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DOMAIN OF EXOTIC LAPLACIAN CONSTRUCTED BY WIENER INTEGRALS OF EXPONENTIAL WHITE NOISE DISTRIBUTIONS

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Abstract. In this paper we introduce a new domain of an exotic Laplacian consisting of some white noise distribution-valued Wiener integrals based on exponential distributions in a Fock space, and give a construction of a stochastic process as an infinite dimensional Brownian motion generated by the exotic Laplacians. The Brownian motion generated by the Gross Laplacian is extended to the stochastic process generated by the Lévy Laplacian on the domain. Moreover we give a relationship between semigroups generated by the exotic Laplacians and the Lévy Laplacian on the new domain.

1. Introduction

An infinite dimensional Laplacian was introduced by Accardi and Smolyanov [7] in terms of a higher order Cesàro mean and called an exotic Laplacian. In Refs. [1] and [16], it was proved that the Lévy Laplacian can be considered as the Gross Laplacian on a space of white noise test functionals based on Cesàro Hilbert space. In Ref. [3], it was proved that all exotic Laplacians can also be considered as Gross Laplacians on the test functionals and generate infinite dimensional Brownian motions. Any exotic Laplacian is the Lévy Laplacian based on higher derivative of white noise. This fact is obtained in Ref. [5] and therefore the exotic Laplacians of order $2p+1$ ($p \in \mathbb{N}$) are the generators of the innite dimensional Brownian motion corresponding to the p-th distributional derivative of the standard Brownian motion. In Ref. [6] the authors proved that the white noise (process) generated by the exotic Laplacian of order $2a$ ($a > 0$) is the a-th distributional derivative of the white noise (process) and that the Lévy Laplacian corresponds to the derivative of order $1/2$.

The main purpose of this paper is to introduce a new domain of the exotic Laplacian consisting of some white noise distribution-valued Wiener integrals based on exponential distributions in a Fock space, and construct a stochastic process as an infinite dimensional Brownian motion generated by the exotic Laplacian. In Refs. [3] and [6], the authors constructed the Brownian motion in the space of white noise distributions which is not in the Cesàro Hilbert space. In this paper, we give

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more clearly constructions of the Brownian motion in the Cesàro Hilbert space introducing a stochastic Fock space as white noise distribution-valued Wiener integrals in the Fock space.

In Sections 2 and 3 we recall some basic notions and results in the white noise theory as a language that prepares the ground for the applications in the following sections. In Section 4 we discuss the similarity between the exotic Laplacian and the Gross Laplacian. In Sections 5 and 6 we prepare an infinite dimensional Brownian motion generated by the Gross Laplacian and the second quantization of the adjoint of higher power of differential operator. The second quantization gives a topological isomorphism on the space of white noise distributions. This also gives a relationship between the exotic Laplacians and the Lévy Laplacian in terms of the higher order derivatives of white noise. In Section 7 we introduce the Fock space valued Wiener integrals and exponential white noise distributions. Based on the exponential distributions we construct a Hilbert space $\mathbb{D}_{c,2a+1}$ with $a \geq 0$ in which the exotic Laplacian becomes a densely defined linear operator. In Section 8 we consider a densely defined self-adjoint operator in $L^2(\mathbb{R})$ and introduce p-norms $|||\cdot|||_{c,2a+1,p}$ in $\mathbb{D}_{c,2a+1}$, the completions $\mathbb{D}_{c,2a+1,p}$ with respect to the p-norms, and the projective limit space $\mathbb{D}_{c,2a+1,\infty}$ of spaces $\mathbb{D}_{c,2a+1,p}$ for $p \geq 0$. We also introduce an infinite dimensional stochastic process $\{\mathbb{B}_a(t);\ t \geq 0\}$ valued in the Cesàro Hilbert space as a-th derivative of white noise process for any $a \geq 0$. Finally we obtain a stochastic process generated by the exotic Laplacian and also give a relationship between the stochastic process and the semigroup generated by the Lévy Laplacian with respect to the higher order derivatives of white noise.

2. Standard Triples

Throughout this paper, the complexification $\mathfrak{X}_{\mathbb{R}} + i\mathfrak{X}_{\mathbb{R}}$ of a real vector space $\mathfrak{X}_{\mathbb{R}}$ is denoted by $\mathfrak{X}_{\mathbb{C}}$. If there is no confusion, then for notational convenience, we use the symbol $\mathfrak{X}$ for the complexification $\mathfrak{X}_{\mathbb{C}}$.

Let $\boldsymbol{e} \equiv \{e_k\}_{k=1}^{\infty}$ be a complete orthonormal basis of a separable real Hilbert space $H_{\mathbb{R}}$ equipped with the inner product $\langle\cdot,\cdot\rangle$ and let $\boldsymbol{\lambda} \equiv \{\lambda_k\}_{k=1}^{\infty}$ be an increasing sequence of positive real numbers such that

$$1 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots, \quad \delta := \sum_{k=1}^{\infty} \lambda_k^{-2} < \infty. \tag{2.1}$$

Then we define a densely defined positive, selfadjoint operator A on $(H, \boldsymbol{e})$ by

$$A\xi := \sum_{k=1}^{\infty} \lambda_k \langle e_k, \xi\rangle e_k, \quad \xi \in H = H_{\mathbb{C}}.$$

From (H, A), by the standard construction [14, 17, 18, 25], we obtain the Gelfand triple:

$$E \subset H \subset E^*, \tag{2.2}$$

where H^* and H are identified and E^* is the strong dual space of E, which now depends on the triple $(H, \boldsymbol{e}, \boldsymbol{\lambda})$ since A depends on $\boldsymbol{e}$ and $\boldsymbol{\lambda}$. More precisely, for any $p \in \mathbb{R}$, we define the p-norm by

$$|\xi|_p = |A^p \xi|_0 \ , \qquad \xi \in H,$$

where $|\cdot|_0$ is the norm on H, and for $p \geq 0$, we define

$$E_p = \{\xi \in H \ : \ |\xi|_p < \infty\}$$

and E_{-p} to be the completion of H with respect to $|\cdot|_{-p}$. Then we obtain a chain of Hilbert spaces:

$$\cdots E_p \subset H \subset E_{-p} \cdots \ ,$$

and finally by taking

$$E := \mathrm{projlim}_{p\to\infty} E_p, \quad E^* \cong \mathrm{indlim}_{p\to\infty} E_{-p},$$

we obtain the Gelfand triple given as in (2.2). The canonical $\mathbb{C}$-bilinear forms on $E^* \times E$ is denoted $(\cdot, \cdot)$, which can be represented by $\langle \bar{\cdot}, \cdot \rangle$.

Let $\alpha \geq 1$ be given. Then the Cesàro semi–norm of order α of $x \in E^*$ is defined by

$$|x|^2_{\mathrm{c},\alpha} := \lim_{N\to\infty} \frac{1}{N^\alpha} \sum_{n=1}^{N} \overline{(x, e_n)} (x, e_n)$$

in the sense that, when the limit exists, the semi–norm is defined by the above limit. The Cesàro pre–scalar product of order α between $x, y \in E^*$ is defined, in the same sense, by

$$\langle x, y \rangle_{\mathrm{c},\alpha} := \lim_{N\to\infty} \frac{1}{N^\alpha} \sum_{n=1}^{N} \overline{(x, e_n)} (y, e_n). \tag{2.3}$$

Let $H^0_{\mathrm{c},\alpha}$ be a maximal linear subspace of E^* such that for any $x, y \in H^0_{\mathrm{c},\alpha}$ the limit given as in (2.3) exists and let $N^0_{\mathrm{c},\alpha} = \{x \in H^0_{\mathrm{c},\alpha} \ : \ \langle x, x \rangle_{\mathrm{c},\alpha} = 0\}$. Then $H^0_{\mathrm{c},\alpha}/N^0_{\mathrm{c},\alpha}$ becomes a (complex) pre-Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathrm{c},\alpha}$ defined as in (2.3). The completion of $H^0_{\mathrm{c},\alpha}/N^0_{\mathrm{c},\alpha}$ with respect to the norm $|\cdot|_{\mathrm{c},\alpha,0}$ induced from the inner product $\langle \cdot, \cdot \rangle_{\mathrm{c},\alpha}$ is denoted by $H_{\mathrm{c},\alpha}$. Then $H_{\mathrm{c},\alpha}$ becomes a Hilbert space. We now assume that

$$C_{\boldsymbol{\lambda},\alpha} = \sup_{n\geq 1} \left\{ \frac{\lambda_n^\delta}{n^\alpha} \right\} < \infty \quad \text{for some } \delta > 0 \tag{2.4}$$

(see [16] and also Theorem 2.7 in [3]). Then we can construct $H_{\mathrm{c},\alpha}$ to be an infinite dimensional separable Hilbert space (see [16]), which is called the Cesàro Hilbert space of order $\alpha > 0$.

