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## QUANTIZATION OF THE FREE POISSON CENTRAL LIMIT THEOREM

YUNGANG LU\*

**ABSTRACT.** This paper is devoted to quantization of the free Poisson central limit theorem. We construct a sequence of free independent binomial random variables on an interacting Fock space in terms of creation–annihilation operators. By using these random variables, one studies a quantization of the free Poisson central limit theorem with respect to the convergence both in mixed–moments and in law.

### 1. Introduction

In the present paper, as a continuation of [6] and [7], we set up a **quantization** of the free Poisson central limit theorem (CLT in short). Its monotone analogue will be dealt with in [8].

The free Poisson CLT (see [12] and references within) can be formulated practically as follows: *Let  $\{p_n\}_{n=1}^\infty \subset [0, 1]$  be such sequence that  $np_n \rightarrow \lambda$ , then, in the weak convergence*

$$\lim_{n \rightarrow \infty} ((1 - p_n) \delta_0 + p_n \delta_1)^{\star n} = P_f(\lambda) \quad (1.1)$$

where, “ $\star$ ” is the free convolution and  $P_f(\lambda)$  is the free Poisson distribution with the parameter  $\lambda$  (see [9], [10], [11] and [12] for the detail), in particular,  $P_f(0) := \delta_0$ . Throughout, for any  $x \in \mathbb{R}$ ,  $\delta_x :=$ the Dirac measure centred on  $x$ .

One recalls that the free convolution is usually defined by means of the Cauchy transform: for any  $\mu, \nu \in \mathcal{P} := \{\text{probability measure on } (\mathbb{R}, \mathcal{B})\}$  with the Cauchy transforms  $G_\mu$  and  $G_\nu$  respectively, the free convolution  $\mu \star \nu$  is the element of  $\mathcal{P}$  with the Cauchy transform  $G_\mu + G_\nu$ . Moreover, the properties of the Cauchy transform and the characterization of the weak convergence of a sequence of  $\mathcal{P}$  in terms of the Cauchy transform have already been studied systematically (see, e.g. [3], [5], [10] and references within).

Clearly, the free Poisson CLT (1.1) can be reformulated in terms of algebraic random variables and the free independence as follows: *Let  $(\mathcal{X}, \psi)$  be an algebraic probability space and  $\{\xi_{n,k} : n \in \mathbb{N}^*$  and  $k \leq n\}$  be a family of algebraic random variables, let  $\{p_n\}_{n=1}^\infty \subset [0, 1]$ . If*

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- $\psi\left(\xi_{n,k}^m\right) = \psi\left(\xi_{n,k}\right) = p_n$  for any  $m, n \in \mathbb{N}^*$  and  $k \leq n$  (therefore, the  $\psi$ -distribution of  $\xi_{n,k}$  is  $(1-p_n)\delta_0 + p_n\delta_1$ , i.e. the binomial distribution with the parameter  $(1, p_n)$ , one writes this fact simply as  $\xi_{n,k} \stackrel{\psi}{\sim} b(1, p_n)$ );
- for any  $n \geq 2$ ,  $\{\xi_{n,1}, \dots, \xi_{n,n}\}$  is a free independent family with respect to  $\psi$ ,

the  $\psi$ -distribution of  $\sum_{k=1}^n \xi_{n,k}$  goes to  $P_f(\lambda)$ , whenever  $np_n \rightarrow \lambda$ .

For any  $n \in \mathbb{N}^*$ , since  $\xi_{n,k} \stackrel{\psi}{\sim} b(1, p_n)$  for any  $k \leq n$  and since  $\{\xi_{n,1}, \dots, \xi_{n,n}\}$  is a free independent family with respect to  $\psi$ ,  $\sum_{k=1}^n \xi_{n,k}$  is indeed a sum of  $n$  (free) independent random variables and each of them has the binomial distribution with the parameter  $(1, p_n)$ . So, one calls naturally its distribution the *free binomial distribution with the parameter  $(n, p_n)$* .

Notice that this reformulation of the free Poisson CLT does **not** depend on the specific construction of the algebraic probability space and random variables. In other words, if we take such a **particular**  $(\mathcal{A}, \phi)$  and  $\{X_{n,k} : n \in \mathbb{N}^* \text{ and } k \leq n\}$  that

- $X_{n,k} \stackrel{\phi}{\sim} b(1, p_n)$  for any  $n \in \mathbb{N}^*$  and  $k \leq n$ ,
  - $\{X_{n,1}, \dots, X_{n,n}\}$  is a free independent family with respect to  $\phi$  for any  $n \geq 2$ ,
- then the  $\phi$ -distribution of  $\sum_{k=1}^n X_{n,k}$  is  $((1-p_n)\delta_0 + p_n\delta_1)^{*n}$  and the usual free Poisson CLT tells us that it goes to  $P_f(\lambda)$  whenever  $np_n \rightarrow \lambda$ .

In this paper, we start from a given Hilbert space  $\mathcal{H}$  with an orthogonal normal base (onb in short)  $\{e_k\}_{k=1}^\infty$  and introduce a particular interacting Fock space (IFS in short)  $\Gamma_{sf}(\mathcal{H})$  as a modification the usual full (free) Fock space over  $\mathcal{H}$  (for the detail, see Section 2). Then we defined, on the IFS  $\Gamma_{sf}(\mathcal{H})$ , the creation (resp. annihilation) operator  $a_f^+$  (resp.  $a_f$ ) with the test function  $f \in \mathcal{H}$ . By denoting, for any  $k \in \mathbb{N}^*$ ,  $a_k^+ := a_{e_k}^+$  (resp.  $a_k := a_{e_k}$ ) and

$$a_k^{(\varepsilon)} := \begin{cases} a_k, & \text{if } \varepsilon = -1 \\ a_k^+, & \text{if } \varepsilon = 1 \\ a_k a_k^+, & \text{if } \varepsilon = 0 \\ a_k^+ a_k, & \text{if } \varepsilon = 2 \end{cases} \quad (1.2)$$

we will take the following particular algebraic probability space  $(\mathcal{A}, \phi)$  and algebraic random variables  $\{X_{n,k} : n \in \mathbb{N}^* \text{ and } k \leq n\}$ : Let

- $\mathcal{A}$  be the algebra generated by  $\mathcal{A}_k$ 's and

$$\mathcal{A}_k := \{\text{polynomial in } a_k \text{ and } a_k^+ \text{ with degree } \geq 1\}, \quad \forall k \quad (1.3)$$

and any monomial in  $a_k$  and  $a_k^+$  with degree  $\geq 1$  will be called a **word** of the algebra  $\mathcal{A}_k$ ;

- $\phi :=$ the vacuum state, i.e.,  $\phi(\cdot) := \langle \Psi, \cdot \Psi \rangle$  with  $\Psi :=$ the vacuum vector of  $\Gamma_{sf}(\mathcal{H})$ ;
- for any  $n \in \mathbb{N}^*$ ,  $k \leq n$  and for any given  $\{p_n\}_{n=1}^\infty \subset [0, 1]$ ,

$$X_{n,k} := \sqrt{p_n(1-p_n)} a_k^{(-1)} + \sqrt{p_n(1-p_n)} a_k^{(+1)} + p_n a_k^{(0)} + (1-p_n) a_k^{(2)} \quad (1.4)$$

*Remark 1.1.* We will say that,  $\sqrt{p_n(1-p_n)}a_k^{(-1)}$  and  $\sqrt{p_n(1-p_n)}a_k^{(+1)}$  are **off-diagonal components** of  $X_{n,k}$  since they are conjugate each other;  $p_na_k^{(0)}$  and  $(1-p_n)a_k^{(2)}$  are **diagonal components** of  $X_{n,k}$  since they are self-adjoint.

In Section 2, we give concrete construction of the IFS  $\Gamma_{sf}(\mathcal{H})$  and show

- the algebras  $\mathcal{A}_k$ 's defined in (1.3) are free independent;
- any  $X_{n,k}$  is a projector and  $\phi(X_{n,k}) = p_n$ ; consequently  $X_{n,k} \stackrel{\phi}{\sim} b(1, p_n)$ .

Thus, we introduce, for any  $n \in \mathbb{N}^*$

$$\begin{aligned} B_n^{(\pm 1)} &:= \sqrt{p_n(1-p_n)} \sum_{k=1}^n a_k^{(\pm 1)}; \quad B_n^{(0)} := p_n \sum_{k=1}^n a_k^{(0)}; \quad B_n^{(2)} := (1-p_n) \sum_{k=1}^n a_k^{(2)} \\ B_n &:= B_n^{(-1)} + B_n^{(+1)} + B_n^{(0)} + B_n^{(2)} = \sum_{k=1}^n X_{n,k} \end{aligned} \quad (1.5)$$

and moreover, for any  $\{c_0, c_1, c_2\} \subset \mathbb{R}$ ,

$$\begin{aligned} &D_F(n, p_n; c_0, c_1, c_2) \\ &:= \phi\text{-distribution of } c_1 \left( B_n^{(-1)} + B_n^{(+1)} \right) + c_0 B_n^{(0)} + c_2 B_n^{(2)} \end{aligned} \quad (1.6)$$

As said above, the usual free Poisson CLT confirms that the  $\phi$ -distribution of  $B_n$  (i.e.,  $D_F(n, p_n; 1, 1, 1)$ ) goes to  $P_f(\lambda)$  whenever  $np_n \rightarrow \lambda$ . Contrast with this, the *quantized* Poisson type CLT is:

- 1) to calculate, for any  $\{c_0, c_1, c_2\} \subset \mathbb{R}$ , the limit

$$\lim_{N \rightarrow \infty} \phi \left( \exp \left( it \left( c_1 \left( B_n^{(-1)} + B_n^{(+1)} \right) + c_0 B_n^{(0)} + c_2 B_n^{(2)} \right) \right) \right), \quad \forall t \in \mathbb{R} \quad (1.7)$$

i.e. to know the weak limit of  $D_F(n, p_n; c_0, c_1, c_2)$  by means of the characteristic function;

- 2) to study the moment version of above limit, i.e., to see the limit

$$\lim_{n \rightarrow \infty} \phi \left( B_n^{(\varepsilon(1))} \dots B_n^{(\varepsilon(m))} \right) \quad (1.8)$$

for any  $m \in \mathbb{N}$  and  $\varepsilon \in \{-1, 0, 1, 2\}^m$ ;

- 3) to give a suitable representation to the above limits.

