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QUANTIZATION OF THE BOOLEAN POISSON CENTRAL LIMIT THEOREM AND A GENERALIZED BOOLEAN BERNOULLI SEQUENCE

YUNGANG LU*

ABSTRACT. This paper is devoted to setting up a quantization of the Boolean Poisson central limit theorem. A sequence of the Boolean independent binomial random variables on an interacting Fock space is constructed in terms of creation–annihilation operators. By using these random variables, we study the Boolean Bernoulli sequence, its generalization and moreover, a quantization of the Poisson central limit theorem with respect to the convergence both in mixed–moments and in law.

1. Introduction

As a continuation of [4], the main goal of the present paper is to get a **quantization** of the Boolean Poisson central limit theorem (CLT in short). Its free and monotone analogues will be dealt with in [5] and [6]

The Boolean Poisson CLT (see [7] and references within) is usually formulated as follows: *Let* $\{p_n\}_{n=1}^{\infty} \subset [0,1]$ *be such sequence that* $\lim_{n\to\infty} np_n = \lambda$, *then*

$$\lim_{n\to\infty} \big((1-p_n)\,\delta_0 + p_n\delta_1 \big)^{\uplus n} = \frac{1}{1+\lambda}\delta_0 + \frac{\lambda}{1+\lambda}\delta_{1+\lambda} \tag{1.1}$$

in the weak convergence, where, $\uplus$ *is the* **Boolean convolution** *and the probability measure* $\frac{1}{1+\lambda}\delta_0 + \frac{\lambda}{1+\lambda}\delta_{1+\lambda}$ *is called the* **Boolean Poisson distribution with the parameter** λ. Throughout, δ_x :=the Dirac measure centred on x for any $x \in \mathbb{R}$.

One recalls that

1) the Boolean convolution is defined by means of "*self–energy*" function (see e.g., [2]): for any μ belonging to $\mathcal{P}$:=the set of all probability measures on $(\mathbb{R}, \mathcal{B})$, its self–energy function is defined as $K_\mu : z \longmapsto z - \frac{1}{G_\mu(z)}$ for any $z \in \mathbb{C}$ with $\mathrm{Im}\,(z) > 0$, where, G_μ is the Cauchy transform of μ;

2) for any $\mu, \nu \in \mathcal{P}$, $\mu = \nu$ if and only $K_\mu = K_\nu$; the Boolean convolution $\mu \uplus \nu$ is the element of $\mathcal{P}$ with the self–energy function $K_\mu + K_\nu$;

3) for any $\{\mu_n\}_{n=1}^{\infty} \subset \mathcal{P}$ and $\mu \in \mathcal{P}$, $\mu_n \xrightarrow{w} \mu$ if and only if $\mathrm{Im}K_{\mu_n}(x+iy) \longrightarrow \mathrm{Im}K_\mu(x+iy)$ for **some** $y > 0$ and **any** $x \in \mathbb{R}$

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Clearly, one can reformulate the above Boolean Poisson CLT in terms of algebraic random variables and the Boolean independence as follows: *Let* $(\mathcal{X},\psi)$ *be an algebraic probability space and* $\{\xi_{n,k}: n\in\mathbb{N}^*$ and $k\le n\}$ *be a family of algebraic random variables, let* $\{p_n\}_{n=1}^{\infty}\subset[0,1]$. *If*

- $\psi\big(\xi_{n,k}^m\big)=\psi(\xi_{n,k})=p_n$ *for any* $m,n\in\mathbb{N}^*$ *and* $k\le n$ *(in this case, one says that the* $\psi-$*distribution of* $\xi_{n,k}$ *is the binomial distribution with the parameter* $(1,p_n)$ *and writes this fact simply as* $\xi_{n,k}\overset{\psi}{\sim}b(1,p_n)$*),*
- *for any* $n\ge 2$, $\{\xi_{n,1},\dots,\xi_{n,n}\}$ *is a Boolean independent family with respect to* ψ.

the $\psi-$*distribution of* $\sum_{k=1}^{n}\xi_{n,k}$ *goes to the Boolean Poisson distribution with the parameter* λ, *(i.e., the two points distribution* $\frac{1}{1+\lambda}\delta_0+\frac{\lambda}{1+\lambda}\delta_{1+\lambda}$*) in the weak convergence, whenever* $np_n\longrightarrow\lambda$.

This reformulation of the Boolean Poisson CLT does **not** depend on the specific forms of the algebraic probability space and random variables. In other words, if we take an algebraic probability space $(\mathcal{A},\phi)$ and a family of algebraic random variables $\{X_{n,k}: n\in\mathbb{N}^*$ and $k\le n\}$ such that

- for any $n\in\mathbb{N}^*$ and $k\le n$, the $\phi-$distribution of $X_{n,k}$ is $b(1,p_n)$ ($X_{n,k}\overset{\phi}{\sim}b(1,p_n)$ in short) and where $np_n\longrightarrow\lambda$,
- for any $n\ge 2$, $\{X_{n,1},\dots,X_{n,n}\}$ is a Boolean independent family with respect to ϕ,

the Boolean Poisson CLT guarantees that $\phi-$distribution of $\sum_{k=1}^{n}X_{n,k}$ goes to the two points distribution $\frac{1}{1+\lambda}\delta_0+\frac{\lambda}{1+\lambda}\delta_{1+\lambda}$ in the weak convergence.

In this paper, we take a concrete algebraic probability space $(\mathcal{A},\phi)$ and a particular family of algebraic random variables $\{X_{n,k}: n\in\mathbb{N}^*$ and $k\le n\}$ based on $\Gamma_{Boolean}(\mathcal{H})$:=the *standard Boolean Fock space* over a given (pre–)Hilbert space $\mathcal{H}$, which is defined as such a special *1–mode–type–interacting Fock space* (1MT–IFS in short) $\Gamma(\mathcal{H},\{\omega_n\}_n)$ that $\omega_1=1$ and $\omega_n=0$ for any $n\ge 2$; the word *standard* refers to the fact $\omega_1=1$. Recall that for any (pre–)Hilbert space $\mathcal{H}$ and $\{\omega_n\}_{n\ge1}\subset[0,+\infty)$, the 1MT–IFS $\Gamma(\mathcal{H},\{\omega_n\}_n)$ is defined as $\oplus_{n\ge0}\mathcal{H}_n$, where

- $\mathcal{H}_0:=\mathbb{C}$;
- for any $n\in\mathbb{N}^*$, $\mathcal{H}_n:=\{\mathcal{H}^{\otimes n},\langle\cdot,\cdot\rangle\}$ is the (pre–)Hilbert space obtained by equipping the following scalar product on $\mathcal{H}^{\otimes n}$:

$$\langle f,g\rangle:=(f,g)\prod_{k=1}^{n}\omega_k\quad\forall f,g\in\mathcal{H}^{\otimes n}$$

  and $(\cdot,\cdot)$ is the usual tensor scalar product.

In particular, $\Gamma(\mathcal{H},\{\omega_n\}_n)$ is called a *1–mode–interacting Fock space* (1M–IFS in short) if $\mathcal{H}=\mathbb{C}$.

Clearly, for any $n\in\mathbb{N}^*$, $\mathcal{H}_n$ is either equal to $\{0\}$ (if and only if $\omega_k=0$ for some $1\le k\le n$) or topologically equivalent to the (pre–)Hilbert space $\{\mathcal{H}^{\otimes n},(\cdot,\cdot)\}$ (if and only if $\omega_k>0$ for any $1\le k\le n$). In particular, $\Gamma_{Boolean}(\mathcal{H})$ is isomorphic to $\mathbb{C}\oplus\mathcal{H}$.

In the sequel, we will take $\mathcal{H}$ to be a (pre–)Hilbert space with an orthogonal normal family $\{e_k\}_{k=1}^{\infty}$. Our concrete algebraic probability space $(\mathcal{A},\phi)$ and random variables $\{X_{n,k} : n \in \mathbb{N}^* \text{ and } k \le n\}$ are taken as follows:

- $\mathcal{A}$ is the algebra generated by $\mathcal{A}_k$'s and for any $k \in \mathbb{N}^*$,
$$\mathcal{A}_k := \text{the set of all polynomials in } a_k \text{ and } a_k^+ \text{ with degree} \ge 1 \tag{1.2}$$
a_k (respectively, a_k^+) is the annihilation (respectively, creation) operator of $\Gamma_{Boolean}(\mathcal{H})$ with the test function e_k (see (2.1) below for the detail);
- ϕ :=the vacuum state, i.e. $\phi(\cdot) := \langle \Psi, \cdot\Psi\rangle$ and $\Psi := 1 \oplus 0$ is the vacuum vector of $\Gamma_{Boolean}(\mathcal{H})$;
- for any $n \in \mathbb{N}^*$ and $k \le n$, for any given $\{p_n\}_{n=1}^{\infty} \subset [0,1]$, the algebraic random variable $X_{n,k}$ is defined as a sum of four terms as follows:
$$\sqrt{p_n(1-p_n)}\, a_k + \sqrt{p_n(1-p_n)}\, a_k^+ + p_n a_k a_k^+ + (1-p_n)\, a_k^+ a_k$$

In the notation
$$a_k^{(\varepsilon)} := \begin{cases} a_k, & \text{if } \varepsilon = -1 \\ a_k^+, & \text{if } \varepsilon = 1 \\ a_k a_k^+, & \text{if } \varepsilon = 0 \\ a_k^+ a_k, & \text{if } \varepsilon = 2 \end{cases}, \qquad \forall k \in \mathbb{N}^* \tag{1.3}$$
one has in fact
$$X_{n,k} = \sqrt{p_n(1-p_n)}\left(a_k^{(-1)} + a_k^{(+1)}\right) + p_n a_k^{(0)} + (1-p_n)\, a_k^{(2)} \tag{1.4}$$
Moreover, we will call $\sqrt{p_n(1-p_n)}\, a_k^{(-1)}$ and $\sqrt{p_n(1-p_n)}\, a_k^{(+1)}$ the **off–diagonal components** of $X_{n,k}$ since they are conjugate each other; $p_n a_k^{(0)}$ and $(1-p_n)\, a_k^{(2)}$ the **diagonal components** of $X_{n,k}$ since they are self–adjoint.

In Section 2, we prove that

- the algebras $\mathcal{A}_k$'s defined in (1.2) are Boolean independent with respect to the vacuum state ϕ;
- any $X_{n,k}$ is a projector and $\phi(X_{n,k}) = p_n$, in particular, $X_{n,k} \overset{\phi}{\sim} b(1,p_n)$;
- both $\left\{B_n^{(\varepsilon)} : n \in \mathbb{N}^* \text{ and } \varepsilon \in \{-1,0,1,2\}\right\}$ and $\left\{B_n\right\}_{n=1}^{\infty}$ are uniformly bounded if $\{np_n\}_{n=1}^{\infty}$ is bounded, hereinafter,

$$B_n^{(\pm 1)} := \sqrt{p_n(1-p_n)}\sum_{k=1}^{n} a_k^{(\pm 1)};\ B_n^{(0)} := p_n \sum_{k=1}^{n} a_k^{(0)};\ B_n^{(2)} := (1-p_n)\sum_{k=1}^{n} a_k^{(2)}$$
$$B_n := B_n^{(-1)} + B_n^{(+1)} + B_n^{(0)} + B_n^{(2)} = \sum_{k=1}^{n} X_{n,k} \tag{1.5}$$

As mentioned above, the particularity of our choice of $(\mathcal{A},\phi)$ and $\left\{X_{n,k} : n \in \mathbb{N}^* \text{ and } k \le n\right\}$ do **not** damage the Boolean Poisson CLT. And what is more, one gets some new information that (1.1) does not supply to us, e.g.,

- the convergence in law of the families $\left\{B_n^{(\varepsilon)} : n \in \mathbb{N}^* \text{ and } \varepsilon \in \{-1,0,1,2\}\right\}$ and $\{B_n\}_{n=1}^{\infty}$ is the same as the convergence of all moments thanks to the uniform boundedness;

- as a generalization of the vacuum distribution of $\{B_n\}_{n=1}^{\infty}$ and its weak limit, one can study, for any $\{c_0, c_1, c_2\} \subset \mathbb{R}$,

$$D(n, p_n; c_0, c_1, c_2)$$
$$:= \text{the vacuum distribution of } c_1\left(B_n^{(-1)} + B_n^{(+1)}\right) + c_0 B_n^{(0)} + c_2 B_n^{(2)} \tag{1.6}$$

and the corresponding weak limit; consequently, one gets *individually* the contributions of $\left\{B_n^{(\varepsilon)}\right\}_{n=1}^{\infty}$ to the Boolean Poisson CLT, not just their sum $\left\{\sum_{\varepsilon\in\{-1,0,1,2\}} B_n^{(\varepsilon)}\right\}_{n=1}^{\infty}$ (i.e. $\{B_n\}_{n=1}^{\infty}$).