Let $A_{\mathrm{c},\alpha}$ be a densely defined positive, selfadjoint operator in $H_{\mathrm{c},\alpha}$ satisfying the condition:

(A): $\inf \mathrm{Spec}(A_{\mathrm{c},\alpha}) > 1$ and $A^{-1}_{\mathrm{c},\alpha}$ is of Hilbert-Schmidt type.

Then there exist a sequence $\{\ell_{\mathrm{c},\alpha,k}\}_{k=1}^\infty$ satisfying that

$$1 < \ell_{\mathrm{c},\alpha,1} \leq \ell_{\mathrm{c},\alpha,2} \leq \ell_{\mathrm{c},\alpha,3} \leq \cdots \ , \qquad \|A^{-1}_{\mathrm{c},\alpha}\|^2_{\mathrm{HS}} = \sum_{k=1}^{\infty} \ell^{-2}_{\mathrm{c},\alpha,k} < \infty,$$

and an orthonormal basis $\{e_{\mathrm{c},\alpha,k}\}_{k=1}^{\infty}$ of $H_{\mathrm{c},\alpha}$ such that $A_{\mathrm{c},\alpha}e_{\mathrm{c},\alpha,k} = \ell_{\mathrm{c},\alpha,k}e_{\mathrm{c},\alpha,k}$. Then from $(H_{\mathrm{c},\alpha}, A_{\mathrm{c},\alpha})$, by the standard construction [14, 17, 18, 25], we obtain the Gelfand triple given as in (2.5) as following. For each $p \in \mathbb{R}$ we define

$$|\xi|_{\mathrm{c},\alpha,p}^2 = |A_{\mathrm{c},\alpha}^p \xi|_{\mathrm{c},\alpha,0}^2 = \sum_{k=1}^{\infty} \ell_{\mathrm{c},\alpha,k}^{2p} |\langle \xi, e_{\mathrm{c},\alpha,k}\rangle_{\mathrm{c},\alpha}|^2, \qquad \xi \in H_{\mathrm{c},\alpha}.$$

Now let $p \geq 0$. We put $\mathcal{N}_{\mathrm{c},\alpha,p} = \{\xi \in H_{\mathrm{c},\alpha} : |\xi|_{\mathrm{c},\alpha,p} < \infty\}$ and define $\mathcal{N}_{\mathrm{c},\alpha,-p}$ to be the completion of $H_{\mathrm{c},\alpha}$ with respect to $|\cdot|_{\mathrm{c},\alpha,-p}$. Thus we obtain a chain of Hilbert spaces $\{\mathcal{N}_p\,;\, p \in \mathbb{R}\}$ and consider their limit spaces:

$$\mathcal{N}_{\mathrm{c},\alpha} = \mathrm{projlim}_{p\to\infty}\mathcal{N}_{\mathrm{c},\alpha,p}, \qquad \mathcal{N}_{\mathrm{c},\alpha}^* \cong \mathrm{indlim}_{p\to\infty}\mathcal{N}_{\mathrm{c},\alpha,-p}$$

which are mutually dual spaces. Note that $\mathcal{N}_{\mathrm{c},\alpha}$ becomes a countably Hilbert nuclear space. Finally, by identifying $H_{\mathrm{c},\alpha}$ with its dual space, we obtain a complex Gelfand triple:

$$\mathcal{N}_{\mathrm{c},\alpha} \subset H_{\mathrm{c},\alpha} \subset \mathcal{N}_{\mathrm{c},\alpha}^*. \tag{2.5}$$

Throughout this paper, we always assume that for any $\alpha > 0$ there exists $\beta > 0$ such that

$$(2\pi\,[(k+1)/2])^{\alpha} \leq \ell_{\mathrm{c},\alpha,k}^{\beta}, \quad (2\pi[(k+1)/2])^{\alpha} \leq \lambda_k^{\beta} \tag{2.6}$$

for all $k \in \mathbb{N}$, which are useful to study relationships between E^* and $\mathcal{N}_{\mathrm{c},\alpha}$ (see Remark 6 in [5], and also Theorem 2.5 in [3]), where the sequence $\boldsymbol{\lambda} \equiv \{\lambda_k\}_{k=1}^{\infty}$ is given as in (2.1).

3. White Noise Functionals

Let $\mathfrak{X} \subset \mathfrak{H} \subset \mathfrak{X}^*$ be the Gelfand triple constructed from the pair $(\mathfrak{H}, \boldsymbol{A})$ by standard method used as in Section 2 via the chain of Hilbert spaces:

$$\cdots \mathfrak{X}_p \subset \mathfrak{H} \subset \mathfrak{X}_{-p} \cdots,$$

where $\mathfrak{H}$ is a separable complex Hilbert space and $\boldsymbol{A}$ is a densely defined, positive selfadjoint operator in $\mathfrak{H}$ such that $\mathfrak{X}$ becomes a countably Hilbert nuclear space. The norm on $\mathfrak{X}_p$ ($p \in \mathbb{R}$) is denoted by $|\cdot|_p$ again.

For each $p \in \mathbb{R}$, let $\Gamma(\mathfrak{X}_p)$ be the Fock space over the Hilbert space $\mathfrak{X}_p$, i.e.,

$$\Gamma(\mathfrak{X}_p) = \left\{ \phi = (f_n)_{n=0}^{\infty} \,:\, f_n \in \mathfrak{X}_p^{\widehat{\otimes} n},\ \|\phi\|_p^2 = \sum_{n=0}^{\infty} n!|f_n|_p^2 < \infty \right\}.$$

Then by identifying $\Gamma(\mathfrak{H})$ with its dual space, we have a chain of Fock spaces:

$$\cdots \subset \Gamma(\mathfrak{X}_p) \subset \Gamma(\mathfrak{X}_0) = \Gamma(\mathfrak{H}) \subset \Gamma(\mathfrak{X}_{-p}) \subset \cdots$$

for $p \geq 0$, and a new triple

$$(\mathfrak{X}) = \mathrm{projlim}_{p\to\infty}\Gamma(\mathfrak{X}_p) \subset \Gamma(\mathfrak{H}) \subset (\mathfrak{X})^* = \mathrm{indlim}_{p\to\infty}\Gamma(\mathfrak{X}_{-p}). \tag{3.1}$$

The canonical $\mathbb{C}$-bilinear form on $(\mathfrak{X})^* \times (\mathfrak{X})$, denoted $\langle\!\langle \cdot, \cdot \rangle\!\rangle$, has the form:

$$\langle\!\langle \Phi, \phi \rangle\!\rangle = \sum_{n=0}^{\infty} n!\,(F_n, f_n)\,, \qquad \Phi = (F_n) \in (\mathfrak{X})^*, \quad \phi = (f_n) \in (\mathfrak{X}), \tag{3.2}$$

where $(\cdot,\cdot)$ is the canonical $\mathbb{C}$-bilinear form on $\mathfrak{X}^* \times \mathfrak{X}$.