As a consequence of the above 1) and 2), the quantized Poisson CLT makes in evidence the *individual* contributions of each  $B_n^{(\varepsilon)}$  in the CLT's procedure, not only their sum. In particular, by noticing that

$$B_n^{(\pm 1)} := \sqrt{p_n(1-p_n)} \sum_{k=1}^n a_k^{(\pm 1)} \stackrel{np_n \rightarrow \lambda}{\approx} \sqrt{\lambda} \frac{1}{\sqrt{n}} \sum_{k=1}^n a_k^{(\pm 1)}$$

and by taking  $c_0 = c_2 = 0$  and  $c_1 = 1$  in the above 1), one gets the free Laplace-de Moivre CLT. Therefore, the quantization of the free Poisson CLT gives a view to understand the relationship between the free Poisson CLT and corresponding Laplace-de Moivre CLT: *the free Laplace-de Moivre CLT is the off-diagonal part of the free Poisson CLT*. Moreover, by using the *representation* mentioned in the

above 3), one can also understand the relationship between the free Poisson distribution and the free Gaussian distribution, namely, the Wigner (e.g. semi-circle) distribution.

In Section 3, we

- show that the limit (1.7) equals to

$$\left\langle \Phi, \exp \left( it \left( c_1 \sqrt{\lambda} \left( b^{(-1)} + b^{(+1)} \right) + \lambda c_0 b^{(0)} + c_2 b^{(2)} \right) \right) \Phi \right\rangle \quad (1.9)$$

in other words, the vacuum distribution of  $\{c_1(B_n^{(-1)} + B_n^{(+1)}) + c_0 B_n^{(0)} + c_2 B_n^{(2)}\}_{n=1}^{\infty}$  (i.e.  $D_n(n, p_n; c_0, c_1, c_2)$ ) goes to the vacuum distribution of  $c_1 \sqrt{\lambda} (b^{(-1)} + b^{(+1)}) + \lambda c_0 b^{(0)} + c_2 b^{(2)}$  in the weak convergence, where and throughout the present paper,

$$b^{(\varepsilon)} := \begin{cases} b, & \text{if } \varepsilon = -1 \\ b^+, & \text{if } \varepsilon = 1 \\ \mathbf{1}, & \text{if } \varepsilon = 0 \\ \mathbf{1} - P_{\Phi}, & \text{if } \varepsilon = 2 \end{cases} \quad (1.10)$$

$b^+$  and  $b$  are the creation–annihilation operators on the 1–mode–interacting Fock space (1M–IFS in short)  $\Gamma_{fr}(\mathbb{C}) := \Gamma(\mathbb{C}, \{\omega_n\}_n)$  with  $\omega_n = 1$  for any  $n \in \mathbb{N}^*$ ,  $\Phi$  is the vacuum vector and  $P_{\Phi}$  is the projector from  $\Gamma_{fr}(\mathbb{C})$  to its vacuum subspace;

- prove the existence of the limit (1.8) and show that it has the form

$$\left\langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(n+1))} \Phi \right\rangle \lambda^{\sum_{k=1}^{n+1} (1 - |\varepsilon(k)|/2)} \quad (1.11)$$

*Remark 1.2.* On the 1M–IFS  $\Gamma_{fr}(\mathbb{C})$ , one has  $bb^+ = \mathbf{1}$  and  $b^+b = \mathbf{1} - P_{\Phi}$ .

## 2. Definition and Some Elementary Properties of $\Gamma_{sf}(\mathcal{H})$

The following notations will be used frequently

$$\begin{aligned} \overline{\mathbb{F}}_n &:= \{\text{functions from } \{1, \dots, n\} \text{ to } \mathbb{N}\} \\ \mathbb{F}_n &:= \{\mathbf{k} \in \overline{\mathbb{F}}_n : \mathbf{k}(i) \neq \mathbf{k}(i+1) \text{ for any } 1 \leq i < n\} \\ \mathbb{F}_n^0 &:= \{\mathbf{k} \in \mathbb{F}_n : \mathbf{k}(i) \neq \mathbf{k}(j) \text{ for any } 1 \leq i \neq j \leq n\} \end{aligned} \quad (2.1)$$

Let  $\mathcal{H}$  be a Hilbert space with an onb  $\{e_k\}_{k=1}^{\infty}$ , one defines  $\Gamma_{sf}(\mathcal{H})$  as  $\bigoplus_{n=0}^{\infty} \mathcal{H}_n$ , where  $\mathcal{H}_0 := \mathbb{C}$ ,  $\mathcal{H}_1 := \mathcal{H}$  and for any  $n \geq 2$ ,  $\mathcal{H}_n$  is the (pre–)Hilbert space obtained by equipping the usual the tensor scalar product  $\langle \cdot, \cdot \rangle_n$  on the vector space

$$\text{lin.sp.} \{e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)} : \mathbf{k} \in \mathbb{F}_n^0\} \quad (2.2)$$

i.e.,

$$\langle e_{\mathbf{h}(n)} \otimes \dots \otimes e_{\mathbf{h}(1)}, e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)} \rangle_n := \prod_{1 \leq j \leq n} \delta_{\mathbf{h}(j), \mathbf{k}(j)}, \quad \forall \mathbf{k}, \mathbf{h} \in \mathbb{F}_n^0 \quad (2.3)$$

As usual, one calls  $\Psi := 1 \oplus 0 \oplus 0 \oplus \dots$  and  $\mathcal{H}_n$  the vacuum vector and the  $n$ -particle space respectively.

$\Gamma_{sf}(\mathcal{H})$  is obviously a subspace of the usual full (free) Fock space over  $\mathcal{H}$  which is  $\bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}$ . Contrast with  $\mathcal{H}_n$ ,  $\mathcal{H}^{\otimes n}$  is obtained by introducing the same scalar product (2.3) on the vector space

$$\text{lin.sp.} \{e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)} : \mathbf{k} \in \overline{\mathbb{F}_n}\}$$

On the other hand,  $\Gamma_{sf}(\mathcal{H})$  can be viewed as a particular IFS over  $\mathcal{H}$  : for any  $n \geq 2$ , one defines  $\Omega_n : \mathcal{H}^{\otimes n} \mapsto \mathcal{H}^{\otimes n}$  by the linearity and

$$\begin{aligned} & \Omega_n (e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)}) \\ & := \prod_{1 \leq j < r \leq n} (1 - \delta_{\mathbf{k}(j), \mathbf{k}(r)}) (e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)}), \quad \forall \mathbf{k} \in \overline{\mathbb{F}_n} \end{aligned} \quad (2.4)$$

It is easy to see that  $\Omega_n$  is a projector and moreover, the following  $(\cdot, \cdot)_n$

$$(x, y)_n := \langle x, \Omega_n y \rangle, \quad \forall x, y \in \mathcal{H}^{\otimes n}$$

defines a positive quadratic form and our  $\mathcal{H}_n$  (more precisely,  $(\mathcal{H}_n, \langle \cdot, \cdot \rangle_n)$ ) is isomorphic to  $(\mathcal{H}^{\otimes n}, \langle \cdot, \cdot \rangle_n) / \ker(\langle \cdot, \cdot \rangle_n)$ . Clearly,

- for any  $n \in \mathbb{N}^*$ ,  $\{e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)} : \mathbf{k} \in \mathbb{F}_n^0\}$  is an onb of  $\mathcal{H}_n$ ;
- a general element of  $\Gamma_{sf}(\mathcal{H})$  has the form  $\sum_{n \in I} x_n$  with  $I \subset \mathbb{N}$  finite and  $x_n \in \mathcal{H}_n$  for any  $n \in I$ .

For any  $k \in \mathbb{N}^*$ , we introduce the linear operator (called the creation operator with the test function  $e_k$ )  $a_k^+$  on  $\Gamma_{sf}(\mathcal{H})$  by linearity and

$$\begin{aligned} a_k^+ \Psi & := e_k \\ a_k^+ e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)} & := \begin{cases} e_k \otimes e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)}, & \text{if } k \notin \text{range}(\mathbf{k}) \\ 0, & \text{if } k \in \text{range}(\mathbf{k}) \end{cases} \\ & = \prod_{j=1}^n (1 - \delta_{k, \mathbf{k}(j)}) e_k \otimes e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)}, \quad \forall n \geq 1, \mathbf{k} \in \mathbb{F}_n^0 \end{aligned} \quad (2.5)$$

**Proposition 2.1.** *For any  $n \in \mathbb{N}$  and  $k \in \mathbb{N}^*$ , one has the following affirmations.*

- 1)  $\|a_k^+|_{\mathcal{H}_n}\| = 1$ ,  $\|a_k^+\| = 1$  and  $(a_k^+)^2 = 0$ .
- 2)  $a_k :=$  the conjugate of  $a_k^+$ , is called the annihilation operator with the test function  $e_k$  and verifies

$$\begin{aligned} a_k \Psi & = 0, \\ a_k (e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)}) & = \delta_{k, \mathbf{k}(n)} e_{\mathbf{k}(n-1)} \otimes \dots \otimes e_{\mathbf{k}(1)}, \quad \forall \mathbf{k} \in \mathbb{F}_n^0 \end{aligned} \quad (2.6)$$

Moreover,  $\|a_k|_{\mathcal{H}_n}\| = 1$  for any  $n \in \mathbb{N}^*$ ,  $\|a_k\| = 1$  and  $(a_k)^2 = 0$ .

