.2.2), (4.2.3) and (4.4.3) imply

\[
\inf_{v \in V_h} \| \nabla \times (\phi - v) \|_{L^2(\Omega)} \leq \| \nabla \times (\phi - \Pi_h \phi) \|_{L^2(\Omega)} \leq C h^\beta \| \xi \|_{L^2(\Omega)}.
\]  
(4.4.21)

Hence, under the assumption that \( f \in [L^2(\Omega)]^2 \), we can use (4.4.14), (4.4.15) and (4.4.21) to obtain the following bound:

\[
\| \nabla \times (\phi - \phi_h) \|_{L^2(\Omega)} \leq C h^\beta \| f \|_{L^2(\Omega)}.
\]  
(4.4.22)

The next result follows from a standard argument using (4.4.2) and Galerkin orthogonality.

**Lemma 4.4.5.** We have, for \( 1 \leq i \leq p \),

\[
| \varphi_i - \varphi_{i,h} |_{H^1(\Omega)} \leq C h^\beta.
\]  
(4.4.23)

**Proof.** Combining the weak formulation of (4.2.11) for \( \varphi_i \) and (4.3.4), we have the Galerkin orthogonality

\[
(\nabla (\varphi_i - \varphi_{i,h}), \nabla v) = 0 \quad \forall v \in V_h \cap H_0^1(\Omega)
\]  
(4.4.24)

and hence

\[
| \varphi_i - \varphi_{i,h} |_{H^1(\Omega)}^2 = (\nabla (\varphi_i - \varphi_{i,h}), \nabla (\varphi_i - v)) \\
\leq | \varphi_i - \varphi_{i,h} |_{H^1(\Omega)} | \varphi_i - v |_{H^1(\Omega)} \quad \forall v \in V_h \cap H_0^1(\Omega),
\]

which implies that

\[
| \varphi_i - \varphi_{i,h} |_{H^1(\Omega)} \leq \inf_{v \in V_h \cap H_0^1(\Omega)} | \varphi_i - v |_{H^1(\Omega)} \leq | \varphi_i - \Pi_h \varphi_i |_{H^1(\Omega)}.
\]  
(4.4.25)

By the interpolation error estimate (cf. Section 2.4) and (4.4.25), we have the estimate (4.4.23).

\[ \square \]

Now we compare \( c_{i,h} \) and \( c_i \).
Lemma 4.4.6. For \( h \) sufficiently small, we have

\[
|c_i - c_{i,h}| \leq C h^\beta \|f\|_{L^2(\Omega)} \quad \text{for} \quad 1 \leq i \leq p. \tag{4.4.26}
\]

Proof. First we observe that (4.4.23) implies

\[
|(f, \nabla \varphi_i) - (f, \nabla \varphi_{i,h})| \leq C h^\beta \|f\|_{L^2(\Omega)} \quad \text{for} \quad 1 \leq i \leq p. \tag{4.4.27}
\]

Furthermore, since \( \varphi_i - \varphi_{i,h} \in H^1_0(\Omega) \) for \( 1 \leq i \leq p \), (4.2.12) implies that, for \( 1 \leq i, j \leq p \),

\[
(\nabla \varphi_i, \nabla \varphi_j) - (\nabla \varphi_{i,h}, \nabla \varphi_{j,h}) = (\nabla \varphi_i, \nabla \varphi_j) + (\nabla (\varphi_{i,h} - \varphi_i), \nabla \varphi_j) \\
- (\nabla \varphi_{i,h}, \nabla \varphi_{j,h}) + (\nabla \varphi_i, \nabla (\varphi_{j,h} - \varphi_j)) \\
= (\nabla (\varphi_i - \varphi_{i,h}), \nabla (\varphi_{j,h} - \varphi_j))
\]

and hence, in view of (4.4.23),

\[
|(\nabla \varphi_i, \nabla \varphi_j) - (\nabla \varphi_{i,h}, \nabla \varphi_{j,h})| \leq C h^{2\beta} \quad \text{for} \quad 1 \leq i, j \leq p. \tag{4.4.28}
\]

We can rewrite (4.2.14) and (4.3.6) as

\[
Ac = b \quad \text{and} \quad A_h c_h = b_h,
\]

where \( c \in \mathbb{R}^p \) (resp. \( c_h \in \mathbb{R}^p \)) is the vector whose \( j \)-th component is \( c_j \) (resp. \( c_{j,h} \)), \( A \in \mathbb{R}^{p \times p} \) (resp. \( A_h \in \mathbb{R}^{p \times p} \)) is the matrix whose \((i,j)\)-th component is \( (\nabla \varphi_j, \nabla \varphi_i) \) (resp. \((\nabla \varphi_{j,h}, \nabla \varphi_{i,h})\)), and \( b \in \mathbb{R}^p \) (resp. \( b_h \in \mathbb{R}^p \)) is the vector whose \( j \)-th component is \( \alpha^{-1}(f, \nabla \varphi_j) \) (resp. \( \alpha^{-1}(f, \nabla \varphi_{j,h}) \)).

Note that

\[
\|b\|_\infty \leq |\alpha|^{-1} \left( \max_{1 \leq i \leq p} \|\nabla \varphi_i\|_{L^2(\Omega)} \right) \|f\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}, \tag{4.4.29}
\]

and the estimates (4.4.27)-(4.4.28) are translated into

\[
\|b - b_h\|_\infty \leq C h^\beta \|f\|_{L^2(\Omega)} \quad \text{and} \quad \|A - A_h\|_\infty \leq C h^{2\beta}. \tag{4.4.30}
\]
The estimate (4.4.26) follows from the identity
\[ c - c_h = A^{-1} b - A_h^{-1} b_h = A^{-1} (b - b_h) + A^{-1} (A_h - A) A_h^{-1} ((b_h - b) + b) \]
and (4.4.29)-(4.4.30).

Putting all the lemmas together, we can deduce the following error estimate for \( \hat{\mathbf{u}}_h \).

**Theorem 4.4.7.** When \( \alpha \geq 0 \) in (4.1.1), we have
\[
\| \hat{\mathbf{u}} - \hat{\mathbf{u}}_h \|_{L^2(\Omega)} \leq C h^\beta \| f \|_{L^2(\Omega)},
\]
(4.4.31)
where \( \beta \) is defined by (4.4.1). When \( \alpha < 0 \) and \( h \) sufficiently small, we also have (4.4.31).

**Proof.** From (4.4.29), we have that the solutions \( c_1, c_2, \cdots, c_p \) of (4.2.14) satisfy
\[
|c_i| \leq C \| f \|_{L^2(\Omega)} \quad \text{for } 1 \leq i \leq p.
\]
(4.4.32)

From (4.2.1), (4.2.14) and (4.3.7), we have that
\[
\| \hat{\mathbf{u}} - \hat{\mathbf{u}}_h \|_{L^2(\Omega)} \leq C |\phi - \phi_h|_{H^1(\Omega)} + \sum_{i=1}^{p} |c_i \phi_i - c_{i,h} \phi_{i,h}|_{H^1(\Omega)}
\]
\[
\leq C |\phi - \phi_h|_{H^1(\Omega)} + \sum_{i=1}^{p} (|c_i - c_{i,h}| \| \phi_i \|_{H^1(\Omega)} + |c_{i,h}| \| \phi_i - \phi_{i,h} \|_{H^1(\Omega)})
\]
(4.4.33)
\[
\leq C |\phi - \phi_h|_{H^1(\Omega)} + \sum_{i=1}^{p} |c_i - c_{i,h}| (|\phi_i|_{H^1(\Omega)} + |\phi_i - \phi_{i,h}|_{H^1(\Omega)})
\]
\[
+ \sum_{i=1}^{p} |c_i| |\phi_i - \phi_{i,h}|_{H^1(\Omega)}.
\]

The estimate (4.4.31) follows from (4.4.22), (4.4.23), (4.4.26), (4.4.32) and (4.4.33).
Remark 4.4.8. In the case where \( c_i = 0 = c_{i,h} \) for \( 1 \leq i \leq p \), it follows from (4.4.15) and (4.4.33) that

\[
\| \hat{\mathbf{u}} - \hat{\mathbf{u}}_h \|_{L^2(\Omega)} \leq C (h^\beta \inf_{v \in V_h} \| \nabla \times (\xi - v) \|_{L^2(\Omega)} + \inf_{v \in V_h} \| \nabla \times (\phi - v) \|_{L^2(\Omega)}).
\]

4.5 Numerical Results

In this section we present the results of several numerical experiments that confirm the theoretical results obtained in Section 4.4.

In the first set of experiments, we examine the convergence behavior of the numerical scheme on the L-shaped domain \((-1, 1)^2 \setminus [0, 1]^2\) with uniform meshes (See Figure 4.1). The exact solution is chosen to be

\[
\hat{\mathbf{u}} = \nabla \times (r^{2/3} \cos(\frac{2}{3} \theta - \frac{\pi}{3})\Phi(x)),
\]

where \((r, \theta)\) are the polar coordinates at the origin and \(\Phi(x) = (1 - x_1^2)^2(1 - x_2^2)^2\). It has the correct Maxwell singularity at the reentrant corner. We solve (4.1.1) for \(\alpha = -1, 0\) and 1, with \(\mathbf{f} = \nabla \times (\nabla \times \hat{\mathbf{u}}) + \alpha \hat{\mathbf{u}} \in H(\text{div}^0; \Omega)\). The results are tabulated in Table 4.1.

Note that the convergence of \(\hat{\mathbf{u}}_h\) to \(\hat{\mathbf{u}}\) is approaching the order of \(\beta = 2/3\), which is predicted by Theorem 4.4.7. On the other hand, since \(\xi = \nabla \times \hat{\mathbf{u}}\) behaves like \(r^{2/3}\) at the origin, the order of convergence for \(\xi_h\) according to (4.4.4) is \((2/3) + (2/3) = 4/3\), which agrees with the observed order of convergence.
<table>
<thead>
<tr>
<th>$\alpha$ = -1</th>
<th>$h$</th>
<th>$\frac{|\nabla \cdot \hat{u} - \xi_h|<em>{L^2}}{|f|</em>{L^2}}$</th>
<th>Order</th>
<th>$h$</th>
<th>$\frac{|\hat{u} - u_h|<em>{L^2}}{|f|</em>{L^2}}$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/8$</td>
<td>3.57E-02</td>
<td>1.43</td>
<td>$1/8$</td>
<td>3.19E-02</td>
<td>1.41</td>
<td></td>
</tr>
<tr>
<td>$1/16$</td>
<td>1.32E-02</td>
<td>1.43</td>
<td>$1/16$</td>
<td>1.23E-02</td>
<td>1.38</td>
<td></td>
</tr>
<tr>
<td>$1/32$</td>
<td>4.98E-03</td>
<td>1.41</td>
<td>$1/32$</td>
<td>5.03E-03</td>
<td>1.28</td>
<td></td>
</tr>
<tr>
<td>$1/64$</td>
<td>1.90E-03</td>
<td>1.39</td>
<td>$1/64$</td>
<td>2.26E-03</td>
<td>1.15</td>
<td></td>
</tr>
<tr>
<td>$1/128$</td>
<td>7.37E-04</td>
<td>1.37</td>
<td>$1/128$</td>
<td>1.13E-03</td>
<td>0.99</td>
<td></td>
</tr>
<tr>
<td>$1/256$</td>
<td>2.87E-04</td>
<td>1.36</td>
<td>$1/256$</td>
<td>6.17E-04</td>
<td>0.87</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$ = 0</th>
<th>$h$</th>
<th>$\frac{|\nabla \cdot \hat{u} - \xi_h|<em>{L^2}}{|f|</em>{L^2}}$</th>
<th>Order</th>
<th>$h$</th>
<th>$\frac{|\hat{u} - u_h|<em>{L^2}}{|f|</em>{L^2}}$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/8$</td>
<td>1.12E-02</td>
<td>1.44</td>
<td>$1/8$</td>
<td>1.35E-02</td>
<td>1.29</td>
<td></td>
</tr>
<tr>
<td>$1/16$</td>
<td>4.24E-03</td>
<td>1.41</td>
<td>$1/16$</td>
<td>6.13E-03</td>
<td>1.14</td>
<td></td>
</tr>
<tr>
<td>$1/32$</td>
<td>1.63E-03</td>
<td>1.38</td>
<td>$1/32$</td>
<td>3.07E-03</td>
<td>0.99</td>
<td></td>
</tr>
<tr>
<td>$1/64$</td>
<td>6.36E-04</td>
<td>1.36</td>
<td>$1/64$</td>
<td>1.66E-03</td>
<td>0.89</td>
<td></td>
</tr>
<tr>
<td>$1/128$</td>
<td>2.56E-04</td>
<td>1.35</td>
<td>$1/128$</td>
<td>9.46E-04</td>
<td>0.81</td>
<td></td>
</tr>
<tr>
<td>$1/256$</td>
<td>9.86E-05</td>
<td>1.34</td>
<td>$1/256$</td>
<td>5.58E-04</td>
<td>0.76</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$ = 1</th>
<th>$h$</th>
<th>$\frac{|\nabla \cdot \hat{u} - \xi_h|<em>{L^2}}{|f|</em>{L^2}}$</th>
<th>Order</th>
<th>$h$</th>
<th>$\frac{|\hat{u} - u_h|<em>{L^2}}{|f|</em>{L^2}}$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/8$</td>
<td>6.77E-03</td>
<td>1.39</td>
<td>$1/8$</td>
<td>1.06E-02</td>
<td>1.14</td>
<td></td>
</tr>
<tr>
<td>$1/16$</td>
<td>2.63E-03</td>
<td>1.36</td>
<td>$1/16$</td>
<td>5.27E-03</td>
<td>1.01</td>
<td></td>
</tr>
<tr>
<td>$1/32$</td>
<td>1.04E-03</td>
<td>1.34</td>
<td>$1/32$</td>
<td>2.80E-03</td>
<td>0.91</td>
<td></td>
</tr>
<tr>
<td>$1/64$</td>
<td>4.14E-04</td>
<td>1.33</td>
<td>$1/64$</td>
<td>1.56E-03</td>
<td>0.84</td>
<td></td>
</tr>
<tr>
<td>$1/128$</td>
<td>1.65E-04</td>
<td>1.33</td>
<td>$1/128$</td>
<td>9.06E-04</td>
<td>0.79</td>
<td></td>
</tr>
<tr>
<td>$1/256$</td>
<td>6.57E-05</td>
<td>1.32</td>
<td>$1/256$</td>
<td>5.38E-04</td>
<td>0.75</td>
<td></td>
</tr>
</tbody>
</table>

TABLE 4.1. Results for (4.1.1) on the L-shaped domain with exact solution given by (4.5.1)
In the second set of experiments, we examine the convergence behavior of the numerical scheme on the doubly connected domain \((0,4)^2 \setminus [1,3]^2\) with uniform meshes (See Figure 4.2).

In this case the solution \(\hat{\mathbf{u}}\) of \((4.1.1)\) can be written as
\[
\hat{\mathbf{u}} = \nabla \times \phi + c \nabla \varphi,
\]
(4.5.2)
where \(c\) is a constant and the harmonic function \(\varphi\) satisfies the boundary conditions
\[
\varphi|_{\Gamma_0} = 0 \quad \text{and} \quad \varphi|_{\Gamma_1} = 1.
\]
Here \(\Gamma_0\) (resp. \(\Gamma_1\)) is the boundary of \((0,4)^2\) (resp. \((1,3)^2\)). The exact solution is chosen to be
\[
\hat{\mathbf{u}} = \begin{bmatrix}
  x_2(1-x_2)(3-x_2)(4-x_2) \\
  x_1(1-x_1)(3-x_1)(4-x_1)
\end{bmatrix}.
\]
(4.5.3)
We solve \((4.1.1)\) for \(\alpha = -1\) and 1, with \(\mathbf{f} = \nabla \times (\nabla \times \hat{\mathbf{u}}) + \alpha \hat{\mathbf{u}} \in H(div^0; \Omega)\). The results are tabulated in Table 4.2.

Note that in this case \(\hat{\mathbf{u}}\) is the curl of a quintic polynomial and hence \(c = 0\) in (4.5.2). In fact, since \(\mathbf{f}\) is also the curl of a polynomial, we have \((\mathbf{f}, \nabla \varphi_h) = 0\) by Lemma 3.2.9, and it is observed that \(c_h = 0\) up to machine error.