For any $n \in \mathbb{N}^*$, by writing p_n in the definition of $B_n^{(\varepsilon)}$'s just as p, then $D(n, p; 1, 1, 1)$ is nothing else than the usual Boolean Bernoulli distribution with the parameter (n, p). Suggested by this, one can call naturally $D(n, p; c_0, c_1, c_2)$ the **generalized Boolean Bernoulli distribution with the parameter** (n, p).

Versus the usual Boolean Poisson CLT which is a study of the weak limit of $D(n, p_n; 1, 1, 1)$ (i.e. the vacuum distribution of B_n), the *quantized* Boolean Poisson CLT is to investigate the weak limit of $D(n, p_n; c_0, c_1, c_2)$ for arbitrary $\{c_0, c_1, c_2\} \subset \mathbb{R}$. In a little more detail:

1) to study, for any $m \in \mathbb{N}^*$ and $\varepsilon \in \{-1, 0, 1, 2\}^m$, the limit

$$\lim_{n\to\infty} \phi\left(B_n^{(\varepsilon(1))} \cdots B_n^{(\varepsilon(m))}\right) \tag{1.7}$$

i.e., to see individually the limits of the mixed–moments of $B_n^{(\varepsilon)}$'s;

2) to calculate, for any $\{c_0, c_1, c_2\} \subset \mathbb{R}$ and $t \in \mathbb{R}$, the limit

$$\lim_{n\to\infty} \phi\left(\exp\left(it\left(c_1\left(B_n^{(-1)} + B_n^{(+1)}\right) + c_0 B_n^{(0)} + c_2 B_n^{(2)}\right)\right)\right) \tag{1.8}$$

i.e., to know the weak limit of $D(n, p_n; c_0, c_1, c_2)$ in terms of the characteristic function;

3) to give a suitable representation to the above limits.

In case $np_n \longrightarrow \lambda$ (recall that this is an elementary assumption for performing the Poisson type CLT), one knows that $\sqrt{np_n(1-p_n)}$ is asymptotically equal to $\sqrt{\lambda}$ and so

$$B_n^{(\pm1)} := \sqrt{p_n(1-p_n)} \sum_{k=1}^{n} a_k^{(\pm1)} \approx \sqrt{\lambda} \frac{1}{\sqrt{n}} \sum_{k=1}^{n} a_k^{(\pm1)}$$

As a consequence, by taking $c_0 = c_2 = 0$ and $c_1 = 1$ in the above 2), one gets the usual Laplace–de Moivre type CLT. Therefore, our quantization of the Boolean Poisson CLT gives a view to understand the relationship between the Boolean Poisson CLT and the corresponding Laplace–de Moivre CLT: **the Boolean Laplace–de Moivre CLT is the off–diagonal part of the Boolean Poisson CLT**. Moreover, by using the representation mentioned in the above 3), one can also understand the relationship between the Boolean Poisson distribution and the Boolean Gaussian distribution (i.e., symmetric two points distribution).

In Section 3, we express the usual Boolean Bernoulli distribution and the usual Boolean Poisson CLT in terms of the sequence of the algebraic random variables $\{X_{n,k} : n \in \mathbb{N}^* \text{ and } k \le n\}$ given in (1.4). Technically, instead of the self–energy function used in [7], we use the characteristic function.

The section 4 is the main part of this paper, in which we

- calculate the generalized Boolean Bernoulli distribution with the parameter (n,p); i.e., the probability measure $D(n,p;c_0,c_1,c_2)$ with $n\in\mathbb{N}^*$, $p\in[0,1]$ and $\{c_0,c_1,c_2\}\subset\mathbb{R}$;
- prove that the limit (1.7) exists and has the form

$$\left\langle \Phi, b^{(\varepsilon(1))}\dots b^{(\varepsilon(m))}\Phi\right\rangle \lambda^{\sum_{k=1}^{m}(1-|\varepsilon(k)|/2)} \tag{1.9}$$

where and throughout the paper,

$$b^{(\varepsilon)} := \begin{cases} b, & \text{if } \varepsilon=-1\\ b^+, & \text{if } \varepsilon=1\\ P_\Phi, & \text{if } \varepsilon=0\\ \mathbf{1}-P_\Phi, & \text{if } \varepsilon=2\end{cases} \tag{1.10}$$

b and b^+ are annihilation–creation operators on $\Gamma_{Boolean}(\mathbb{C})$:=the 1M–IFS $\Gamma\left(\mathbb{C},\{\omega_n\}_n\right)$ with $\omega_1=1$ and $\omega_n=0$ for any $n\geq 2$; Φ is its vacuum vector and P_Φ is the projector from $\Gamma_{Boolean}(\mathbb{C})$ to its vacuum subspace;
- show that the limit (1.8) exists and equals to

$$\left\langle \Phi, \exp\left(it\left(c_1\sqrt{\lambda}\left(b^{(-1)}+b^{(+1)}\right)+\lambda c_0 b^{(0)}+c_2 b^{(2)}\right)\right)\Phi\right\rangle \tag{1.11}$$

in other words, the vacuum distribution of $c_1\left(B_n^{(-1)}+B_n^{(+1)}\right)+c_0B_n^{(0)}+c_2B_n^{(2)}$ goes to the vacuum distribution of

$$c_1\sqrt{\lambda}\left(b^{(-1)}+b^{(+1)}\right)+\lambda c_0 b^{(0)}+c_2 b^{(2)}$$

in the weak convergence as $n\longrightarrow\infty$.

Remark 1.1. On the 1M–IFS $\Gamma_{Boolean}(\mathbb{C})$, the projectors P_Φ and $\mathbf{1}-P_\Phi$ equal to nothing else than bb^+ and b^+b respectively. So (1.10) can be alternatively rewritten to

$$b^{(\varepsilon)} := \begin{cases} b, & \text{if } \varepsilon=-1\\ b^+, & \text{if } \varepsilon=1\\ bb^+, & \text{if } \varepsilon=0\\ b^+b, & \text{if } \varepsilon=2\end{cases}$$

Moreover, it is natural to call, in the expression $c_1\sqrt{\lambda}\left(b^{(-1)}+b^{(+1)}\right)+\lambda c_0b^{(0)}+c_2b^{(2)}$, $b^{(+1)}$ and $b^{(-1)}$ the *off–diagonal* components since they are conjugate each other; $b^{(0)}$ and $b^{(2)}$ the *diagonal* components since they are self–adjoint.

2. A Brief Discussion to $\Gamma_{Boolean}(\mathcal{H})$

On the standard Boolean Fock space $\Gamma_{Boolean}(\mathcal{H})$, one denotes by Ψ and a_f^+ (respectively, a_f) the vacuum vector $1\oplus 0$, the creation (respectively, annihilation) operator with the test function $f\in\mathcal{H}$, namely,

$$a_f^+ x := \begin{cases} 0, & \text{if } x \in \mathcal{H} \\ f, & \text{if } x = \Psi \end{cases}, \quad a_f x := \left(a_f^+\right)^* x = \begin{cases} \langle f, x\rangle \Psi, & \text{if } x \in \mathcal{H} \\ 0, & \text{if } x = \Psi \end{cases} \tag{2.1}$$

Proposition 2.1. *$\{\Psi, e_k\}_k$ is an orthogonal normal family of $\Gamma_{Boolean}(\mathcal{H})$ whenever $\{e_k\}_k$ is an orthogonal normal family of $\mathcal{H}$. Moreover,*

1) for any $\{f, g\} \subset \mathcal{H}$,

$$a_f a_g = a_f^+ a_g^+ = 0 \tag{2.2}$$

and so, for any $n \geq 2$ and $\{f_1, \dots, f_n\} \subset \mathcal{H}$, the product $a_{f_1}^{\varepsilon(1)} \dots a_{f_n}^{\varepsilon(n)}$ differs from zero only if $\varepsilon(1) \neq \varepsilon(2) \neq \dots \neq \varepsilon(n)$;

2) for any $\{f, g\} \subset \mathcal{H}$,

$$a_f a_g^+ = \langle f, g\rangle P_\Psi; \quad a_f^+ a_g = 0_{\mathbb{C}} \oplus |f\rangle\langle g|; \quad a_f^+ a_f = \|f\|^2 P_f \tag{2.3}$$

where, $0_{\mathbb{C}}$ is the zero operator on $\mathbb{C}$; $|f\rangle\langle g| x := \langle g, x\rangle f$ for any $x \in \mathcal{H}$; P_Ψ :=the projector form $\Gamma_{Boolean}(\mathcal{H})$ to the vacuum subspace and P_f :=the projector form $\Gamma_{Boolean}(\mathcal{H})$ to the subspace $\{cf : c \in \mathbb{C}\}$; in particular,

$$a_k^{(0)} := a_k a_k^+ = P_\Psi, \quad a_k^{(2)} := a_k^+ a_k = P_{e_k}, \quad \forall k \tag{2.4}$$

(recall that, $a_k := a_{e_k}$ and $a_k^+ := a_{e_k}^+$ for any $\varepsilon \in \{-1, 1, 0, 2\}$) and

$$a_k e_h = \delta_{k,h}\Psi; \quad a_k^+ e_h = 0 = P_\Psi e_h; \quad P_{e_k} e_h = \delta_{k,h} e_h, \quad \forall k, h \tag{2.5}$$

3) for any k, $\mathcal{A}_k$ introduced in (1.2) has finite dimensional, more precisely

$$\mathcal{A}_k = lin-sp.\left\{a_k, a_k^+, P_\Psi, P_{e_k}, a_k^+ P_\Psi, P_\Psi a_k, a_k P_{e_k}, P_{e_k} a_k^+\right\} \tag{2.6}$$

4) $\mathcal{A}_k$ is unital and in fact, $(P_\Psi + P_{e_k}) x = x = x (P_\Psi + P_{e_k})$ for any $x \in \mathcal{A}_k$, in other words, $P_\Psi + P_{e_k}$ is identity of $\mathcal{A}_k$ (but not necessarily to be the identity of $\Gamma_{Boolean}(\mathcal{H})$);

5) the identity of $\Gamma_{Boolean}(\mathcal{H})$ belongs to $\mathcal{A}_k$ if and only if $\mathcal{H} = \{ce_k : c \in \mathbb{C}\}$;

6) the family $\{\mathcal{A}_k\}_k$ is Boolean independent with respect to the vacuum state.

Proof. Above all, $\{\Psi, e_k\}_k$ is clearly an orthogonal normal family of $\Gamma_{Boolean}(\mathcal{H})$ if $\{e_k\}_k \subset \mathcal{H}$ is orthogonal normal. Now we are going to prove the six affirmations.

1) (2.2) is just a consequence of (2.1). Consequently, $a_{f_1}^{\varepsilon(1)} \dots a_{f_n}^{\varepsilon(n)} = 0$ for any $\{f_1, \dots, f_n\} \subset \mathcal{H}$ whenever $\varepsilon(k) = \varepsilon(k+1)$ for some k.