3) By denoting,

$$\begin{aligned} \mathcal{H}_{n,k}^{(s)} & := \text{lin.sp.} \{e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)} : \mathbf{k} \in \mathbb{F}_n^0 \text{ with } \mathbf{k}(n) = k\} \\ \mathcal{H}_{n,k}^{(d)} & := \text{lin.sp.} \{e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)} : \mathbf{k} \in \mathbb{F}_n^0 \text{ with } k \notin \text{range}(\mathbf{k})\} \end{aligned} \quad (2.7)$$

then  $\mathcal{H}_{n,k}^{(s)}$  and  $\mathcal{H}_{n,k}^{(d)}$  are closed,

$$\begin{aligned} \left(\mathcal{H}_{n,k}^{(s)}\right)^\perp &= \text{lin.sp.} \{e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)} : \mathbf{k} \in \mathbb{F}_n^0 \text{ with } \mathbf{k}(n) \neq k\} \\ \left(\mathcal{H}_{n,k}^{(d)}\right)^\perp &= \text{lin.sp.} \{e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)} : \mathbf{k} \in \mathbb{F}_n^0 \text{ with } k \in \text{range}(\mathbf{k})\} \end{aligned} \quad (2.8)$$

and

$$\mathcal{H}_n = \mathcal{H}_{n,k}^{(s)} \oplus \left(\mathcal{H}_{n,k}^{(s)}\right)^\perp = \mathcal{H}_{n,k}^{(d)} \oplus \left(\mathcal{H}_{n,k}^{(d)}\right)^\perp$$

*Proof.* All affirmations can be checked easily and we omit their detail.  $\square$

**Proposition 2.2.** 1) For any  $k \in \mathbb{N}^*$ ,

$$\begin{aligned} a_k^+ a_k &= P_k^{(s)} := \bigoplus_{n=1}^{\infty} (\text{the projector to } \mathcal{H}_{n,k}^{(s)}) \\ a_k a_k^+ &= P_k^{(d)} := (\text{the projector to } \mathcal{H}_0) \oplus \bigoplus_{n=1}^{\infty} (\text{the projector to } \mathcal{H}_{n,k}^{(d)}) \end{aligned} \quad (2.9)$$

and

$$a_k^+ P_k^{(d)} = P_k^{(s)} a_k^+; \quad a_k P_k^{(d)} = P_k^{(s)} a_k = 0 \quad (2.10)$$

For any different  $h, k \in \mathbb{N}^*$ ,

$$a_k a_h^+ = 0, \quad P_k^{(s)} P_h^{(s)} = P_h^{(s)} P_k^{(s)} = 0 \quad (2.11)$$

and

$$\begin{aligned} &a_h^+ a_k (e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)}) \\ &= \delta_{k,\mathbf{k}(n)} \prod_{j=1}^{n-1} (1 - \delta_{h,\mathbf{k}(j)}) (e_h \otimes e_{\mathbf{k}(n-1)} \otimes \dots \otimes e_{\mathbf{k}(1)}), \quad \forall n \geq 1, \mathbf{k} \in \mathbb{F}_n^0 \end{aligned} \quad (2.12)$$

2) For any  $n \in \mathbb{N}^*$ ,

$$\left\| \sum_{k=1}^n a_k^+ a_k \right\| = 1, \quad \left\| \frac{1}{\sqrt{n}} \sum_{k=1}^n a_k^{(\pm 1)} \right\| \leq 1, \quad \left\| \frac{1}{n} \sum_{k=1}^n a_k a_k^+ \right\| \leq 1$$

and so  $\{B_n^{(-1)}, B_n^{(+1)}, B_n^{(0)}, B_n^{(2)} : n \in \mathbb{N}^*\}$  is a uniform bounded family whenever there exists the limit  $\lim_{n \rightarrow \infty} np_n$ .

*Proof.* Let  $J \subset \mathbb{N}$  be finite, let  $n_r \in \mathbb{N}^*$ ,  $\alpha_r \in \mathbb{C}$  and  $\mathbf{k}_r \in \mathbb{F}_{n_r}^0$  for any  $r \in J$ , by writing  $\sum_{r \in J} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \dots \otimes e_{\mathbf{k}_r(1)}$  to

$$\left( \sum_{r \in J: \mathbf{k}_r(n_r)=k} + \sum_{r \in J: \mathbf{k}_r(n_r) \neq k} \right) \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \dots \otimes e_{\mathbf{k}_r(1)} \quad (2.13)$$

(2.5) and (2.6) tell us that

$$a_k^+ a_k \sum_{r \in J: \mathbf{k}_r(n_r)=k} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \dots \otimes e_{\mathbf{k}_r(1)} = \sum_{r \in J: \mathbf{k}_r(n_r)=k} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \dots \otimes e_{\mathbf{k}_r(1)}$$

and

$$a_k^+ a_k \Psi = 0 = a_k^+ a_k \sum_{r \in J: \mathbf{k}_r(n_r) \neq k} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \dots \otimes e_{\mathbf{k}_r(1)}$$

therefore,  $a_k^+ a_k = P_k^{(s)}$ .

Similarly, by writing  $\sum_{r \in J} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \dots \otimes e_{\mathbf{k}_r(1)}$  to

$$\left( \sum_{r \in J: k \in \text{range}(\mathbf{k}_r)} + \sum_{r \in J: k \notin \text{range}(\mathbf{k}_r)} \right) \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \dots \otimes e_{\mathbf{k}_r(1)} \quad (2.14)$$

(2.5) and (2.6) say that

$$a_k a_k^+ \sum_{r \in J: k \in \text{range}(\mathbf{k}_r)} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \dots \otimes e_{\mathbf{k}_r(1)} = 0$$

and

$$\begin{aligned} a_k a_k^+ \Psi &= \Psi \\ a_k a_k^+ \sum_{r \in J: k \notin \text{range}(\mathbf{k}_r)} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \dots \otimes e_{\mathbf{k}_r(1)} \\ &= \sum_{r \in J: k \notin \text{range}(\mathbf{k}_r)} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \dots \otimes e_{\mathbf{k}_r(1)} \end{aligned}$$

therefore,  $a_k a_k^+ = P_k^{(d)}$ .

Clearly, in order to obtain (2.10), one needs only to prove the first equality and others are trivial: in fact,  $a_k P_k^{(d)} = a_k a_k a_k^+ = 0$ ,  $P_k^{(s)} a_k = a_k^+ a_k a_k = 0$ .

For any  $k \in \mathbb{N}^*$ , (2.9) and the definition of creation operator tell us that

$$\begin{aligned} P_k^{(d)} &= \sum_{r \in J: k \in \text{range}(\mathbf{k}_r)} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \dots \otimes e_{\mathbf{k}_r(1)} \\ &= 0 = a_k^+ \sum_{r \in J: k \in \text{range}(\mathbf{k}_r)} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \dots \otimes e_{\mathbf{k}_r(1)} \end{aligned}$$

So both  $a_k^+ P_k^{(d)}$  and  $P_k^{(s)} a_k^+$  bring the vector in the form of

$$\sum_{r \in J: k \in \text{range}(\mathbf{k}_r)} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \dots \otimes e_{\mathbf{k}_r(1)}$$

to zero. Moreover, it follows just from the definitions of  $a_k^+$ ,  $P_k^{(d)}$  and  $P_k^{(s)}$  that

$$\begin{aligned} a_k^+ P_k^{(d)} \left( \alpha \Psi + \sum_{r: k \notin \text{range}(\mathbf{k}_r)} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \dots \otimes e_{\mathbf{k}_r(1)} \right) \\ = \alpha e_k + \sum_{r: k \notin \text{range}(\mathbf{k}_r)} \alpha_r e_k \otimes e_{\mathbf{k}_r(n_r)} \otimes \dots \otimes e_{\mathbf{k}_r(1)} \end{aligned}$$

and



$$\begin{aligned}
& P_k^{(s)} a_k^+ \left( \alpha \Psi + \sum_{r: k \notin \text{range}(\mathbf{k}_r)} \alpha_r e_{\mathbf{k}_r(n_r)} \otimes \dots \otimes e_{\mathbf{k}_r(1)} \right) \\
&= P_k^{(s)} \left( \alpha e_k + \sum_{r: k \notin \text{range}(\mathbf{k}_r)} \alpha_r e_k \otimes e_{\mathbf{k}_r(n_r)} \otimes \dots \otimes e_{\mathbf{k}_r(1)} \right) \\
&= \alpha e_k + \sum_{r: k \notin \text{range}(\mathbf{k}_r)} \alpha_r e_k \otimes e_{\mathbf{k}_r(n_r)} \otimes \dots \otimes e_{\mathbf{k}_r(1)}
\end{aligned}$$

So one gets the first equality of (2.10).