According to Remark 4.4.8, the order of convergence for \(\hat{\mathbf{u}}_h\) is 1 (since \(\xi\) and \(\phi\) are smooth), which is observed. The order of convergence for \(\xi_h\) is found to be 2,
\[ \| \nabla \times \hat{u} - \xi_h \|_{L^2} \| f \|_{L^2} \]

| $h$ | $\| \nabla \times \hat{u} - \xi_h \|_{L^2} \| f \|_{L^2}$ | Order | $\| c \|_h$ | $\| \hat{u} - u_h \|_{L^2} \| f \|_{L^2}$ | Order |
|---|---|---|---|---|
| $\alpha = -1$ | | | | |
| 1/8 | 3.71E-03 | 2.01 | 7.93E-17 | 1.13E-02 | 1.05 |
| 1/16 | 9.26E-04 | 2.00 | 1.36E-16 | 5.61E-03 | 1.01 |
| 1/32 | 2.31E-04 | 2.00 | 1.49E-16 | 2.80E-03 | 1.00 |
| 1/64 | 5.78E-05 | 2.00 | 7.69E-16 | 1.39E-03 | 1.00 |
| 1/128 | 1.44E-05 | 2.00 | 7.43E-16 | 6.99E-04 | 1.00 |
| $\alpha = 1$ | | | | |
| 1/8 | 1.69E-03 | 1.98 | 9.25E-16 | 9.50E-03 | 1.00 |
| 1/16 | 4.25E-04 | 1.99 | 1.11E-15 | 4.75E-03 | 1.00 |
| 1/32 | 1.06E-04 | 2.00 | 1.35E-15 | 2.38E-03 | 1.00 |
| 1/64 | 2.66E-05 | 2.00 | 3.27E-15 | 1.19E-03 | 1.00 |
| 1/128 | 6.64E-06 | 2.00 | 4.96E-15 | 5.94E-04 | 1.00 |

TABLE 4.2. Results for (4.1.1) on the doubly connected domain with exact solution given by (4.5.3)

which is better than the order of $\beta + 1 = 5/3$ predicted by (4.4.4). This is likely due to the effects of superconvergence since we use uniform meshes in computing $\xi_h$ and the exact solution $\xi$ is smooth.

Finally we take the right-hand side of (4.1.1) to be the piecewise smooth vector field

\[
 f = \begin{cases} 
 \begin{bmatrix} 1 + x_1 \\ 0 \\ 0 \\ 1 + x_2 
\end{bmatrix} & \text{if } x_1 < x_2 \text{ and } 3 < x_1 < 4, \\
 \begin{bmatrix} 0 \\ 0 \\ 1 + x_2 
\end{bmatrix} & \text{otherwise.}
\end{cases}
\] (4.5.4)

The results are tabulated in Table 4.3.

The observed orders of convergence are consistent with the theoretical results. In particular, the order of convergence for $\hat{u}_h$ is $2/3$ for $\alpha = 1$ and approaching $2/3$ for $\alpha = -1$, which agrees with the estimate (4.4.31). The order of convergence for $c_h$ is $2/3 + 1/2 = 7/6$. This is because $f$ is piecewise smooth and hence the estimate (4.4.26) can be improved (cf. [14, Remark 4.8]). The order of convergence for $\xi_h$ in
| $h$ | $\frac{\|\nabla \times \mathbf{u} - \mathbf{G}_h\|_{L^2}}{\|f\|_{L^2}}$ | Order | $|c_h|$ | Order | $\frac{\|\mathbf{u} - \mathbf{u}_h\|_{L^2}}{\|f\|_{L^2}}$ | Order |
|-----|---------------------------------|-------|-------|-------|---------------------------------|-------|
| $\alpha = -1$ | | | | | | |
| 1/8 | 1.72E-01 | 1.26 | 0.763918 | 1.05 | 2.68E-01 | 1.00 |
| 1/16 | 5.28E-02 | 1.70 | 0.765285 | 0.87 | 1.28E-01 | 1.06 |
| 1/32 | 1.49E-02 | 1.83 | 0.765991 | 0.95 | 6.93E-02 | 0.89 |
| 1/64 | 4.29E-03 | 1.80 | 0.766332 | 1.05 | 4.04E-02 | 0.78 |
| 1/128 | 1.13E-03 | 1.69 | 0.766489 | 1.12 | 2.42E-02 | 0.73 |
| $\alpha = 1$ | | | | | | |
| 1/8 | 1.03E-02 | 1.33 | -0.763918 | 1.05 | 8.60E-02 | 0.71 |
| 1/16 | 4.04E-03 | 1.35 | -0.765285 | 0.87 | 5.30E-02 | 0.70 |
| 1/32 | 1.58E-03 | 1.35 | -0.765991 | 0.95 | 3.29E-02 | 0.69 |
| 1/64 | 6.21E-04 | 1.35 | -0.766332 | 1.05 | 2.05E-02 | 0.68 |
| 1/128 | 2.44E-04 | 1.34 | -0.766489 | 1.12 | 1.28E-02 | 0.67 |

TABLE 4.3. Results for (4.1.1) on the doubly connected domain with right-hand side given by (4.5.4)  

both cases is higher than the order predicted by (4.4.3). This is probably due to the fact that the mesh size $h$ is not small enough and the asymptotic behavior has not been reached.
Chapter 5
Multigrid Methods for Maxwell Equations in Heterogeneous Media

5.1 Introduction

Let $\Omega$ be a bounded simply connected polygonal domain in $\mathbb{R}^2$, and $\Omega_j$, $1 \leq j \leq J$, be open polygonal subdomains of $\Omega$ that form a partition of $\Omega$, i.e.,

$$\Omega_{j_1} \cap \Omega_{j_2} = \emptyset \quad \text{for} \quad j_1 \neq j_2 \quad \text{and} \quad \cup_{j=1}^{J} \bar{\Omega}_j = \bar{\Omega}.$$

Let $\Gamma = \cup_{j_1,j_2} \Gamma_{j_1,j_2}$ be the interface of $\Omega$, where $\Gamma_{j_1,j_2} = \bar{\Omega}_{j_1} \cap \bar{\Omega}_{j_2}$, if $\bar{\Omega}_{j_1} \cap \bar{\Omega}_{j_2} \neq \emptyset$.

Let $f \in [L^2(\Omega)]^2$ and $\epsilon, \mu$ be piecewise constant functions in the domain $\Omega$ such that $\epsilon(x) = \epsilon_j$ and $\mu(x) = \mu_j$ for $x \in \Omega_j$, with the assumption that $\epsilon_j$ and $\mu_j$ are positive numbers.

We will consider the following weak formulation of the Maxwell interface problem:

Find $\hat{u} \in H_0(curl; \Omega) \cap H(div^0; \Omega; \epsilon)$ such that

$$(\mu^{-1} \nabla \times \hat{u}, \nabla \times v) + \alpha(\epsilon \hat{u}, v) = (f, v) \quad \forall v \in H_0(curl; \Omega) \cap H(div^0; \Omega; \epsilon), \quad (5.1.1)$$

where the space $H(div^0; \Omega; \epsilon)$ is defined in Subsection 3.2.2.

Remark 5.1.1. The variational formulation here actually implies the interface conditions

$$[n \times \hat{u}] = n_- \times u_- + n_+ \times u_+ = 0$$

and

$$[n \cdot (\epsilon \hat{u})] = n_- \cdot (\epsilon_- u_-) + n_+ \cdot (\epsilon_+ u_+) = 0$$

in the sense of $H^{-1/2}(\Gamma)$, where $\Gamma$ is the interface. The first interface condition above comes from the assumption that $\hat{u} \in H(curl; \Omega)$. The second interface condition
above means that there is no charge density distribution on the interface. It comes from the assumption that $\hat{u} \in H(div^0; \Omega; \epsilon)$.

Under the assumption that $\Omega$ is simply connected, we can write (cf. Lemma 3.2.12)

$$\epsilon \hat{u} = \nabla \times \phi,$$

where $\phi \in H^1(\Omega)$ satisfies $(\phi, 1) = 0$. Then we can show (cf. Section 5.2) that the function $\phi$ in (5.1.2) is determined by

$$(\nabla \times \phi, \epsilon^{-1} \nabla \times \psi) = (\mu \xi, \psi) \quad \forall \psi \in H^1(\Omega)$$

(5.1.3)

and the constraint

$$(\phi, 1) = 0,$$

(5.1.4)

where the function $\xi = \mu^{-1} \nabla \times \hat{u} \in H^1(\Omega)$ is determined by

$$(\nabla \times \xi, \epsilon^{-1} \nabla \times \psi) + \alpha (\mu \xi, \psi) = (f, \epsilon^{-1} \nabla \times \psi) \quad \forall \psi \in H^1(\Omega),$$

(5.1.5)

when $\alpha \neq 0$ and by the equation (5.1.5) with the constraint

$$(\mu \xi, 1) = 0,$$

(5.1.6)

when $\alpha = 0$.

We can therefore solve (5.1.1) by the following numerical procedure under the assumption that $-\alpha$ is not a Maxwell eigenvalue.

Step 1. Compute a numerical approximation $\tilde{\xi}$ of $\xi$ by solving the interface problem (5.1.5) when $\alpha \neq 0$, and by solving (5.1.5) with the constraint $(\xi, 1) = 0$ when $\alpha = 0$.

Step 2. Compute a numerical approximation $\tilde{\phi}$ of $\phi$ by solving the interface problem (5.1.3) under the constraint $(\phi, 1) = 0$, where $\xi$ is replaced by $\tilde{\xi}$.

Step 3. The numerical approximation $\tilde{u}$ for $\hat{u}$ is given by

$$\tilde{u} = \nabla \times \tilde{\phi}.$$
Because equations (5.1.5) and (5.1.3) are elliptic interface problems, we already know that the solutions of these equations have very low regularity (cf. Subsection 2.2.2) and hence the solution of Maxwell’s interface problem (5.1.1) can have very low regularity. Therefore the $P_1$ finite element method does not work well in this case. However we can take advantage of the singular function representations of elliptic interface problems and the extraction formulas for the stress intensity factors (cf. the discussion in Subsection 2.3.2) to recover the optimal convergence of the $P_1$ finite element method on quasi-uniform grids by using a full multigrid approach for the interface problems (5.1.5) and (5.1.3).

5.2 Equation for $\xi$ and $\phi$

Our goal in this section is to justify the equations (5.1.3)-(5.1.6).

**Lemma 5.2.1.** Given $v \in H(\text{div}^0; \Omega; \epsilon)$, there exists a unique $\phi \in H^1(\Omega)$ such that $(\phi, 1) = 0$ and

$$\epsilon v = \nabla \times \phi.$$  

*Proof.* By Lemma 3.2.12, we have

$$v = \epsilon^{-1} \nabla \times \varphi,$$

for some $\varphi \in H^1(\Omega)$. Let $\phi = \varphi - \frac{1}{|\Omega|} \int_{\Omega} \varphi \, dx$, then it satisfies the constraint $(\phi, 1) = 0$. The uniqueness comes from Friedrichs’ inequality (cf. [20, Lemma 4.3.14]). \qed

Now we use Lemma 5.2.1 to justify the equation (5.1.3). For any $\psi \in H^1(\Omega)$ we have

$$(\nabla \times \phi, \epsilon^{-1} \nabla \times \psi) = (\epsilon \dot{u}, \epsilon^{-1} \nabla \times \psi) = (\dot{u}, \nabla \times \psi) = (\nabla \times \dot{u}, \psi) = (\mu \xi, \psi),$$

76
since $\mathbf{u} \in H_0(curl; \Omega)$ satisfies Lemma 3.1.12. To justify the equation (5.1.5), we need another lemma (cf. [14]).

**Lemma 5.2.2.** The solution $\mathbf{u}$ of (5.1.1) satisfies

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{u}) + \alpha(\epsilon \mathbf{u}) = \epsilon Q(\epsilon^{-1} \mathbf{f})$$

(5.2.1)

in the sense of distributions, where $Q : [L^2(\Omega; \epsilon)]^2 \to H(div^0; \Omega; \epsilon)$ is the orthogonal projection.

**Proof.** Let $\zeta \in [D(\Omega)]^2$ be a $C^\infty$ vector field with compact support in $\Omega$. So $\zeta \in H_0(curl; \Omega)$, $Q\zeta \in H(div^0; \Omega; \epsilon)$ and $\zeta - Q\zeta \in \nabla H^1_0(\Omega)$ because of Theorem 3.2.7.

Because of Corollary 3.2.10, we have $\zeta - Q\zeta \in H_0(curl; \Omega)$ and hence $Q\zeta \in H_0(curl; \Omega)$. Therefore

$$Q\zeta \in H_0(curl; \Omega) \cap H(div^0; \Omega; \epsilon).$$

(5.2.2)

Furthermore, we have

$$\nabla \times (\zeta - Q\zeta) = 0,$$

(5.2.3)

since $\nabla \times (\nabla H^1_0(\Omega)) = \{0\}$, and for $\mathbf{u} \in H(div^0; \Omega; \epsilon)$, we have

$$\langle \epsilon \mathbf{u}, \zeta - Q\zeta \rangle = 0.$$  

(5.2.4)

Using (5.1.1), (5.2.2), (5.2.3) and (5.2.4), we have

$$(\mu \mathbf{\nabla} \times \mathbf{u}, \nabla \times \zeta) + \alpha(\epsilon \mathbf{u}, \zeta) = (\mu \mathbf{\nabla} \times \mathbf{u}, \nabla \times (Q\zeta + \zeta - Q\zeta))$$

$$+ \alpha(\epsilon \mathbf{u}, Q\zeta + (\zeta - Q\zeta))$$

$$= (\mathbf{\nabla} \times \mathbf{u}, \nabla \times Q\zeta) + \alpha(\epsilon \mathbf{u}, Q\zeta)$$

$$= \langle \mathbf{f}, Q\zeta \rangle$$

$$= \langle Q(\epsilon^{-1} \mathbf{f}), \zeta \rangle_{L^2(\Omega; \epsilon)}$$

77
\[(\epsilon Q(\epsilon^{-1} f), \zeta),\]

which completes the proof. \qed

Remark 5.2.3. Lemma 5.2.2 is a generalization of Lemma 4.2.2.

With the help of Lemma 3.2.2, equation (5.1.5) can be justified by the same argument as in Chapter 4.

Let \( \psi \in H^1(\Omega) \) be arbitrary. Since \( \epsilon^{-1} \nabla \times \psi \in H(\text{div}^0; \Omega; \epsilon) \) by Theorem 3.2.2 and Definition 3.2.5, we have

\[
(f, \epsilon^{-1} \nabla \times \psi) = (\epsilon^{-1} f, \epsilon^{-1} \nabla \times \psi)_{L^2(\Omega; \epsilon)} \\
= (Q(\epsilon^{-1} f), \epsilon^{-1} \nabla \times \psi)_{L^2(\Omega; \epsilon)} \\
= (\epsilon Q(\epsilon^{-1} f), \epsilon^{-1} \nabla \times \psi) \\
= (\nabla \times \xi + \alpha(\epsilon \hat{u}), \epsilon^{-1} \nabla \times \psi) \quad \text{(by Lemma 5.2.2)} \\
= (\nabla \times \xi, \epsilon^{-1} \nabla \times \psi) + \alpha(\nabla \times \hat{u}, \psi) \quad \text{(by Lemma 3.2.9)} \\
= (\nabla \times \xi, \epsilon^{-1} \nabla \times \psi) + \alpha(\mu \xi, \psi),
\]

which gives (5.1.5). The constraint \((\xi, 1) = 0\) follows immediately from Lemma 3.1.12.

### 5.3 Regularity, Stress Intensity Factors and Extraction Formulas

For simplicity we will assume from here on that there is only one interface vertex \( p_* \) of the subdomains near which the solution \( \phi \) of (5.1.3) and/or the solution \( \xi \) of (5.1.5) are singular.