The *exponential vector* associated with $x \in \mathfrak{X}^*$ is defined by

$$\phi_\xi = \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \dots, \frac{\xi^{\otimes n}}{n!}, \dots\right). \tag{3.3}$$

In fact, for each $x \in \mathfrak{X}^*$, there exists $p \in \mathbb{R}$ such that $x \in \mathfrak{X}_p$ and so we obtain that

$$\|\phi_x\|_p^2 = \sum_{n=0}^{\infty} n! \left|\frac{\xi^{\otimes n}}{n!}\right|_p^2 = \sum_{n=0}^{\infty} \frac{|x|_p^{2n}}{n!} = e^{|x|_p^2},$$

which implies that $\phi_x \in \Gamma(\mathfrak{X}_p)$ if and only if $x \in \mathfrak{X}_p$ for $p \in \mathbb{R}$. Therefore, $\phi_\xi \in (\mathfrak{X})$ for any $\xi \in \mathfrak{X}$. It is well-known that the exponential vectors ϕ_ξ for $\xi \in \mathfrak{X}$ span a dense subspace of $(\mathfrak{X})$. Hence every element $\Phi \in (\mathfrak{X})^*$ is uniquely specified by its $S_{\mathfrak{X}}$*-transform* $S_{\mathfrak{X}}\Phi : \mathfrak{X} \to \mathbb{C}$ defined by

$$S_{\mathfrak{X}}\Phi(\xi) = \langle\langle \Phi, \phi_\xi \rangle\rangle, \qquad \xi \in \mathfrak{X}$$

A complex-valued function F on $\mathfrak{X}$ is called a U*-functional* if F is Gâteaux entire and there exist constants $C, K \geq 0$ and $p \geq 0$ such that

$$|F(\xi)| \leq C \exp\left(K|\xi|_p^2\right), \qquad \xi \in \mathfrak{X}.$$

Theorem 3.1 ([26, 6])**.** *A $\mathbb{C}$-valued function F on $\mathfrak{X}$ is the $S_{\mathfrak{X}}$-transform of an element in $(\mathfrak{X})^*$ if and only if F is a U-functional.*

Remark 3.2. Let $\mathfrak{X}_{\mathbb{R}}^*$ be a real locally convex space such that $\mathfrak{X}^* = \mathfrak{X}_{\mathbb{R}}^* + i\mathfrak{X}_{\mathbb{R}}^*$ (the complexification of $\mathfrak{X}_{\mathbb{R}}^*$). Then the Bochner–Minlos theorem implies the existence of a probability measure $\boldsymbol{\mu}$ on $\mathfrak{X}_{\mathbb{R}}^*$ such that

$$\int_{\mathfrak{X}_{\mathbb{R}}^*} e^{i\langle x, \xi\rangle} d\boldsymbol{\mu}(x) = e^{-\frac{1}{2}\langle \xi, \xi\rangle}, \qquad \xi \in \mathfrak{X}.$$

The Wiener-Itô-Segal isomorphism between $\Gamma(\mathfrak{H})$ and $L^2(\mathfrak{X}_{\mathbb{R}}^*, \boldsymbol{\mu})$ is the unitary isomorphism uniquely determined by the correspondence:

$$\phi_\xi = \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \dots, \frac{\xi^{\otimes n}}{n!}, \dots\right) \quad \longleftrightarrow \quad \phi_\xi(\cdot) = e^{\langle \cdot, \xi\rangle - \langle \xi, \xi\rangle/2}, \quad \xi \in \mathfrak{X}. \tag{3.4}$$

The standard triple obtained from (3.1) through the Wiener-Itô-Segal isomorphism is denoted also by

$$(\mathfrak{X}) \subset L^2(\mathfrak{X}_{\mathbb{R}}^*, \boldsymbol{\mu}) \subset (\mathfrak{X})^*,$$

which is referred to as the *Hida–Kubo–Takenaka space.* An element of $(\mathfrak{X})$ (resp. $(\mathfrak{X})^*$) is called a test (resp. generalized) white noise functional.

From the Gelfand triples given as in (2.2) and (2.5), we obtain the Gelfand triples for Fock spaces by above constructions which are denoted by

$$(E) \subset \Gamma(H) \subset (E)^*, \quad (\mathcal{N}_{\mathrm{c},\alpha}) \subset \Gamma(H_{\mathrm{c},\alpha}) \subset (\mathcal{N}_{\mathrm{c},\alpha})^*,$$

respectively.

4. Infinite Dimensional Laplacians

Let $\mathcal{L}(\mathfrak{Y},\mathfrak{Z})$ be the space of all continuous linear operators from a locally convex space $\mathfrak{Y}$ into another locally convex space $\mathfrak{Z}$. Let $\mathfrak{X}\subset\mathfrak{H}\subset\mathfrak{X}^*$ be the Gelfand triple which is considered in Section 3.

A function $F:\mathfrak{X}\to\mathbb{C}$ is said to be of class $C^2:=C^2(\mathfrak{X})$ if it is twice (continuously) Fréchet differentiable, i.e., there exist two continuous maps

$$\xi\longmapsto F'(\xi)\in\mathfrak{X}^*,\quad \xi\longmapsto F''(\xi)\in\mathcal{L}(\mathfrak{X},\mathfrak{X}^*),\qquad \xi\in\mathfrak{X}$$

such that

$$F(\xi+\eta)=F(\xi)+(F'(\xi),\eta)+\frac{1}{2}(F''(\xi)\eta,\eta)+\varepsilon(\eta)$$

for any $\eta\in\mathfrak{X}$, where the error terms satisfy

$$\lim_{t\to0}\frac{\varepsilon(t\eta)}{t}\to0,\qquad \eta\in\mathfrak{X}.$$

We denote $\widetilde{D}_\eta$ the Gateaux differentiation in the direction $\eta\in\mathfrak{X}$, i.e.,

$$\widetilde{D}_\eta F(\xi)=\lim_{t\to0}\frac{1}{t}\left[F(\xi+t\eta)-F(\xi)\right]=\left.\frac{d}{dt}F(\xi+t\eta)\right|_{t=0}.$$

Under general regularity conditions one has

$$\widetilde{D}_\eta F(\xi)=(F'(\xi),\eta)$$

The kernel theorem identifies $\mathcal{L}(\mathfrak{X},\mathfrak{X}^*)$ with $\mathfrak{X}\otimes\mathfrak{X}$ since $\mathfrak{X}$ is a countably Hilbert nuclear space. Occasionally, we will use indifferently the notations

$$(F''(\xi)\eta,\eta)=(F''(\xi),\eta\otimes\eta)=F''(\xi)(\eta,\eta)=\widetilde{D}_\eta^2F(\xi)$$

for $\xi,\eta\in\mathfrak{X}$.

For arbitrarily fixed $\alpha\geq0$ and let $\{e_{\mathfrak{H},k}\}_{k=1}^\infty\subset\mathfrak{X}$ be a fixed complete orthonormal basis of $\mathfrak{H}$. Let $\mathrm{Dom}(\boldsymbol{\Delta}_{\mathfrak{H},\mathrm{c},\alpha})$ denote the set of all $\Phi\in(\mathfrak{X})^*$ such that for each $\xi\in\mathfrak{X}$, the limit

$$\widetilde{\boldsymbol{\Delta}}_{\mathfrak{H},\mathrm{c},\alpha}S_{\mathfrak{X}}\Phi(\xi)=\lim_{N\to\infty}\frac{1}{N^\alpha}\sum_{k=1}^N((S_{\mathfrak{X}}\Phi)''(\xi),e_{\mathfrak{H},k}\otimes e_{\mathfrak{H},k})$$

exists and a functional $\widetilde{\boldsymbol{\Delta}}_{\mathfrak{H},\mathrm{c},\alpha}(S_{\mathfrak{X}}\Phi)$ is the $S_{\mathfrak{X}}$-transform of an element of $(\mathfrak{X})^*$. Then the *exotic Laplacian* $\boldsymbol{\Delta}_{\mathfrak{H},\mathrm{c},\alpha}$ on $\mathrm{Dom}(\boldsymbol{\Delta}_{\mathfrak{H},\mathrm{c},\alpha})$ is defined by

$$\boldsymbol{\Delta}_{\mathfrak{H},\mathrm{c},\alpha}\Phi=S_{\mathfrak{X}}^{-1}(\widetilde{\boldsymbol{\Delta}}_{\mathfrak{H},\mathrm{c},\alpha}S_{\mathfrak{X}}\Phi),\qquad \Phi\in\mathrm{Dom}(\boldsymbol{\Delta}_{\mathfrak{H},\mathrm{c},\alpha}).$$

The operators $\boldsymbol{\Delta}_{\mathfrak{H},\mathrm{c},1}$ and $\boldsymbol{\Delta}_{\mathfrak{H},\mathrm{c},0}$ are called the *Lévy Laplacian* and the *Gross Laplacian*, respectively, and are also written by $\boldsymbol{\Delta}_{\mathrm{L}}:=\boldsymbol{\Delta}_{\mathfrak{H},\mathrm{L}}$ and $\boldsymbol{\Delta}_{\mathrm{G}}:=\boldsymbol{\Delta}_{\mathfrak{H},\mathrm{G}}$, respectively.