In the case of  $h \neq k$ , the equalities in (2.11) are obtained just from (2.5), (2.6) and the fact  $\mathcal{H}_{n,k}^{(s)} \cap \mathcal{H}_{n,h}^{(s)} = \emptyset$ . Moreover, (2.12) is proved simply as follows:

$$\begin{aligned}
a_h^+ a_k (e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)}) &= \delta_{k,\mathbf{k}(n)} a_h^+ (e_{\mathbf{k}(n-1)} \otimes \dots \otimes e_{\mathbf{k}(1)}) \\
&= \delta_{k,\mathbf{k}(n)} \prod_{j=1}^{n-1} (1 - \delta_{h,\mathbf{k}(j)}) (e_h \otimes e_{\mathbf{k}(n-1)} \otimes \dots \otimes e_{\mathbf{k}(1)})
\end{aligned}$$

Summing up, the affirmation 1) is proved.

The first equality of (2.9) says that  $a_k^+ a_k$  is a non-zero projector for any  $k$ ; then (2.11) tells us that  $\sum_{k=1}^n a_k^+ a_k$  is a non-zero projector and so  $\|\sum_{k=1}^n a_k^+ a_k\| = 1$ . The second equality of (2.9) guarantees that  $\|a_k a_k^+\| = 1$  for any  $k$ . Consequently

$$\left\| \frac{1}{n} \sum_{k=1}^n a_k a_k^+ \right\| \leq \frac{1}{n} \sum_{k=1}^n \|a_k a_k^+\| = 1$$

Therefore, the first equality of (2.11) makes sure that

$$\left( \sum_{k=1}^n a_k^+ \right)^* \left( \sum_{k=1}^n a_k^+ \right) = \sum_{k,h=1}^n a_k a_h^+ = \sum_{k=1}^n a_k a_k^+$$

and so

$$\begin{aligned}
& \left\| \left( \frac{1}{\sqrt{n}} \sum_{k=1}^n a_k^+ \right)^* \right\|^2 = \left\| \left( \frac{1}{\sqrt{n}} \sum_{k=1}^n a_k^+ \right) \right\|^2 \\
&= \left\| \left( \frac{1}{\sqrt{n}} \sum_{k=1}^n a_k^+ \right)^* \left( \frac{1}{\sqrt{n}} \sum_{k=1}^n a_k^+ \right) \right\| = \left\| \frac{1}{n} \sum_{k=1}^n a_k a_k^+ \right\|^2 \leq 1 \quad \square
\end{aligned}$$

*Remark 2.3.* One has clearly the conjugate formulation of (2.10):

$$P_k^{(d)} a_k = a_k P_k^{(s)}, \quad P_k^{(d)} a_k^+ = a_k^+ P_k^{(s)} = 0$$

**Proposition 2.4.** 1) For any  $k$ ,

$$\mathcal{A}_k = \text{lin.sp.} \left\{ a_k^+, a_k, P_k^{(d)}, a_k^+ P_k^{(d)}, P_k^{(s)}, a_k P_k^{(s)} \right\} \quad (2.15)$$

i.e., any word of  $\mathcal{A}_k$  is either the following:  $a_k^+$ ,  $a_k$ ,  $P_k^{(d)}$ ,  $a_k^+ P_k^{(d)}$ ,  $P_k^{(s)}$ ,  $a_k P_k^{(s)}$ .

2) For any  $k$  and  $x_k$  being a word of  $\mathcal{A}_k$ ,

$$x_k \Psi = \begin{cases} e_k, & \text{if } x_k \in \{a_k^+, a_k^+ P_k^{(d)}\} \\ \Psi, & \text{if } x_k = P_k^{(d)} \\ 0, & \text{if } x_k \in \{a_k, P_k^{(s)}, a_k P_k^{(s)}\} \end{cases} \quad (2.16)$$

and for any  $n \in \mathbb{N}^*$  and  $\mathbf{k} \in \mathbb{F}_n^0$

$$\begin{aligned} & x_k (e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)}) \\ &= \begin{cases} \prod_{h=1}^n (1 - \delta_{k, \mathbf{k}(h)}) e_k \otimes e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)}, & \text{if } x_k \in \{a_k^+, a_k^+ P_k^{(d)}\} \\ \prod_{h=1}^n (1 - \delta_{k, \mathbf{k}(h)}) e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)}, & \text{if } x_k = P_k^{(d)} \\ \delta_{k, \mathbf{k}(n)} e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)}, & \text{if } x_k = P_k^{(s)} \\ \delta_{k, \mathbf{k}(n)} e_{\mathbf{k}(n-1)} \otimes \dots \otimes e_{\mathbf{k}(1)}, & \text{if } x_k \in \{a_k, a_k P_k^{(s)}\} \end{cases} \end{aligned} \quad (2.17)$$

hereinafter,

$$e_{\mathbf{k}(r)} \otimes \dots \otimes e_{\mathbf{k}(1)} \Big|_{r=0} := \Psi \quad (2.18)$$

3) For any  $k, n \in \mathbb{N}^*$ ,  $\mathbf{k} \in \mathbb{F}_n$  (not necessarily belongs to  $\mathbb{F}_n^0$ ) and  $j \in \{1, \dots, n\}$ , for any  $x_j \in \{a_{\mathbf{k}(j)}^+, a_{\mathbf{k}(j)}, a_{\mathbf{k}(j)}^+ P_{\mathbf{k}(j)}^{(d)}, P_{\mathbf{k}(j)}^{(s)}, a_{\mathbf{k}(j)} P_{\mathbf{k}(j)}^{(s)}\}$  (notice that  $x_j$  is NOT be taken as  $P_{\mathbf{k}(j)}^{(d)}$ ),

$$\begin{aligned} & x_n \dots x_1 \Psi \\ &= \begin{cases} e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)}, & \text{if } \mathbf{k} \in \mathbb{F}_n^0 \text{ and } x_j \in \{a_{\mathbf{k}(j)}^+, a_{\mathbf{k}(j)}^+ P_{\mathbf{k}(j)}^{(d)}\}, \forall j \in \{1, \dots, n\} \\ 0, & \text{otherwise} \end{cases} \end{aligned} \quad (2.19)$$

*Proof.* Just by applying definitions of operators  $a_k^+, a_k, P_k^{(d)}$  and  $P_k^{(s)}$ , one gets easily (2.16) and (2.17) for  $\mathbf{k} \in \mathbb{F}_n^0$ .

The affirmation 3) is a consequence of the affirmation 2), more precisely, is the consequence of the fact:

$$x_k (e_{\mathbf{k}(m)} \otimes \dots \otimes e_{\mathbf{k}(1)}) = \prod_{h=1}^m (1 - \delta_{k, \mathbf{k}(h)}) e_k \otimes e_{\mathbf{k}(m)} \otimes \dots \otimes e_{\mathbf{k}(1)}$$

for any  $m \in \mathbb{N}$ ,  $\mathbf{k} \in \mathbb{F}_m^0$  and  $x_k \in \{a_k^+, a_k^+ P_k^{(d)}\}$ . Now we turn to show the affirmation 1).

By the definition,

$$\mathcal{A}_k = \text{lin.sp.} \left\{ a_k^{\varepsilon(1)} \dots a_k^{\varepsilon(n)} : n \in \mathbb{N}^*, \varepsilon \in \{-1, 1\}^n \right\}$$

where, one prefers to write  $\{a_k^{\varepsilon(1)} \dots a_k^{\varepsilon(n)} : n \in \mathbb{N}^*, \varepsilon \in \{-1, 1\}^n\}$  to

$$\begin{aligned} & \{a_k, a_k^+\} \cup \{a_k^{\varepsilon(1)} \dots a_k^{\varepsilon(2m)} : m \in \mathbb{N}^*, \varepsilon \in \{-1, 1\}^{2m}\} \\ & \cup \{a_k^{\varepsilon(1)} \dots a_k^{\varepsilon(2m+1)} : m \in \mathbb{N}^*, \varepsilon \in \{-1, 1\}^{2m+1}\} \end{aligned}$$

It follows from the fact  $a_k^{+2} = a_k^2 = 0$  (see Proposition 2.1) that a general word of  $\mathcal{A}_k$ , i.e.,  $a_k^{\varepsilon(1)} \dots a_k^{\varepsilon(n)}$  with  $n \in \mathbb{N}^*$  and  $\varepsilon \in \{-1, 1\}^n$ , differs from zero only if  $\varepsilon$  is *consecutively different*:  $\varepsilon(1) \neq \varepsilon(2) \neq \dots \neq \varepsilon(n-1) \neq \varepsilon(n)$ . Moreover

- if  $n = 2m$  and  $1 = \varepsilon(1) \neq \varepsilon(2) \neq \dots \neq \varepsilon(n-1) \neq \varepsilon(n)$ , one has

$$a_k^{\varepsilon(1)} \dots a_k^{\varepsilon(n)} = a_k^{\varepsilon(1)} \dots a_k^{\varepsilon(2m)} = (a_k^+ a_k)^m = P_k^{(s)}$$

- if  $n = 2m$  and  $-1 = \varepsilon(1) \neq \varepsilon(2) \neq \dots \neq \varepsilon(n-1) \neq \varepsilon(n)$ , one has

$$a_k^{\varepsilon(1)} \dots a_k^{\varepsilon(n)} = a_k^{\varepsilon(1)} \dots a_k^{\varepsilon(2m)} = (a_k a_k^+)^m = P_k^{(d)}$$

- if  $n = 2m + 1$  and  $1 = \varepsilon(1) \neq \varepsilon(2) \neq \dots \neq \varepsilon(n-1) \neq \varepsilon(n)$ , one has

$$a_k^{\varepsilon(1)} \dots a_k^{\varepsilon(n)} = a_k^{\varepsilon(1)} \dots a_k^{\varepsilon(2m+1)} = a_k^+ (a_k a_k^+)^m = a_k^+ P_k^{(d)}$$

- if  $n = 2m + 1$  and  $-1 = \varepsilon(1) \neq \varepsilon(2) \neq \dots \neq \varepsilon(n-1) \neq \varepsilon(n)$ , one has

$$a_k^{\varepsilon(1)} \dots a_k^{\varepsilon(n)} = a_k^{\varepsilon(1)} \dots a_k^{\varepsilon(2m+1)} = a_k (a_k^+ a_k)^m = a_k P_k^{(s)}$$

Summing up, the thesis is obtained.  $\square$

*Remark 2.5.* Among all words of  $\mathcal{A}_k$ ,  $P_k^{(d)}$  has a particular feature: its vacuum expectation equals to 1 while anyone of other words has zero vacuum expectation.