2) In (2.3), the last equality is clearly a particular case of the second; the first two are obtained as follows: for any $f, g \in \mathcal{H}$,

- $a_f a_g^+ \Psi = a_f\, g = \langle f, g\rangle \Psi$ and $a_f a_g^+ : \mathcal{H} \longmapsto \{0\}$ (since $a_g^+ : \mathcal{H} \longmapsto \{0\}$), i.e., $a_f a_g^+ = \langle f, g\rangle P_\Psi$;
- $a_f^+ a_g \Psi = 0$ (since $a_g \Psi = 0$); $a_f^+ a_g x = a_f^+ \langle g, x\rangle \Psi = \langle g, x\rangle f$ for any $x \in \mathcal{H}$, i.e., $a_f^+ a_g = 0_{\mathbb{C}} \oplus |f\rangle\langle g|$.

3) The affirmation 2) tells us that for any $n \geq 2$, $f \in \mathcal{H}$ and such $\varepsilon \in \{-1, 1\}^n$ that $\varepsilon(1) \neq \varepsilon(2) \neq \dots \neq \varepsilon(n)$,

$$a_f^{\varepsilon(1)} \dots a_f^{\varepsilon(n)} = \begin{cases} \|f\|^{2m} P_\Psi, & \text{if } n = 2m \text{ and } \varepsilon(n) = 1 \\ \|f\|^{2m} P_f, & \text{if } n = 2m \text{ and } \varepsilon(n) = -1 \\ \|f\|^{2m} a_f^+ P_\Psi, & \text{if } n = 2m+1 \text{ and } \varepsilon(n) = 1 \\ \|f\|^{2m} a_f P_f, & \text{if } n = 2m+1 \text{ and } \varepsilon(n) = -1 \end{cases}$$

By taking $f = e_k$ and noticing that $a_k, a_k^+ \in \mathcal{A}_k$, one gets (2.6).

4) For any k, we show first of all

$$(P_\Psi + P_{e_k})\, a_k^{(\varepsilon)} = a_k^{(\varepsilon)} = a_k^{(\varepsilon)} (P_\Psi + P_{e_k}), \quad \forall \varepsilon \in \{-1, 1\} \tag{2.7}$$

It is clear that a general element of $\Gamma_{Boolean}(\mathcal{H})$ has surely the form $\alpha_1 \Psi + \alpha_2 e_k + \alpha_3 e_k^\perp$ with $\{\alpha_1, \alpha_2, \alpha_3\} \subset \mathbb{C}$ and $\langle e_k^\perp, \Psi\rangle = \langle e_k^\perp, e_k\rangle = \langle e_k, \Psi\rangle = 0$. So,

$$a_k^+ e_k^\perp = 0 = a_k^+ e_k \text{ and } a_k^+ \Psi = e_k\,; \quad a_k e_k^\perp = 0 = a_k \Psi \text{ and } a_k e_k = \Psi$$
$$P_\Psi e_k = 0 = P_\Psi e_k^\perp \text{ and } P_\Psi \Psi = \Psi\,; \quad P_{e_k} \Psi = 0 = P_{e_k} e_k^\perp \text{ and } P_{e_k} e_k = e_k$$

Consequently, one gets the first equality of (2.7) as follows:

$$\begin{aligned} &(P_\Psi + P_{e_k})\, a_k \left(\alpha_1 \Psi + \alpha_2 e_k + \alpha_3 e_k^\perp\right) \\ =& (P_\Psi + P_{e_k})\, \alpha_2 \Psi = \alpha_2 \Psi = a_k \left(\alpha_1 \Psi + \alpha_2 e_k + \alpha_3 e_k^\perp\right) \end{aligned}$$

and

$$\begin{aligned} &(P_\Psi + P_{e_k})\, a_k^+ \left(\alpha_1 \Psi + \alpha_2 e_k + \alpha_3 e_k^\perp\right) \\ =& (P_\Psi + P_{e_k})\, \alpha_1 e_k = \alpha_1 e_k = a_k^+ \left(\alpha_1 \Psi + \alpha_2 e_k + \alpha_3 e_k^\perp\right) \end{aligned}$$

Similarly, one gets the second equality of (2.7): for any $\varepsilon \in \{-1, 1\}$,

$$\begin{aligned} &a_k^{(\varepsilon)} (P_\Psi + P_{e_k}) \left(\alpha_1 \Psi + \alpha_2 e_k + \alpha_3 e_k^\perp\right) \\ =& a_k^{(\varepsilon)} (\alpha_1 \Psi + \alpha_2 e_k) = a_k^{(\varepsilon)} \left(\alpha_1 \Psi + \alpha_2 e_k + \alpha_3 e_k^\perp\right) \end{aligned}$$

Secondly, the fact $P_\Psi P_{e_k} = P_{e_k} P_\Psi = 0$ tells us

$$\begin{aligned} (P_\Psi + P_{e_k})\, P_\Psi &= P_\Psi = P_\Psi (P_\Psi + P_{e_k}) \\ (P_\Psi + P_{e_k})\, P_{e_k} &= P_{e_k} = P_{e_k} (P_\Psi + P_{e_k}) \end{aligned} \tag{2.8}$$

Finally, by combining (2.7) and (2.8), one knows that, for any $\varepsilon \in \{-1, 1\}$

$$\begin{aligned} (P_\Psi + P_{e_k})\, a_k^{(\varepsilon)} P_\Psi &= a_k^{(\varepsilon)} P_\Psi = a_k^{(\varepsilon)} P_\Psi (P_\Psi + P_{e_k}) \\ (P_\Psi + P_{e_k})\, P_\Psi a_k^{(\varepsilon)} &= P_\Psi a_k^{(\varepsilon)} = P_\Psi a_k^{(\varepsilon)} (P_\Psi + P_{e_k}) \end{aligned} \tag{2.9}$$

Summing up, one finds $(P_\Psi + P_{e_k})\, x = x = x (P_\Psi + P_{e_k})$ for any $x \in \{a_k, a_k^+, P_\Psi, P_{e_k}, a_k^+ P_\Psi, P_\Psi a_k, a_k P_{e_k}, P_{e_k} a_k^+\}$, so does for any $x \in \mathcal{A}_k$.

5) In the case of $\mathcal{H} = \{c e_k : c \in \mathbb{C}\}$, $P_\Psi + P_{e_k}$ is nothing else than the identity of $\Gamma_{Boolean}(\mathcal{H})$. In the case of $\mathcal{H} \neq \{c e_k : c \in \mathbb{C}\}$, one takes such element $e_k^\perp \in \mathcal{H} \setminus \{0\}$ that $\langle e_k^\perp, e_k\rangle = 0$. This choice gives us $a_k e_k^\perp = 0 = a_k^+ e_k^\perp$ and $P_\Psi e_k^\perp = 0 = P_{e_k} e_k^\perp$. Consequently, $x e_k^\perp = 0$ for any $x \in \mathcal{A}_k$, and so the identity of $\Gamma_{Boolean}(\mathcal{H})$, which must bring $e_k^\perp \in \mathcal{H} \setminus \{0\}$ to itself, does not belong to $\mathcal{A}_k$.

6) Recall that the Boolean independence with respect to the vacuum state $\langle \Psi, \cdot \Psi\rangle$ means that for any n and $\mathbf{k}$ belonging to $\mathbb{F}_n := \{\mathbf{k}$: function from

$\{1,\ldots,n\}$ to $\mathbb{N}$ and $\mathbf{k}(i)\neq\mathbf{k}(i+1)\ \ \forall i\in\{1,\ldots,n-1\}\big\}$, for any $x_j\in\mathcal{A}_{\mathbf{k}(j)}$ with $j\in\{1,\ldots,n\}$, one has

$$\langle\Psi,x_1\ldots x_n\Psi\rangle=\prod_{j=1}^{n}\langle\Psi,x_j\Psi\rangle \tag{2.10}$$

For any j, the affirmation 3) and the fact $x_j\in\mathcal{A}_{\mathbf{k}(j)}$ tell us that there must be $\{\alpha_{r,j}\}_{r=1}^{8}\subset\mathbb{C}$ such that

$$\begin{aligned}x_j=&\ \alpha_{1,j}a_{\mathbf{k}(j)}+\alpha_{2,j}a^+_{\mathbf{k}(j)}+\alpha_{3,j}P_\Psi+\alpha_{4,j}P_{e_{\mathbf{k}(j)}}+\alpha_{5,j}a^+_{\mathbf{k}(j)}P_\Psi\\&+\alpha_{6,j}P_\Psi a_{\mathbf{k}(j)}+\alpha_{7,j}a_{\mathbf{k}(j)}P_{e_{\mathbf{k}(j)}}+\alpha_{8,j}P_{e_{\mathbf{k}(j)}}a^+_{\mathbf{k}(j)}\end{aligned} \tag{2.11}$$

Therefore,

$$x_j\Psi=(\alpha_{2,j}+\alpha_{5,j}+\alpha_{8,j})\,e_{\mathbf{k}(j)}+\alpha_{3,j}\Psi\ \text{and}\ \langle\Psi,x_j\Psi\rangle=\alpha_{3,j},\quad\forall j \tag{2.12}$$

and this implies that for any $n\geq 2$,

$$\begin{aligned}\langle\Psi,x_1\ldots x_n\Psi\rangle&=\Big\langle\Psi,x_1\ldots x_{n-1}\Big((\alpha_{2,n}+\alpha_{5,n}+\alpha_{9,n})\,e_{\mathbf{k}(n)}+\alpha_{3,n}\Psi\Big)\Big\rangle\\&=\big\langle\Psi,x_1\ldots x_{n-1}e_{\mathbf{k}(n)}\big\rangle(\alpha_{2,n}+\alpha_{5,n}+\alpha_{9,n})+\langle\Psi,x_1\ldots x_{n-1}\Psi\rangle\,\alpha_{3,n}\end{aligned} \tag{2.13}$$

Moreover, by combining (2.5) and (2.11), one gets $x_{n-1}e_{\mathbf{k}(n)}=0$ whenever $\mathbf{k}(n)\neq\mathbf{k}(n-1)$. So, (2.13) becomes to

$$\langle\Psi,x_1\ldots x_n\Psi\rangle=\langle\Psi,x_1\ldots x_{n-1}\Psi\rangle\,\alpha_{3,n}=\langle\Psi,x_1\ldots x_{n-1}\Psi\rangle\,\langle\Psi,x_n\Psi\rangle$$

The induction argument gives the thesis. □

Proposition 2.2. *On the standard Boolean Fock space $\Gamma_{Boolean}(\mathcal{H})$, the following affirmations hold:*

1) for any $k\neq h$, $a_ka_h^+=0$ and $a_k^2=\left(a_k^+\right)^2=0$; for any $k\in\mathbb{N}$ and $p\in[0,1]$,

$$s_{k,p}:=\sqrt{p(1-p)}\big(a_k^{(-1)}+a_k^{(+1)}\big)+pa_k^{(0)}+(1-p)\,a_k^{(2)}$$

is a projector and $s_{k,p}\overset{\phi}{\sim}b(1,p)$ with $\phi(\cdot):=\langle\Psi,\cdot\Psi\rangle$; in particular, $\phi(s_{k,p})=p$;

2) for any $f\in\mathcal{H}$, $\|a_f\|=\|a_f^+\|=\|f\|$; $\|a_fa_f^+\|=\|a_f^+a_f\|=\|f\|^2$, in particular, $\|a_k^{(\varepsilon)}\|=1$ for any k and $\varepsilon\in\{-1,0,1,2\}$;

3) for any $n\in\mathbb{N}^*$ and $\varepsilon\in\{-1,0,1,2\}$

$$\Big\|n^{\left(\frac{|\varepsilon|}{2}-1\right)}\sum_{k=1}^{n}a_k^{(\varepsilon)}\Big\|=1 \tag{2.14}$$

Proof. 1) For any $k\neq h$, (2.2) and the first equality of (2.3) give us $a_k^2=\left(a_k^+\right)^2=0$ and $a_ka_h^+=0$. As a consequence, $s_{k,p}$ is a projector and so $s_{k,p}\overset{\phi}{\sim}b(1,p)$ since $\phi\big(s_{k,p}^m\big)=\phi(s_{k,p})=p$ for any $m\in\mathbb{N}^*$.