The *symbol* of a continuous linear operator $\Xi\in\mathcal{L}((\mathfrak{X}),(\mathfrak{X})^*)$ is defined by

$$\widehat{\Xi}(\xi,\eta)=\langle\!\langle\Xi\phi_\xi,\phi_\eta\rangle\!\rangle,\qquad \xi,\eta\in\mathfrak{X}.$$

An operator $\Xi\in\mathcal{L}((\mathfrak{X}),(\mathfrak{X})^*)$ is uniquely specified by the symbol since $\{\phi_\xi:\xi\in\mathfrak{X}\}$ spans a dense subspace of $(\mathfrak{X})$. Moreover, we have an analytic characterization of symbols.

Theorem 4.1 ([24]). *A $\mathbb{C}$-valued function Θ on $\mathfrak{X}\times\mathfrak{X}$ is the symbol of an operator $\Xi \in \mathcal{L}((\mathfrak{X}),(\mathfrak{X})^*)$ if and only if*

(i) *Θ is Gâteaux entire,*
(ii) *there exist $C \geq 0$, $K \geq 0$ and $p \geq 0$ such that*

$$|\Theta(\xi,\eta)| \leq C \exp\, K\left(|\xi|_p^2 + |\eta|_p^2\right), \qquad \xi,\eta \in \mathfrak{X}.$$

Moreover, Θ is the symbol of an operator $\Xi \in \mathcal{L}((\mathfrak{X}),(\mathfrak{X}))$ if and only if Θ satisfies (i) and

(ii′) *for any $p \geq 0$ and $\epsilon > 0$, there exist constants $C \geq 0$ and $q \geq 0$ such that*

$$|\Theta(\xi,\eta)| \leq C \exp\epsilon\left(|\xi|_{p+q}^2 + |\eta|_{-p}^2\right), \qquad \xi,\eta \in \mathfrak{X}.$$

By applying Theorem 4.1 we can easily see that $\mathbf{\Delta}_{\mathrm{G}} \in \mathcal{L}((\mathfrak{X}),(\mathfrak{X}))$ (see [25, 18]). From (2.3) by applying the kernel theorem, we define $\tau_{\mathrm{c},\alpha} \in (\mathcal{N}_{\mathrm{c},\alpha} \otimes \mathcal{N}_{\mathrm{c},\alpha})^*$ by

$$(\tau_{\mathrm{c},\alpha}, \xi\otimes\eta) = (\eta,\xi)_{\mathrm{c},\alpha} := \lim_{N\to\infty} \frac{1}{N^\alpha}\sum_{n=1}^{N}(\eta,e_n)(\xi,e_n). \tag{4.1}$$

Theorem 4.2. *Any element $\phi \in (E)^* \cap (\mathcal{N}_{\mathrm{c},\alpha})$ is in $\mathrm{Dom}(\mathbf{\Delta}_{H,\mathrm{c},\alpha})$, where H is the Hilbert space considered as in Section 2. Moreover, if $\phi = (f_n)_{n=0}^\infty$, then we have*

$$\mathbf{\Delta}_{H,\mathrm{c},\alpha}\phi = \Big((n+2)(n+1)\tau_{\mathrm{c},\alpha}\widehat{\otimes}^2 f_{n+2}\Big)_{n=0}^{\infty}. \tag{4.2}$$

Proof. The proof is a simple modification of the proof of Theorem 4.2 in [16]. Let $\phi = (f_n) \in (\mathcal{N}_{\mathrm{c},\alpha})$. Then we can easily show that

$$(S_E\phi)''(\xi)(e_k,e_k) = \sum_{n=0}^{\infty}(n+2)(n+1)((e_k\otimes e_k)\widehat{\otimes}^2 f_{n+2}, \xi^{\otimes n}),$$

where $\widehat{\otimes}^2$ is the left 2-contraction (see [25]). which implies from (4.1) that

$$\begin{aligned}\widetilde{\mathbf{\Delta}}_{H,\mathrm{c},\alpha}S_E\phi(\xi) &= \lim_{N\to\infty}\frac{1}{N^\alpha}\sum_{k=1}^{N}\sum_{n=0}^{\infty}(n+2)(n+1)((e_k\otimes e_k)\widehat{\otimes}^2 f_{n+2},\xi^{\otimes n})\\ &= \sum_{n=0}^{\infty}(n+2)(n+1)(\tau_{\mathrm{c},\alpha}\widehat{\otimes}^2 f_{n+2},\xi^{\otimes n}).\end{aligned}$$

Therefore, by applying (3.2) we prove (4.2). □

From Theorem 4.2 we see that

$$\mathbf{\Delta}_{H,\mathrm{c},\alpha}\phi = \mathbf{\Delta}_{H_{\mathrm{c},\alpha},\mathrm{G}}\phi, \quad \phi \in (E)^*\cap(\mathcal{N}_{\mathrm{c},\alpha}).$$

5. Infinite Dimensional Stochastic Process Generated by Gross Laplacian

Let $\mathfrak{X} \subset \mathfrak{H} \subset \mathfrak{X}^*$ be the Gelfand triple which is considered in Section 3. Then there exist an increasing sequence $\{\ell_{\mathfrak{H},k}\}_{k=1}^\infty$ of positive real numbers with $\ell_{\mathfrak{H},1} > 1$,

$\sum_{k=1}^{\infty} \ell_{\mathfrak{H},k}^{-2} < \infty$, and an orthonormal basis $\{e_{\mathfrak{H},k}\}_{k=1}^{\infty} \subset \mathfrak{X}_{\mathbb{R}}$ of $\mathfrak{H}$ such that for each $p \in \mathbb{R}$, the norm $|\cdot|_p$ is constructed by

$$|\xi|_p^2 = \sum_{k=1}^{\infty} \ell_{\mathfrak{H},k}^{2p} |\langle e_{\mathfrak{H},k}, \xi\rangle|^2, \qquad \xi \in \mathfrak{H}.$$

Let $\{\{b_k(t)\}_{t\geq 0}\}_{k=1}^{\infty}$ be a sequence of independent (1-dimensional) Brownian motions on a probability space $(\Omega, \mathcal{F}, P)$. Then for each $t \geq 0$, we put

$$\boldsymbol{B}(t) = \sum_{k=1}^{\infty} b_k(t) e_{\mathfrak{H},k}. \tag{5.1}$$

In fact, for any $p \geq 0$, we obtain that

$$E\left[|\boldsymbol{B}(t)|_{-p}^2\right] = E\left[\left|\sum_{k=1}^{\infty} b_k(t) e_{\mathfrak{H},k}\right|_{-p}^2\right] = E\left[\sum_{k=1}^{\infty} \ell_{\mathfrak{H},k}^{-2p} b_k(t)^2\right] = \left(\sum_{k=1}^{\infty} \ell_{\mathfrak{H},k}^{-2p}\right) t,$$

which implies that for all $p > 1/2$ and for almost all $\omega \in \Omega$, $\boldsymbol{B}(t,\omega) \in \mathfrak{X}_{-p}$. Therefore, $\{\boldsymbol{B}(t)\}_{t\geq 0}$ is a $\mathfrak{X}^*$-valued stochastic process.