For any  $k, n \in \mathbb{N}^*$  and  $\mathbf{k} \in \mathbb{F}_n^0$ , we have the following easily proved result:

$$1 - \prod_{h=1}^n (1 - \delta_{k, \mathbf{k}(h)}) = \sum_{j=1}^n \delta_{k, \mathbf{k}(j)} = \chi_{\text{range}(\mathbf{k})}(k) \quad (2.20)$$

where,  $\chi_E(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E \end{cases}$  for any set  $E$ .

In fact, for any  $\mathbf{k} \in \mathbb{F}_n^0$ ,  $\{\mathbf{k}^{-1}(\{j\})\}$ 's are 2-2 disjoint and so one gets the second equality of (2.20):

$$\chi_{\text{range}(\mathbf{k})}(k) = \chi_{\cup_{j=1}^n \mathbf{k}^{-1}(\{j\})}(k) = \sum_{j=1}^n \chi_{\mathbf{k}^{-1}(\{j\})}(k) = \sum_{j=1}^n \delta_{k, \mathbf{k}(j)}$$

Moreover, the first equality of (2.20) is obtained by noticing that

- both  $1 - \prod_{h=1}^n (1 - \delta_{k, \mathbf{k}(h)})$  and  $\sum_{j=1}^n \delta_{k, \mathbf{k}(j)}$  take values in  $\{0, 1\}$ ;
- $1 - \prod_{h=1}^n (1 - \delta_{k, \mathbf{k}(h)}) = 0$  iff  $\prod_{h=1}^n (1 - \delta_{k, \mathbf{k}(h)}) = 1$  iff  $\delta_{k, \mathbf{k}(h)} = 0$  for all  $h \in \{1, \dots, n\}$  iff  $\sum_{h=1}^n \delta_{k, \mathbf{k}(h)} = 0$ .

**Proposition 2.6.** *For any  $k, n \in \mathbb{N}^*$  and  $\mathbf{k} \in \mathbb{F}_n^0$ , one has the following affirmations:*

- 1) as a particular case of (2.16) and (2.17),

$$(1 - P_k^{(d)})\Psi = 0$$

$$(1 - P_k^{(d)}) (e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)}) = \chi_{\text{range}(\mathbf{k})}(k) e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)} \quad (2.21)$$

- 2) let

$$\tilde{\mathcal{A}}_k := \left\{ a_k^+, a_k, a_k^+ P_k^{(d)}, P_k^{(s)}, a_k P_k^{(s)}, 1 - P_k^{(d)} \right\} \quad (2.22)$$

and

$$\mathcal{A}_k^{(+1)} := \{a_k^+, a_k^+ P_k^{(d)}\}, \mathcal{A}_k^{(-1)} := \{a_k, a_k P_k^{(s)}\}, \mathcal{A}_k^{(0)} := \{P_k^{(s)}, 1 - P_k^{(d)}\} \quad (2.23)$$

Then

$$\mathcal{A}_k^{(+1)}, \mathcal{A}_k^{(-1)} \text{ and } \mathcal{A}_k^{(0)} \text{ are 2-2 disjoint; } \mathcal{A}_k^{(+1)} \cup \mathcal{A}_k^{(-1)} \cup \mathcal{A}_k^{(0)} = \tilde{\mathcal{A}}_k \quad (2.24)$$

and

$$x_k(e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)}) \in \begin{cases} \{ce_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)} : c \in \mathbb{C}\}, & \text{if } x_k \in \mathcal{A}_k^{(0)} \\ \{ce_k \otimes e_{\mathbf{k}(n)} \otimes \dots \otimes e_{\mathbf{k}(1)} : c \in \mathbb{C}\}, & \text{if } x_k \in \mathcal{A}_k^{(+1)} \\ \{ce_{\mathbf{k}(n-1)} \otimes \dots \otimes e_{\mathbf{k}(1)} : c \in \mathbb{C}\}, & \text{if } x_k \in \mathcal{A}_k^{(-1)} \end{cases} \quad (2.25)$$

*Proof.* The proof is completed easily as follows:

- the first equality of (2.21) is trivial in virtue of (2.16);
- the second equality of (2.21) is obtained just by combining (2.17) and (2.20);
- (2.24) is just a consequence of (2.22) and (2.23);
- finally, one gets (2.25) thanks to (2.16), (2.17) and (2.23).  $\square$

**Proposition 2.7.** *For any  $m \in \mathbb{N}^*$  and  $\mathbf{k} \in \mathbb{F}_m^0$ , for any  $x_j \in \mathcal{A}_{\mathbf{k}(j)}^{(+1)} \cup \mathcal{A}_{\mathbf{k}(j)}^{(0)}$  with  $j \in \{1, \dots, m\}$ , by denoting  $r := |\{j \in \{1, \dots, m\} : x_j \in \mathcal{A}_{\mathbf{k}(j)}^{(+1)}\}|$ , there exist a constant  $C$  (depending on  $\mathbf{k}$  in general) and  $1 \leq i_1 < \dots < i_r \leq m$  such that*

$$x_m \dots x_1 \Psi = C e_{\mathbf{k}(i_r)} \otimes \dots \otimes e_{\mathbf{k}(i_1)} \quad (2.26)$$

hereinafter, we have adopted the conventions (2.18) and  $\{i_1, \dots, i_r\}|_{r=0} := \emptyset$ .

*Proof.* (2.16) and the first equality of (2.21) say that

$$x_1 \Psi = \begin{cases} e_{\mathbf{k}(1)}, & \text{if } x_1 \in \mathcal{A}_{\mathbf{k}(1)}^{(+1)} \\ 0, & \text{if } x_1 \in \mathcal{A}_{\mathbf{k}(1)}^{(0)} \cup \mathcal{A}_{\mathbf{k}(1)}^{(-1)} \end{cases} \quad (2.27)$$

So we obtain (2.26) by taking (recall that we have assumed  $x_1 \in \mathcal{A}_{\mathbf{k}(1)}^{(+1)} \cup \mathcal{A}_{\mathbf{k}(1)}^{(0)}$ )

- $C = 0$  if  $x_1 \in \mathcal{A}_{\mathbf{k}(1)}^{(0)}$ ;
- $C = 1$ ,  $r = 1$  and  $i_1 = 1$  if  $x_1 \in \mathcal{A}_{\mathbf{k}(1)}^{(+1)}$ .

By assuming the validity of our thesis for  $m = n$ , one has  $n + 1 > n \geq i_r$  and

$$\begin{aligned} r' &:= |\{j \in \{1, \dots, n+1\} : x_j \in \mathcal{A}_{\mathbf{k}(j)}^{(+1)}\}| \\ &= \begin{cases} |\{j \in \{1, \dots, n\} : x_j \in \mathcal{A}_{\mathbf{k}(j)}^{(+1)}\}| + 1, & \text{if } x_{n+1} \in \mathcal{A}_{\mathbf{k}(n+1)}^{(+1)} \\ |\{j \in \{1, \dots, n\} : x_j \in \mathcal{A}_{\mathbf{k}(j)}^{(+1)}\}|, & \text{if } x_{n+1} \in \mathcal{A}_{\mathbf{k}(n+1)}^{(0)} \end{cases} \\ &= \begin{cases} r + 1, & \text{if } x_{n+1} \in \mathcal{A}_{\mathbf{k}(n+1)}^{(+1)} \\ r, & \text{if } x_{n+1} \in \mathcal{A}_{\mathbf{k}(n+1)}^{(0)} \end{cases} \end{aligned}$$

So, by applying (2.25) and noticing  $x_{n+1} \in \mathcal{A}_{\mathbf{k}(n+1)}^{(+1)} \cup \mathcal{A}_{\mathbf{k}(n+1)}^{(0)}$ ,

$$\begin{aligned} x_{n+1} x_n \dots x_1 \Psi &= C x_{n+1} e_{\mathbf{k}(i_r)} \otimes \dots \otimes e_{\mathbf{k}(i_1)} \\ &= C \begin{cases} c_1 e_{\mathbf{k}(n+1)} \otimes e_{\mathbf{k}(i_r)} \otimes \dots \otimes e_{\mathbf{k}(i_1)}, & \text{if } x_{n+1} \in \mathcal{A}_{\mathbf{k}(n+1)}^{(+1)} \\ c_2 e_{\mathbf{k}(i_r)} \otimes \dots \otimes e_{\mathbf{k}(i_1)}, & \text{if } x_{n+1} \in \mathcal{A}_{\mathbf{k}(n+1)}^{(0)} \end{cases} \quad (2.28) \end{aligned}$$

with  $c_1, c_2 \in \mathbb{C}$ . The induction principle gives the thesis.  $\square$

In the following, we denote, for any  $k \in \mathbb{N}^*$ ,

$$\overline{\mathcal{A}}_k := \text{lin.sp.} \left\{ \mathbf{1}, a_k^+, a_k, P_k^{(d)}, a_k^+ P_k^{(d)}, P_k^{(s)}, a_k P_k^{(s)} \right\} \quad (2.29)$$

**Theorem 2.8.** *With respect to the vacuum state,  $\{\overline{\mathcal{A}}_k\}_k$  and  $\{\mathcal{A}_k\}_k$  are free independent families.*

*Proof.* Clearly we need only to prove the free independence of  $\{\overline{\mathcal{A}}_k\}_k$ . That is, by the definition, to show

$$\langle \Psi, x_n \dots x_1 \Psi \rangle = 0 \quad (2.30)$$

for any  $n \geq 2$ ,  $\mathbf{k} \in \mathbb{F}_n$  and for any such  $\{x_j\}_{j=1}^n$  that  $x_j \in \overline{\mathcal{A}}_{\mathbf{k}(j)}$  and  $\langle \Psi, x_j \Psi \rangle = 0$  for all  $j \in \{1, \dots, n\}$ .