We further assume that \( f \in H^1(\Omega_j) \) and \( \Omega_j \) is convex for \( 1 \leq j \leq J \). Then by integration by parts, the weak problems (5.1.5) and (5.1.3) are equivalent to the following strong problems:
Find \( \xi \in H^1(\Omega) \) such that

\[
-\epsilon_j^{-1} \Delta \xi + \alpha \mu_j \xi = \epsilon_j^{-1} \nabla \times f \quad \text{in } \Omega_j, 1 \leq j \leq J, \quad (5.3.1a)
\]

\[
\epsilon_j^{-1} \frac{\partial \xi}{\partial n} = -\epsilon_j^{-1} n \times f \quad \text{on the boundary } \partial \Omega, \quad (5.3.1b)
\]

\[
\left[\epsilon_j^{-1} \frac{\partial \xi}{\partial n}\right] = -[\epsilon_j^{-1} n \times f] \quad \text{on the interface } \Gamma, \quad (5.3.1c)
\]

and find \( \phi \in H^1(\Omega) \) such that

\[
-\epsilon_j^{-1} \Delta \phi = \mu_j \xi \quad \text{in } \Omega_j, 1 \leq j \leq J, \quad (5.3.2a)
\]

\[
\epsilon_j^{-1} \frac{\partial \phi}{\partial n} = 0 \quad \text{on the boundary } \partial \Omega, \quad (5.3.2b)
\]

\[
\left[\epsilon_j^{-1} \frac{\partial \phi}{\partial n}\right] = 0 \quad \text{on the interface } \Gamma. \quad (5.3.2c)
\]

We can rewrite the problem (5.3.2) as a weak problem of the form (2.2.8). For the problem (5.3.1), we can find a function \( U \) satisfying the boundary and interface conditions and \( U|_{\Omega_j} \in H^2(\Omega_j) \) for \( 1 \leq j \leq J \). By Theorem 3.2.7, there exist \( U_{j,1} \in H^1_0(\Omega_j) \) and \( U_{j,2} \in H^1(\Omega_j) \) such that

\[
f|_{\Omega_j} = \nabla U_{j,1} + \nabla \times U_{j,2},
\]

where \( U_{j,1} \) is the solution of

\[
(\nabla U_{j,1}, \nabla v) = -(\nabla \cdot f, v) \quad \forall v \in H^1_0(\Omega_j).
\]

We have \( U_{j,1} \in H^2(\Omega_j) \) since \( \Omega_j \) is convex and \( \nabla \cdot f \in L^2(\Omega_j) \). Therefore, \( \nabla \times U_{j,2} = f - \nabla U_{j,1} \) is in \( H^1(\Omega_{j,1}) \) and hence \( U_{j,2} \) is in \( H^2(\Omega_{j}) \). On the boundary \( \partial \Omega_j \), we have

\[
\epsilon_j^{-1} n \times (\nabla \times U_{j,2}) = \epsilon_j^{-1} n \times f = -\epsilon_j^{-1} \frac{\partial \xi}{\partial n},
\]

since \( U_{j,1} \in H^1_0(\Omega_j) \) implies \( n \times \nabla U_{j,1} = 0 \) on the boundary \( \partial \Omega_j \). Now we define \( U = U_{j,2} \) for \( x \in \Omega_j \) and \( 1 \leq j \leq J \). The function \( U \) satisfies the corresponding
boundary and interface conditions and $U|_{\Omega_j} \in H^2(\Omega_j)$ for $1 \leq j \leq J$. So the function $\xi - U$ satisfies homogeneous boundary and interface conditions and hence is the solution of a weak problem described by (2.2.8). According to the discussion in Subsection 2.2.2, we let $\lambda_l = (\sigma_l)^2, l \geq 1$, be the positive eigenvalues of the Sturm-Liouville problem at $p_*$ and the functions $\Theta_l$ be the corresponding eigenfunctions.

We define the singular functions $s_l$ by the formula

$$s_l(r, \theta) = q_l(r)r^{\sigma_l}\Theta_l(\theta),$$

where $q_l$ is a smooth cut-off function that equals 1 identically near $r = 0$ and vanishes for $r \geq \delta$.

We have the singular function representations (2.2.11)/(2.2.14) for the solution $\xi$ of (5.1.5):

$$\xi = \sum_{0 < \sigma_l < 1} \kappa_l^\xi s_l + w_\xi,$$  \hspace{1cm} (5.3.3)

and the solution $\phi$ of (5.1.3):

$$\phi = \sum_{0 < \sigma_l < 1} \kappa_l^\phi s_l + w_\phi,$$  \hspace{1cm} (5.3.4)

where

$$w_\xi|_{\Omega_j}, w_\phi|_{\Omega_j} \in H^2(\Omega_j) \text{ for } 1 \leq j \leq J.$$  

Moreover, the solution $\xi$ and $\phi$ satisfy the following elliptic regularity estimates (cf. Subsection 2.3.2):

$$\sum_{0 < \sigma_l < 1} |\kappa_l^\xi| + \sum_{j=1}^J \|w_\xi\|_{H^2(\Omega_j)} \leq C\|\nabla \times f\|_{L^2(\Omega)}$$  \hspace{1cm} (5.3.5)

and

$$\sum_{0 < \sigma_l < 1} |\kappa_l^\phi| + \sum_{j=1}^J \|w_\phi\|_{H^2(\Omega_j)} \leq C\|\nabla \times f\|_{L^2(\Omega)}.$$  \hspace{1cm} (5.3.6)
The stress intensity factors $\kappa_\xi$ and $\kappa_\phi$ can be computed by the following extraction formulas (Lemma 2.3.4 and Lemma 2.3.5)

$$\kappa_\xi = \frac{1}{2\sigma_l} \left\{ \int_{\Omega} (\epsilon^{-1}(\nabla \times f)s_l^* - \alpha \mu \xi s_l^* + \epsilon^{-1} \xi \Delta s_l^*) dx - \int_{\partial \Omega} \epsilon^{-1}(n \times f)s_l^* ds - \int_{\Gamma} [\epsilon^{-1}n \times f]s_l^* ds \right\},$$

and

$$\kappa_\phi = \frac{1}{2\sigma_l} \int_{\Omega} (\mu \xi s_l^* + \epsilon^{-1} \phi \Delta s_l^*) dx.$$  \hspace{1cm} (5.3.7)

Now we present the basic idea of our algorithm by focusing on (5.1.5) with $\alpha \neq 0$. For (5.1.3) the idea will be essentially the same, so we omit it here. By (5.1.5) and (5.3.3), $w_\xi$ (the regular part of $\xi$) is the solution of

$$(\nabla \times w_\xi, \epsilon^{-1} \nabla \times \psi) + \alpha(\mu w_\xi, \psi) = (f, \epsilon^{-1} \nabla \times \psi) + \sum_{0<\sigma_l<1} \kappa_\xi^\xi [(\epsilon^{-1} \Delta s_l, \psi) - \alpha(\mu s_l, \psi)] \quad \forall \psi \in H^1(\Omega).$$ \hspace{1cm} (5.3.9)

We can then solve (5.3.9) using a $P_1$ finite element method and the convergence rate in $H^1$-norm would be of order $O(h)$ since $w \in H^2(\Omega_j)$ for $1 \leq j \leq J$. But of course we do not know the stress intensity factors $\kappa_\xi^\xi$ and therefore we consider the following problem instead:

$$(\nabla \times \hat{w}, \epsilon^{-1} \nabla \times \psi) + \alpha(\mu \hat{w}, \psi) = (f, \epsilon^{-1} \nabla \times \psi) + \sum_{0<\sigma_l<1} \kappa_\xi_{\hat{w},k} [(\epsilon^{-1} \Delta s_l, \psi) - \alpha(\mu s_l, \psi)] \quad \forall \psi \in H^1(\Omega).$$ \hspace{1cm} (5.3.10)

Here the numbers $\kappa_\xi_{\hat{w},k}$ are the approximate stress intensity factors computed through the extraction formula (5.3.7) where $\xi$ is replaced by an approximation from the previous level. This strategy can be implemented naturally through the full multigrid methodology.

In the resulting algorithm we are really computing the regular part $w$ of the solution and therefore the improvement in the convergence rate is possible because
has better regularity than $\xi$. Actually, the optimal convergence rate of the standard $P_1$ finite element is recovered, i.e., the convergence rate of $w_k$ to $w$ in the $H^1$-norm will be $O(h)$ (cf. Theorem 5.5.3).

5.4 The Algorithm

Consider a sequence of triangulations $\{\mathcal{T}_1, \cdots, \mathcal{T}_N\}$ of $\Omega$, where the triangulations are aligned with the interface between subdomains. Suppose $\mathcal{T}_1$ is given and let $\mathcal{T}_k, \ k \geq 2$, be obtained from $\mathcal{T}_{k-1}$ via a regular subdivision, i.e., edge midpoints in $\mathcal{T}_{k-1}$ are connected by new edges to form $\mathcal{T}_k$. Let $V_k$ be the $P_1$ finite element space associated with $\mathcal{T}_k$ (cf. Section 2.4), and $\tilde{V}_k$ be the subspace of $V_k$ such that

$$v_k \in \tilde{V}_k \iff (\mu v_k, 1) = 0.$$ 

Let $h_k = \max_{T \in \mathcal{T}_k} \text{diam } T$. We introduce a discrete inner product $(\cdot, \cdot)_k$ on $V_k$ by

$$(v_1, v_2)_k = h^2_k \sum_{v} v_1(p)v_2(p) \quad \forall v_1, v_2 \in V_k, \quad (5.4.1)$$

where the summation is taken over all the vertices $p$ of $\mathcal{T}_k$.

The operators $M_k : V_k \to V_k$, $A_k : V_k \to V_k$, $I^{k-1}_k : V_k \to V_{k-1}$, $Q_{a,k} : V_k \to \tilde{V}_k$, where $a = \mu$ or 1, are defined by

$$(A_k v_1, v_2)_k = \int_{\Omega} \epsilon^{-1} \nabla \times v_1 \cdot \nabla \times v_2 dx \quad \forall v_1, v_2 \in V_k, \quad (5.4.2)$$

$$(I^{k-1}_k v, w)_{k-1} = (v, w)_k \quad \forall v \in V_k, w \in V_{k-1}(\subset V_k), \quad (5.4.3)$$

$$(M_k v_1, v_2)_k = \int_{\Omega} \mu v_1 v_2 dx \quad \forall v_1, v_2 \in V_k, \quad (5.4.4)$$

and $Q_{a,k} : V_k \to \tilde{V}_k$, where

$$\tilde{V}_k = \{v \in V_k : (\mu v, 1) = 0\}$$

or

$$\tilde{V}_k = \{v \in V_k : (v, 1) = 0\},$$

82
is the orthogonal projection with respect to the inner product $(\cdot, \cdot)_k$, i.e.,

$$(Q_{a,k}v_1, v_2)_k = (v_1, v_2)_k \quad \forall v_1 \in V_k, v_2 \in \tilde{V}_k. \quad (5.4.5)$$

For the convergence analysis, we also define the Ritz projection operators $P_k : H^1(\Omega) \to \tilde{V}_k$ and $P_{k,\mu} : H^1(\Omega) \to V_k$. If $\alpha = 0$ in (5.1.5), then $P_k : H^1(\Omega) \to \tilde{V}_k$ is defined by

$$(\epsilon^{-1} \nabla \times (\zeta - P_k \zeta), \nabla \times v) = 0 \quad \forall \zeta \in H^1(\Omega), v \in \tilde{V}_k. \quad (5.4.6)$$

If $\alpha \neq 0$ in (5.1.5), then $P_{k,\mu} : H^1(\Omega) \to V_k$ is defined by

$$(\epsilon^{-1} \nabla \times (\zeta - P_k \zeta), \nabla \times v) + \alpha (\mu (\zeta - P_k \zeta), v) = 0 \quad \forall \zeta \in H^1(\Omega), v \in V_k. \quad (5.4.7)$$

The following is the standard two-sided symmetric $k^{th}$ level multigrid iteration scheme. For $p = 1$ it is the V-cycle algorithm with $m$ presmoothing steps and $m$ postsmoother steps, and for $p = 2$ it is the W-cycle algorithm with $m$ presmoother steps and $m$ postsmoother steps.

### 5.4.1 The $k^{th}$ Level Iteration

The $k^{th}$ level iteration with initial guess $z_0$ yields $MG(k, z_0, g)$ as an approximate solution to the equation

$$
\begin{cases}
A_k z = g, \\
(\mu z, 1) = 0,
\end{cases}
$$

or

$$A_k z + \alpha M_k z = g \quad (5.4.9)$$

when $\alpha \neq 0$. For $k = 1$, $MG(1, z_0, g)$ is the solution obtained from an exact solver.

For $k > 1$, $MG(k, z_0, g)$ is obtained recursively in three steps.

**Presmoother Step.** For $1 \leq l \leq m$,

$$
\begin{cases}
\tilde{z}_{l} = z_{l-1} + \frac{1}{\Lambda_k} (g - A_k z_{l-1}), \\
z_l = Q_{\mu,k} \tilde{z}_l,
\end{cases}
$$

83
or
\[ z_l = z_{l-1} + \frac{1}{\Lambda_k} (g - \alpha M_k z_{l-1} - A_k z_{l-1}) \]  \hspace{1cm} (5.4.11)

when \( \alpha \neq 0 \), where \( m \) is a positive integer independent of \( k \), and \( \Lambda_k \) dominates the spectral radius of \( A_k + \alpha M_k \).

Correction Step. Let \( \bar{g} = I^{k-1}_k (g - A_k z_m) \) or \( \bar{g} = I^{k-1}_k (g - \alpha M_k z_m - A_k z_m) \) and \( q_i \in V_{k-1} (0 \leq i \leq p, p = 1 \text{ or } 2) \) be defined recursively by

\[ q_0 = 0 \]  \hspace{1cm} (5.4.12)

and

\[ q_i = MG(k - 1, q_{i-1}, \bar{g}) \text{ for } 1 \leq i \leq p. \]  \hspace{1cm} (5.4.13)

Then
\[ z_{m+1} = z_m + I^{k}_{k-1} q_p. \]  \hspace{1cm} (5.4.14)

Postsmoothing Step. For \( m + 2 \leq l \leq 2m + 1 \), let

\[
\begin{cases}
\tilde{z}_l = z_{l-1} + \frac{1}{\Lambda_k} (g - A_k z_{l-1}), \\
z_l = Q_{a,k} \tilde{z}_l,
\end{cases}
\]  \hspace{1cm} (5.4.15)

or
\[ z_l = z_{l-1} + \frac{1}{\Lambda_k} (g - \alpha M_k z_{l-1} - A_k z_{l-1}) \]  \hspace{1cm} (5.4.16)

if \( \alpha \neq 0 \). Then the final output of the \( k^{th} \) level iteration is

\[ MG(k, z_0, g) = z_{2m+1}. \]  \hspace{1cm} (5.4.17)

5.4.2 The Full Multigrid Algorithms

We use a nested iteration to compute \( \kappa^{\xi}_{l,k} \) and \( w_k \in V_k \) so that \( \kappa^{\xi}_{k,l} \) approximates the stress intensity factor \( \kappa^{\xi}_{l} \), and \( \xi_k = \sum_{0 < i < 1} \kappa^{\xi}_{l,k} s_i + w_k \) approximates the solution \( \xi \) of (5.1.5). Then we approximate \( \xi \) in (5.1.3) by \( \xi_N \) (the approximation of \( \xi \) on the finest level), and use a nested iteration to compute \( \kappa^{\phi}_{l,k} \) and \( v_k \in V_k \) so that
\[ \phi_k = \sum_{0<\sigma_l<1} \kappa_{l,k}^\xi s + v_k \] approximates \( \phi \). The full multigrid algorithm is described as follows.

**Algorithm 5.4.1.** (Full Multigrid Algorithm for \( \xi \)) Let \( Q_{a,k} = Q_{\mu,k} \) in (5.4.10) and (5.4.15) when \( \alpha \neq 0 \). For \( k = 1 \), \( \xi_1 \) is the exact solution of (5.4.8), where \( g_1 \in V_1 \) is defined by

\[ (g_1, v)_1 = (f, \epsilon^{-1} \nabla \times v) \quad \forall v \in V_1, \]

and we set \( w_1 = \xi_1 \).