Let $y \in \mathfrak{X}^*$ be given. Then by applying Theorem 4.1, we see that there exists a unique operator $T_y \in \mathcal{L}((\mathfrak{X}), (\mathfrak{X}))$ such that

$$\widehat{T_y}(\xi, \eta) = e^{(y,\xi)+(\xi,\eta)}, \quad \xi, \eta \in \mathfrak{X}.$$

In fact, for any $\xi \in \mathfrak{X}$, we obtain that

$$\phi_\xi(x+y) = e^{(x+y,\xi) - \frac{1}{2}(\xi,\xi)} = e^{(y,\xi)} \phi_\xi(x),$$

see (3.4), and so for any $\xi, \eta \in \mathfrak{X}$ it holds that

$$\langle\!\langle \phi_\xi(\cdot + y), \phi_\eta \rangle\!\rangle = e^{(y,\xi)+(\xi,\eta)} = \widehat{T_y}(\xi, \eta) = \langle\!\langle T_y \phi_\xi, \phi_\eta \rangle\!\rangle,$$

which implies that for any $\phi \in (\mathfrak{X}) \subset L^2(\mathfrak{X}_{\mathbb{R}}^*, \boldsymbol{\mu})$, it holds that $T_y\phi(x) = \phi(x+y)$. Hence T_y is called the *translation operator*. Moreover, for the annihilation operator $a(y) \in \mathcal{L}((\mathfrak{X}), (\mathfrak{X}))$ defined by $a(y)\phi_\xi = (y,\xi)\phi_\xi$ for all $\xi \in \mathfrak{X}$, it holds that $T_y = e^{a(y)}$ (see [25]).

Theorem 5.1. *The stochastic process $\{\mathbf{B}(t)\}_{t\in\mathbb{R}}$ is generated by $\frac{1}{2}\boldsymbol{\Delta}_{\mathrm{G}} \in \mathcal{L}((\mathfrak{X}), (\mathfrak{X}))$, where $\boldsymbol{\Delta}_{\mathrm{G}}$ is the Gross Laplacian. More precisely, it holds that*

$$E\left[T_{\boldsymbol{B}(t)}\phi\right] = e^{\frac{t}{2}\boldsymbol{\Delta}_{\mathrm{G}}}\phi, \quad \phi \in (\mathfrak{X}). \tag{5.2}$$

Proof. For any $\xi \in \mathfrak{X}$, we obtain that

$$E\left[T_{\boldsymbol{B}(t)}\phi_\xi\right] = E\left[e^{(\boldsymbol{B}(t),\xi)}\right]\phi_\xi = e^{\frac{t}{2}(\xi,\xi)}\phi_\xi = e^{\frac{t}{2}\boldsymbol{\Delta}_{\mathrm{G}}}\phi_\xi,$$

and so by continuities of translation operator, expectation and the exponential of the Gross Laplacian, we see that (5.2) holds. □

For any $\phi \in (\mathfrak{X})$, from (5.2) we obtain that

$$E\left[T_{\boldsymbol{B}(s)} E\left[T_{\boldsymbol{B}(t)}\phi\right]\right] = e^{(s+t)\frac{1}{2}\boldsymbol{\Delta}_{\mathrm{G}}}\phi = E\left[T_{\boldsymbol{B}(s+t)}\phi\right]$$

for all $s, t \geq 0$.

6. Higher Order Derivatives of WN and Laplacians

In the sequel, $\mathbb{N}_0$ will denote the set of all nonnegative integers, i.e., $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\alpha \in \mathbb{N}_0$ be given. The differential operator of order α is denoted by ∂^α, i.e. $\partial^\alpha \xi = \xi^{(\alpha)}$ for any α-th differentiable function ξ, and the differential operator of order 1 is denoted by ∂.

In the following, we take $H = \{1\}^\perp$ (the orthogonal complement of constant functions) to be the closed subspace of $L^2([0,1])$ generated by the orthonormal basis $\{e_n\}_{n=1}^\infty$ given by

$$\begin{aligned} e_{2k}(t) &= \sqrt{2}\sin(2k\pi t) \\ e_{2k-1}(t) &= \sqrt{2}\cos(2k\pi t), \;\; k = 1, 2, \ldots. \end{aligned} \tag{6.1}$$

Then we have the following Lemma.

Lemma 6.1 (Lemma 3.1 in [4]). *For any $\alpha \in \mathbb{N}_0$ and $k \in \mathbb{N}$, we have*

$$e_{2k}^{(\alpha)}(t) = \begin{cases} (-1)^{\alpha/2}(2k\pi)^\alpha e_{2k}(t), & \text{if } \alpha \text{ is even,} \\ (-1)^{(\alpha+1)/2}(2k\pi)^\alpha e_{2k-1}(t), & \text{if } \alpha \text{ is odd} \end{cases}$$

$$e_{2k-1}^{(\alpha)}(t) = \begin{cases} (-1)^{\alpha/2}(2k\pi)^\alpha e_{2k-1}(t), & \text{if } \alpha \text{ is even,} \\ (-1)^{(\alpha+1)/2}(2k\pi)^\alpha e_{2k}(t), & \text{if } \alpha \text{ is odd.} \end{cases}$$

From now on, we consider the Gelfand triple $E \subset H \subset E^*$ constructed in Section 2.

Proposition 6.2 (Lemma 3.1 in [4]). *Let $\alpha \in \mathbb{N}_0$ be given. Then the differential operator ∂^α of order α is a topological isomorphism from E onto itself.*

Proof. Suppose that α is even. The linearity of ∂^α is obvious. Let $p \geq 0$ be given. For given $\alpha \geq 0$, from (2.6) there exists $\beta \geq 0$ such that $(2\pi[\nu/2])^\alpha \leq \lambda_\nu^\beta$ for all $\nu \in \mathbb{N}$. Therefore, for any $\xi \in E$, we obtain that

$$\begin{aligned} \left|\xi^{(\alpha)}\right|_p^2 &= \left|\sum_{\nu=1}^\infty \langle e_\nu, \xi\rangle e_\nu^{(\alpha)}\right|_p^2 = \left|\sum_{\nu=1}^\infty \langle e_\nu, \xi\rangle (2\pi[\nu/2])^\alpha e_\nu\right|_p^2 \\ &= \sum_{\nu=1}^\infty \lambda_\nu^{2p}(2\pi[\nu/2])^{2\alpha}|\langle e_\nu, \xi\rangle|^2 \leq \sum_{\nu=1}^\infty \lambda_\nu^{2(p+\beta)}|\langle e_\nu, \xi\rangle|^2 \\ &= |\xi|_{p+\beta}^2, \end{aligned}$$

which implies that $\partial^\alpha \in \mathcal{L}(E, E)$. The bijectivity of ∂^α is clear from Lemma 6.1. On the other hand, for any $p \geq 0$ and $\xi \in E$, we obtain that

$$\begin{aligned} \left|(\partial^\alpha)^{-1}\xi\right|_p^2 &= \left|\sum_{\nu=1}^\infty \frac{1}{(2\pi[\nu/2])^\alpha}\langle e_\nu, \xi\rangle e_\nu\right|_p^2 = \left(\sup_v \frac{1}{(2\pi[\nu/2])^\alpha}\right)\sum_{\nu=1}^\infty \lambda_v^{2p}|\langle e_\nu, \xi\rangle|^2 \\ &\leq \left(\sup_v \frac{1}{(2\pi[\nu/2])^\alpha}\right)|\xi|_p^2, \end{aligned}$$

which implies that $(\partial^\alpha)^{-1}$ is continuous. Hence ∂^α is a topological isomorphism from E onto itself. The case of odd α is similar. $\square$

Let $K \in \mathcal{L}(E, E)$ be given. The second quantization $\Gamma(K) \in \mathcal{L}((E), (E))$ of K defined by

$$\Gamma(K)\phi = (K^{\otimes n} f_n), \quad \phi = (f_n) \in (E). \tag{6.2}$$

Then for any $\Phi = (F_n) \in (E)^*$ and $\phi = (f_n) \in (E)$, we obtain that

$$\langle\!\langle \Gamma(K)^* \Phi, \phi \rangle\!\rangle = \langle\!\langle \Phi, \Gamma(K)\phi \rangle\!\rangle = \sum_{n=0}^{\infty} n!(F_n, K^{\otimes n} f_n) = \langle\!\langle \Gamma(K^*)\Phi, \phi \rangle\!\rangle ,$$

which implies that $\Gamma(K)^* = \Gamma(K^*) \in \mathcal{L}((E), (E))$ and $S_E(\Gamma(K)^*\Phi)(\xi) = S_E\Phi(K\xi)$ for all $\xi \in E$, from which we define

$$\widetilde{K} := S_E \Gamma(K^*) S_E^{-1} : S_E[(E)^*] \ni S_E\Phi \mapsto S_E\Phi \circ K \in S_E[(E)^*]. \tag{6.3}$$