(2.29) says that any  $x \in \overline{\mathcal{A}}_k$  must have the form

$$\alpha_1 a_k^+ + \alpha_2 a_k + \alpha_3 a_k^+ P_k^{(d)} + \alpha_4 P_k^{(s)} + \alpha_5 a_k P_k^{(s)} + \alpha_6 P_k^{(d)} + \alpha_7 \mathbf{1}$$

By combining this with the following facts

- $\langle \Psi, \mathbf{1} \Psi \rangle = 1 = \langle \Psi, P_k^{(d)} \Psi \rangle$  for any  $k \in \mathbb{N}^*$ ,
  - $\langle \Psi, x \Psi \rangle = 0$  for any  $x \in \{a_k^+, a_k, a_k^+ P_k^{(d)}, P_k^{(s)}, a_k P_k^{(s)}\}$  and  $k \in \mathbb{N}^*$ ,
- one knows that any such  $x_j \in \overline{\mathcal{A}}_{\mathbf{k}(j)}$  that  $\langle \Psi, x_j \Psi \rangle = 0$  must have the form:

$$\begin{aligned} x_j &= \alpha_{\mathbf{k}(j),1} a_{\mathbf{k}(j)}^+ + \alpha_{\mathbf{k}(j),2} a_{\mathbf{k}(j)} + \alpha_{\mathbf{k}(j),3} a_{\mathbf{k}(j)}^+ P_{\mathbf{k}(j)}^{(d)} \\ &\quad + \alpha_{\mathbf{k}(j),4} P_{\mathbf{k}(j)}^{(s)} + \alpha_{\mathbf{k}(j),5} a_{\mathbf{k}(j)} P_{\mathbf{k}(j)}^{(s)} + \alpha_{\mathbf{k}(j),6} (\mathbf{1} - P_{\mathbf{k}(j)}^{(d)}) \end{aligned}$$

and so  $x_n \dots x_1$  equals to a sum of finite products in the form  $y_n \dots y_1$ , where  $y_j$  belongs to  $\tilde{\mathcal{A}}_{\mathbf{k}(j)}$  introduced in (2.22) for any  $j \in \{1, \dots, n\}$ . So, (2.30) will be proved if one gets

$$\langle \Psi, y_n \dots y_1 \Psi \rangle = 0, \quad \forall y_j \in \tilde{\mathcal{A}}_{\mathbf{k}(j)}, \quad \forall j \in \{1, \dots, n\} \quad (2.31)$$

and now we turn to show it.

If  $n = 1$ , (2.27) tells us that  $y_1 \Psi \in \{0, e_{\mathbf{k}(1)}\}$  and so  $\langle \Psi, y_1 \Psi \rangle = 0$ . Now we examine (2.31) for  $n \geq 2$ . Moreover, since  $y_1 \Psi = 0$  if  $y_1 \in \mathcal{A}_{\mathbf{k}(1)}^{(-1)} \cup \mathcal{A}_{\mathbf{k}(1)}^{(0)}$ , we need only to prove (2.31) with the additional condition  $y_1 \in \mathcal{A}_{\mathbf{k}(1)}^{(+1)}$ . Now we perform it in two cases.

The 1<sup>st</sup> case:  $\{j : y_j \in \mathcal{A}_{\mathbf{k}(j)}^{(-1)}\} \neq \emptyset$ . In this case, let

$$m + 1 := \min \{j : y_j \in \mathcal{A}_{\mathbf{k}(j)}^{(-1)}\}$$

then,

- $m \leq n - 1$ ,  $y_{m+1} \in \mathcal{A}_{\mathbf{k}(m+1)}^{(-1)}$  and  $y_j \in \mathcal{A}_{\mathbf{k}(j)}^{(0)} \cup \mathcal{A}_{\mathbf{k}(j)}^{(+1)}$  for all  $j \leq m$ ;
- the additional condition  $y_1 \in \mathcal{A}_{\mathbf{k}(1)}^{(+1)}$  mentioned in above tells us that  $m \geq 1$ .

For any  $m \in \{1, \dots, n - 1\}$ , Proposition 2.7 says that  $y_m \dots y_1 \Psi$  has the form  $C e_{\mathbf{k}'(r)} \otimes \dots \otimes e_{\mathbf{k}'(1)}$  with  $C \in \mathbb{C}$ ,  $1 \leq i_1 < \dots < i_r \leq m$  and  $\mathbf{k}'(j) := \mathbf{k}(i_j)$  for any  $j \in \{1, \dots, r\}$ . Moreover, the facts  $\mathbf{k} \in \mathbb{F}_n^0$  and  $1 \leq i_1 < \dots < i_r < m + 1$

make sure that  $\mathbf{k}' \in \mathbb{F}_r^0$  and  $\mathbf{k}(m+1) \neq \mathbf{k}'(r)$ , where  $r := |\{j \leq m : y_j \in \mathcal{A}_{\mathbf{k}(j)}^{(+1)}\}|$ . Therefore, thank to the fact  $y_{m+1} \in \mathcal{A}_{\mathbf{k}(m+1)}^{(-1)}$  and the formula (2.17), one gets

$$y_{m+1}y_m \dots y_1 \Psi = C y_{m+1} e_{\mathbf{k}'(r)} \otimes \dots \otimes e_{\mathbf{k}'(1)} = 0 \quad (2.32)$$

The 2<sup>nd</sup> case:  $\{j : y_j \in \mathcal{A}_{\mathbf{k}(j)}^{(-1)}\} = \emptyset$ . In this case, Proposition 2.7 says that the vector  $y_n \dots y_1 \Psi$  has the form  $C e_{\mathbf{k}(i_r)} \otimes \dots \otimes e_{\mathbf{k}(i_1)}$ , where, it follows from the fact  $y_1 \in \mathcal{A}_{\mathbf{k}(j)}^{(+1)}$  that  $i_1 = 1$  and  $r \geq 1$ . Consequently one gets the equality in (2.31).

Summing up, the proof is completed.  $\square$

*Remark 2.9.* We do not take the standard free Fock space over  $\mathcal{H}$  and the corresponding creation–annihilation operators, even if on any free Fock space, the family  $\{X_{n,k} : k \leq n\}$  is free independent for any  $n \geq 2$ . The reason is, on the standard free Fock space, for any annihilation (resp. creation) operator  $a$  (resp.  $a^+$ ) with uni–norm test function, for any  $p \in (0, 1)$ , the vacuum distribution of  $X := \sqrt{p(1-p)}(a + a^+) + paa^+ + (1-p)a^+a$  can not be  $b(1, p)$ .

### 3. Quantization of the Free Poisson CLT

Now we are ready to see the quantized free Poisson CLT, which says essentially that if  $np_n \rightarrow \lambda$ , then for any  $\varepsilon \in \{1, -1, 0, 2\}$ ,  $B_n^{(\varepsilon)}$  introduced in (1.5) equals asymptotically to  $\lambda^{1-|\varepsilon|/2} b^{(\varepsilon)}$ , where  $b^{(\varepsilon)}$  is defined in (1.10).

**Proposition 3.1.** 1) On the IFS  $\Gamma_{sf}(\mathcal{H})$ , one has, for any  $\{p_N\}_{N=1}^\infty \subset [0, 1]$  and  $N \in \mathbb{N}^*$ ,

$$\begin{aligned} B_N^{(-1)} \Psi &= B_N^{(2)} \Psi = 0; \quad B_N^{(0)} \Psi = N p_N \Psi; \quad \langle \Psi, (B_N^{(+1)})^n \Psi \rangle = 0 \\ B_N^{(-1)} (B_N^{(+1)})^n \Psi &= \chi_{N|}(n) (N - n + 1) p_N (1 - p_N) (B_N^{(+1)})^{n-1} \Psi \\ B_N^{(0)} (B_N^{(+1)})^n \Psi &= \chi_{N|}(n) (N - n) p_N (B_N^{(+1)})^n \Psi \\ B_N^{(2)} (B_N^{(+1)})^n \Psi &= \chi_{N|}(n) (1 - p_N) (B_N^{(+1)})^n \Psi, \quad \forall n \in \mathbb{N}^* \end{aligned} \quad (3.1)$$

hereinafter,  $\chi_{N|}(n) := \begin{cases} 1, & \text{if } n \leq N \\ 0, & \text{if } n > N \end{cases}$ .

2) On the 1M-IFS  $\Gamma_{fr}(\mathbb{C})$ , one has

$$\begin{aligned} b^{(-1)} \Phi &= b^{(2)} \Phi = 0; \quad \langle \Phi, (b^{(+1)})^n \Phi \rangle = 0 \\ b^{(2)} (b^{(+1)})^n \Phi &= (b^{(+1)})^n \Phi; \quad b^{(-1)} (b^{(+1)})^n \Phi = (b^{(+1)})^{n-1} \Phi, \quad \forall n \in \mathbb{N}^* \end{aligned} \quad (3.2)$$

*Proof.* Clearly, we need only to show the last three equalities of (3.1) and others are trivial. Moreover, one uses simply  $p$  instead of  $p_N$ .