For \( 2 \leq k \leq N \), the stress intensity factors \( \kappa_{l,k}^\xi \) are computed by the following extraction formula:

\[
\kappa_{l,k}^\xi = \frac{1}{2\sigma_l} \left\{ \int_{\Omega} (\epsilon^{-1} (\nabla \times f))s_l^* - \alpha \mu \xi_{k-1}s_l^* + \epsilon^{-1} \xi_{k-1} \Delta s_l^* \right\} dx \\
- \int_{\partial\Omega} \epsilon^{-1} (n \times f)s_l^* ds - \int_{\Gamma} [\epsilon^{-1} n \times f]s_l^* ds \right\}, 
\]

(5.4.18)

and \( w_k \in V_k \) is obtained recursively by

\[
\begin{cases}
  w_{k,0} = w_{k-1}, \\
  w_{k,l} = MG(k, w_{k,l-1}, g_k) \quad \text{for} \quad 1 \leq l \leq n, \\
  w_k = w_{k,n},
\end{cases}
\]

(5.4.19)

where \( n \) is a positive integer independent of \( k \), and \( g_k \in V_k \) is defined by

\[
(g_k, v)_k = \int_{\Omega} \{ f \cdot (\epsilon^{-1} \nabla \times v) - \epsilon^{-1} \sum_{0<\sigma_l<1} \kappa_{l,k}^\xi \nabla s_l \cdot \nabla v - \alpha \mu v \sum_{0<\sigma_l<1} \kappa_{l,k}^\xi s_l \} dx \quad \forall v \in V_k.
\]

(5.4.20)

Then we define

\[
\xi_k = w_k + \sum_{0<\sigma_l<1} \kappa_{l,k}^\xi s_l. 
\]

(5.4.21)

Similarly, we define a full multigrid algorithm for \( \phi \).
Algorithm 5.4.2. (Full Multigrid Algorithm for $\phi$.) Let $Q_{a,k} = Q_{1,k}$ in (5.4.10) and (5.4.15). For $k = 1$, $\phi_1$ is the exact solution of (5.4.8), where $g_1 \in V_1$ is defined by

$$(g_1, v)_1 = (\mu \xi_N, v) \quad \forall v \in V_1,$$

and we set $v_1 = \phi_1$.

For $2 \leq k \leq N$, the stress intensity factors $\kappa_{l,k}^\phi$ are computed by the extraction formula:

$$\kappa_{l,k}^\phi = \frac{1}{2 \sigma_l} \int_\Omega (\mu \xi_N s_l^* + \epsilon^{-1} \phi_{k-1} \Delta s_l^*) dx$$

(5.4.22)

and $v_k \in V_k$ is obtained recursively by

$$
\begin{cases}
    v_{k,0} = v_{k-1}, \\
    v_{k,l} = MG(k, v_{k,l-1}, g_k) \text{ for } 1 \leq l \leq n, \\
    v_k = v_{k,n},
\end{cases}
$$

(5.4.23)

where $g_k \in V_k$ is defined by

$$(g_k, v)_k = \int_\Omega (\mu \xi_N v - \epsilon^{-1} \sum_{0<\sigma_l<1} \kappa_{l,k}^\phi \nabla s_l \cdot \nabla v) dx \quad \forall v \in V_k.$$  

(5.4.24)

Then we define

$$\phi_k = v_k + \sum_{0<\sigma_l<1} \kappa_{l,k}^\phi s_l$$

(5.4.25)

and the approximation of $\hat{u}$ by

$$\hat{u}_h = \epsilon^{-1} \nabla \times v_k + \sum_{0<\sigma_l<1} \kappa_{l,k}^\phi \epsilon^{-1} \nabla \times s_l.$$

5.5 Convergence Analysis

In this section we prove the convergence of Algorithm 5.4.1 and Algorithm 5.4.2 for the case when $\alpha \geq 0$.

In order to avoid the proliferation of constants, we will use the notation $A \lesssim B$ to represent the statement that $A$ is bounded by $B$ multiplied by a constant which
is independent of the variables, the mesh sizes and the grid levels. The notation $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$, and the notation $A \lesssim_{\epsilon} B$ indicates that the constant may depend on $\epsilon$.

First we derive a convergence result for the $k^{th}$ level symmetric W-cycle multigrid algorithm applied to our interface problems. Recall that $MG(k, z_0, g)$ is the approximate solution of (5.4.8) obtained by the $k^{th}$ level iteration scheme with initial guess $z_0$.

**Lemma 5.5.1.** Let $p = 2$ in the $k^{th}$ level iteration scheme. Let $\delta \in (0, 1)$, $\epsilon \in (0, \sigma_1)$ and $\sigma_\epsilon = 1 - \sigma_1 + \epsilon$, we have

$$\|z - MG(k, z_0, g)\|_{H^1(\Omega)} \leq \delta \|z - z_0\|_{H^1(\Omega)} \quad \forall k \geq 1,$$

(5.5.1)

and

$$\|z - MG(k, z_0, g)\|_{H^{\sigma_\epsilon}(\Omega)} \leq \delta \|z - z_0\|_{H^{\sigma_\epsilon}(\Omega)} \quad \forall k \geq 1,$$

(5.5.2)

provided that the number of smoothing steps $m$ is sufficiently large.

**Proof.** The proof of (5.5.1) is essentially identical to the proof of Theorem 1 in the paper [4]. The proof of (5.5.2) below is similar to the proof of Lemma 3.2 in [22], where the boundary condition is different.

We follow the methodology in [22]. For simplicity, we assume that $\alpha = 1$ in (5.1.5). Let $\tilde{A}_k = A_k + M_k$. First we consider the two-grid algorithm where $q_p$ in (5.4.14) is replaced by

$$q = (\tilde{A}_{k-1})^{-1} \tilde{g} = P_{k-1}(z - z_m),$$

(5.5.3)

and there is no postsmoothing steps.

Let $R_k : V_k \longrightarrow V_k$ be defined by

$$R_k = I - \frac{1}{\tilde{A}_k} \tilde{A}_k.$$

(5.5.4)
Then, by (5.5.4) and (5.4.11), we have

\[ z - z_m = R_k^m (z - z_0). \]  
(5.5.5)

It follows from (5.5.3) and (5.5.5) that the error of the output \( \tilde{z} = z_m + q \) of the two-grid method is given by

\[ z - \tilde{z} = z - z_m - q = (I - P_{k-1})(z - z_m) = (I - P_{k-1})R_k^m (z - z_0). \]  
(5.5.6)

For \( 0 \leq s \leq 1 \), we define

\[ \|v\|_s = \sqrt{\langle \tilde{A}_k^s v, v \rangle_k} \quad \forall v \in V_k. \]  
(5.5.7)

By the spectral decomposition for positive definite operators, (5.5.7) and a slight modification of [20, Lemma 6.2.8], we have

\[ \|v\|_1 \leq [\rho(\tilde{A}_k^{1-s})]^{1/2} \|v\|_s \lesssim h^{-s-1}_k \|v\|_s \quad \forall v \in V_k, 0 \leq s \leq 1, \]  
(5.5.8)

and since \( \Lambda_k = Ch_k^{-2} \) dominates the spectral radius of \( \tilde{A}_k \),

\[ \|R_k v\|_s \leq \|v\|_s \quad \forall v \in V_k, 0 \leq s \leq 1. \]  
(5.5.9)

The effect of the smoothing step is measured by the following smoothing property:

\[ \|R_k^m v\|_1 = ((\tilde{A}_k R_k^m v, R_k^m v)_k)^{1/2} = ((\tilde{A}_k R_k^m v, v)_k)^{1/2} \]
\[ \leq \Lambda_k^{s/2} \left[ \rho(\tilde{A}_k^{1-s} A_k R_k^{2m}) \right]^{1/2} \|v\|_{1-s} \quad \text{(by spectral decomposition)} \]
\[ \lesssim h^{-s}_k (\sup_{0 \leq t \leq 1} t^s (1 - t)^{2m})^{1/2} \|v\|_{1-s} \]
\[ \lesssim h^{-s}_k m^{-s/2} \|v\|_{1-s} \quad \forall v \in V_k, 0 \leq s \leq 1. \]  
(5.5.10)

The effect of the correction step is given by the following approximation property:

\[ \|v - P_{k-1} v\|_{H^\sigma(\Omega)} \lesssim h^{\sigma - \epsilon}_k \|v - P_{k-1} v\|_{H^1(\Omega)} \quad \forall v \in H^1(\Omega). \]  
(5.5.11)
We will establish (5.5.11) by a duality argument. Let \( L \in [H^{\sigma}(\Omega)]' \) and \( \zeta \in H^1(\Omega) \) satisfy

\[
(\epsilon^{-1} \nabla \times \zeta, \nabla \times v) + (\mu \zeta, v) = L(v) \quad \forall v \in H^1(\Omega).
\]

(5.5.12)

It is well known (cf. [53]) that \( \zeta \in H^{1+\sigma_1-\epsilon}(\Omega) \) and

\[
\|\zeta\|_{H^{1+\sigma_1-\epsilon}(\Omega)} \lesssim \epsilon \|L\|_{[H^{\sigma}(\Omega)]'}.
\]

(5.5.13)

Let \( \Pi_k \) be the nodal interpolation operator associated with \( V_k \). It follows from standard interpolation error estimates (cf. Section 2.4) that

\[
\|\zeta - \Pi_k \zeta\|_{H^1(\Omega)} \lesssim \epsilon \|L\|_{[H^{\sigma}(\Omega)]'} h_k^\sigma \|\zeta\|_{H^{1+\sigma_1-\epsilon}(\Omega)}.
\]

(5.5.14)

We can modify the proof for Friedrichs’ inequality (cf. [20, Lemma 4.3.14]) and have

\[
\|u - \bar{u}\|_{H^1(\Omega)} \lesssim |u|_{H^1(\Omega)},
\]

(5.5.15)

where \( u \in H^1(\Omega) \) and \( \bar{u} = \frac{1}{|\Omega|} \int_\Omega \mu u \, dx \).

Using (5.4.7), (5.5.12), (5.5.15) and (5.5.14), we have

\[
L(v - P_{k-1}v) = (\epsilon^{-1} \nabla \times \zeta, \nabla \times (v - P_{k-1}v)) + (\mu \zeta, v - P_{k-1}v)
\]

\[
= (\epsilon^{-1} \nabla \times (\zeta - \Pi_k \zeta), \nabla \times (v - P_{k-1}v)) + (\mu (\zeta - \Pi_k \zeta), v - P_{k-1}v)
\]

\[
\lesssim \|\zeta - \Pi_k \zeta\|_{H^1(\Omega)} \|v - P_{k-1}v\|_{H^1(\Omega)}
\]

\[
\lesssim h_k^{\sigma_1-\epsilon} \|L\|_{[H^{\sigma}(\Omega)]'} |v - P_{k-1}v|_{H^1(\Omega)}.
\]

(5.5.16)

The estimate (5.5.11) follows from (5.5.16) and the duality formula

\[
\|\eta\|_{H^{\sigma}(\Omega)} = \sup[L(\eta)/\|L\|_{[H^{\sigma}(\Omega)]'}] \quad \forall \eta \in H^{\sigma}(\Omega),
\]

(5.5.17)

where the supremum is taken over all \( L \in [H^{\sigma}(\Omega)]' \setminus \{0\} \).
The final ingredient is the relation between the mesh dependent norm $\|\cdot\|_s$ and the Sobolev norm $\|\cdot\|_{H^s(\Omega)}$ on $V_k$. First of all, we have $|v|_{H^1(\Omega)} \approx \|v\|_1$ and $\|v\|_{L^2(\Omega)} \lesssim \|v\|_0$ for all $v \in V_k$. Interpolating these estimates (cf. [20, Proposition 14.1.5]), we have
\[
\|v\|_{H^s(\Omega)} \lesssim \|v\|_s \quad \forall v \in V_k, 0 \leq s \leq 1. \tag{5.5.18}
\]
Meanwhile, there exists (cf. [20, 59]) an interpolation operator $\pi_k : L^2(\Omega) \rightarrow V_k$ such that
\[
\|\pi_k v\|_0 \lesssim \|\pi_k v\|_{L^2(\Omega)} \lesssim \|v\|_{L^2(\Omega)} \quad \forall v \in L^2(\Omega), \tag{5.5.19}
\]
\[
\|\pi_k v\|_1 \lesssim \|\pi_k v\|_{H^1(\Omega)} \lesssim \|v\|_{H^1(\Omega)} \quad \forall v \in H^1(\Omega), \tag{5.5.20}
\]
\[
\pi_k v = v \quad \forall v \in V_k. \tag{5.5.21}
\]
For $0 < s < 1$, we can interpolate (5.5.19) and (5.5.20) (cf. [20, Proposition 14.1.5, Theorem 14.2.3]) to obtain
\[
\|\pi_k v\|_s \lesssim \|v\|_{H^s(\Omega)} \quad \forall v \in H^s(\Omega), 0 \leq s \leq 1. \tag{5.5.22}
\]
Combining (5.5.21) and (5.5.22), we find
\[
\|v\|_s \lesssim \|v\|_{H^s(\Omega)} \quad v \in V_k, 0 \leq s \leq 1. \tag{5.5.23}
\]
Therefore, from (5.5.6), (5.5.10), (5.5.11), (5.5.18) and (5.5.23), we have the following error estimate for the two-grid algorithm:
\[
\|z - \tilde{z}\|_{H^{s_\epsilon}(\Omega)} \lesssim h_k^{s_\epsilon - \epsilon} \|R^m_k(z - z_m)\|_{H^1(\Omega)} \lesssim m^{[-s_\epsilon + \epsilon]/2} \|z - z_0\|_{H^{s_\epsilon}(\Omega)}. \tag{5.5.24}
\]
Now we estimate the error for the one-sided W-cycle $k^{th}$ level iteration. Let $\gamma_m = m^{[-s_\epsilon + \epsilon]/2}$ and suppose that the error of the $(k - 1)^{st}$ level iteration in the
$H^{\alpha_\epsilon}(\Omega)$ norm is reduced by a factor $\eta$. Then it follows from (5.4.14), (5.5.6) and (5.5.24) that
\[
\|z - MG(k, z_0, g)\|_{H^{\alpha_\epsilon}(\Omega)} \leq \|z - z\|_{H^{\alpha_\epsilon}(\Omega)} + \|q - q_2\|_{H^{\alpha_\epsilon}(\Omega)} \\
\leq C_\epsilon \gamma_m \|z - z_0\|_{H^{\alpha_\epsilon}(\Omega)} + \eta^2 \|q\|_{H^\sigma(\Omega)}.
\] (5.5.25)

From (5.5.8), (5.5.11), (5.5.18) and (5.5.23), we have
\[
\|P_{k-1}v\|_{H^{\alpha_\epsilon}(\Omega)} \leq \|v - P_{k-1}v\|_{H^{\alpha_\epsilon}(\Omega)} + \|v\|_{H^{\alpha_\epsilon}(\Omega)} \\
\lesssim \epsilon \eta_k \|v\|_{H^1(\Omega)} + \|v\|_{H^{\alpha_\epsilon}(\Omega)} \\
\lesssim \epsilon \|v\|_{H^{\alpha_\epsilon}(\Omega)} \quad \forall v \in V_k.
\] (5.5.26)

Combining (5.5.5), (5.5.9), (5.5.18), (5.5.22) and (5.5.26), we obtain
\[
\|q\|_{H^{\alpha_\epsilon}(\Omega)} = \|P_{k-1}(z - z_m)\|_{H^{\alpha_\epsilon}(\Omega)} \leq C'_\epsilon \|z - z_0\|_{H^{\alpha_\epsilon}(\Omega)}.
\] (5.5.27)

The estimates (5.5.23), (5.5.25) and (5.5.27) together imply that
\[
\|z - MG(k, z_0, g)\|_{H^{\alpha_\epsilon}(\Omega)} \leq (C_\epsilon \gamma_m + C'_\epsilon \eta^2) \|z - z_0\|_{H^{\alpha_\epsilon}(\Omega)}.
\] (5.5.28)

For $m$ sufficiently large, we have $\gamma_m < (4C_\epsilon C'_\epsilon)^{-1}$ and
\[
\eta_m = \left[ 1 - (1 - 4C_\epsilon C'_\epsilon \gamma_m)^{1/2} \right] / (2C'_\epsilon)
\] (5.5.29)
is a fixed point of the map $T(\eta) = C_\epsilon \gamma_m + C'_\epsilon \eta^2$. Since the first level iteration is an exact solver, it follows from (5.5.29) and mathematical induction that
\[
\|z - MG(k, z_0, g)\|_{H^{\alpha_\epsilon}(\Omega)} \leq \eta_m \|z - z_0\|_{H^{\alpha_\epsilon}(\Omega)} \quad \text{for} \quad k \geq 1.
\] (5.5.30)

As $\lim_{m \to \infty} \eta_m = 0$, the estimate (5.5.2) follows from (5.5.30).

\[\Box\]

Remark 5.5.2. For the symmetric W-cycle $k^{th}$ level iteration, we have
\[
z - MG(k, z_0, g) = R^n_{k}(z - z_{m+1}).
\] (5.5.31)
Combining (5.5.9) and (5.5.31), we can prove (5.5.2) for the symmetric W-cycle $k^{th}$ level iteration.