2) For any $\{f,g\}\subset\mathcal{H}$ and $\{\alpha,\beta\}\subset\mathbb{C}$, one has $a_f^+(\alpha\Psi+\beta g)\overset{(2.1)}{=}\alpha f$ and $\|\alpha\Psi+\beta g\|^2=|\alpha|^2+|\beta|^2\cdot\|g\|^2$ in virtue of the fact $\langle\Psi,g\rangle=0$. So,

$$\left\| a_f^+ \right\| = \sup_{(\alpha,\beta,g):\ |\alpha|^2+|\beta|^2=1, \|g\|=1} \left\| a_f^+ \left(\alpha\Psi + \beta g\right) \right\| = \sup_{(\alpha,\beta,g):\ |\alpha|^2+|\beta|^2=1, \|g\|=1} |\alpha| \, \|f\| = \|f\|$$

Consequently, $\left\| a_f \right\| = \left\| \left(a_f^+\right)^* \right\| = \|f\|$ and $\left\| a_f a_f^+ \right\| = \left\| a_f^+ a_f \right\| = \|f\|^2$.

3) (2.14) holds for $\varepsilon = 0$ since $a_k^{(0)} = P_\Psi$ for any k and so $n^{\left(\frac{|0|}{2}-1\right)} \sum_{k=1}^n a_k^{(0)} = \frac{1}{n} \sum_{k=1}^n a_k^{(0)} = P_\Psi$.

(2.14) holds for $\varepsilon = 2$ since the facts $a_k^{(2)} = P_{e_k}$ and $P_{e_k} \perp P_{e_h}$ whenever $k \neq h$ give that $n^{\left(\frac{|2|}{2}-1\right)} \sum_{k=1}^n a_k^{(2)} = \sum_{k=1}^n a_k^{(2)} = \sum_{k=1}^n P_{e_k}$, which equals to the projector of the subspace generated by $\{e_1, \ldots, e_n\}$.

Finally, (2.14) holds for $\varepsilon = \pm 1$ because

$$\left\| n^{\left(\frac{|+1|}{2}-1\right)} \sum_{k=1}^n a_k^{(+1)} \right\|^2 = \left\| \frac{1}{\sqrt{n}} \sum_{h=1}^n a_h^+ \right\|^2$$

$$= \left\| \Big(\frac{1}{\sqrt{n}} \sum_{k=1}^n a_k \Big) \Big(\frac{1}{\sqrt{n}} \sum_{h=1}^n a_h^+ \Big) \right\| = \left\| \frac{1}{n} \sum_{k=1}^n a_k a_k^+ \right\| = \left\| \frac{1}{n} \sum_{k=1}^n a_k^{(0)} \right\| = 1$$

and

$$\left\| n^{\left(\frac{|-1|}{2}-1\right)} \sum_{k=1}^n a_k^{(-1)} \right\|^2 = \left\| \frac{1}{\sqrt{n}} \sum_{h=1}^n a_h \right\|^2 = \left\| \frac{1}{\sqrt{n}} \sum_{h=1}^n a_h^+ \right\|^2 = 1$$

□

Corollary 2.3. *For any $n \in \mathbb{N}^*$ and $p_n \in [0,1]$,*

$$\|B_n^{(\pm 1)}\| \le \sqrt{np_n\,(1-p_n)}; \quad \|B_n^{(0)}\| \le np_n; \quad \|B_n^{(2)}\| \le 1 - p_n \tag{2.15}$$

Moreover, by denoting $r_n := \max\left\{1, np_n, \sqrt{np_n}\right\}$, one has

$$\max_{\varepsilon \in \{-1,0,1,2\}} \|B_n^{(\varepsilon)}\| \le r_n\,; \qquad \sup_{n\ge 1, \varepsilon \in \{-1,0,1,2\}} \|B_n^{(\varepsilon)}\| \le \sup_{n \ge 1} r_n \tag{2.16}$$

and consequently, both $\left\{B_n^{(\varepsilon)} : n \in \mathbb{N}^* \text{ and } \varepsilon \in \{-1,0,1,2\}\right\}$ and $\left\{B_n\right\}_{n=1}^\infty$ are uniformly bounded whenever $\{np_n\}_{n=1}^\infty$ is bounded.

Proof. If $\{np_n\}_{n=1}^\infty$ is bounded, one has $\sup_{n\ge1} r_n < +\infty$ and so the uniform boundedness of $\left\{B_n^{(\varepsilon)} : n \in \mathbb{N}^* \text{ and } \varepsilon \in \{-1,0,1,2\}\right\}$ and $\left\{B_n\right\}_{n=1}^\infty$ follows from the second inequality of (2.16), which is a trivial consequence of the first. The affirmation 3) of Proposition 2.2 and the definition of $B_n^{(\varepsilon)}$'s guarantee the inequalities in (2.15) and which make sure the first inequality of (2.16). □

As the end of this section, we formulate some easily checked equalities: for any $n\in\mathbb{N}^*$, by denoting p_n simply as p in the definition of $B_n^{(\varepsilon)}$'s,

$$\begin{aligned}&B_n^{(-1)}\Psi=B_n^{(2)}\Psi=0,\quad B_n^{(0)}\Psi=np\Psi,\\ &B_n^{(-1)}B_n^{(+1)}\Psi=np\,(1-p)\,\Psi,\quad B_n^{(2)}B_n^{(+1)}\Psi=(1-p)\,B_n^{(+1)}\Psi,\\ &B_n^{(0)}B_n^{(+1)}=B_n^{(0)}B_n^{(2)}=B_n^{(+1)}B_n^{(+1)}=0\end{aligned}\tag{2.17}$$

In fact these results are consequences of the following facts: for any $k,h\in\mathbb{N}^*$,

$$\begin{aligned}&a_k^{(-1)}\Psi=a_k^{(2)}\Psi=0,\quad a_k^{(0)}\Psi=\Psi,\quad a_k^{(-1)}a_h^{(+1)}\Psi=\delta_{k,h}\Psi,\\ &a_k^{(2)}a_h^{(+1)}\Psi=\delta_{k,h}a_k^{(+1)}\Psi,\quad a_k^{(0)}a_k^{(+1)}=a_k^{(0)}a_k^{(2)}=a_k^{(+1)}a_h^{(+1)}=0\end{aligned}$$

By applying (2.17), one finds, for any $\{c_0,c_1,c_2\}\subset\mathbb{R}$ and $m\ge1$,

$$\begin{aligned}&\Big(c_1\big(B_n^{(-1)}+B_n^{(+1)}\big)+c_0B_n^{(0)}+c_2B_n^{(2)}\Big)^m\Psi\\ =&\Big(c_1\big(B_n^{(-1)}+B_n^{(+1)}\big)+c_0B_n^{(0)}+c_2B_n^{(2)}\Big)^{m-1}\Big(c_1B_n^{(+1)}+c_0B_n^{(0)}\Big)\Psi\\ =&c_0np\Big(c_1\big(B_n^{(-1)}+B_n^{(+1)}\big)+c_0B_n^{(0)}+c_2B_n^{(2)}\Big)^{m-1}\Psi\\ &+c_1\Big(c_1\big(B_n^{(-1)}+B_n^{(+1)}\big)+c_0B_n^{(0)}+c_2B_n^{(2)}\Big)^{m-1}B_n^{(+1)}\Psi\end{aligned}\tag{2.18}$$

and

$$\begin{aligned}&\Big(c_1\big(B_n^{(-1)}+B_n^{(+1)}\big)+c_0B_n^{(0)}+c_2B_n^{(2)}\Big)^m B_n^{(+1)}\Psi\\ =&\Big(c_1\big(B_n^{(-1)}+B_n^{(+1)}\big)+c_0B_n^{(0)}+c_2B_n^{(2)}\Big)^{m-1}\Big(c_1B_n^{(-1)}+c_2B_n^{(2)}\Big)B_n^{(+1)}\Psi\\ =&c_1np(1-p)\Big(c_1\big(B_n^{(-1)}+B_n^{(+1)}\big)+c_0B_n^{(0)}+c_2B_n^{(2)}\Big)^{m-1}\Psi\\ &+c_2(1-p)\Big(c_1\big(B_n^{(-1)}+B_n^{(+1)}\big)+c_0B_n^{(0)}+c_2B_n^{(2)}\Big)^{m-1}B_n^{(+1)}\Psi\end{aligned}\tag{2.19}$$

In particular, one has

- by taking $c_0=c_1=c_2=1$,

$$B_n^m\Psi=np\,B_n^{m-1}\Psi+B_n^{m-1}B_n^{(+1)}\Psi\tag{2.20}$$

and

$$B_n^mB_n^{(+1)}\Psi=np(1-p)B_n^{m-1}\Psi+(1-p)B_n^{m-1}B_n^{(+1)}\Psi\tag{2.21}$$

- by taking $c_2=0$,

$$\begin{aligned}&\Big(c_1\big(B_n^{(-1)}+B_n^{(+1)}\big)+c_0B_n^{(0)}\Big)^m\Psi\\ =&c_0np\Big(c_1\big(B_n^{(-1)}+B_n^{(+1)}\big)+c_0B_n^{(0)}\Big)^{m-1}\Psi\\ &+c_1\Big(c_1\big(B_n^{(-1)}+B_n^{(+1)}\big)+c_0B_n^{(0)}\Big)^{m-1}B_n^{(+1)}\Psi\end{aligned}\tag{2.22}$$

and

$$\begin{aligned}&\left(c_1\left(B_n^{(-1)}+B_n^{(+1)}\right)+c_0B_n^{(0)}\right)^m B_n^{(+1)}\Psi\\ =&c_1np(1-p)\left(c_1\left(B_n^{(-1)}+B_n^{(+1)}\right)+c_0B_n^{(0)}\right)^{m-1}\Psi \end{aligned}\tag{2.23}$$

- by taking $c_1=0$,

$$\left(c_0B_n^{(0)}+c_2B_n^{(2)}\right)^m\Psi=(c_0np)^m\Psi=\left(c_0B_n^{(0)}\right)^m\Psi \tag{2.24}$$

and

$$\left(c_0B_n^{(0)}+c_2B_n^{(2)}\right)^m B_n^{(+1)}\Psi=(c_2(1-p))^m B_n^{(+1)}\Psi=\left(c_2B_n^{(2)}\right)^m B_n^{(+1)}\Psi \tag{2.25}$$

3. A Reformulation of the Usual Boolean Poisson CLT

Our main goal in this section is to reformulate the usual Boolean Poisson CLT and give a new proof by using the characteristic function rather than the self–energy function as did in [7].

Proposition 3.1. *The following affirmations hold.*

1) For any $n\in\mathbb{N}^*$ and $p\in[0,1]$, the Boolean Bernoulli distribution with the parameter (n,p) (i.e., the probability measure $D(n,p;1,1,1)$) is $\frac{1-p}{1+(n-1)p}\delta_0+\frac{np}{1+(n-1)p}\delta_{1+(n-1)p}$.

2) In the case of $\lim_{n\to\infty}np_n=\lambda$, for any $m\in\mathbb{N}$

$$\lim_{n\to\infty}\langle\Psi,B_n^m\Psi\rangle=\lambda(1+\lambda)^{m-1}=\int x^m d\left(\frac{1}{1+\lambda}\delta_0+\frac{\lambda}{1+\lambda}\delta_{1+\lambda}\right) \tag{3.1}$$

and moreover,

$$\begin{aligned}\lim_{n\to\infty}\langle\Psi,\exp(itB_n)\Psi\rangle&=\frac{1+\lambda e^{it(1+\lambda)}}{1+\lambda}\\ &=\int e^{itx}d\Big(\frac{1}{1+\lambda}\delta_0+\frac{\lambda}{1+\lambda}\delta_{1+\lambda}\Big),\quad\forall t\in\mathbb{R}\end{aligned}\tag{3.2}$$

i.e., $D(n,p_n;1,1,1)\longrightarrow\frac{1}{1+\lambda}\delta_0+\frac{\lambda}{1+\lambda}\delta_{1+\lambda}$ in the weak convergence;

3) By using the notations introduced in Section 1 (more concrete, (1.10) *and Remark 1.1), the expressions in the right hand side of (3.1) and (3.2) equal to*

$$\left\langle\Phi,\left(\sqrt{\lambda}\left(b+b^+\right)+\lambda P_\Phi+1-P_\Phi\right)^m\Phi\right\rangle \tag{3.3}$$

and

$$\left\langle\Phi,\exp\left(it\left(\sqrt{\lambda}\left(b+b^+\right)+\lambda P_\Phi+1-P_\Phi\right)\right)\Phi\right\rangle \tag{3.4}$$

respectively. In other words, the vacuum distribution of B_n (i.e., the probability measure $D_n(n,p_n;1,1,1)$) tends, in the the weak convergence, to $\frac{1}{1+\lambda}\delta_0+\frac{\lambda}{1+\lambda}\delta_{1+\lambda}$, i.e., the vacuum distribution of $\sqrt{\lambda}(b+b^+)+\lambda P_\Phi+1-P_\Phi$.