It follows from the definitions of  $a^{(\varepsilon)}$ 's and  $B_N^{(\varepsilon)}$ 's that

$$\begin{aligned}
& \left(B_N^{(+1)}\right)^n \Psi = \left(\sqrt{p(1-p)}\right)^n \sum_{\mathbf{k} \in \overline{\mathbb{F}}_n, \mathbf{k}(j) \leq N \text{ for any } j} a_{\mathbf{k}(n)}^+ \cdots a_{\mathbf{k}(1)}^+ \Psi \\
& = \chi_{N|}(n) \left(\sqrt{p(1-p)}\right)^n \sum_{\mathbf{k} \in \mathbb{F}_n^0, \mathbf{k}(j) \leq N \text{ for any } j} e_{\mathbf{k}(n)} \otimes \cdots \otimes e_{\mathbf{k}(1)} \quad (3.3)
\end{aligned}$$

So the last three equalities of (3.1) are trivial for  $n > N$ . For  $n \leq N$ , one gets

$$\begin{aligned}
& B_N^{(0)} \left(B_N^{(+1)}\right)^n \Psi \\
& \stackrel{(3.3)}{=} p \left(\sqrt{p(1-p)}\right)^n \sum_{k=1}^N \sum_{\mathbf{k} \in \mathbb{F}_n^0, \mathbf{k}(j) \leq N \text{ for any } j} a_k a_k^+ (e_{\mathbf{k}(n)} \otimes \cdots \otimes e_{\mathbf{k}(1)}) \\
& \stackrel{(2.9)}{=} p \left(\sqrt{p(1-p)}\right)^n \sum_{\mathbf{k} \in \mathbb{F}_n^0, \mathbf{k}(j) \leq N \text{ for any } j} \sum_{1 \leq k \leq N, k \notin \text{range}(\mathbf{k})} e_{\mathbf{k}(n)} \otimes \cdots \otimes e_{\mathbf{k}(1)} \\
& = (N-n)p \left(\sqrt{p(1-p)}\right)^n \sum_{\mathbf{k} \in \mathbb{F}_n^0, \mathbf{k}(j) \leq N \text{ for any } j} e_{\mathbf{k}(n)} \otimes \cdots \otimes e_{\mathbf{k}(1)} \\
& \stackrel{(3.3)}{=} (N-n)p \left(B_N^{(+1)}\right)^n \Psi \quad (3.4)
\end{aligned}$$

and

$$\begin{aligned}
& B_N^{(2)} \left(B_N^{(+1)}\right)^n \Psi \\
& \stackrel{(3.3)}{=} p \left(\sqrt{p(1-p)}\right)^n \sum_{k=1}^N \sum_{\mathbf{k} \in \mathbb{F}_n^0, \mathbf{k}(j) \leq N \text{ for any } j} a_k^+ a_k (e_{\mathbf{k}(n)} \otimes \cdots \otimes e_{\mathbf{k}(1)}) \\
& = (1-p) \left(\sqrt{p(1-p)}\right)^n \sum_{k=1}^N \sum_{\mathbf{k} \in \mathbb{F}_n^0, \mathbf{k}(j) \leq N \text{ for any } j} \delta_{k, \mathbf{k}(n)} (e_{\mathbf{k}(n)} \otimes \cdots \otimes e_{\mathbf{k}(1)}) \\
& = (1-p) \left(\sqrt{p(1-p)}\right)^n \sum_{\mathbf{k} \in \mathbb{F}_n^0, \mathbf{k}(j) \leq N \text{ for any } j} e_{\mathbf{k}(n)} \otimes \cdots \otimes e_{\mathbf{k}(1)} \\
& \stackrel{(3.3)}{=} (1-p) \left(B_N^{(+1)}\right)^n \Psi \quad (3.5)
\end{aligned}$$

Finally, one has

$$\begin{aligned}
& B_N^{(-1)} \left(B_N^{(+1)}\right)^n \Psi \\
& \stackrel{(3.3)}{=} \left(\sqrt{p(1-p)}\right)^{n+1} \sum_{k=1}^N \sum_{\mathbf{k} \in \mathbb{F}_n^0, \mathbf{k}(j) \leq N \text{ for any } j} a_k (e_{\mathbf{k}(n)} \otimes \cdots \otimes e_{\mathbf{k}(1)}) \\
& = \left(\sqrt{p(1-p)}\right)^{n+1} \sum_{k=1}^N \sum_{\mathbf{k} \in \mathbb{F}_n^0, \mathbf{k}(j) \leq N \text{ for any } j} \delta_{k, \mathbf{k}(n)} (e_{\mathbf{k}(n-1)} \otimes \cdots \otimes e_{\mathbf{k}(1)}) \quad (3.6)
\end{aligned}$$

Since

$$\begin{aligned}
 & \sum_{k=1}^N \sum_{\mathbf{k} \in \mathbb{F}_n^0, \mathbf{k}(j) \leq N \forall j \leq n} \delta_{k, \mathbf{k}(n)} e_{\mathbf{k}(n-1)} \otimes \dots \otimes e_{\mathbf{k}(1)} \\
 = & \sum_{\mathbf{k} \in \mathbb{F}_{n-1}^0, \mathbf{k}(j) \leq N \forall j \leq n-1} e_{\mathbf{k}(n-1)} \otimes \dots \otimes e_{\mathbf{k}(1)} \sum_{k=1}^N \sum_{1 \leq \mathbf{k}(n) \leq N, \mathbf{k}(n) \neq \mathbf{k}(j) \forall j \leq n-1} \delta_{k, \mathbf{k}(n)} \\
 = & (N - n + 1) \sum_{\mathbf{k} \in \mathbb{F}_{n-1}^0, \mathbf{k}(j) \leq N \forall j \leq n-1} e_{\mathbf{k}(n-1)} \otimes \dots \otimes e_{\mathbf{k}(1)}
 \end{aligned}$$

one obtains, by applying this formula to (3.6),

$$\begin{aligned}
 & B_N^{(-1)} \left( B_N^{(+1)} \right)^n \Psi \\
 = & (N - n + 1) \left( \sqrt{p(1-p)} \right)^{n+1} \sum_{\mathbf{k} \in \mathbb{F}_{n-1}^0, \mathbf{k}(j) \leq N \text{ for any } j} e_{\mathbf{k}(n-1)} \otimes \dots \otimes e_{\mathbf{k}(1)} \\
 \stackrel{(3.3)}{=} & (N - n + 1) p(1-p) \left( B_N^{(+1)} \right)^{n-1} \Psi
 \end{aligned} \tag{3.7}$$

□

**Theorem 3.2.** *On the IFS  $\Gamma_{sf}(\mathcal{H})$ , if  $\{p_N\}_{N=1}^\infty \subset [0, 1]$  verifies  $\lim_{N \rightarrow \infty} Np_N = \lambda$ , then for any  $n \in \mathbb{N}$  and  $\varepsilon \in \{-1, 0, 1, 2\}^n$ , the limits, as  $N \rightarrow \infty$ , of (1.7) and (1.8) exist; moreover, they equal to the expressions (1.9) and (1.11) respectively. Consequently, the Jacobi coefficients of the free Poisson distribution with the parameter  $\lambda$  (more precisely,  $(\lambda, 1)$ ) is*

$$\alpha_0 = \omega_n = \lambda, \quad \alpha_n = \lambda + 1, \quad \forall n \in \mathbb{N}^* \tag{3.8}$$

*Proof.* (3.1) and (3.2) tell us

$$\begin{aligned}
 \langle \Psi, B_N^{(\varepsilon)} \Psi \rangle &= \begin{cases} Np_N, & \text{if } \varepsilon = 0 \\ 0, & \text{otherwise} \end{cases} \longrightarrow \begin{cases} \lambda, & \text{if } \varepsilon = 0 \\ 0, & \text{otherwise} \end{cases} \\
 &= \lambda \langle \Phi, b^{(\varepsilon)} \Phi \rangle = \langle \Phi, b^{(\varepsilon)} \Phi \rangle \lambda^{1-|\varepsilon|/2}
 \end{aligned}$$

and

$$\begin{aligned}
 & \langle \Psi, B_N^{(\varepsilon(1))} B_N^{(\varepsilon(2))} \Psi \rangle \\
 = & \begin{cases} Np_N(1-p_N), & \text{if } \varepsilon(1) = -1 \text{ and } \varepsilon(2) = 1 \\ 0, & \text{otherwise} \end{cases} \\
 \longrightarrow & \begin{cases} \lambda, & \text{if } \varepsilon(1) = -1 \text{ and } \varepsilon(2) = 1 \\ 0, & \text{otherwise} \end{cases} = \lambda \langle \Phi, b^{(\varepsilon(1))} b^{(\varepsilon(2))} \Phi \rangle \\
 = & \langle \Phi, b^{(\varepsilon(1))} b^{(\varepsilon(2))} \Phi \rangle \lambda^{\sum_{k=1}^2 (1-|\varepsilon(k)|/2)}
 \end{aligned}$$

i.e. the thesis is proved for  $n = 1$  and  $2$ .



Suppose that the thesis is proved up to  $n$ , let's see it for  $n + 1$ . Since we are considering the limits of (1.7) and (1.8) as  $N \rightarrow \infty$ , one can assume that  $N \geq n + 1$ .