Then by definition it holds that

$$\Gamma(K^*) = S_E^{-1} \widetilde{K} S_E.$$

Proposition 6.3. *Let $K \in \mathcal{L}(E, E)$ be given. Then K is a topological isomorphism from E onto itself if and only if K^* is a topological isomorphism from E^* onto itself.*

Proof. The proof is straightforward from the duality. □

Proposition 6.4. *Let $K \in \mathcal{L}(E, E)$ be a topological isomorphism from E onto itself. Then $\Gamma(K)$ is a topological isomorphism from (E) onto itself.*

Proof. Since K be a topological isomorphism from E onto itself, $K^{\otimes n}$ $(n \in \mathbb{N})$ be a topological isomorphism from $E^{\otimes n}$ onto itself. Therefore, by the definition of the second quantization $\Gamma(K)$ of K given as in (6.2), $\Gamma(K)$ be a topological isomorphism from (E) onto itself. □

Theorem 6.5 (Theorem 3.2 in [4])**.** *The operator $\Gamma((\partial^\alpha)^*)$ is a topological isomorphism on $(E)^*$ into itself.*

Proof. By Proposition 6.2 ∂^α is a topological isomorphism from E onto itself. Therefore, by Proposition 6.4, $\Gamma(\partial^\alpha)$ is a topological isomorphism from (E) onto itself. Hence, by applying Proposition 6.4, we see that $\Gamma((\partial^\alpha)^*)$ is a topological isomorphism from $(E)^*$ onto itself. □

Theorem 6.6 ([5])**.** *Let the sequence $a = (a_n)_{n=1}^\infty \in \mathbb{C}^\infty$ be such that, for some $p > 0$ the limit*

$$\lim_{N\to\infty} \frac{1}{N^p} \sum_{n=1}^{N} a_n =: A_p(a)$$

exists. Then for each $\alpha \in \mathbb{R}_+$, one has

$$\lim_{N\to\infty} \frac{1}{N^{p+\alpha}} \sum_{n=1}^{N} n^\alpha a_n = \frac{p}{p+\alpha} A_p(a) \tag{6.4}$$

in the sense that the limit on the left hand side exists and the equality holds.

Let $R \in \mathcal{L}(E,E)$ be given. Then for each $\Phi \in (E)^*$, from (6.3), we obtain that

$$[S_E\Gamma(R^*)\Phi]''(\xi)(\eta,\zeta) = \left[\widetilde{R}S_E\Phi\right]''(\xi)(\eta,\zeta) = [S_E\Phi]''(R\xi)(R\eta,R\zeta) \tag{6.5}$$

for all $\xi,\eta,\zeta \in E$.

Theorem 6.7. *Let $\gamma > 0$ and $\Phi \in \mathrm{Dom}(\mathbf{\Delta}_{H,\mathrm{c},\gamma})$. Then for any $\alpha \in \mathbb{N}$,*

$$\Gamma((\partial^\alpha)^*)\Phi \in \mathrm{Dom}(\mathbf{\Delta}_{H,\mathrm{c},2\alpha+\gamma}),$$

and it holds that

$$\mathbf{\Delta}_{H,\mathrm{c},2\alpha+\gamma}\Gamma((\partial^\alpha)^*)\Phi = \frac{\gamma(2\pi)^\alpha}{\gamma+2\alpha}\Gamma((\partial^\alpha)^*)\mathbf{\Delta}_{H,\mathrm{c},\gamma}\Phi. \tag{6.6}$$

Proof. Let α be even and $\Phi \in \mathrm{Dom}(\mathbf{\Delta}_{H,\mathrm{c},\gamma})$. Then from (6.5) and Lemma 6.1, we obtain that

$$[S_E\Gamma((\partial^\alpha)^*)\Phi]''(\xi)(e_l,e_l) = [S_E\Phi]''(\partial^\alpha\xi)\left(e_l^{(\alpha)},e_l^{(\alpha)}\right) = c_l^2[S_E\Phi]''(\partial^\alpha\xi)\left(e_l,e_l\right),$$

where $c_l = (-1)^{\alpha/2}(2k\pi)^\alpha$ for $l = 2k$ or $l = 2k-1$. Hence for all $\xi \in E$, we obtain that

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N^{2\alpha+\gamma}}\sum_{l=1}^N [S_E\Gamma((\partial^\alpha)^*)\Phi]''(\xi)(e_l,e_l)\\
&= \lim_{N\to\infty}\frac{1}{N^{2\alpha+\gamma}}\sum_{l=1}^N c_l^2[S_E\Phi]''(\partial^\alpha\xi)\left(e_l,e_l\right)\\
&= \lim_{N\to\infty}\frac{1}{N^{2\alpha+\gamma}}\sum_{l=1}^N (l\pi)^{2\alpha}[S_E\Phi]''(\partial^\alpha\xi)\left(e_l,e_l\right),
\end{aligned}$$

where the second equality holds since $\alpha > 0$. Hence by applying Theorem 6.6, for all $\xi \in E$, we obtain that

$$\begin{aligned}
S_E\left(\mathbf{\Delta}_{H,\mathrm{c},2\alpha+\gamma}\Gamma((\partial^\alpha)^*)\Phi\right)(\xi) &= (2\pi)^\alpha\left(\lim_{N\to\infty}\frac{1}{N^{2\alpha+\gamma}}\sum_{l=1}^N l^{2\alpha}[S_E\Phi]''(\partial^\alpha\xi)\left(e_l,e_l\right)\right)\\
&= \frac{\gamma(2\pi)^\alpha}{\gamma+2\alpha}\left(\lim_{N\to\infty}\frac{1}{N^\gamma}\sum_{l=1}^N [S_E\Phi]''(\partial^\alpha\xi)\left(e_l,e_l\right)\right)\\
&= \frac{\gamma(2\pi)^\alpha}{\gamma+2\alpha}S_E\left(\Gamma((\partial^\alpha)^*)\mathbf{\Delta}_{H,\mathrm{c},\gamma}\Phi\right)(\xi),
\end{aligned}$$

from which we obtain (6.6). The case of odd α is similar. □

The result is closed related to the result of Theorem 5.1 in [5], see also Theorem 6.1 in [4].

We introduce the *Lévy Laplacian* $\mathbf{\Delta}_{\mathrm{L},\alpha}$ $(\alpha > 0)$ *in terms of the higher order derivative of white noise* by

$$\mathbf{\Delta}_{\mathrm{L},\alpha}\Phi = \Gamma((\partial^\alpha)^*)\mathbf{\Delta}_{\mathrm{L}}\Gamma((\partial^\alpha)^*)^{-1}\Phi$$

for $\Phi \in \Gamma((\partial^\alpha)^*)(\mathrm{Dom}(\mathbf{\Delta}_{\mathrm{L}}))$.

Corollary 6.8 (Corollary 5.2 in [5]; Theorem 5.3 in [4]). *Let $\Phi \in \mathrm{Dom}(\mathbf{\Delta}_{\mathrm{L}})$ be given. Then for any $\alpha > 0$, $\Gamma((\partial^\alpha)^*)\Phi \in \mathrm{Dom}(\mathbf{\Delta}_{H,\mathrm{c},2\alpha+1})$ and the identity:*

$$\mathbf{\Delta}_{H,\mathrm{c},2\alpha+1}\Gamma((\partial^\alpha)^*)\Phi = \frac{\pi^{2\alpha}}{2\alpha+1}\Gamma((\partial^\alpha)^*)\mathbf{\Delta}_{\mathrm{L}}\Phi = \frac{\pi^{2\alpha}}{2\alpha+1}\mathbf{\Delta}_{\mathrm{L},\alpha}\Gamma((\partial^\alpha)^*)\Phi$$

holds.