With the help of Lemma 5.5.1, we can prove the following result by following the methodology of [23].

**Theorem 5.5.3.** Let $p = 2$ in the $k^{th}$ level iteration scheme, $\epsilon \in (0, \sigma_1)$, $\sigma_\epsilon = 1 - \sigma_1 + \epsilon$, and the number of smoothing steps $m$ be sufficiently large so that the $k^{th}$ level iteration scheme is a contraction. If the number of nested iterations $n$ is sufficiently large, then we have

$$\sum_{0 < \sigma_1 < 1} |\kappa_1^\xi - \kappa_{i,k}^\xi| \leq C h_k^{1+\sigma_1-\epsilon} \|\nabla \times f\|_{L^2(\Omega)} \quad \forall k \geq 1$$  \hspace{1cm} (5.5.32)

and

$$\|w_\xi - w_k\|_{H^{\sigma_\epsilon}(\Omega)} \leq C h_k^{1+\sigma_1-\epsilon} \|\nabla \times f\|_{L^2(\Omega)} \quad \forall k \geq 1.$$  \hspace{1cm} (5.5.33)

Moreover we have the following estimates in the $H^1$ norm:

$$\|w_\xi - w_k\|_{H^1(\Omega)} \leq C h_k \|\nabla \times f\|_{L^2(\Omega)} \quad \forall k \geq 1.$$  \hspace{1cm} (5.5.34)

Let

$$\xi_k = \sum_{0 < \sigma_1 < 1} \kappa_{i,k}^\xi s_l + w_k,$$  \hspace{1cm} (5.5.35)

then (5.5.33) and (5.5.34) lead to the following estimate:

$$\|\xi - \xi_k\|_{H^1(\Omega)} \leq C h_k \|\nabla \times f\|_{L^2(\Omega)} \quad \forall k \geq 1.$$  \hspace{1cm} (5.5.36)

**Proof.** We will establish (5.5.32) and (5.5.33) through recursive estimates. By (5.3.7) and (5.4.18) along with (5.3.3) and (5.4.21), we know that, for $1 \leq l \leq L$,

$$|\kappa_1^\xi - \kappa_{i,k}^\xi| = \frac{1}{2\sigma_1} \int_{\Omega} e^{-1}(\xi - \xi_{k-1}) \Delta s_l^* dx$$  \hspace{1cm} (5.5.37)

$$\lesssim \|\xi - \xi_{k-1}\|_{L^2(\Omega)} \|\Delta s_l^*\|_{L^2(\Omega)}$$
where

$$\hat{w}_k = \xi - \sum_{0<\sigma_l<1} \kappa_{l,k}^\xi s_l = \sum_{0<\sigma_l<1} (\kappa_{l,k}^\xi - \kappa_{l,k}^\xi) s_l + w_\xi = (\xi - \xi_k) + w_k. \quad (5.5.38)$$

Let $a_k = \sum_{0<\sigma_l<1} |\kappa_{l,k}^\xi - \kappa_{l,k}^\xi|$, and $b_k = \|\hat{w}_k - w_k\|_{H^\sigma(\Omega)}$. So (5.5.37) says that

$$a_k \lesssim b_{k-1}. \quad (5.5.39)$$

To estimate $b_k$, we begin with

$$b_k \leq \|\hat{w}_k - P_k \hat{w}_k\|_{H^\sigma(\Omega)} + \|P_k \hat{w}_k - w_k\|_{H^\sigma(\Omega)}. \quad (5.5.40)$$

For the first term on the right-hand side of equation (5.5.40), by using the elliptic regularity estimate (5.3.5) and standard finite element tools (cf. Section 2.4), we have the following estimate

$$\|\hat{w}_k - P_k \hat{w}_k\|_{H^\sigma(\Omega)} \leq \|w_\xi - P_k w_\xi\|_{H^\sigma(\Omega)} + \sum_{0<\sigma_l<1} |\kappa_{l,k}^\xi - \kappa_{l,k}^\xi| \|s_l - P_k s_l\|_{H^\sigma(\Omega)}$$

$$\lesssim h_k^{1+\sigma_1-\epsilon} \|\nabla \times f\|_{L_2(\Omega)} + h_k^{2(\sigma_1-\epsilon)} a_k. \quad (5.5.41)$$

The second term on the right-hand side of (5.5.40) can be estimated as follows:

$$\|P_k \hat{w}_k - w_k\|_{H^\sigma(\Omega)} \leq \delta^n \|P_k \hat{w}_k - w_{k-1}\|_{H^\sigma(\Omega)}, \quad (5.5.42)$$

by Lemma 5.5.1, since $P_k \hat{w}$ is the exact solution of (5.4.9) whose right-hand side is given by (5.4.19) and $w_k$ is its approximate solution.

Now we estimate $\|P_k \hat{w}_k - w_{k-1}\|_{H^\sigma(\Omega)}$ as follows:

$$\|P_k \hat{w}_k - w_{k-1}\|_{H^\sigma(\Omega)} \leq \|P_k \hat{w}_k - \hat{w}_k\|_{H^\sigma(\Omega)} + \|\hat{w}_k - w_\xi\|_{H^\sigma(\Omega)}$$

$$+ \|w_\xi - \hat{w}_{k-1}\|_{H^\sigma(\Omega)} + \|\hat{w}_{k-1} - w_{k-1}\|_{H^\sigma(\Omega)}$$
\[ b_k \lesssim \epsilon h_k^{1+\sigma_1-\epsilon} \| \nabla \times f \|_{L^2(\Omega)} + a_k + a_{k-1} + b_{k-1}, \] (5.5.43)

by (5.5.38), (5.5.41) and the definitions of \( a_k \) and \( b_k \).

Combining (5.5.39)-(5.5.43) we find, for \( k \geq 2 \),

\[
\begin{align*}
\quad b_k & \lesssim (h_k^{1+\sigma_1-\epsilon} \| \nabla \times f \|_{L^2(\Omega)} + a_k + a_{k-1} + b_{k-1}) \\
& \quad + a_k + a_{k-1} + b_{k-1) \\
& \leq C_\epsilon (h_k^{2(\sigma_1-\epsilon)} + \delta^n) b_{k-1} + C_\epsilon \delta^n b_{k-2} + C_\epsilon h_k^{1+\sigma_1-\epsilon} \| \nabla \times f \|_{L^2(\Omega)},
\end{align*}
\] (5.5.44)

where \( C_\epsilon \) is a positive constant.

Therefore (5.5.44) leads to the estimate

\[
\begin{align*}
b_k & \leq \beta b_{k-1} + \beta b_{k-2} + C_* h_k^{1+\sigma_1-\epsilon},
\end{align*}
\] (5.5.45)

provided that \( C_* h_k^{2(\sigma_1-\epsilon)} + C_\epsilon \delta^n < \beta \) for some positive constant \( \beta \) and

\[
C_* = C_\epsilon \| \nabla \times f \|_{L^2(\Omega)}. \tag{5.5.46}
\]

Later we will identify the choice of \( \beta \).

We reformulate (5.5.45) as

\[
\begin{bmatrix} b_{k-1} \\ b_k \end{bmatrix} \leq \begin{bmatrix} 0 & 1 \\ \beta & \beta \end{bmatrix} \begin{bmatrix} b_{k-2} \\ b_{k-1} \end{bmatrix} + C_* h_k^{1+\sigma_1-\epsilon} \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\] (5.5.47)

where the vector inequality is interpreted component-wise.

Let \( M = \begin{bmatrix} 0 & 1 \\ \beta & \beta \end{bmatrix} \) and rewrite (5.5.47) as

\[
\begin{bmatrix} b_{k-1} \\ b_k \end{bmatrix} \leq M \begin{bmatrix} b_{k-2} \\ b_{k-1} \end{bmatrix} + C_* h_k^{1+\sigma_1-\epsilon} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{5.5.48}
\]

For any given \( \epsilon \in (0, \sigma_1) \) and \( \beta > 0 \), there exist sufficiently large \( k^* \) and \( n^* \) depending on \( \beta \) such that \( C_* h_k^{2(\sigma_1-\epsilon)} + C_\epsilon \delta^n < \beta \) for \( k \geq k^* \) and \( n \geq n^* \). So for \( k \geq k^* \) and \( n \geq n^* \), (5.5.48) is valid.
By iterating (5.5.48), we obtain
\[
\begin{bmatrix}
 b_{k-1} \\
 b_k
\end{bmatrix} \leq M^{k-k^*} \begin{bmatrix}
 b_{k^*-1} \\
 b_{k^*}
\end{bmatrix} + C_s(h_k^{1+\sigma_1-\epsilon} \begin{bmatrix}
 0 \\
 1
\end{bmatrix} + h_k^{1+\sigma_1-\epsilon} M \begin{bmatrix}
 0 \\
 1
\end{bmatrix}) \tag{5.5.49}
\]
\[
+ \ldots + h_{k^*+1}^{1+\sigma_1-\epsilon} M^{k-k^*-1} \begin{bmatrix}
 0 \\
 1
\end{bmatrix}.
\]

By a direct computation, we have
\[
M^2 = \beta(I + M), \tag{5.5.50}
\]
where \( I \) is the \( 2 \times 2 \) identity matrix.

By (5.5.50), we have
\[
\|M^2\|_\infty \leq 2\beta
\]
and hence
\[
\|M^{2^t}\|_\infty \leq (2\beta)^t
\]
and
\[
\|M^{2^{t+1}}\|_\infty \leq (2\beta)^t,
\]
where \( t \) is a positive integer. So for sufficiently small \( \beta \), i.e., for \( k \) and \( n \) sufficiently large, we have
\[
\|M^t\|_\infty \leq 2^{-(1+\sigma_1)t} \text{ for } t = 2, \ldots. \tag{5.5.51}
\]

Then (5.5.49) and (5.5.51) implies that, for \( k \geq k^*+1 \) and \( n \geq n^* \),
\[
b_k \leq 2^{-(1+\sigma_1)(k-k^*)} b_{k^*} + C_s h_k^{1+\sigma_1-\epsilon}(1 + 2^{-\epsilon} + \ldots) \tag{5.5.52}
\]
\[
+ 2^{-k^{k^*+1}}\|\nabla \times f\|_{L_2(\Omega)},
\]
\[
\leq 2^{-(1+\sigma_1)(k-k^*)} b_{k^*} + \frac{C_s}{1-2^{-\epsilon}} h_k^{1+\sigma_1-\epsilon} \|\nabla \times f\|_{L_2(\Omega)}.
\]

On the other hand, we have
\[
a_1 = \sum_{0<\sigma_1<1} |\kappa_l| \lesssim \|\nabla \times f\|_{L^2(\Omega)}, \tag{5.5.53}
\]
95
\[ b_1 = \| \xi - P_1 \xi \|_{H^{s1}(\Omega)} \lesssim \| \xi \|_{H^1(\Omega)} \lesssim \| \nabla \times f \|_{L^2(\Omega)}. \]  

Combining (5.5.44), (5.5.53) and (5.5.54), we obtain, for \( 1 \leq k \leq k^* \),

\[ b_k \lesssim \epsilon \| \nabla \times f \|_{L^2(\Omega)} \lesssim h_k^{1+s_1-\epsilon} \| \nabla \times f \|. \tag{5.5.55} \]

Combining (5.5.52) and (5.5.55), we conclude that

\[ b_k \lesssim h_k^{1+s_1-\epsilon} \| \nabla \times f \|_{L^2(\Omega)} \quad \text{for} \quad k \geq 1. \tag{5.5.56} \]

We have established (5.5.33), and (5.5.32) follows directly from (5.5.39).

Now we consider the error estimate (5.5.34). First note that if we let \( \epsilon \) be \( \frac{s_1}{2} \) in (5.5.32), then we obtain

\[ a_k \leq Ch_k^{1+s_1/2} \| \nabla \times f \|_{L^2(\Omega)}, \tag{5.5.57} \]

where the constant \( C \) no longer depends on \( \epsilon \).

Let us denote

\[ \bar{b}_k := \| \hat{w}_k - w_k \|_{H^1(\Omega)}. \tag{5.5.58} \]

We have the following estimate

\[ \| w_\xi - \hat{w}_k \|_{H^{1-s_1+s_1}(\Omega)} \lesssim \| w_\xi - \hat{w}_k \|_{H^1(\Omega)} \lesssim a_k \] \tag{5.5.59}

by (5.5.38).

Also, by the elliptic regularity estimate assumption, we have the estimate

\[ \bar{b}_1 = \| \hat{w}_1 - w_1 \|_{H^1(\Omega)} = \| \xi - P_1 \xi \|_{H^1(\Omega)} \lesssim \| \nabla \times f \|_{L^2(\Omega)}. \tag{5.5.60} \]

We may also obtain the analogue of (5.5.40)-(5.5.43). From (5.5.58), we have

\[ \bar{b}_k \leq \| \hat{w}_k - P_k \hat{w}_k \|_{H^1(\Omega)} + \| P_k \hat{w}_k - w_k \|_{H^1(\Omega)}. \tag{5.5.61} \]

By (5.5.38), (5.5.57) and the analogue of (5.5.41), we have

\[ \| \hat{w}_k - P_k \hat{w}_k \|_{H^1(\Omega)} \lesssim h_k \| \nabla \times f \|_{L^2(\Omega)} + h_k^{s_1-\epsilon} \tag{5.5.62} \]
\[ \lesssim h_k \| \nabla \times f \|_{L^2(\Omega)}. \]

By Lemma 5.5.1, we have

\[ \| P_k \hat{w}_k - w_k \|_{H^1(\Omega)} \leq \delta^n \| P_k \hat{w}_k - w_{k-1} \|_{H^1(\Omega)}. \tag{5.5.63} \]

By the analogue of (5.5.44), we have

\[ \| P_k \hat{w}_k - w_{k-1} \|_{H^1(\Omega)} \lesssim h_k \| \nabla \times f \|_{L^2(\Omega)} + a_k + a_{k-1} + \bar{b}_{k-1} \lesssim h_k \| \nabla \times f \|_{L^2(\Omega)} + \bar{b}_{k-1}. \tag{5.5.64} \]

So (5.5.61)-(5.5.64) imply that

\[ \bar{b}_k \lesssim \delta^n \bar{b}_{k-1} + h_k \| \nabla \times f \|_{L^2(\Omega)} \tag{5.5.65} \]

or

\[ \bar{b}_k \leq C_\dagger (\delta^n \bar{b}_{k-1} + h_k \| \nabla \times f \|_{L^2(\Omega)}) \tag{5.5.66} \]

for some constant \( C_\dagger \). For \( n \) sufficiently large so that \( C_\dagger \delta^n < 1/4 \), we can iterate (5.5.66) and apply (5.5.60) to get the estimate

\[ \bar{b}_k \leq (C_\dagger \delta^n)^{k-1} \bar{b}_1 + C_\dagger \| \nabla \times f \|_{L^2(\Omega)} \sum_{i=2}^{k} (C_\dagger \delta^n)^{k-i} h_i \]

\[ \lesssim h_k \| \nabla \times f \|_{L^2(\Omega)} \text{ for } k \geq 1, \tag{5.5.67} \]

which is (5.5.34).

\[ \square \]

**Remark 5.5.4.** Because of (5.3.3), (5.5.35), (5.5.32), and (5.5.33), we also have the estimate

\[ \| \xi - \xi_k \|_{L^2(\Omega)} \leq C h_k^{1+\sigma_1-\epsilon} \| \nabla \times f \|_{L^2(\Omega)} \quad \forall k \geq 1. \tag{5.5.68} \]

**Remark 5.5.5.** For the case \( \alpha < 0 \), it is a symmetric indefinite problem and there are several multigrid schemes developed [3, 47, 48, 9, 63, 8] in the literature. We will not discuss this case here.
We use $\xi_N = \sum_{0<\sigma_1<1} \kappa_{l,N}^\xi s_l + v_N$, where $\mathcal{T}_N$ is the finest triangulation, as the approximation to the solution $\xi$ of (5.1.5) in (5.1.3). Applying Algorithm 5.4.2 to the equation (5.1.3), we obtain the approximation $\kappa_{l,k}^\phi$ and $v_k$, for $1 \leq k \leq N$.