If  $\varepsilon(n + 1) \in \{-1, 2\}$ , (3.1) and (3.2) tell us that

$$\begin{aligned} \langle \Psi, B_N^{(\varepsilon(1))} \dots B_N^{(\varepsilon(n+1))} \Psi \rangle &= 0 = \langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(n+1))} \Phi \rangle \\ &= \langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(n+1))} \Phi \rangle \lambda^{\sum_{k=1}^{n+1} (1 - |\varepsilon(k)|/2)} \end{aligned}$$

If  $\varepsilon(n + 1) = 0$ , (3.1) gives

$$\begin{aligned} \langle \Psi, B_N^{(\varepsilon(1))} \dots B_N^{(\varepsilon(n+1))} \Psi \rangle &= \langle \Psi, B_N^{(\varepsilon(1))} \dots B_N^{(\varepsilon(n))} B_N^{(0)} \Psi \rangle \\ &= N p_N \langle \Psi, B_N^{(\varepsilon(1))} \dots B_N^{(\varepsilon(n))} \Psi \rangle \end{aligned}$$

The assumption of induction and the fact  $N p_N \rightarrow \lambda$  make sure that

$$\langle \Psi, B_N^{(\varepsilon(1))} \dots B_N^{(\varepsilon(n+1))} \Psi \rangle \rightarrow \lambda \langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(n))} \Phi \rangle \lambda^{\sum_{k=1}^n (1 - |\varepsilon(k)|/2)}$$

On the other hand, (3.2) gives

$$\begin{aligned} &\left\langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(n))} b^{(\varepsilon(n+1))} \Phi \right\rangle \lambda^{\sum_{k=1}^{n+1} (1 - |\varepsilon(k)|/2)} \Big|_{\varepsilon(n+1)=0} \\ &= \left\langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(n))} b^{(0)} \Phi \right\rangle \lambda^{1 + \sum_{k=1}^n (1 - |\varepsilon(k)|/2)} \\ &= \lambda \left\langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(n))} \Phi \right\rangle \lambda^{\sum_{k=1}^n (1 - |\varepsilon(k)|/2)} \end{aligned}$$

i.e.

$$\begin{aligned} &\left\langle \Psi, B_N^{(\varepsilon(1))} \dots B_N^{(\varepsilon(n))} B_N^{(0)} \Psi \right\rangle \\ &\rightarrow \left\langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(n))} b^{(\varepsilon(n+1))} \Phi \right\rangle \lambda^{\sum_{k=1}^{n+1} (1 - |\varepsilon(k)|/2)} \Big|_{\varepsilon(n+1)=0} \end{aligned}$$

Finally, one sees the case of  $\varepsilon(n + 1) = 1$ .

If  $\varepsilon(j) = 1$  for all  $j \in \{1, \dots, n + 1\}$ , one gets, thanks to (3.1) and (3.2),

$$\begin{aligned} \langle \Psi, B_N^{(\varepsilon(1))} \dots B_N^{(\varepsilon(n+1))} \Psi \rangle &= \langle \Psi, (B_N^{(+1)})^{n+1} \Psi \rangle \\ = 0 &= \langle \Phi, (b^+)^{n+1} \Phi \rangle = \left\langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(n))} b^{(\varepsilon(n+1))} \Phi \right\rangle \lambda^{\sum_{k=1}^{n+1} (1 - |\varepsilon(k)|/2)} \end{aligned}$$

So one needs only to see the case of  $\{k : \varepsilon(k) \neq 1\} \neq \emptyset$ . In this case, one denotes  $m := \max\{k : \varepsilon(k) \neq 1\}$ , then  $\varepsilon(m) \in \{-1, 0, 2\}$ ,  $\varepsilon(m + 1) = \dots = \varepsilon(n + 1) = 1$ ,

$$B_N^{(\varepsilon(1))} \dots B_N^{(\varepsilon(n+1))} = B_N^{(\varepsilon(1))} \dots B_N^{(\varepsilon(m-1))} B_N^{(\varepsilon(m))} \left(B_N^{(+1)}\right)^{n+1-m} \quad (3.9)$$

In virtue of the formula (3.1), the formula (3.9), the assumption of induction and the fact  $N p_N \rightarrow \lambda$ , one gets the following:

If  $\varepsilon(m) = 0$ ,

$$\begin{aligned}
 & \left\langle \Psi, B_N^{(\varepsilon(1))} \dots B_N^{(\varepsilon(n+1))} \Psi \right\rangle \Big|_{\varepsilon(m)=0, \varepsilon(j)=1 \ \forall j > m} \\
 &= \left\langle \Psi, B_N^{(\varepsilon(1))} \dots B_N^{(\varepsilon(m-1))} B_N^{(0)} \left( B_N^{(+1)} \right)^{n+1-m} \Psi \right\rangle \\
 &= (N - (n+1-m)) p_N \left\langle \Psi, B_N^{(\varepsilon(1))} \dots B_N^{(\varepsilon(m-1))} \left( B_N^{(+1)} \right)^{n+1-m} \Psi \right\rangle \\
 &\longrightarrow \lambda \left\langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(m-1))} \left( b^{(+1)} \right)^{n+1-m} \Phi \right\rangle \lambda^{\sum_{k=1}^{m-1} (1-|\varepsilon(k)|/2) + (n+1-m)/2} \\
 &= \left\langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(n+1))} \Phi \right\rangle \lambda^{\sum_{k=1}^{n+1} (1-|\varepsilon(k)|/2)} \Big|_{\varepsilon(m)=0, \varepsilon(j)=1 \ \forall j > m}
 \end{aligned}$$

If  $\varepsilon(m) = -1$ ,

$$\begin{aligned}
 & \left\langle \Psi, B_N^{(\varepsilon(1))} \dots B_N^{(\varepsilon(n+1))} \Psi \right\rangle \Big|_{\varepsilon(m)=-1, \varepsilon(j)=1 \ \forall j > m} \\
 &= \left\langle \Psi, B_N^{(\varepsilon(1))} \dots B_N^{(\varepsilon(m-1))} B_N^{(-1)} \left( B_N^{(+1)} \right)^{n+1-m} \Psi \right\rangle \\
 &= N p_N (1 - p_N) \left\langle \Psi, B_N^{(\varepsilon(1))} \dots B_N^{(\varepsilon(m-1))} \left( B_N^{(+1)} \right)^{n-m} \Psi \right\rangle \\
 &\longrightarrow \lambda \left\langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(m-1))} \left( b^{(+1)} \right)^{n-m} \Phi \right\rangle \lambda^{\sum_{k=1}^{m-1} (1-|\varepsilon(k)|/2) + (n-m)/2}
 \end{aligned}$$

On the other hand, (3.2) gives

$$\begin{aligned}
 & \left\langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(n+1))} \Phi \right\rangle \lambda^{\sum_{k=1}^{n+1} (1-|\varepsilon(k)|/2)} \Big|_{\varepsilon(m)=-1, \varepsilon(j)=1 \ \forall j > m} \\
 &= \left\langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(m-1))} b^{(-1)} \left( b^{(+1)} \right)^{n+1-m} \Phi \right\rangle \lambda^{\sum_{k=1}^{m-1} (1-|\varepsilon(k)|/2) + (n-m+2)/2}
 \end{aligned}$$

So

$$\begin{aligned}
 & \left\langle \Psi, B_N^{(\varepsilon(1))} \dots B_N^{(\varepsilon(n+1))} \Psi \right\rangle \Big|_{\varepsilon(m)=-1, \varepsilon(j)=1 \ \forall j > m} \\
 &\longrightarrow \left\langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(n+1))} \Phi \right\rangle \lambda^{\sum_{k=1}^{n+1} (1-|\varepsilon(k)|/2)} \Big|_{\varepsilon(m)=-1, \varepsilon(j)=1 \ \forall j > m}
 \end{aligned}$$

If  $\varepsilon(m) = 2$ ,

$$\begin{aligned}
 & \left\langle \Psi, B_N^{(\varepsilon(1))} \dots B_N^{(\varepsilon(n+1))} \Psi \right\rangle \Big|_{\varepsilon(m)=2, \varepsilon(j)=1 \ \forall j > m} \\
 &= \left\langle \Psi, B_N^{(\varepsilon(1))} \dots B_N^{(\varepsilon(m-1))} B_N^{(2)} \left( B_N^{(+1)} \right)^{n+1-m} \Psi \right\rangle \\
 &= (1 - p_N) \left\langle \Psi, B_N^{(\varepsilon(1))} \dots B_N^{(\varepsilon(m-1))} \left( B_N^{(+1)} \right)^{n+1-m} \Psi \right\rangle \\
 &\longrightarrow \left\langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(m-1))} \left( b^{(+1)} \right)^{n+1-m} \Phi \right\rangle \lambda^{\sum_{k=1}^{m-1} (1-|\varepsilon(k)|/2) + (n-m)/2} \\
 &= \left\langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(m-1))} b^{(2)} \left( b^{(+1)} \right)^{n+1-m} \Phi \right\rangle \lambda^{\sum_{k=1}^{m-1} (1-|\varepsilon(k)|/2) + (n-m)/2} \\
 &= \left\langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(n+1))} \Phi \right\rangle \lambda^{\sum_{k=1}^{n+1} (1-|\varepsilon(k)|/2)} \Big|_{\varepsilon(m)=2, \varepsilon(j)=1 \ \forall j > m}
 \end{aligned}$$

Summing up, we have proved that the limit of (1.8) equals to the expression (1.11). By combining together this and the boundedness of  $\{B_n^{(\varepsilon)}\}_{n=1}^{\infty}$ , we know that the limit of (1.7) equals to the expression (1.9).  $\square$

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