Proof. The proof is immediate from Theorem 6.7 by taking $\gamma = 0$. □

7. An Exponential Distribution in $\Gamma(E^*)$

Lemma 7.1 (Lemma 5.1 in [4]). *For any $\alpha \in \mathbb{N}_0$ and $t, s \in [0,1]$ with $t - s \notin \mathbb{Q} \setminus \{0\}$, we have*

$$\frac{1}{N^{2\alpha+1}} \sum_{n=1}^{N} e_n^{(\alpha)}(t) e_n^{(\alpha)}(s) \to \frac{\pi^{2\alpha}}{2\alpha+1}\delta_{t,s} \tag{7.1}$$

as $N \to \infty$, where $\delta_{s,t}$ is the Kronecker delta.

Proof. The proof is a simple modification of the proof of Lemma 5.1 in [4]. □

Remark 7.2. In Lemma 7.1, if we consider the function e_k as trigonometric functions defined on $\mathbb{R}$, then (7.1) holds for all $s, t \in \mathbb{R}$ with $t - s \notin \mathbb{Q} \setminus \{0\}$. However, if we consider $\{e_k\}_{k=1}^\infty$ as a complete orthonormal basis of $\{1\}^\perp$ in $L^2([0,1])$, then our attention is limited to $[0,1]$.

Lemma 7.3. *For any $s, t \in [0,1]$ with $t - s \notin \mathbb{Q} \setminus \{0\}$ and $a \in \mathbb{N}$, we have*

$$\langle \delta_s^{(a)}, \delta_t^{(a)} \rangle_{\mathrm{c},2a+1} = \frac{\pi^{2a}}{2a+1}\delta_{s,t}.$$

Proof. The proof is straightforward by applying Lemma 7.1. In fact, for any $p \geq 0$, $a \in \mathbf{N} \cup \{0\}$ and $s \in [0,1]$, since from (2.6) there exists $\beta \geq 0$ such that $(2\pi[\nu/2])^a \leq \lambda_\nu^\beta$ for all $\nu \in \mathbb{N}$, we obtain that

$$\begin{aligned}
\left|\delta_s^{(a)}\right|_{-p}^2 &= \left|\sum_{\nu=1}^\infty \langle e_\nu, \delta_s \rangle e_\nu^{(a)}\right|_{-p}^2 = \sum_{\nu=1}^\infty (2\pi[\nu/2])^{2a} \lambda_\nu^{-2p} |e_\nu(t)|^2 \\
&\leq 2\sum_{\nu=1}^\infty \lambda_\nu^{-2(p-\beta)},
\end{aligned}$$

which implies that $\delta_s^{(a)} \in E_{-p}$ for $p - \beta \geq 1$. Then by applying Lemma 7.1, we also obtain that

$$\begin{aligned}
\langle \delta_s^{(a)}, \delta_t^{(a)} \rangle_{\mathrm{c},2a+1} &= \lim_{N\to\infty} \frac{1}{N^{2a+1}} \sum_{k=1}^N \langle \delta_s^{(a)}, e_k \rangle \langle e_k, \delta_t^{(a)} \rangle \\
&= \lim_{N\to\infty} \frac{1}{N^{2a+1}} \sum_{k=1}^N e_k^{(a)}(s) e_k^{(a)}(t) \\
&= \frac{\pi^{2a}}{2a+1}\delta_{s,t}.
\end{aligned}$$

The proof is complete. □

For each $a \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$, we put

$$e_{a,\lambda} := \frac{\sqrt{2a+1}}{\pi^a} \delta_\lambda^{(a)} \in E^*. \tag{7.2}$$

Remark 7.4. Consider the equivalence relation $\sim$ defined by $s \sim t$ for $s, t \in [0,1]$ if and only if $s - t \in \mathbb{Q}$. Put

$$\boldsymbol{S} = \{c_s \in [s] \,:\, s \in [0,1]\},$$

where $[s]$ is the equivalence class containing s. Then by Lemma 7.3, $\{e_{a,\lambda}\}_{\lambda \in \boldsymbol{S}}$ is an orthonormal family with respect to $\langle \cdot, \cdot \rangle_{c,2a+1}$. We note that $\boldsymbol{S}$ is not a Lebesgue measurable set (and so not countable).

Let $\{B(t)\}_{t \geq 0}$ be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. For a complex Hilbert space $\mathcal{K}$ and $\Phi \in L^2(\mathbb{R}; \Gamma(\mathcal{K}))$, we define the $\Gamma(\mathcal{K})$*–valued Wiener integral* $\int_{\mathbb{R}} \Phi(t) dB(t)$ as the element of $L^2(\Omega, \mathcal{F}, P; \Gamma(\mathcal{K}))$ characterized by

$$\int_{\mathbb{R}} \Phi(t) dB(t) = \left(\int_{\mathbb{R}} f_n(t) dB(t) \right)_{n=0}^{\infty}$$

for $\Phi(t) = (f_n(t))_{n=0}^{\infty} \in \Gamma(\mathcal{K})$.

For each $x \in E^*$, the exponential vector associated with x is denoted by $\Phi_x := \phi_x \in (E)^*$, which is defined as in (3.3). Therefore, if $a \in \mathbb{N}_0$ and $\lambda \in [0,1]$, then for sufficiently large $p \geq 0$ such that $|e_{a,\lambda}|_{-p} \leq 1$, we obtain that

$$e^{-t^2} \left\| \Phi_{ite_{a,\lambda}} \right\|_{-p}^2 = e^{-t^2} e^{t^2 |e_{a,\lambda}|_{-p}^2} \leq 1,$$

and so for each $f \in L^2(\mathbb{R})$,

$$e^{-\frac{t^2}{2}} \widehat{f}(t) \Phi_{ite_{a,\lambda}} \in L^2(\mathbb{R}; \Gamma(E_{-p})),$$

where $\widehat{f}$ is the Fourier transform of f. For $a \in \mathbb{N}_0$, $\lambda \in [0,1]$ and $f \in L^2(\mathbb{R})$, we put

$$\mathbf{I}_\lambda(f) := \int_{\mathbb{R}} e^{-\frac{t^2}{2}} \widehat{f}(t) \Phi_{ite_{a,\lambda}} dB(t) \in (E)^*.$$

In fact, it holds that

$$\begin{aligned} \mathbf{I}_\lambda(f) &= \int_{\mathbb{R}} e^{-\frac{t^2}{2}} \widehat{f}(t) \left((it)^n \frac{e_{a,\lambda}^{\otimes n}}{n!} \right)_{n=0}^{\infty} dB(t) \\ &= \left(\left(i^n \int_{\mathbb{R}} e^{-\frac{t^2}{2}} \widehat{f}(t) t^n dB(t) \right) \frac{e_{a,\lambda}^{\otimes n}}{n!} \right)_{n=0}^{\infty}, \end{aligned}$$

and then for sufficiently large $p \geq 0$ such that $|e_{a,\lambda}|_{-p} \leq 1$, we obtain that

$$\begin{aligned} E\left[\|\mathbf{I}_\lambda(f)\|_{-p}^2\right] &= E\left[\sum_{n=0}^{\infty}\left|\int_{\mathbb{R}} e^{-\frac{t^2}{2}}\widehat{f}(t)t^n dB(t)\right|^2 \frac{|e_{a,\lambda}|_{-p}^{2n}}{n!}\right] \\ &= \sum_{n=0}^{\infty}\left(\int_{\mathbb{R}} e^{-t^2}|\widehat{f}(t)|^2|t|^{2n}dt\right)\frac{|e_{a,\lambda}|_{-p}^{2n}}{n!} \\ &= \int_{\mathbb{R}} e^{-t^2(1-|e_{a,\lambda}|_{-p}^2)}|\widehat{f}(t)|^2 dt \qquad (7.3) \\ &\leq \int_{\mathbb{R}} |\widehat{f}(t)|^2 dt. \end{aligned}$$