**Theorem 5.5.6.** Let $p = 2$ in the $k^{th}$ level iteration scheme, $\epsilon \in (0, \sigma_1)$, $\sigma_\epsilon = 1 - \sigma_1 + \epsilon$, and the number of smoothing steps $m$ be sufficiently large so that $k^{th}$ level iteration scheme is a contraction scheme. If the number of nested steps $n$ is sufficiently large, then we have

$$\sum_{0<\sigma_1<1} |\kappa_{l,k}^\phi - \kappa_{l,k}^\phi| \leq C h_k^{1+\sigma_1-\epsilon} \|\nabla \times f\|_{L^2(\Omega)} \quad \forall k \geq 1, \quad (5.5.69)$$

$$\|w_\phi - v_k\|_{H^1(\Omega)} \leq C h_k \|\nabla \times f\|_{L^2(\Omega)} \quad \forall k \geq 1. \quad (5.5.70)$$

Let $\phi_k = \sum_{0<\sigma_1<1} \kappa_{l,k}^\phi s_l + v_k$, then we have the following estimates in the $H^1$ norm:

$$\|\phi - \phi_k\|_{H^1(\Omega)} \leq C h_k \|\nabla \times f\|_{L^2(\Omega)} \quad \forall k \geq 1. \quad (5.5.71)$$

**Proof.** Let $\hat{\phi} \in H^1(\Omega)$ be the exact solution of the following problem:

$$\begin{align*}
(\epsilon^{-1} \nabla \times \hat{\phi}, \nabla \times v) &= (\mu \xi, v) \quad \forall v \in H^1(\Omega), \\
(\hat{\phi}, 1) &= 0.
\end{align*} \quad (5.5.72a, b)$$

Then $\hat{\phi}$ is an approximation of $\phi$, which is the exact solution of the problem:

$$\begin{align*}
(\epsilon^{-1} \nabla \times \phi, \nabla \times v) &= (\mu \xi, v) \quad \forall v \in H^1(\Omega), \\
(\phi, 1) &= 0.
\end{align*} \quad (5.5.73a, b)$$

We denote the singular representation for $\hat{\phi}$ by

$$\hat{\phi} = \sum_{0<\sigma_1<1} \kappa_{l,k}^\phi s_l + w_\phi.$$ 

Because of the estimate (5.5.68) the error between $\phi$ and $\hat{\phi}$ can be estimated by

$$\|\phi - \hat{\phi}\|_{H^1(\Omega)} \leq C \|\xi - \xi_N\|_{L^2(\Omega)} \leq h_N^{1+\sigma_1-1} \|\nabla \times f\|_{L^2(\Omega)}. \quad (5.5.74)$$
Moreover, from (5.5.74) and the extraction formulas for $\kappa_i^\phi$ and $\kappa_i^\hat{\phi}$, we have

$$|\kappa_i^\phi - \kappa_i^\hat{\phi}| \leq \frac{1}{2\sigma_1} \left| \int_{\Omega} \mu(\xi - \xi_N)s_i^* + (\phi - \hat{\phi}_N)\Delta s_i^* dx \right|$$

$$\lesssim h_N^{1+\sigma_1-\epsilon} \|\nabla \times f\|_{L^2(\Omega)}$$

(5.5.75)

and

$$\|w_\phi - w_{\hat{\phi}}\|_{H^1(\Omega)} \lesssim \|\xi - \xi_N\|_{L^2(\Omega)} + \sum_{0<\sigma_i<1} |\kappa_i^\phi - \kappa_i^\hat{\phi}|$$

$$\lesssim h_N^{1+\sigma_1-1} \|\nabla \times f\|_{L^2(\Omega)}.$$  

(5.5.76)

We know that $\phi_k$ is an approximation of $\hat{\phi}$. A similar argument to the proof of Theorem 5.5.3 gives the following estimates:

$$\sum_{0<\sigma_i<1} |\kappa_i^\phi - \kappa_i^\hat{\phi}| \leq Ch_k^{1+\sigma_1-\epsilon} \|\nabla \times f\|_{L^2(\Omega)} \quad \forall k \geq 1,$$  

(5.5.77)

$$\|w_\phi - v_k\|_{H^1(\Omega)} \leq Ch_k \|\nabla \times f\|_{L^2(\Omega)} \quad \forall k \geq 1.$$  

(5.5.78)

Let $\phi_k = \sum_{0<\sigma_i<1} \kappa_i^\phi s_i + v_k$, then we have the following estimates in the $H^1$ norm:

$$\|\hat{\phi} - \phi_k\|_{H^1(\Omega)} \leq Ch_k \|\nabla \times f\|_{L^2(\Omega)} \quad \forall k \geq 1.$$  

(5.5.79)

Now (5.5.74)-(5.5.79) imply (5.5.69)-(5.5.71).

5.6 Numerical Results

In this section we present the results of several numerical tests that illustrate the performance of our algorithm. The first two numerical examples are performed on the L-shaped domain $\Omega = (-1, 1)^2 \setminus [0, 1]^2$, where the subdomains are $\Omega_1 = (-1, 0) \times (0, 1)$, $\Omega_2 = (-1, 0)^2$, and $\Omega_3 = (0, 1) \times (-1, 0)$ (See Figure 5.1). We use the $P_1$ finite element in the experiments. The mesh size $h_k$ for the $k^{th}$ level grid is taken to be $1/(3 \cdot 2^k)$. All the computations are done using a $W$-cycle $k^{th}$ level iteration with 50 smoothing steps, and the number of nested iterations in both full
multigrid algorithms (See Algorithm 5.4.1 and Algorithm 5.4.2 in Subsection 5.4.2) is also 50.

We will consider the case where $\epsilon = [1/350, 1, 1/350]$. In this situation, the contraction numbers for the symmetric W-cycle algorithm are given in Table 5.1.

**Example 5.6.1.** We solve equations (5.1.5), (5.1.3) on the domain $\Omega$ where $\alpha = 0, \epsilon = [1/350, 1, 1/350]$ and $\mu = [350, 1, 350]$. The vector function $f$ is given by

$$f(x) = \begin{bmatrix} r^{\sigma_1} g'_{\text{cut}}(A_j \sin((\sigma_1 + 1)\theta) - B_j \cos((\sigma_1 + 1)\theta)) \\ r^{\sigma_1} g'_{\text{cut}}(-A_j \cos((\sigma_1 + 1)\theta) - B_j \sin((\sigma_1 + 1)\theta)) \end{bmatrix}, \quad (5.6.1)$$

for $x$ in the subdomain $\Omega_j$, $1 \leq j \leq J$, where $(r, \theta)$ are the polar coordinates of the point $(0, 0)$, $\lambda_1 = \sigma_1^2$ is the first eigenvalue of the related Sturm-Liouville problem (cf. Subsection 2.2.2), $\sigma_1 = 0.048066746316346...$, $A_j, B_j$ are coefficients appearing

![FIGURE 5.1. The domain $\Omega$ and its subdomains.](image)

<table>
<thead>
<tr>
<th>$m$</th>
<th>$k = 4$</th>
<th>$k = 5$</th>
<th>$k = 6$</th>
<th>$k = 7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>0.4662</td>
<td>0.6128</td>
<td>0.8082</td>
<td>0.9803</td>
</tr>
<tr>
<td>12</td>
<td>0.4461</td>
<td>0.5874</td>
<td>0.7528</td>
<td>0.9545</td>
</tr>
<tr>
<td>13</td>
<td>0.4286</td>
<td>0.5645</td>
<td>0.7028</td>
<td>0.9236</td>
</tr>
<tr>
<td>14</td>
<td>0.4131</td>
<td>0.5437</td>
<td>0.5824</td>
<td>0.8903</td>
</tr>
<tr>
<td>15</td>
<td>0.3993</td>
<td>0.5249</td>
<td>0.6416</td>
<td>0.8560</td>
</tr>
<tr>
<td>16</td>
<td>0.3870</td>
<td>0.5079</td>
<td>0.6218</td>
<td>0.8217</td>
</tr>
</tbody>
</table>

**TABLE 5.1.** Contraction numbers for the symmetric W-cycle algorithm on the L-shaped domain with $m$ smoothing steps for the case $\epsilon = [1/350, 1, 1/350]$. 
in the eigenfunction $\Theta$, and $\varrho_{\text{cut}}$ is the cut-off function defined by

$$
\varrho_{\text{cut}}(r) = \left\{ \begin{array}{ll}
1, & 0 \leq r \leq 1/4 \\
-192r^5 + 480r^4 - 440r^3 + 180r^2 - \frac{125}{4} r + \frac{27}{8}, & 1/4 < r < 3/4 \\
0, & r \geq 3/4.
\end{array} \right.
$$

(5.6.2)

In this case the exact solution $\xi = s = r^{\sigma_1}\Theta(\theta)\varrho_{\text{cut}}$. The numerical results are tabulated in Table 5.2. For comparison, we solved the same problem by full multigrid without using the extraction formula. The numerical results are tabulated in Table 5.3.

From Table 5.2, we see that the approximate stress intensity factor $\kappa^\xi_8$ is very accurate when using the new full multigrid with the extraction formula. Actually, the relative error between $\kappa^\xi_8$ and the exact one $\kappa^\xi = 1$ is less than 0.1%. However, when using the full multigrid without the extraction formula, we see from Table 5.3 that the relative error between $\kappa^\xi_8$ and the exact one $\kappa^\xi = 1$ is larger than 30%. Because the equation (5.1.3) has $\xi$ as the right-hand side input function, we need a good approximation to $\xi$ to obtain a good approximation to $\phi$. Therefore, the numerical results tell us that the standard full multigrid method can not obtain a reliable approximation to $\phi$, but the new full multigrid method with extraction formula has a much better performance than the standard and it can give us a reliable approximation to $\phi$.

Furthermore we consider the error between the values of the exact solution and the numerical solution at a particular nodal point. Also, we consider the $L_\infty$ error between the values of the exact solution and the numerical solution. Those errors provide another way to check the accuracy of our algorithm.

We choose two particular nodal points $(1/3, -1/3)$ and $(2/3, -2/3)$ and use the following notations to denote the errors between the values of the exact solution
and the numerical solutions $\xi_k = w_k + \kappa \xi_k$ (obtained by full multigrid using the extraction formula) or $\xi_k = w_k$ (obtained by full multigrid without using the extraction formula) at the two points:

$$e_1^k = |\xi_k(1/3, -1/3) - \xi(1/3, -1/3)|$$

(5.6.3)

and

$$e_2^k = |\xi_k(2/3, -2/3) - \xi(2/3, -2/3)|.$$  

(5.6.4)

The $k^{th}$ level convergence rates are computed and denoted by:

$$\eta_1^k = \log_2 \left( \frac{e_1^k}{e_1^{k+1}} \right)$$

(5.6.5)

and

$$\eta_2^k = \log_2 \left( \frac{e_2^k}{e_2^{k+1}} \right).$$

(5.6.6)

We also denote the errors in $L_\infty$ norm and their convergence rates by:

$$e_\infty^k = \|\xi_k - \Pi_h \xi\|_\infty = \max_{p \text{ is a nodal point}} |\xi_k(p) - \xi(p)|$$

(5.6.7)

and

$$\eta_\infty^k = \log_2 \left( \frac{e_\infty^k}{e_\infty^{k+1}} \right),$$

(5.6.8)

where $\Pi_h$ is the nodal interpolant (cf. Subsection 2.4). The numerical results are tabulated in Table 5.4 and Table 5.5. From Table 5.4 we clearly see that the numerical solutions obtained by the full multigrid methods using the extraction formulas converge quickly. Meanwhile, from the results in Table 5.5 we see that the numerical solutions obtained by the full multigrid methods without using the extraction formulas converge very slowly.

**Example 5.6.2.** We solve equations (5.1.5) and (5.1.3) on the domain $\Omega$ where $\alpha = 1$, $\epsilon = [1/350, 1, 1/350]$, $\mu = [1, 1, 1]$, and the right-hand side vector function
Table 5.2. Results of the full multigrid method with exactation formulas for Example 5.6.1

| $k$ | $\|w_{k+1} - w_k\|_{L_2}/\|f\|_{L_2}$ | Order | $|w_{k+1} - w_k|_{H^1}/\|f\|_{L_2}$ | Order | $\kappa^\epsilon_k$ | Order |
|-----|---------------------------------|-------|---------------------------------|-------|----------------|-------|
| $k=3$ | 4.13089E-002 | 2.32 | 9.54251E-002 | 2.27 | 1.33663 | 2.40 |
| $k=4$ | 8.25511E-003 | 2.70 | 1.97396E-002 | 2.65 | 1.06363 | 1.65 |
| $k=5$ | 1.26805E-003 | 1.75 | 3.13929E-003 | 1.69 | 1.02034 | 1.55 |
| $k=6$ | 3.7619E-004 | 1.60 | 9.71532E-004 | 1.53 | 1.00659 | 1.58 |

Table 5.3. Results of the full multigrid method without extraction formula for Example 5.6.1

| $k$ | $|v_{k+1} - v_k|_{L_2}/\|f\|_{L_2}$ | Order | $|v_{k+1} - v_k|_{H^1}/\|f\|_{L_2}$ | Order | $\kappa^\phi_k$ | Order |
|-----|---------------------------------|-------|---------------------------------|-------|----------------|-------|
| $k=3$ | 4.97390E-002 | 2.24 | 1.12834E-001 | 2.08 | 1.55132 | 0.54 |
| $k=4$ | 1.05195E-002 | 0.04 | 2.65966E-002 | 0.04 | 1.87351 | 0.96 |
| $k=5$ | 1.02509E-002 | 0.88 | 2.59071E-002 | 0.81 | 2.21573 | 1.13 |
| $k=6$ | 5.58506E-003 | 1.17 | 1.48107E-002 | 1.08 | 2.41218 | 1.18 |
| $k=7$ | 2.48641E-003 | - | 6.99230E-003 | - | 2.50392 | 1.05 |
| $k=8$ | - | - | - | - | 2.54140 | - |

| $h$ | $|v_{k+1} - v_k|_{L_2}/\|f\|_{L_2}$ | Order | $|v_{k+1} - v_k|_{H^1}/\|f\|_{L_2}$ | Order | $\kappa^\phi_k$ | Order |
|-----|---------------------------------|-------|---------------------------------|-------|----------------|-------|
| $k=3$ | 8.76793E-003 | 0.02 | 2.39732E-002 | -0.06 | -3.6208 | - |
| $k=4$ | 8.61832E-003 | 0.03 | 2.49235E-002 | -0.05 | -3.7190 | - |
| $k=5$ | 8.46529E-003 | 0.03 | 2.57451E-002 | -0.04 | -3.8143 | - |
| $k=6$ | 8.30104E-003 | 0.03 | 2.64454E-002 | -0.03 | -3.9076 | - |
| $k=7$ | 8.12447E-003 | - | 2.70319E-002 | - | -3.9989 | - |
| $k=8$ | - | - | - | - | -4.0882 | - |
TABLE 5.4. Pointwise errors for Example 5.6.1 using the full multigrid algorithms with extraction formulas, where $e_1^k$, $\eta_1^k$, $e_2^k$, $\eta_2^k$, $e_\infty^k$ and $\eta_\infty^k$ are defined by (5.6.3)-(5.6.8)