Proof. We prove first of all the affirmation 1). Clearly, $D(n,p;1,1,1)$ is the vacuum distribution of B_n introduced in (1.5) with $p_n = p$.

One denotes, for any $n,m\in\mathbb{N}$,

$$u_{n,m} := \langle \Psi, B_n^m \Psi\rangle \ ; \quad w_{n,m} := \left\langle \Psi, B_n^m B_n^{(+1)}\Psi\right\rangle \tag{3.5}$$

(notice that $u_{n,m}$ is nothing else than the $m-$th moment of the distribution $D(n,p;1,1,1)$). It is trivial to have

$$u_{n,0}=1, \quad u_{n,1}=np, \quad w_{n,0}=0 \tag{3.6}$$

and in virtue of the formulae (2.20) and (2.21), one finds

$$\begin{cases} u_{n,m} & = np\, u_{n,m-1} + w_{n,m-1}, \\ w_{n,m} & = np(1-p)\, u_{n,m-1} + (1-p) w_{n,m-1} \end{cases} \quad \forall m\in\mathbb{N}^* \tag{3.7}$$

The second equation in (3.7) and the initial condition (3.6) make sure that

$$w_{n,m} = np\sum_{k=1}^{m} (1-p)^k u_{n,m-k}\,, \quad \forall m\in\mathbb{N}^* \tag{3.8}$$

With the help of this result and the initial condition (3.6), one finds the following recursion relation among $u_{n,m}$'s:

$$\begin{aligned} u_{n,m+1} &\overset{(3.7)}{=} np\, u_{n,m} + w_{n,m} \overset{(3.8)}{=} npu_{n,m} + np\sum_{k=1}^{m}(1-p)^k u_{n,m-k} \\ &= np\sum_{k=0}^{m}(1-p)^k u_{n,m-k}\,, \qquad \forall m\in\mathbb{N}^* \end{aligned} \tag{3.9}$$

This gives the moment–generating function of $D(n,p;1,1,1)$ as follows:

$$\begin{aligned} M(t) &:= \sum_{m=0}^{\infty} t^m u_{n,m} = 1 + t\sum_{m=0}^{\infty} t^m u_{n,m+1} \\ &\overset{(3.9)}{=} 1 + npt\sum_{m=0}^{\infty} t^m \sum_{k=0}^{m}(1-p)^k u_{n,m-k} = 1 + \frac{npt}{1-(1-p)\,t} M(t) \end{aligned} \tag{3.10}$$

where, $\sum_{m=0}^{\infty} t^m u_{n,m}$ converges for $t\in(-\frac{1}{4r},\frac{1}{4r})$ with $r := \max\left\{1, np_n, \sqrt{np_n}\right\}$ since (2.15) implies that $|u_{n,m}|\le (4r)^m$ for any m. It follows from (3.10) that

$$M(t) = \frac{1-(1-p)\,t}{1-(1+(n-1)\,p)\,t} \tag{3.11}$$

and the corresponding distribution is clearly

$$D(n,p;1,1,1) = \frac{1-p}{1+(n-1)\,p}\delta_0 + \frac{np}{1+(n-1)\,p}\delta_{1+(n-1)p} \tag{3.12}$$

As a consequence, one gets explicitly

$$u_{n,m} = \langle \Psi, B_n^m\Psi\rangle = np(1+(n-1)p)^{m-1}, \quad \forall m\ge 1 \tag{3.13}$$

Now we turn to prove the affirmation 2). By the definition, the vacuum distribution B_n is $D(n,p_n;1,1,1)$ and where $\lim_{n\to\infty} np_n = \lambda$. So

$$\begin{aligned}\langle\Psi, B_n^m\Psi\rangle =& \text{the } m\text{-moment of the distribution } D\left(n,p_n;1,1,1\right)\\ \overset{(3.13)}{=}& \frac{np_n}{1+(n-1)\,p_n}\left(1+(n-1)\,p_n\right)^m \longrightarrow \lambda\left(1+\lambda\right)^{m-1}\\ =& \text{the } m\text{-moment of the distribution } \frac{1}{1+\lambda}\delta_0+\frac{\lambda}{1+\lambda}\delta_{1+\lambda} \qquad (3.14)\end{aligned}$$

The assumption $np_n\to\lambda$ makes sure that $\sup_{n\geq 1}\max\left\{1,np_n,\sqrt{np_n}\right\}<+\infty$. So, thanks to Corollary 2.3, the families $\left\{B_n^{(\varepsilon)} : n\in\mathbb{N}^* \text{ and } \varepsilon\in\{-1,0,1,2\}\right\}$ and $\left\{B_n\right\}_{n=1}^{\infty}$ are uniformly bounded. For such a sequence, the convergence of all moments is equivalent to the weak convergence and therefore, (3.14) gives

$$D\left(n,p_n;1,1,1\right) := \text{the vacuum distribution of } B_n \overset{w}{\longrightarrow} \frac{1}{1+\lambda}\delta_0+\frac{\lambda}{1+\lambda}\delta_{1+\lambda}$$

Finally we prove the affirmation 3). By recalling the definition of $b^{(\varepsilon)}$'s given in (1.10) on the 1M–IFS $\Gamma_{Boolean}(\mathbb{C})$, one has the following analogy of (2.17):

$$\begin{aligned}&b^{(-1)}\Phi=b^{(2)}\Phi=0,\quad b^{(0)}\Phi=\Phi,\quad b^{(-1)}b^{(+1)}\Phi=\Phi\\ &b^{(2)}b^{(+1)}\Phi=b^{(+1)}\Phi,\quad b^{(0)}b^{(+1)}=b^{(0)}b^{(2)}=b^{(+1)}b^{(+1)}=0 \qquad (3.15)\end{aligned}$$

So, by denoting

$$\begin{aligned}u_m :=& \left\langle\Phi,\left(\sqrt{\lambda}\left(b+b^+\right)+\lambda P_\Phi+1-P_\Phi\right)^m\Phi\right\rangle\\ =& \left\langle\Phi,\left(\sqrt{\lambda}\big(b^{(-1)}+b^{(+1)}\big)+\lambda b^{(0)}+b^{(2)}\right)^m\Phi\right\rangle\end{aligned}$$

one has $u_0=1$, and moreover it follows from (3.15) that, for any $m\in\mathbb{N}$

$$\begin{aligned}u_{m+1} =& \left\langle\Phi,\left(\sqrt{\lambda}\big(b^{(-1)}+b^{(+1)}\big)+\lambda b^{(0)}+b^{(2)}\right)^{m+1}\Phi\right\rangle\\ =& \left\langle\Phi,\left(\sqrt{\lambda}\big(b^{(-1)}+b^{(+1)}\big)+\lambda b^{(0)}+b^{(2)}\right)^m\left(\sqrt{\lambda}b^{(+1)}+\lambda b^{(0)}\right)\Phi\right\rangle\\ =& \lambda u_m+\sqrt{\lambda}\left\langle\Phi,\left(\sqrt{\lambda}\big(b^{(-1)}+b^{(+1)}\big)+\lambda b^{(0)}+b^{(2)}\right)^m b^{(+1)}\Phi\right\rangle\\ =& \lambda u_m+\sqrt{\lambda}\Big\langle\Phi,\left(\sqrt{\lambda}\big(b^{(-1)}+b^{(+1)}\big)+\lambda b^{(0)}+b^{(2)}\right)^{m-1}\left(\sqrt{\lambda}b^{(-1)}+b^{(2)}\right)b^{(+1)}\Phi\Big\rangle\\ =& \lambda u_m+\lambda u_{m-1}+\sqrt{\lambda}\Big\langle\Phi,\left(\sqrt{\lambda}\big(b^{(-1)}+b^{(+1)}\big)+\lambda b^{(0)}+b^{(2)}\right)^{m-1}b^{(+1)}\Phi\Big\rangle\\ =& \ldots=\lambda\sum_{k=0}^m u_k\end{aligned}$$

The system

$$\begin{cases}u_{m+1}=\lambda\sum_{k=0}^m u_k\\ u_0=1\end{cases}$$

guarantees that

$$u_m=\lambda\left(1+\lambda\right)^{m-1}=\int x^m d\Big(\frac{1}{1+\lambda}\delta_0+\frac{\lambda}{1+\lambda}\delta_{1+\lambda}\Big),\quad \forall m\in\mathbb{N}^* \qquad \square$$

4. Quantization of the Boolean Poisson CLT

This section is devoted to

• calculating explicitly, for any $n \in \mathbb{N}^*$, $p \in [0,1]$ and $\{c_0, c_1, c_2\} \subset \mathbb{R}$, the generalized Boolean Bernoulli distribution (with the parameter (n,p)) $D(n,p;c_0,c_1,c_2)$, i.e., the vacuum distribution of $c_1\big(B_n^{(-1)} + B_n^{(+1)}\big) + c_0 B_n^{(0)} + c_2 B_n^{(2)}$ with $p_n = p$ in definition of $B_n^{(\varepsilon)}$'s;

• getting a quantization of the usual Boolean Poisson CLT;

First of all, we treat some particular cases.

Proposition 4.1. *For any $n \in \mathbb{N}^*$,*

$$D(n,p;c_0,0,c_2) = \delta_{c_0 np}, \quad \forall p \in [0,1] \text{ and } \{c_0,c_2\} \subset \mathbb{R} \tag{4.1}$$

and

$$D(n,1;c_0,c_1,c_2) = \delta_{c_0 n}, \quad D(n,0;c_0,c_1,c_2) = \delta_0, \quad \forall \{c_0,c_1,c_2\} \subset \mathbb{R} \tag{4.2}$$

Proof. It follows from (2.24) that for any $m \in \mathbb{N}$ (recall that $p_n = p$),

$$\text{the } m\text{−th moment of } D(n,p;c_0,0,c_2) = \langle \Psi, \big(c_0 B_n^{(0)} + c_2 B_n^{(2)}\big)^m \Psi\rangle = \big(c_0 np\big)^m$$

and this is clearly the same as (4.1).

The definition of $B_n^{(\varepsilon)}$'s makes sure that for any $\{c_0, c_1, c_2\} \subset \mathbb{R}$ and $n \in \mathbb{N}^*$,

$$B_n^{(\pm)} = 0 = B_n^{(2)} \text{ if } p = 1; \quad B_n^{(\pm)} = 0 = B_n^{(0)} \text{ if } p = 0$$

So, in the case of $p \in \{0,1\}$, for any $m, n \in \mathbb{N}^*$,

$$\Big\langle \Psi, \Big(c_1\big(B_n^{(-1)} + B_n^{(+1)}\big) + c_0 B_n^{(0)} + c_2 B_n^{(2)}\Big)^m \Psi\Big\rangle$$
$$= \begin{cases} \big\langle \Psi, \big(c_0 B_n^{(0)}\big)^m \Psi\big\rangle, & \text{if } p = 1 \\ \big\langle \Psi, \big(c_2 B_n^{(2)}\big)^m \Psi\big\rangle, & \text{if } p = 0 \end{cases} = \begin{cases} \big(c_0 n\big)^m, & \text{if } p = 1 \\ 0, & \text{if } p = 0 \end{cases}$$

and the corresponding distributions are those given in (4.2) □

Thanks to Proposition 4.1, we will concentrate our discussion to the case

$$p \in (0,1), \qquad c_1 \neq 0$$

Lemma 4.2. *For any $n \in \mathbb{N}^*$, $p \in (0,1)$, hold the following affirmations.*

1) For any $\{x_0, x_1\} \subset \mathbb{R}$, by denoting

$$E(x_0,x_1,0;n,p) := \sqrt{(x_0 np)^2 + 4x_1^2 np(1-p)}$$
$$E(x_0,x_1,1;n,p) := \sqrt{(x_0 np - 1 + p)^2 + 4x_1^2 np(1-p)} \tag{4.3}$$

one has

• $E(x_0,x_1,0;n,p) > 0$ *whenever* $|x_0| + |x_1| > 0$;

• $E(x_0,x_1,1;n,p) > 0$ *whenever* $x_1 \neq 0$.