<table>
<thead>
<tr>
<th>$k$</th>
<th>$e_1^k$</th>
<th>$\eta_1^k$</th>
<th>$e_2^k$</th>
<th>$\eta_2^k$</th>
<th>$e_\infty^k$</th>
<th>$\eta_\infty^k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2.44230E-002</td>
<td>1.64</td>
<td>2.44371E-002</td>
<td>1.64</td>
<td>2.49002E-002</td>
<td>1.64</td>
</tr>
<tr>
<td>4</td>
<td>7.87278E-003</td>
<td>2.46</td>
<td>7.89542E-002</td>
<td>2.46</td>
<td>7.96934E-003</td>
<td>1.45</td>
</tr>
<tr>
<td>5</td>
<td>1.42442E-003</td>
<td>1.71</td>
<td>1.42349E-003</td>
<td>1.71</td>
<td>1.45365E-003</td>
<td>1.70</td>
</tr>
<tr>
<td>6</td>
<td>4.36182E-004</td>
<td>1.61</td>
<td>4.35866E-004</td>
<td>1.61</td>
<td>4.48301E-004</td>
<td>1.60</td>
</tr>
<tr>
<td>7</td>
<td>1.42605E-004</td>
<td>1.64</td>
<td>1.42499E-004</td>
<td>1.64</td>
<td>1.47737E-004</td>
<td>1.63</td>
</tr>
<tr>
<td>8</td>
<td>4.58088E-005</td>
<td>-</td>
<td>4.57748E-005</td>
<td>-</td>
<td>4.78813E-005</td>
<td>-</td>
</tr>
</tbody>
</table>

TABLE 5.5. Pointwise errors for Example 5.6.1 using the full multigrid algorithms without extraction formulas, where $e_1^k$, $\eta_1^k$, $e_2^k$, $\eta_2^k$, $e_\infty^k$ and $\eta_\infty^k$ are defined by (5.6.3)-(5.6.8)

<table>
<thead>
<tr>
<th>$k$</th>
<th>$e_1^k$</th>
<th>$\eta_1^k$</th>
<th>$e_2^k$</th>
<th>$\eta_2^k$</th>
<th>$e_\infty^k$</th>
<th>$\eta_\infty^k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2.44230E-002</td>
<td>-</td>
<td>2.44371E-002</td>
<td>-</td>
<td>2.49002E-002</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>2.33570E-002</td>
<td>-</td>
<td>2.33482E-002</td>
<td>-</td>
<td>2.36740E-002</td>
<td>-</td>
</tr>
<tr>
<td>5</td>
<td>2.23848E-002</td>
<td>-</td>
<td>2.23700E-002</td>
<td>-</td>
<td>2.28435E-002</td>
<td>-</td>
</tr>
<tr>
<td>6</td>
<td>2.14447E-002</td>
<td>-</td>
<td>2.14313E-002</td>
<td>-</td>
<td>2.20428E-002</td>
<td>-</td>
</tr>
<tr>
<td>7</td>
<td>2.05315E-002</td>
<td>-</td>
<td>2.05161E-002</td>
<td>-</td>
<td>2.12710E-002</td>
<td>-</td>
</tr>
<tr>
<td>8</td>
<td>1.96366E-002</td>
<td>-</td>
<td>1.96218E-002</td>
<td>-</td>
<td>2.05272E-002</td>
<td>-</td>
</tr>
</tbody>
</table>

is given by

$$f(x) = \begin{cases} 
1 & \text{if } x \in \Omega_1 \text{ or } \Omega_2 \\
0 & \text{otherwise.}
\end{cases} \quad (5.6.9)$$

The numerical results are tabulated in Table 5.6. For comparison we solved the same problem by full multigrid without using the extraction formula. The numerical results are tabulated in Table 5.7.

Comparing Table 5.6 and Table 5.7, we can clearly see the improvement of the order of convergence for $\kappa_\xi_k$ and $w_k$ while using the algorithm with extraction formulas. The order of convergence for $\kappa_\xi_k$ and $w_k$ matches the estimates (5.5.32) and (5.5.33).
<table>
<thead>
<tr>
<th>$k$</th>
<th>$\frac{|w_{k+1} - w_k|<em>{L^2}}{|f|</em>{L^2}}$</th>
<th>Order</th>
<th>$\frac{|w_{k+1} - w_k|<em>{H^1}}{|f|</em>{L^2}}$</th>
<th>Order</th>
<th>$\kappa_k^\xi$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$1.30236E-002$</td>
<td>2.32</td>
<td>$8.84351E-002$</td>
<td>0.96</td>
<td>0.40726</td>
<td>0.54</td>
</tr>
<tr>
<td>4</td>
<td>$2.60059E-003$</td>
<td>0.01</td>
<td>$4.54224E-002$</td>
<td>0.87</td>
<td>0.48641</td>
<td>0.99</td>
</tr>
<tr>
<td>5</td>
<td>$2.58557E-003$</td>
<td>0.91</td>
<td>$2.48881E-002$</td>
<td>0.91</td>
<td>0.57346</td>
<td>1.17</td>
</tr>
<tr>
<td>6</td>
<td>$1.37815E-003$</td>
<td>1.20</td>
<td>$1.32454E-002$</td>
<td>0.94</td>
<td>0.62234</td>
<td>1.20</td>
</tr>
<tr>
<td>7</td>
<td>$5.99051E-004$</td>
<td>-</td>
<td>$6.91438E-003$</td>
<td>-</td>
<td>0.64457</td>
<td>1.05</td>
</tr>
<tr>
<td>8</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\frac{|v_{k+1} - v_k|<em>{L^2}}{|f|</em>{L^2}}$</th>
<th>Order</th>
<th>$\frac{|v_{k+1} - v_k|<em>{H^1}}{|f|</em>{L^2}}$</th>
<th>Order</th>
<th>$\kappa_k^\phi$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$1.65040E-003$</td>
<td>0.69</td>
<td>$8.33942E-002$</td>
<td>0.89</td>
<td>0.40726</td>
<td>-0.05</td>
</tr>
<tr>
<td>4</td>
<td>$1.02427E-003$</td>
<td>0.12</td>
<td>$4.50228E-002$</td>
<td>0.90</td>
<td>0.41681</td>
<td>0.01</td>
</tr>
<tr>
<td>5</td>
<td>$9.40852E-004$</td>
<td>0.04</td>
<td>$2.41734E-002$</td>
<td>0.89</td>
<td>0.42670</td>
<td>0.03</td>
</tr>
<tr>
<td>6</td>
<td>$9.14799E-004$</td>
<td>0.04</td>
<td>$1.30572E-002$</td>
<td>0.83</td>
<td>0.43647</td>
<td>0.03</td>
</tr>
<tr>
<td>7</td>
<td>$8.92539E-004$</td>
<td>-</td>
<td>$7.34235E-003$</td>
<td>-</td>
<td>0.44611</td>
<td>-</td>
</tr>
<tr>
<td>8</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.45551</td>
<td>-</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\frac{|v_{k+1} - v_k|<em>{L^2}}{|f|</em>{L^2}}$</th>
<th>Order</th>
<th>$\frac{|v_{k+1} - v_k|<em>{H^1}}{|f|</em>{L^2}}$</th>
<th>Order</th>
<th>$\kappa_k^\phi$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$1.00389E-003$</td>
<td>0.06</td>
<td>$5.63034E-002$</td>
<td>0.59</td>
<td>-0.40462</td>
<td>-0.05</td>
</tr>
<tr>
<td>4</td>
<td>$9.65823E-004$</td>
<td>0.03</td>
<td>$3.73602E-003$</td>
<td>0.26</td>
<td>-0.41468</td>
<td>0.01</td>
</tr>
<tr>
<td>5</td>
<td>$9.44524E-004$</td>
<td>0.03</td>
<td>$3.13045E-003$</td>
<td>0.06</td>
<td>-0.42507</td>
<td>0.02</td>
</tr>
<tr>
<td>6</td>
<td>$9.25241E-004$</td>
<td>0.03</td>
<td>$3.01696E-003$</td>
<td>-0.01</td>
<td>-0.43541</td>
<td>0.03</td>
</tr>
<tr>
<td>7</td>
<td>$9.05325E-004$</td>
<td>-</td>
<td>$3.02828E-003$</td>
<td>-</td>
<td>-0.44557</td>
<td>-</td>
</tr>
<tr>
<td>8</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-0.45551</td>
<td>-</td>
</tr>
</tbody>
</table>

**TABLE 5.6.** Results of the full multigrid method with extraction formulas for Example 5.6.2

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\frac{|v_{k+1} - v_k|<em>{L^2}}{|f|</em>{L^2}}$</th>
<th>Order</th>
<th>$\frac{|v_{k+1} - v_k|<em>{H^1}}{|f|</em>{L^2}}$</th>
<th>Order</th>
<th>$\kappa_k^\phi$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$1.00389E-003$</td>
<td>0.06</td>
<td>$5.63034E-002$</td>
<td>0.59</td>
<td>-0.40462</td>
<td>-0.05</td>
</tr>
<tr>
<td>4</td>
<td>$9.65823E-004$</td>
<td>0.03</td>
<td>$3.73602E-003$</td>
<td>0.26</td>
<td>-0.41468</td>
<td>0.01</td>
</tr>
<tr>
<td>5</td>
<td>$9.44524E-004$</td>
<td>0.03</td>
<td>$3.13045E-003$</td>
<td>0.06</td>
<td>-0.42507</td>
<td>0.02</td>
</tr>
<tr>
<td>6</td>
<td>$9.25241E-004$</td>
<td>0.03</td>
<td>$3.01696E-003$</td>
<td>-0.01</td>
<td>-0.43541</td>
<td>0.03</td>
</tr>
<tr>
<td>7</td>
<td>$9.05325E-004$</td>
<td>-</td>
<td>$3.02828E-003$</td>
<td>-</td>
<td>-0.44557</td>
<td>-</td>
</tr>
<tr>
<td>8</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-0.45551</td>
<td>-</td>
</tr>
</tbody>
</table>

**TABLE 5.7.** Results of the full multigrid method without extraction formulas for Example 5.6.2
The last numerical example is performed on the domain $\Omega = (-1,1)^2$, where the subdomains are $\Omega_1 = (0,1) \times (0,1)$, $\Omega_2 = (-1,0) \times (0,1)$, $\Omega_3 = (-1,0)^2$, and $\Omega_4 = (0,1) \times (-1,0)$ (See Figure 5.2). We use the $P_1$ finite element in the experiments. The mesh size $h_k$ for the $k^{th}$ level grid is taken to be $1/2^{k-1}$. All the computations are done using a $W$-cycle $k^{th}$ level iteration with 50 smoothing steps, and the number of nested iterations in both full multigrid algorithms (See Algorithm 5.4.1 and Algorithm 5.4.2 in Subsection 5.4.2) is also 50.

**Example 5.6.3.** We solve equations (5.1.5) and (5.1.3) on the domain $\Omega$ where $\alpha = 1$, $\epsilon = [1/10, 1/10^3, 1, 1/10^4]$, $\mu = [1,1,1,1]$, and the right-hand side vector function is given by

$$f(x) = \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \text{if } x \in \Omega_1 \text{ or } \Omega_2 \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \text{otherwise.} \end{cases}$$

(5.6.10)

In this case, the square root of the first eigenvalue of the related Sturm-Liouville problem (cf. Subsection 2.2.2) is $\sigma_1 = 0.069817020390924...$. The numerical results are tabulated in Table 5.8. For comparison we solved the same problem by full multigrid without using the extraction formula. The numerical results are tabulated in Table 5.9.
| k  | $\|w_{k+1} - w_k\|_{L^2} / \|f\|_{L^2}$ | Order | $|w_{k+1} - w_k|_{H^1} / \|f\|_{L^2}$ | Order | $\kappa_k^{\xi}$ | Order |
|----|----------------------------------|-------|----------------------------------|-------|----------------|-------|
| k=4| 2.50818e-002                     | 1.93  | 1.04075e-001                     | 0.92  | 1.57342        | 1.92  |
| k=5| 6.59066e-003                     | 0.66  | 5.48533e-002                     | 0.53  | 1.99063        | 0.55  |
| k=6| 4.16937e-003                     | 1.07  | 3.79186e-002                     | 0.69  | 2.27528        | 1.01  |
| k=7| 1.98659e-003                     | 1.32  | 2.35356e-002                     | 0.77  | 2.41677        | 1.29  |
| k=8| 7.97897e-004                     | -     | 1.37912e-002                     | -     | 2.47466        | 1.49  |
| k=9| -                                | -     | -                                | -     | 2.49524        | -     |

| h  | $\|v_{k+1} - v_k\|_{L^2} / \|f\|_{L^2}$ | Order | $|v_{k+1} - v_k|_{H^1} / \|f\|_{L^2}$ | Order | $\kappa_k^\phi$ | Order |
|----|----------------------------------|-------|----------------------------------|-------|----------------|-------|
| k=4| 2.63827e-003                     | 1.83  | 6.47826e-003                     | 1.02  | -0.13092       | 1.89  |
| k=5| 7.39617e-004                     | 0.65  | 3.19478e-003                     | 0.33  | -0.16625       | 0.56  |
| k=6| 4.71923e-004                     | 1.11  | 2.54380e-003                     | 0.79  | -0.19026       | 1.01  |
| k=7| 2.18638e-004                     | 1.42  | 1.47385e-003                     | 1.01  | -0.20221       | 1.29  |
| k=8| 8.15293e-005                     | -     | 7.32761e-004                     | -     | -0.20710       | 1.49  |
| k=9| -                                | -     | -                                | -     | -0.20885       | -     |

**TABLE 5.8.** Results of the full multigrid method with extraction formulas for Example 5.6.3

Comparing Table 5.8 and Table 5.9, we can clearly see the improvement of the order of convergence for $\kappa_k^{\xi}$ and $w_k$ while using the algorithm with extraction formulas. The order of convergence for $\kappa_k^{\xi}$ and $w_k$ matches the estimates (5.5.32) and (5.5.33).
\[
\begin{array}{cccccc}
\text{k} & \frac{\|w_{k+1} - w_k\|_{L^2}}{\|f\|_{L^2}} & \text{Order} & \frac{|w_{k+1} - w_k|_{H^1}}{\|f\|_{L^2}} & \text{Order} & \kappa_k^\xi & \text{Order} \\
n=4 & 3.43595e-003 & 0.40 & 7.52331e-002 & 0.55 & 1.57342 & 4.67 \\
n=5 & 2.60679e-003 & 0.17 & 5.12986e-002 & 0.51 & 1.63532 & 0.16 \\
n=6 & 2.32044e-003 & 0.11 & 3.60078e-002 & 0.59 & 1.69081 & 0.07 \\
n=7 & 2.15047e-003 & 0.08 & 2.39138e-002 & 0.56 & 1.74351 & 0.06 \\
n=8 & 2.04033e-003 & - & 1.62005e-002 & - & 1.79395 & 0.06 \\
n=9 & - & - & - & 1.84220 & - & - \\
\hline \\
n=4 & 3.35034e-004 & -0.03 & 2.35234e-003 & -0.27 & -0.13441 & 4.59 \\
n=5 & 3.41577e-004 & 0.09 & 2.82980e-003 & 0.22 & -0.13997 & 0.21 \\
n=6 & 3.20896e-004 & 0.18 & 2.42614e-003 & 0.62 & -0.14479 & 0.09 \\
n=7 & 2.82526e-004 & 0.13 & 1.57991e-003 & 0.58 & -0.14934 & 0.06 \\
n=8 & 2.57405e-004 & - & 1.05698e-003 & - & -0.15369 & 0.06 \\
n=9 & - & - & 0.00000e+000 & - & 0.15787 & - \\
\end{array}
\]

**TABLE 5.9.** Results of the full multigrid method without extraction formulas for Example 5.6.3
Appendix A: Proof for Theorem 2.4.7

Let $C$ denote a generic constant independent of the mesh size $h$.

Let $s = r^\sigma \sin(\sigma \theta)$, where $(r, \theta)$ are the polar coordinates with respect to a corner $c$ and $\sigma$ is a number between 0 and 1.