2) For any $c_0 \in \mathbb{R}$ and $c_1 \in \mathbb{R} \setminus \{0\}$,

$$P_\pm^{(0)} := \frac{E(c_0,c_1,0;n,p) \pm c_0 np}{2E(c_0,c_1,0;n,p)} \geq 0 \tag{4.4}$$

and in addition, for any $c_2 \in \mathbb{R} \setminus \{0\}$,

$$P_{\pm}^{(1)} := \frac{E\left(c_0', c_1', 1; n, p\right) \pm \left(c_0' np - 1 + p\right)}{2E\left(c_0', c_1', 1; n, p\right)} \geq 0 \tag{4.5}$$

whereafter,

$$c_0' := \frac{c_0}{c_2}, \quad c_1' := \frac{c_1}{c_2} \tag{4.6}$$

3) For any $\varepsilon \in \{0, 1\}$,

$$P_-^{(\varepsilon)} + P_+^{(\varepsilon)} = 1 \tag{4.7}$$

4) For any such $\{p_n\}_{n=1}^{\infty} \subset [0, 1]$ *that* $np_n \longrightarrow \lambda$,

$$\lim_{n\to+\infty} E\left(x_0, x_1, 0; n, p_n\right) = \sqrt{\left(\lambda x_0\right)^2 + 4x_1^2\lambda} =: E\left(\lambda; x_0, x_1, 0\right)$$
$$\lim_{n\to+\infty} E\left(x_0, x_1, 1; n, p_n\right) = \sqrt{\left(\lambda x_0 - 1\right)^2 + 4x_1^2\lambda} =: E\left(\lambda; x_0, x_1, 1\right) \tag{4.8}$$

Proof. The definitions of $P_{\pm}^{(\varepsilon)}$'s, $E\left(x_0, x_1, 1; n, p\right)$ and $E\left(x_0, x_1, 0; n, p\right)$ make sure (4.7) and the affirmation 4).

For any $x_0, x_1 \in \mathbb{R}$, the definition of $E\left(x_0, x_1, 1; n, p\right)$ and $E\left(x_0, x_1, 0; n, p\right)$ gives their non–negativity. Moreover, in virtue of the assumption $p \in (0, 1)$, one has

$$E\left(x_0, x_1, 0; n, p\right) \geq \max\left\{|x_0 np|, 2|x_1|\sqrt{np\left(1-p\right)}\right\} > 0, \quad \text{if } |x_0| + |x_1| > 0$$
$$E\left(x_0, x_1, 1; n, p\right) \geq 2|x_1|\sqrt{np\left(1-p\right)} > 0, \quad \text{if } x_1 \neq 0$$

Now we turn to prove the affirmation 2). With the given conditions, the affirmation 1) guarantees that $P_{\pm}^{(\varepsilon)}$'s are well–defined. Moreover,

$$E\left(x_0, x_1, 0; n, p\right) \pm x_0 np \geq \sqrt{\left(x_0 np\right)^2} \pm x_0 np \geq 0$$

and

$$E\left(x_0, x_1, 1; n, p\right) \pm \left(x_0 np - 1 + p\right) \geq \sqrt{\left(x_0 np - 1 + p\right)^2} \pm \left(x_0 np - 1 + p\right) \geq 0$$

So $P_{\pm}^{(\varepsilon)} \geq 0$ for any $\varepsilon \in \{0, 1\}$. □

Theorem 4.3. *For any* $n \in \mathbb{N}^*$, $p \in (0, 1)$ *and* $\{c_0, c_1, c_2\} \subset \mathbb{R}$, *the distribution* $D\left(n, p; c_0, c_1, c_2\right)$ *equals to the following probability measure:*

$$\begin{cases} \delta_{c_0 np}, & \text{if } c_1 = 0 \\ P_+^{(0)}\delta_{p_+^{(0)}} + P_-^{(0)}\delta_{p_-^{(0)}}, & \text{if } c_1 \neq 0,\ c_2 = 0 \\ P_+^{(1)}\delta_{p_+^{(1)}} + P_-^{(1)}\delta_{p_-^{(1)}}, & \text{if } c_1 \neq 0,\ c_2 \neq 0 \end{cases} \tag{4.9}$$

where, $P_{\pm}^{(0)}$ *and* $P_{\pm}^{(1)}$ *are those introduced in* (4.4) *and* (4.5) *respectively;* $p_{\pm}^{(0)}$ *and* $p_{\pm}^{(1)}$ *are defined as follows:*

$$p_+^{(0)} := \frac{1}{2}\big(c_0 np + E\,(c_0, c_1, 0; n, p)\,\big)$$
$$p_-^{(0)} := \frac{1}{2}\big(c_0 np - E\,(c_0, c_1, 0; n, p)\,\big)$$
$$p_+^{(1)} := \frac{c_2}{2}\big(c_0' np + 1 - p + E\,(c_0', c_1', 1; n, p)\,\big)$$
$$p_-^{(1)} := \frac{c_2}{2}\big(c_0' np + 1 - p - E\,(c_0', c_1', 1; n, p)\,\big)$$

Proof. First of all, the affirmation 3) of Lemma 4.2 says that (4.9) defines indeed probability measures. Moreover, Proposition 4.1 tells us that $D\,(n, p; c_0, 0, c_2)$ equals to $\delta_{c_0 np}$. We will complete the proof according to the following three cases:

$$c_1 \neq 0 \text{ and } c_2 = 0\,; \quad c_1 \neq 0 \text{ and } c_2 = 1\,; \quad c_1 \neq 0 \text{ and } c_2 \neq 0 \tag{4.10}$$

Clearly, the third case is a generalization of the second.

For any $n \in \mathbb{N}^*$, $m \in \mathbb{N}$ and $\{c_0, c_1, c_2\} \subset \mathbb{R}$, we introduce an analogue of (3.5):

$$v_{n,m} := \left\langle \Psi, \left(c_1 \left(B_n^{(-1)} + B_n^{(+1)}\right) + c_0 B_n^{(0)} + c_2 B_n^{(2)}\right)^m \Psi \right\rangle \tag{4.11}$$

and

$$w_{n,m} := c_1 \left\langle \Psi, \left(c_1 \left(B_n^{(-1)} + B_n^{(+1)}\right) + c_0 B_n^{(0)} + c_2 B_n^{(2)}\right)^m B_n^{(+1)} \Psi \right\rangle \tag{4.12}$$

then (2.17) gives trivially the *initial conditions*:

$$v_{n,0} = 1, \quad v_{n,1} = c_0 np, \quad w_{n,0} = 0 \tag{4.13}$$

We are going to get the explicit expression of the moment–generating function of $c_1\left(B_n^{(-1)} + B_n^{(+1)}\right) + c_0 B_n^{(0)} + c_2 B_n^{(2)}$, i.e.

$$V\,(t) := 1 + \sum_{m=1}^{\infty} t^m v_{n,m} = 1 + t v_{n,1} + \sum_{m=1}^{\infty} t^{m+1} v_{n,m+1} \tag{4.14}$$

in the three cases mentioned in (4.10) and find in further the corresponding distribution. Where, $V(t)$ is well defined on the interval $(-\frac{1}{4r}, \frac{1}{4r})$ with $r := \max\{c_0, c_1, c_2\} \cdot \max\left\{1, np_n, \sqrt{np_n}\right\}$ since $|v_{n,m}| \leq (4r)^m$ for any m thanks to Corollary 2.3.

The 1^{st} case: $c_1 \neq 0$ and $c_2 = 0$. In this case, (2.23) gives us

$$\begin{aligned} w_{n,m} =& c_1 \left\langle \Psi, \left(c_1 \left(B_n^{(-1)} + B_n^{(+1)}\right) + c_0 B_n^{(0)}\right)^m B_n^{(+1)} \Psi \right\rangle \\ =& c_1^2 np\,(1-p) \left\langle \Psi, \left(c_1 \left(B_n^{(-1)} + B_n^{(+1)}\right) + c_0 B_n^{(0)}\right)^{m-1} \Psi \right\rangle \\ =& c_1^2 np\,(1-p)\, v_{m-1} \end{aligned} \tag{4.15}$$

By applying this and (2.22), we obtain

$$\begin{aligned}
v_{n,m+1} &= \left\langle \Psi, \left(c_1 \left(B_n^{(-1)} + B_n^{(+1)} \right) + c_0 B_n^{(0)} \right)^{m+1} \Psi \right\rangle \\
&= c_0 np \left\langle \Psi, \left(c_1 \left(B_n^{(-1)} + B_n^{(+1)} \right) + c_0 B_n^{(0)} \right)^{m} \Psi \right\rangle \\
&\quad + c_1 \left\langle \Psi, \left(c_1 \left(B_n^{(-1)} + B_n^{(+1)} \right) + c_0 B_n^{(0)} \right)^{m} B_n^{(+1)} \Psi \right\rangle \\
&= c_0 np v_{n,m} + w_{n,m} \overset{(4.15)}{=} c_0 np v_{n,m} + c_1^2 np(1-p) v_{n,m-1}, \ \forall m \ge 1 \quad (4.16)
\end{aligned}$$

By combining (4.13), (4.14) and (4.16) together, one obtains

$$\begin{aligned}
V(t) &= 1 + c_0 npt + t \sum_{m=1}^{\infty} t^m \big(c_0 np v_{n,m} + c_1^2 np(1-p) v_{n,m-1} \big) \\
&= 1 + c_0 npt \Big(1 + \sum_{m=1}^{\infty} t^m v_{n,m} \Big) + c_1^2 np(1-p) t^2 \sum_{m=1}^{\infty} t^{m-1} v_{n,m-1} \\
&= 1 + c_0 nptV(t) + c_1^2 np(1-p) t^2 V(t) \quad (4.17)
\end{aligned}$$

i.e.,

$$V(t) = \frac{1}{1 - c_0 npt - c_1^2 np(1-p) t^2} \quad (4.18)$$

Equivalently, corresponding Cauchy transform is, by using $E(x_0, x_1, 0; n, p)$ introduced in (4.3) and noticing that $c_1 \neq 0$,

$$\begin{aligned}
G(t) &= \frac{1}{t} V\left(\frac{1}{t}\right) = \frac{t}{t^2 - c_0 npt - c_1^2 np(1-p)} \\
&= \frac{E(c_0, c_1, 0; n, p) + c_0 np}{2E(c_0, c_1, 0; n, p)} \cdot \frac{1}{t - \frac{1}{2}(c_0 np + E(c_0, c_1, 0; n, p))} \\
&\quad + \frac{E(c_0, c_1, 0; n, p) - c_0 np}{2E(c_0, c_1, 0; n, p)} \cdot \frac{1}{t - \frac{1}{2}(c_0 np - E(c_0, c_1, 0; n, p))} \quad (4.19)
\end{aligned}$$

So, the corresponding distribution is

$$\begin{aligned}
&\frac{E(c_0, c_1, 0; n, p) + c_0 np}{2E(c_0, c_1, 0; n, p)} \cdot \delta_{\frac{1}{2}(c_0 np + E(c_0, c_1, 0; n, p))} \\
&+ \frac{E(c_0, c_1, 0; n, p) - c_0 np}{2E(c_0, c_1, 0; n, p)} \cdot \delta_{\frac{1}{2}(c_0 np - E(c_0, c_1, 0; n, p))} \quad (4.20)
\end{aligned}$$

i.e., the measure $D(n, p; c_0, c_1, c_2)$ given in (4.9) with $c_1 \neq 0$ and $c_2 = 0$.

The 2^{nd} case: $c_1 \neq 0$ and $c_2 = 1$. In this case, (2.18) makes sure that

$$\begin{aligned}
v_{n,m+1} &= c_0 np \left\langle \Psi, \left(c_1 \left(B_n^{(-1)} + B_n^{(+1)} \right) + c_0 B_n^{(0)} + B_n^{(2)} \right)^{m} \Psi \right\rangle \\
&\quad + c_1 \left\langle \Psi, \left(c_1 \left(B_n^{(-1)} + B_n^{(+1)} \right) + c_0 B_n^{(0)} + B_n^{(2)} \right)^{m} B_n^{(+1)} \Psi \right\rangle \\
&= c_0 np v_{n,m} + w_{n,m}, \qquad \forall m \ge 1 \quad (4.21)
\end{aligned}$$

and (2.19) gives

$$\begin{aligned} w_{n,m} &= c_1^2 \left\langle \Psi, \left(c_1 \left(B_n^{(-1)} + B_n^{(+1)} \right) + c_0 B_n^{(0)} + B_n^{(2)} \right)^{m-1} B_n^{(-1)} B_n^{(+1)} \Psi \right\rangle \\ &\quad + c_1 \left\langle \Psi, \left(c_1 \left(B_n^{(-1)} + B_n^{(+1)} \right) + c_0 B_n^{(0)} + B_n^{(2)} \right)^{m-1} B_n^{(2)} B_n^{(+1)} \Psi \right\rangle \\ &= c_1^2 np\,(1-p)\, v_{n,m-1} + (1-p)\, w_{n,m-1}, \qquad \forall m \geq 1 \end{aligned} \tag{4.22}$$

By applying (4.22) to (4.21), one finds

$$\begin{aligned} v_{n,m+1} &= c_0 np v_{n,m} + w_{n,m} \\ &= c_0 np v_{n,m} + c_1^2 np\,(1-p)\, v_{n,m-1} + (1-p)\, w_{n,m-1} \\ &= c_0 np v_{n,m} + c_1^2 np\,(1-p)\, v_{n,m-1} + c_1^2 np\,(1-p)^2\, v_{n,m-2} \\ &\quad + (1-p)^2\, w_{n,m-2} \\ &= \ldots = c_0 np v_{n,m} + c_1^2 np \sum_{k=0}^{m-1} (1-p)^{m-k}\, v_{n,k}, \qquad \forall m \geq 1 \end{aligned} \tag{4.23}$$

So the moment–generating function of the measure $D\,(n, p; c_0, c_1, 1)$ is

$$\begin{aligned} V\,(t) &= 1 + c_0 npt + \sum_{m=1}^{\infty} t^{m+1} \Big(c_0 np v_{n,m} + c_1^2 np \sum_{k=0}^{m-1} (1-p)^{m-k}\, v_{n,k} \Big) \\ &= 1 + c_0 npt \Big(1 + \sum_{m=1}^{\infty} t^m v_{n,m} \Big) + c_1^2 np \sum_{m=1}^{\infty} t^{m+1} \sum_{k=0}^{m-1} (1-p)^{m-k}\, v_{n,k} \\ &= 1 + c_0 nptV\,(t) + c_1^2 npt^2 \sum_{m=1}^{\infty} t^{m-1} \sum_{k=0}^{m-1} (1-p)^{m-k}\, v_{n,k} \\ &= 1 + c_0 nptV\,(t) + c_1^2 np(1-p) t^2 \sum_{k=0}^{\infty} t^k v_{n,k} \sum_{j=k}^{\infty} t^{j-k}\, (1-p)^{j-k} \\ &= 1 + c_0 nptV\,(t) + \frac{c_1^2 np\,(1-p)\, t^2}{1 - (1-p)\, t} V\,(t) \end{aligned} \tag{4.24}$$

This gives us

$$\begin{aligned} V\,(t) =& \frac{1}{1 - c_0 npt - \frac{c_1^2 np(1-p)t^2}{1-(1-p)t}} \\ =& \frac{1 - (1-p)\, t}{1 - (c_0 np + 1 - p)\, t - (c_1^2 - c_0)\, np\,(1-p)\, t^2} \end{aligned} \tag{4.25}$$

and therefore, the corresponding Cauchy transform is

$$G\,(t) = \frac{1}{t} V \left(\frac{1}{t} \right) = \frac{t - (1-p)}{t^2 - (c_0 np + 1 - p)\, t - (c_1^2 - c_0)\, np\,(1-p)} \tag{4.26}$$

The definition of $E\,(x_0, x_1, 1; n, p)$ given in (4.3) tells us that

$$\left(c_1^2 - c_0 \right) np\,(1-p) = \frac{1}{4} \Big(\big(E\,(c_0, c_1, 1; n, p) \big)^2 - (c_0 np + 1 - p)^2 \Big)$$

so one can rewrite the denominator term of the expression in the right hand side of (4.26) as follows:

$$\begin{aligned}
&t^2-(c_0np+1-p)\,t-\left(c_1^2-c_0\right)np\,(1-p)\\
=&t^2-(c_0np+1-p)\,t-\frac{1}{4}\Big(\big(E\,(c_0,c_1,1;n,p)\,\big)^2-(c_0np+1-p)^2\Big)\\
=&\Big(t-\frac{1}{2}\big(c_0np+1-p-E\,(c_0,c_1,1;n,p)\,\big)\Big)\cdot\\
&\cdot\Big(t-\frac{1}{2}\,(c_0np+1-p+E(c_0,c_1,1;n,p))\Big) \qquad (4.27)
\end{aligned}$$

By introducing

$$c:=1-p,\quad c_\pm:=\frac{1}{2}\big(c_0np+1-p\pm E\,(c_0,c_1,1;n,p)\,\big) \qquad (4.28)$$

one finds

$$\begin{aligned}
c_+-c&=\frac{1}{2}\big(E\,(c_0,c_1,1;n,p)+c_0np+p-1\big)\\
c-c_-&=\frac{1}{2}\big(E\,(c_0,c_1,1;n,p)-(c_0np+p-1)\big)\\
c_+-c_-&=E\,(c_0,c_1,1;n,p)>0 \qquad (4.29)
\end{aligned}$$

where, the last inequality is obtained thanks to the fact $c_1\neq 0$ and Lemma 4.2.

By using (4.27), (4.28) and (4.29), the right hand side of (4.26) is in fact

$$\frac{t-c}{(t-c_+)(t-c_-)}$$

and so (4.26) becomes to

$$G\,(t)=\frac{c_+-c}{c_+-c_-}\cdot\frac{1}{t-c_+}+\frac{c-c_-}{c_+-c_-}\cdot\frac{1}{t-c_-} \qquad (4.30)$$

Consequently, the corresponding probability measure $D\,(n,p;c_0,c_1,1)$, i.e., the vacuum distribution of $c_1\big(B_n^{(-1)}+B_n^{(+1)}\big)+c_0B_n^{(0)}+B_n^{(2)}$, equals to

$$\frac{c_+-c}{c_+-c_-}\delta_{c_+}+\frac{c-c_-}{c_+-c_-}\delta_{c_-}$$

which is, thanks to (4.28) and (4.29),

$$\begin{aligned}
&\frac{E\,(c_0,c_1,1;n,p)+c_0np-1+p}{2E\,(c_0,c_1,1;n,p)}\delta_{\frac{1}{2}(c_0np+1-p+E(c_0,c_1,1;n,p))}\\
+&\frac{E\,(c_0,c_1,1;n,p)-(c_0np-1+p)}{2E\,(c_0,c_1,1;n,p)}\delta_{\frac{1}{2}(c_0np+1-p-E(c_0,c_1,1;n,p))} \qquad (4.31)
\end{aligned}$$

i.e., the measure $D\,(n,p;c_0,c_1,c_2)$ given in (4.9) with $c_1\neq 0$ and $c_2=1$.

The 3^{rd} case: $c_1\neq 0$ and $c_2\neq 0$. In this case, as has been proved above, the vacuum distribution of $c_1'\left(B_n^{(-1)}+B_n^{(+1)}\right)+c_0'B_n^{(0)}+B_n^{(2)}$ (recall that c_0' and c_1' are those introduced in (4.6)) is nothing more than replacing the c_0 and c_1

in the measure given in (4.31) with c_0' and c_1'. Therefore, the random variable $c_1\left(B_n^{(-1)}+B_n^{(+1)}\right)+c_0B_n^{(0)}+c_2B_n^{(2)}$, which is in fact

$$c_2\left(c_1'\left(B_n^{(-1)}+B_n^{(+1)}\right)+c_0'B_n^{(0)}+B_n^{(2)}\right)$$

has the vacuum distribution given in (4.9), i.e.,

$$\begin{aligned}&\frac{E\left(c_0',c_1',1;n,p\right)+c_0'np-1+p}{2E\left(c_0',c_1',1;n,p\right)}\delta_{\frac{c_2}{2}\left(c_0'np+1-p+E(c_0',c_1',1;n,p)\right)}\\ &+\frac{E\left(c_0',c_1',1;n,p\right)-\left(c_0'np-1+p\right)}{2E\left(c_0',c_1',1;n,p\right)}\delta_{\frac{c_2}{2}\left(c_0'np+1-p-E(c_0',c_1',1;n,p)\right)}\end{aligned} \tag{4.32}$$

The proof is completed. □

Corollary 4.4. *For any $n\in\mathbb{N}^*$ and $\{c_0,c_1\}\in\mathbb{R}$, we introduce $F_0:=0$ and*

$$F_m:=\textit{the }(m-1)-\textit{th vacuum moment of } c_1\left(B_n^{(-1)}+B_n^{(+1)}\right)+c_0B_n^{(0)}$$

for any $m\in\mathbb{N}^*$. In case of $p_n=p$ in the definition of $B_n^{(\varepsilon)}$'s, $\{F_m\}_{m=0}^{\infty}$ is the ***generalized Fibonacci sequence*** *with the parameter $\left(c_0np,c_1^2np\left(1-p\right)\right)$ (see* [1], [3] *and the references within).*

Proof. The definition of F_m's and (4.16) tell us

$$F_0=0,\quad F_1=1,\quad F_{m+2}=c_0npF_{m+1}+c_1^2np\left(1-p\right)F_m$$

i.e., by the definition, $\{F_m\}_{m=0}^{\infty}$ is the generalized Fibonacci sequence with the parameter $\left(c_0np,c_1^2np\left(1-p\right)\right)$. □

Remark 4.5. The distribution $D\left(n,p;c_0,c_1,c_2\right)$ with $c_1\neq 0$ can be rewritten in a little more compact form. In fact

- the measure given in (4.20), i.e., $D\left(n,p;c_0,c_1,0\right)$ with $c_1\neq 0$, equals to

$$\sum_{\theta\in\{0,1\}}\frac{E\left(c_0,c_1,0;n,p\right)+(-1)^{\theta}c_0np}{2E\left(c_0,c_1,0;n,p\right)}\cdot\delta_{\frac{1}{2}\left(c_0np+(-1)^{\theta}E(c_0,c_1,0;n,p)\right)} \tag{4.33}$$

- the measure given in (4.32), i.e., $D\left(n,p;c_0,c_1,c_2\right)$ with $c_1\neq 0\neq c_2$, equals to

$$\sum_{\theta\in\{0,1\}}\frac{E\left(c_0',c_1',1;n,p\right)+(-1)^{\theta}\left(c_0'np-1+p\right)}{2E\left(c_0',c_1',1;n,p\right)}\cdot\delta_{\frac{c_2}{2}\left(c_0'np+1-p+(-1)^{\theta}E(c_0',c_1',1;n,p)\right)} \tag{4.34}$$

Theorem 4.6. *If $\lim_{n\to\infty}np_n=\lambda$, the following affirmations hold:*

1) For any $m\in\mathbb{N}$ and $\varepsilon\in\{-1,0,1,2\}^m$, as $n\longrightarrow\infty$, the limit of

$$\left\langle\Psi,B_n^{(\varepsilon(1))}\ldots B_n^{(\varepsilon(m))}\Psi\right\rangle \tag{4.35}$$

(i.e. mixed–moment) exists and equals to the expression (1.9).