Around the corner $c$, we have

$$\sum_{T \in \mathcal{T}, c \in \bar{T}} \|s - \Pi_h s\|_{L^2(T)}^2 \leq \sum_{T \in \mathcal{T}, c \in \bar{T}} \int_T (2s^2 + 2|\Pi_h s|^2)dx$$

$$= 2 \sum_{T \in \mathcal{T}, c \in \bar{T}} \int_T s^2 dx + 2 \sum_{T \in \mathcal{T}, c \in \bar{T}} \int_T |\Pi_h s|^2 dx \quad (5.6.11)$$

and

$$\sum_{T \in \mathcal{T}, c \in \bar{T}} \|\nabla s - \nabla \Pi_h s\|_{L^2(T)}^2 \leq \sum_{T \in \mathcal{T}, c \in \bar{T}} \int_T (2|\nabla s|^2 + 2|\nabla \Pi_h s|^2)dx$$

$$= 2 \sum_{T \in \mathcal{T}, c \in \bar{T}} \int_T |\nabla s|^2 dx + 2 \sum_{T \in \mathcal{T}, c \in \bar{T}} \int_T |\nabla \Pi_h s|^2 dx. \quad (5.6.12)$$

For a triangle $T$ around the corner $c$, i.e. $c \in \bar{T}$, let $x_j, j = 1, 2, 3$ be the coordinates of the vertices of the triangle $T$, and $e_j, j = 1, 2, 3$ be the local basis functions. Then we have

$$\int_T s^2 dx \leq \int_T r^{2\sigma} dx \leq C \int_0^h r^{2\sigma+1} dr \leq C h^{2\sigma+2}, \quad (5.6.13)$$

$$\int_T |\Pi_h s|^2 dx = \int_T (\sum_{j=1}^3 s(x_j)e_j)^2 dx \leq \int_T (\sum_{j=1}^3 s(x_j)^2)(\sum_{j=1}^3 e_j^2)dx$$

$$\leq C h^{2\sigma} \int_T (\sum_{j=1}^3 e_j^2)dx \leq C h^{2\sigma+2}, \quad (5.6.14)$$

$$\int_T |\nabla s|^2 dx \leq C \int_T r^{2\sigma-2} dx \leq C \int_0^h r^{2\sigma-1} dr \leq C h^{2\sigma}, \quad (5.6.15)$$

and

$$\int_T |\nabla \Pi_h s|^2 dx = \int_T |\sum_{j=1}^3 s(x_j)\nabla e_j|^2 dx \leq \int_T (\sum_{j=1}^3 s(x_j)^2)(\sum_{j=1}^3 |\nabla e_j|^2)dx.$$
\[ \leq Ch^{2\sigma} \int_T (\sum_{j=1}^3 |\nabla e_j|^2) \, dx \leq Ch^{2\sigma}. \]  
(5.6.16)

By (5.6.11), (5.6.13) and (5.6.14), we have

\[ \sum_{T \in \mathcal{T}_h, c \in \bar{T}} \| s - \Pi_h s \|^2_{L^2(T)} \leq Ch^{2\sigma+2}. \]  
(5.6.17)

By (5.6.12), (5.6.15) and (5.6.16), we have

\[ \sum_{T \in \mathcal{T}_h, c \in \bar{T}} \| \nabla s - \nabla \Pi_h s \|^2_{L^2(T)} \leq Ch^{2\sigma}. \]  
(5.6.18)

Let \( \Omega_h = \{ x \in \Omega : |x - c| > \delta h \} \) for some \( \delta \) between 0 and 1 so that

\[ T \subset \Omega_h \text{ if } T \in \mathcal{T}_h, c \notin \bar{T}. \]

Therefore,

\[ \sum_{T \in \mathcal{T}_h, c \notin \bar{T}} \| s - \Pi_h s \|^2_{L^2(T)} \leq \| s - \Pi_h s \|^2_{L^2(\Omega_h)}, \]  
(5.6.19)

and

\[ \sum_{T \in \mathcal{T}_h, c \notin \bar{T}} \| s - \Pi_h s \|^2_{H^1(T)} \leq \| s - \Pi_h s \|^2_{H^1(\Omega_h)}. \]  
(5.6.20)

Since \( s \in H^2(\Omega_h) \), by [20, Theorem (4.4.4)], we have

\[ \| s - \Pi_h s \|_{L^2(\Omega_h)} + h \| s - \Pi_h s \|_{H^1(\Omega_h)} \leq Ch^2 |s|_{H^2(\Omega_h)}. \]  
(5.6.21)

Note that

\[ |s|_{H^2(\Omega_h)}^2 \leq C \int_{\Omega_h} r^{2\sigma-4} \, dx \leq C \int_h^R r^{2\sigma-3} \, dr \leq Ch^{2\sigma-2}, \]  
(5.6.22)

where \( R \) is the diameter of the domain \( \Omega \).

Therefore (5.6.21) and (5.6.22) imply

\[ \| s - \Pi_h s \|_{L^2(\Omega_h)} + h |s - \Pi_h s|_{H^1(\Omega_h)} \leq Ch^{\sigma+1}. \]  
(5.6.23)
By (5.6.19), (5.6.20), and (5.6.23) we have

\[ \sum_{T \in \mathcal{T}, c \notin T} \| s - \Pi_h s \|_{L^2(T)}^2 \leq C h^{2\sigma + 2}, \]  

(5.6.24)

and

\[ \sum_{T \in \mathcal{T}, c \notin T} \| s - \Pi_h s \|_{H^1(T)}^2 \leq C h^{2\sigma}. \]  

(5.6.25)

Now by (5.6.17), (5.6.18), (5.6.24), and (5.6.25), we have

\[ \| s - \Pi_h s \|_{L^2(\Omega)} + h|s - \Pi_h s|_{H^1(\Omega)} \leq C h^{\sigma + 1}. \]  

(5.6.26)

If \( \beta = 1 \), then \( u \in H^2(\Omega) \). So (2.4.8) is true by [20, Theorem (4.4.4)].

If \( \beta < 1 \), then, from Section 2.2.1, we know that

\[ u = u_S + u_R, \]  

(5.6.27)

where \( u_R \in H^2(\Omega) \) and \( u_S = \sum_{\omega_i > \pi} \kappa_i s_i \). The regular part \( u_R \in H^2(\Omega) \) and hence

\[ \| u_R - \Pi_h u_R \|_{L^2(\Omega)} + h|u_R - \Pi_h u_R|_{H^1(\Omega)} \leq C h^2 \| u_R \|_{H^2(\Omega)}. \]  

(5.6.28)

by [20, Theorem (4.4.4)].

For \( s_i = r^{\pi/\omega_i} \sin((\pi/\omega_i)\theta) \varrho_{\text{cut}} \) with \( \omega_i > \pi \), we have

\[ s_i := r^{\pi/\omega_i} \sin(\pi/\omega_i)\theta)(\varrho_{\text{cut}} - 1) + r^{\pi/\omega_i} \sin(\pi/\omega_i)\theta). \]

Let \( s_{i,1} = r^{\pi/\omega_i} \sin((\pi/\omega_i)\theta) (\varrho_{\text{cut}} - 1) \) and \( s_{i,2} = r^{\pi/\omega_i} \sin((\pi/\omega_i)\theta). \) Then \( s_{i,1} \in H^2(\Omega) \), so

\[ \| s_{i,1} - \Pi_h s_{i,1} \|_{L^2(\Omega)} + h|s_{i,1} - \Pi_h s_{i,1}|_{H^1(\Omega)} \leq C h^2 \| s_{i,1} \|_{H^2(\Omega)} \leq C h^2. \]  

(5.6.29)

By (5.6.26), we have

\[ \| s_{i,2} - \Pi_h s_{i,2} \|_{L^2(\Omega)} + h|s_{i,2} - \Pi_h s_{i,2}|_{H^1(\Omega)} \leq C h^{\sigma + 1}. \]  

(5.6.30)

Now (5.6.27), (5.6.28), (5.6.29), and (5.6.30) together imply (2.4.9).
Appendix B: Proof for (2.2.17)

We will follow the methodology of Example 1 in [54, Section 5] and consider a general Sturm-Liouville problem:

\[
\Theta''(\theta) + \sigma^2 \Theta(\theta) = 0 \quad \text{for} \quad \theta_{i-1} < \theta < \theta_i, \quad i = 1, \ldots, n, \quad (5.6.31a)
\]

\[
\Theta'(\theta_0) = \Theta'(\theta_n) = 0, \quad (5.6.31b)
\]

\[
\Theta(\theta_i-) = \Theta(\theta_i+) \quad \text{for} \quad i = 1, \ldots, n-1, \quad (5.6.31c)
\]

\[
\rho_{i+1}\Theta'(\theta_i+) = \rho_i\Theta'(\theta_i-) \quad \text{for} \quad i = 1, \ldots, n-1, \quad (5.6.31d)
\]

where \(0 \leq \theta_0 < \theta_1 < \ldots < \theta_n \leq 2\pi, \theta_n - \theta_0 \neq 2\pi\), and \(\rho_i > 0\) for \(i = 1, 2, \ldots, n\).

Our goal is to find the eigenvalues \(\lambda = \sigma^2\) of (5.6.31).

The solutions of (5.6.31a) have the general form

\[
\Theta(\theta) = A_i \cos(\sigma \theta) + B_i \sin(\sigma \theta) \quad \text{for} \quad \theta_{i-1} \leq \theta \leq \theta_i, \quad i = 1, \ldots, n.
\]

Substituting the general solution of \(\Theta\) into the boundary condition (5.6.31b) and the interface conditions (5.6.31c)-(5.6.31d), we obtain a linear system about the variables \(A_i\) and \(B_i\). Denote the determinant of the coefficient matrix of this linear system by \(D_N^N(\sigma)\). We also denote, by the symbol \(D_M^N(\sigma)\), the determinant of the coefficient matrix of the linear system obtained from the Sturm-Liouville problem with mixed boundary condition on the external boundary, i.e., we replace the Neumann boundary condition \(\Theta'(\theta_n) = 0\) in (5.6.31b) by the Dirichlet boundary condition \(\Theta(\theta_n) = 0\). When \(n = 1\), the determinants \(D_N^N(\sigma)\) and \(D_M^M(\sigma)\) are given by

\[
D_N^N = \begin{vmatrix}
-\sin(\sigma \theta_0) & \cos(\sigma \theta_0) \\
-\sin(\sigma \theta_1) & \cos(\sigma \theta_1)
\end{vmatrix}
\]

(5.6.32)
and

\[
D_1^M = \begin{vmatrix}
-\sin(\sigma \theta_0) & \cos(\sigma \theta_0) \\
\cos(\sigma \theta_1) & \sin(\sigma \theta_1)
\end{vmatrix}.
\] (5.6.33)

From (5.6.32) and (5.6.33), a simple computation implies

\[
D_1^N(\sigma) = \sin(\sigma \omega_1)
\] (5.6.34a)

and

\[
D_1^M(\sigma) = -\cos(\sigma \omega_1),
\] (5.6.34b)

where \( \omega_1 = \theta_1 - \theta_0 \).

When \( n = 2 \), the determinants \( D_2^N(\sigma) \) and \( D_2^M(\sigma) \) are given by

\[
D_2^N = \begin{vmatrix}
-\sin(\sigma \theta_0) & \cos(\sigma \theta_0) & 0 & 0 \\
\cos(\sigma \theta_1) & \sin(\sigma \theta_1) & -\cos(\sigma \theta_1) & -\sin(\sigma \theta_1) \\
-\rho_1 \sin(\sigma \theta_1) & \rho_1 \cos(\sigma \theta_1) & \rho_2 \sin(\sigma \theta_1) & -\rho_2 \cos(\sigma \theta_1) \\
0 & 0 & -\sin(\sigma \theta_2) & \cos(\sigma \theta_2)
\end{vmatrix}
\] (5.6.35)

and

\[
D_2^M = \begin{vmatrix}
-\sin(\sigma \theta_0) & \cos(\sigma \theta_0) & 0 & 0 \\
\cos(\sigma \theta_1) & \sin(\sigma \theta_1) & -\cos(\sigma \theta_1) & -\sin(\sigma \theta_1) \\
-\rho_1 \sin(\sigma \theta_1) & \rho_1 \cos(\sigma \theta_1) & \rho_2 \sin(\sigma \theta_1) & -\rho_2 \cos(\sigma \theta_1) \\
0 & 0 & \cos(\sigma \theta_2) & \sin(\sigma \theta_2)
\end{vmatrix}.
\] (5.6.36)

Using expansion by minors with respect to the last row of \( D_2^N(\sigma) \), \( D_2^N(\sigma) \) can be computed by

\[
D_2^N(\sigma) = \cos(\sigma \theta_2) \ast N_{44} + \sin(\sigma \theta_2) \ast N_{43},
\] (5.6.37)

where

\[
N_{44} = \begin{vmatrix}
-\sin(\sigma \theta_0) & \cos(\sigma \theta_0) & 0 \\
\cos(\sigma \theta_1) & \sin(\sigma \theta_1) & -\cos(\sigma \theta_1) \\
-\rho_1 \sin(\sigma \theta_1) & \rho_1 \cos(\sigma \theta_1) & \rho_2 \sin(\sigma \theta_1)
\end{vmatrix}
\] (5.6.38)
and

\[
N_{43} = \begin{vmatrix}
-\sin(\sigma \theta_0) & \cos(\sigma \theta_0) & 0 \\
\cos(\sigma \theta_1) & \sin(\sigma \theta_1) & -\sin(\sigma \theta_1) \\
-\rho_1 \sin(\sigma \theta_1) & \rho_1 \cos(\sigma \theta_1) & -\rho_2 \cos(\sigma \theta_1)
\end{vmatrix}.
\] (5.6.39)

Using expansion by minors with respect to the last column of \(N_{43}\) and the last column of \(N_{44}\), we obtain

\[
N_{44} = \rho_2 \sin(\sigma \theta_1) D^M_1(\sigma) + \rho_1 \cos(\sigma \theta_1) D^N_1(\sigma)
\] (5.6.40)

and

\[
N_{43} = -\rho_2 \cos(\sigma \theta_1) D^M_1(\sigma) + \rho_1 \sin(\sigma \theta_1) D^N_1(\sigma)
\] (5.6.41)

By (5.6.37), (5.6.41), (5.6.40) and trigonometry identity, we have

\[
D^N_2(\sigma) = -\rho_2 \sin(\sigma \omega_2) D^M_1(\sigma) + \rho_1 \cos(\sigma \omega_2) D^N_1(\sigma),
\] (5.6.42)

where \(\omega_2 = \theta_2 - \theta_1\). Similarly, we have

\[
D^M_2(\sigma) = \rho_2 \cos(\sigma \omega_2) D^M_1(\sigma) + \rho_1 \sin(\sigma \omega_2) D^N_1(\sigma).
\] (5.6.43)

When \(n \geq 3\), similar to the case when \(n = 2\), we apply expansion by minors with respect to the last row of \(D^N_n(\sigma)\) and the last row of \(D^M_n(\sigma)\). We find that \(D^N_n(\sigma)\) can be determined by the following recurrence formula:

\[
D^N_n(\sigma) = -\rho_n \sin(\sigma \omega_n) D^M_{n-1}(\sigma) + \rho_{n-1} \cos(\sigma \omega_n) D^N_{n-1}(\sigma),
\] (5.6.44a)

\[
D^M_n(\sigma) = \rho_n \cos(\sigma \omega_n) D^M_{n-1}(\sigma) + \rho_{n-1} \sin(\sigma \omega_n) D^N_{n-1}(\sigma).
\] (5.6.44b)

Here \(\omega_n = \theta_n - \theta_{n-1}\). The square of the solutions of the equations \(D^N_n(\sigma) = 0\) are the eigenvalues for the Sturm-Liouville problem (5.6.30).

For Example 2.2.4 in Subsection 2.2.2, we have \(n = 3\), \(\omega_1 = \omega_2 = \omega_3 = \pi/2\). So using the recurrence formulas (5.6.34) and (5.6.44), we have

\[
\rho_2 (\rho_1 + \rho_2 + \rho_3) \sin(\sigma \omega) - (\rho_2 (\rho_1 + \rho_2 + \rho_3) + \rho_1 \rho_3) \sin^3(\sigma \omega) = 0,
\] (5.6.45)
where $\omega = \pi/2$. For $0 < \sigma < 1$, we have that $0 < \sin(\sigma \omega) < 1$ and hence (5.6.45) implies that (2.2.17) holds.
References


Vita

Zhe Nan was born in 1983 in Yueqing, Zhejiang Province, People’s Republic of China. He went to Yueqing High School in 1998, and then went to University of Science and Technology of China in 2001 and graduated with a Bachelor of Science in Mathematics. In August 2006, he came to the Department of Mathematics, Louisiana State University at Baton Rouge to pursue graduate studies in mathematics. He is currently a candidate for the degree of Doctor of Philosophy in Mathematics, which will be awarded in Spring 2013. He met his wife Wanli Xu at LSU and they were happily married on July 16, 2010.