2) For any $t \in \mathbb{R}$ and $\{c_0, c_1, c_2\} \subset \mathbb{R}$, the limit (1.8) equals to (1.11), i.e.

$$\lim_{n\to\infty} \left\langle \Psi, \exp\left(it\left(c_1\left(B_n^{(-1)} + B_n^{(+1)}\right) + c_0 B_n^{(0)} + c_2 B_n^{(2)}\right)\right)\Psi\right\rangle$$
$$= \left\langle \Phi, \exp\left(it\left(c_1\sqrt{\lambda}\left(b + b^+\right) + c_0\lambda P_\Phi + c_2\left(\mathbf{1} - P_\Phi\right)\right)\right)\Phi\right\rangle \tag{4.36}$$

In other words, the weak limit of $D(n, p_n; c_0, c_1, c_2)$ equals to $D(\lambda; c_0, c_1, c_2)$:=the vacuum distribution of $c_1\sqrt{\lambda}(b + b^+) + c_0\lambda P_\Phi + c_2(\mathbf{1} - P_\Phi)$. Moreover, for any $c_0 \in \mathbb{R}$, by using the notations introduced in (4.8),

- *in case $c_1 = 0$, the distribution $D(\lambda; c_0, 0, c_2)$ is $\delta_{c_0\lambda}$;*
- *in case $c_1 \neq 0$ and $c_2 = 0$, the distribution $D(\lambda; c_0, c_1, 0)$ is*

$$\sum_{\theta\in\{0,1\}} \frac{E(\lambda; c_0, c_1, 0) + (-1)^\theta c_0\lambda}{2E(\lambda; c_0, c_1, 0)} \delta_{\frac{1}{2}(c_0\lambda + (-1)^\theta E(\lambda; c_0, c_1, 0))} \tag{4.37}$$

- *in case $c_1 \neq 0$ and $c_2 \neq 0$, by using c_0' and c_1' given in (4.6), the distribution $D(\lambda; c_0, c_1, c_2)$*

$$\sum_{\theta\in\{0,1\}} \frac{E(\lambda; c_0', c_1', 1) + (-1)^\theta (c_0\lambda - 1)}{2E(\lambda; c_0', c_1', 1)} \delta_{\frac{c_2}{2}\left(c_0\lambda + 1 + (-1)^\theta E(\lambda; c_0', c_1', 1)\right)} \tag{4.38}$$

Proof. We prove firstly the affirmation 1) by the induction argument.

In the case of $m = 1$, (2.17) and (3.15) give us

$$\left\langle \Psi, B_n^{(\varepsilon)}\Psi\right\rangle = \begin{cases} np_n, & \text{if } \varepsilon = 0 \\ 0, & \text{otherwise}\end{cases} \longrightarrow \begin{cases} \lambda, & \text{if } \varepsilon = 0 \\ 0, & \text{otherwise}\end{cases}$$
$$= \lambda\left\langle \Phi, b^{(\varepsilon)}\Phi\right\rangle = \left\langle \Phi, b^{(\varepsilon)}\Phi\right\rangle \lambda^{1-|\varepsilon|/2}$$

In the case of $m = 2$, (2.17) and (3.15) guarantee that

$$\left\langle \Psi, B_n^{(\varepsilon(1))} B_n^{(\varepsilon(2))}\Psi\right\rangle$$
$$= \begin{cases} np_n(1 - p_n), & \text{if } \varepsilon(1) = -1 \text{ and } \varepsilon(2) = 1 \\ 0, & \text{otherwise}\end{cases}$$
$$\longrightarrow \begin{cases} \lambda, & \text{if } \varepsilon(1) = -1 \text{ and } \varepsilon(2) = 1 \\ 0, & \text{otherwise}\end{cases}$$
$$= \lambda\left\langle \Phi, b^{(\varepsilon(1))} b^{(\varepsilon(2))}\Phi\right\rangle = \left\langle \Phi, b^{(\varepsilon(1))} b^{(\varepsilon(2))}\Phi\right\rangle \lambda^{\sum_{k=1}^{2}(1-|\varepsilon(k)|/2)}$$

Suppose that the thesis is proved up to m, let's show

$$\left\langle \Psi, B_n^{(\varepsilon(1))} \dots B_n^{(\varepsilon(m+1))}\Psi\right\rangle$$
$$\longrightarrow \left\langle \Phi, b^{(\varepsilon(1))} \dots b^{(\varepsilon(m+1))}\Phi\right\rangle \lambda^{\sum_{k=1}^{m+1}(1-|\varepsilon(k)|/2)} \tag{4.39}$$

If $\varepsilon(m+1) \in \{-1, 2\}$, (4.39) is trivial since (2.17) and (3.15) tell us that both sides of (4.39) are zero.

If $\varepsilon(m+1) = 0$, by applying (2.17), (3.15), the fact $np_n \longrightarrow \lambda$ and the assumption of induction, one gets (4.39) as follows:

$$\left\langle\Psi,B_n^{(\varepsilon(1))}\dots B_n^{(\varepsilon(m+1))}\Psi\right\rangle=\left\langle\Psi,B_n^{(\varepsilon(1))}\dots B_n^{(\varepsilon(m))}B_n^{(0)}\Psi\right\rangle$$
$$=np_n\left\langle\Psi,B_n^{(\varepsilon(1))}\dots B_n^{(\varepsilon(m))}\Psi\right\rangle\longrightarrow\lambda\left\langle\Phi,b^{(\varepsilon(1))}\dots b^{(\varepsilon(m))}\Phi\right\rangle\lambda^{\sum_{k=1}^m(1-|\varepsilon(k)|/2)}$$
$$=\left\langle\Phi,b^{(\varepsilon(1))}\dots b^{(\varepsilon(m))}b^{(0)}\Phi\right\rangle\lambda^{1+\sum_{k=1}^m(1-|\varepsilon(k)|/2)}$$
$$=\left\langle\Phi,b^{(\varepsilon(1))}\dots b^{(\varepsilon(m))}b^{(\varepsilon(m+1))}\Phi\right\rangle\lambda^{\sum_{k=1}^{m+1}(1-|\varepsilon(k)|/2)}$$

For proving (4.39) in the case of $\varepsilon(m+1)=1$, one needs to see all possible values of $\varepsilon(m)$.

If $\varepsilon(m+1)=1$ and $\varepsilon(m)\in\{0,1\}$, (2.17) and (3.15) guarantee that

$$B_n^{(\varepsilon(m))}B_n^{(\varepsilon(m+1))}\Psi=B_n^{(\varepsilon(m))}B_n^{(+1)}\Psi=0=b^{(\varepsilon(m))}b^{(+1)}\Phi=b^{(\varepsilon(m))}b^{(\varepsilon(m+1))}\Phi$$

and so both sides of (4.39) are zero.

If $\varepsilon(m+1)=1$ and $\varepsilon(m)=2$, it follows from (2.17), (3.15), the fact $np_n\longrightarrow\lambda$ (so $p_n\longrightarrow 0$) and the assumption of induction that

$$\left\langle\Psi,B_n^{(\varepsilon(1))}\dots B_n^{(\varepsilon(m+1))}\Psi\right\rangle=\left\langle\Psi,B_n^{(\varepsilon(1))}\dots B_n^{(\varepsilon(m-1))}B_n^{(2)}B_n^{(+1)}\Psi\right\rangle$$
$$=(1-p_n)\left\langle\Psi,B_n^{(\varepsilon(1))}\dots B_n^{(\varepsilon(m-1))}B_n^{(+1)}\Psi\right\rangle$$
$$\longrightarrow\left\langle\Phi,b^{(\varepsilon(1))}\dots b^{(\varepsilon(m-1))}b^{(1)}\Phi\right\rangle\lambda^{(1-|1|/2)+\sum_{k=1}^{m-1}(1-|\varepsilon(k)|/2)}$$
$$=\left\langle\Phi,b^{(\varepsilon(1))}\dots b^{(\varepsilon(m-1))}b^{(2)}b^{(1)}\Phi\right\rangle\lambda^{(1-|1|/2)+(1-|2|/2)+\sum_{k=1}^{m-1}(1-|\varepsilon(k)|/2)}$$
$$=\left\langle\Phi,b^{(\varepsilon(1))}\dots b^{(\varepsilon(m-1))}b^{(\varepsilon(m))}b^{(\varepsilon(m+1))}\Phi\right\rangle\lambda^{\sum_{k=1}^{m+1}(1-|\varepsilon(k)|/2)}$$

If $\varepsilon(m+1)=1$ and $\varepsilon(m)=-1$, it follows from (2.17), (3.15), the fact $np_n\longrightarrow\lambda$ and the assumption of induction that

$$\left\langle\Psi,B_n^{(\varepsilon(1))}\dots B_n^{(\varepsilon(m+1))}\Psi\right\rangle=\left\langle\Psi,B_n^{(\varepsilon(1))}\dots B_n^{(\varepsilon(m-1))}B_n^{(-1)}B_n^{(+1)}\Psi\right\rangle$$
$$=np_n(1-p_n)\left\langle\Psi,B_n^{(\varepsilon(1))}\dots B_n^{(\varepsilon(m-1))}\Psi\right\rangle$$
$$\longrightarrow\lambda\left\langle\Phi,b^{(\varepsilon(1))}\dots b^{(\varepsilon(m-1))}\Phi\right\rangle\lambda^{\sum_{k=1}^{m-1}(1-|\varepsilon(k)|/2)}$$
$$=\left\langle\Phi,b^{(\varepsilon(1))}\dots b^{(\varepsilon(m-1))}b^{(-1)}b^{(+1)}\Phi\right\rangle\lambda^{(1-|1|/2)+(1-|-1|/2)+\sum_{k=1}^{m-1}(1-|\varepsilon(k)|/2)}$$
$$=\left\langle\Phi,b^{(\varepsilon(1))}\dots b^{(\varepsilon(m-1))}b^{(\varepsilon(m))}b^{(\varepsilon(m+1))}\Phi\right\rangle\lambda^{\sum_{k=1}^{m+1}(1-|\varepsilon(k)|/2)}$$

Summing up, the affirmation 1) is proved.

Now we turn to prove the affirmation 2).

First of all, the affirmation 1) and the uniform boundedness of the family $\left\{B_n^{(\varepsilon)}:n\in\mathbb{N}^*,\,\varepsilon\in\{-1,0,1,2\}\right\}$ guarantee the formula (4.36).

Secondly, Theorem 4.3 permits us to complete the proof as follows:

• In the case of $c_1=0$, the vacuum distribution of $c_0B_n^{(0)}+c_2B_n^{(2)}$ is $\delta_{c_0np_n}$, which goes to $\delta_{c_0\lambda}$ in the weak convergence since $np_n\longrightarrow\lambda$.

• In the case of $c_1 \neq 0$ and $c_2 = 0$, by using notations introduced in Remark 4.5, the vacuum distribution of $c_1\big(B_n^{(-1)} + B_n^{(+1)}\big) + c_0 B_n^{(0)}$ is the probability measure given in (4.33), which goes to $D(\lambda; c_0, c_1, 0)$ given in (4.37) in the weak convergence, since $np_n \longrightarrow \lambda$ and since $E(c_0, c_1, 0; n, p_n) \longrightarrow E(\lambda; c_0, c_1, 0)$ as said in (4.8).

• In the case of $c_1 \neq 0 \neq c_2$, by using notations introduced in Remark 4.5, the vacuum distribution of $c_1\left(B_n^{(-1)} + B_n^{(+1)}\right) + c_0 B_n^{(0)} + c_2 B_n^{(2)}$ is the probability measure given in (4.34), which goes to $D(\lambda; c_0, c_1, c_2)$ given in (4.38) in the weak convergence, since $np_n \longrightarrow \lambda$ and since $E(c_0', c_1', 1; n, p_n) \longrightarrow E(\lambda; c_0', c_1', 1)$ as said in (4.8).

Summing up, the proof is completed. □

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