Let $a \in \mathbb{N}_0$ be given. For any $x \in E^*$ with $|x|_{c,2a+1} < \infty$, $\Phi_{ix} \in \mathrm{Dom}(\Delta_{H,c,a})$ and

$$\Delta_{H,c,a}\Phi_{ix} = -|x|_{c,2a+1}^2\Phi_{ix}.$$

For any $\lambda \in [0,1]$ and $f \in L^2(\mathbb{R})$, we have $\mathbf{I}_\lambda(f) \in \mathrm{Dom}(\Delta_{H,c,a})$ and

$$\Delta_{H,c,a}\mathbf{I}_\lambda(f) = -\int_{\mathbb{R}} t^2 e^{-\frac{t^2}{2}}\hat{f}(t)\Phi_{ite_{a,\lambda}}dB(t).$$

For each $a \in \mathbb{N}_0$ and $\lambda \in [0,1]$, we define a space $\mathcal{E}_{2a+1}^\lambda$ by

$$\mathcal{E}_{2a+1}^\lambda := \left\{\mathbf{I}_\lambda(f) \,:\, f \in L^2(\mathbb{R})\right\} \subset L^2(\Omega, P; (E)^*)$$

with the inductive limit topology given by the norms:

$$|||\mathbf{I}_\lambda(f)|||_{-p} := E[\|\mathbf{I}_\lambda(f)\|_{-p}^2]^{1/2}, \quad p \geq 0.$$

Then the space $\mathcal{E}_{2a+1}^\lambda$ is a closed linear subspace of $L^2(\Omega, P; (E)^*)$.

Lemma 7.5. *Let $a \in \mathbb{N} \cup \{0\}$ and $\lambda \in [0,1]$ be given. Then the distribution $\Phi_{i(\cdot)e_{a,\lambda}}$ is an element of $C(\mathbb{R}; \Gamma(H_{\mathrm{c},2a+1,\mathbb{C}}))$.*

Proof. We obtain that

$$\begin{aligned} &\|\Phi_{ite_{a,\lambda}} - \Phi_{ise_{a,\lambda}}\|_{\mathrm{c},2a+1}^2 \\ &= \|\Phi_{ite_{a,\lambda}}\|_{\mathrm{c},2a+1}^2 + \|\Phi_{ise_{a,\lambda}}\|_{\mathrm{c},2a+1}^2 - 2\langle\langle\Phi_{ite_{a,\lambda}}, \Phi_{ise_{a,\lambda}}\rangle\rangle_{\mathrm{c},2a+1} \\ &= e^{t^2} + e^{s^2} - 2e^{ts} \to 0 \quad (\text{as } |t-s| \to 0), \end{aligned}$$

where $\|\cdot\|_{\mathrm{c},2a+1}$ is the norm on $\Gamma(H_{\mathrm{c},2a+1,\mathbb{C}})$, which implies the assertion. $\square$

By the proof of Lemma 7.5, we see that $\|\Phi_{ite_{a,\lambda}}\|_{\mathrm{c},2a+1} = e^{t^2/2}$, and so it holds that

$$e^{-(\cdot)^2/2}f(\cdot)\Phi_{i(\cdot)e_{a,\lambda}} \in L^2(\mathbb{R}; \Gamma(H_{\mathrm{c},2a+1,\mathbb{C}}))$$

for all $f \in L^2(\mathbb{R})$. Therefore, the integral:

$$\mathbf{I}_\lambda(f) = \int_{\mathbb{R}} e^{-t^2/2}\widehat{f}(t)\Phi_{ite_{a,\lambda}}dB(t)$$

has a meaning as a $\Gamma(H_{\mathrm{c},2a+1,\mathbb{C}})$-valued Wiener integral, and becomes an element of

$$L^2\left(\Omega, P;\ \Gamma(H_{\mathrm{c},2a+1,\mathbb{C}})\right) = \bigoplus_{n=0}^{\infty}\sqrt{n!}\,L^2\left(\Omega, P;\ H_{\mathrm{c},2a+1,\mathbb{C}}^{\widehat{\otimes}n}\right).$$

Lemma 7.6. *For any* $f \in L^2(\mathbb{R})$ *we have* $|||\mathbf{I}_\lambda(f)|||_{c,2a+1} = |f|_0$*, where* $|||\cdot|||_{c,2a+1} = E\left[\|\cdot\|_{c,2a+1}\right]$.

Proof. By same arguments used in (7.3) with the property that $|e_{a,\lambda}|_{c,2a+1} = 1$, the proof is straightforward. In fact, we obtain that

$$\begin{aligned}
|||\mathbf{I}_\lambda(f)|||^2_{c,2a+1} &= E[\|\mathbf{I}_\lambda(f)\|^2_{c,2a+1}] \\
&= E\left[\sum_{n=0}^{\infty} n! \left| \int_{\mathbb{R}} e^{-t^2/2}\widehat{f}(t)\frac{(ite_{a,\lambda})^{\otimes n}}{n!}dB(t)\right|^2_{c,2a+1}\right] \\
&= E\left[\sum_{n=0}^{\infty} \frac{1}{n!} \left| \int_{\mathbb{R}} t^n e^{-t^2/2}\widehat{f}(t)dB(t)(e_{a,\lambda})^{\otimes n}\right|^2_{c,2a+1}\right] \\
&= E\left[\sum_{n=0}^{\infty} \frac{1}{n!} \left| \int_{\mathbb{R}} t^n e^{-t^2/2}\widehat{f}(t)dB(t)\right|^2 |e_{a,\lambda}|^{2n}_{c,2a+1}\right] \\
&= \int_{\mathbb{R}} |\widehat{f}(t)|^2 dt = \int_{\mathbb{R}} |f(t)|^2 dt = |f|_0^2
\end{aligned}$$

for all $f \in L^2(\mathbb{R})$. $\square$

Now, for each $\lambda \in [0,1]$ and $a \in \mathbb{N}_0$, we define $\widetilde{\mathcal{E}^\lambda_{2a+1}}$ by

$$\widetilde{\mathcal{E}^\lambda_{2a+1}} := \{\mathbf{I}_\lambda(f) \ : \ f \in L^2(\mathbb{R})\} \subset L^2(\Omega, P;\ \Gamma(H_{c,2a+1,\mathbb{C}}))$$

with norm $|||\cdot|||_{c,2a+1}$ given by

$$|||\mathbf{I}_\lambda(f)|||^2_{c,2a+1} := E[\|\mathbf{I}_\lambda(f)\|^2_{c,2a+1}],\ f \in L^2(\mathbb{R}).$$

Then, by Lemma 7.6, we see that $\widetilde{\mathcal{E}^\lambda_{2a+1}}$ is a closed subspace of $L^2(\Omega, P;\ \Gamma(H_{c,2a+1,\mathbb{C}}))$.

Lemma 7.7. *Let* $\Phi = \int_0^1 \mathbf{I}_\lambda(f_\lambda)d\lambda$*,* $\mathbf{I}_\lambda(f_\lambda) \in \widetilde{\mathcal{E}^\lambda_{2a+1}}$ *for* $f_{(\cdot)}(\cdot) \in L^2([0,1]\times\mathbb{R})$*. If*

$$\int_0^1 |||\mathbf{I}_\lambda(f_\lambda)|||^2_{c,2a+1}d\lambda = 0,$$

then it holds that $\Phi = 0$.

Proof. It follows from Lemma 7.6 that

$$\int_0^1 |||\mathbf{I}_\lambda(f_\lambda)|||^2_{c,2a+1}d\lambda = \int_0^1 |f_\lambda|_0^2 d\lambda,$$

which implies the assertion. $\square$

Define the space $\mathbb{D}_{c,2a+1}$ by

$$\mathbb{D}_{c,2a+1} := \left\{\int_0^1 \Psi_\lambda d\lambda \ : \ \Psi_\lambda \in \widetilde{\mathcal{E}^\lambda_{2a+1}}, \int_0^1 |||\Psi_\lambda|||^2_{c,2a+1}d\lambda < \infty\right\}$$

with the norm $|||\Psi|||_{c,2a+1}$ given by

$$|||\Psi|||^2_{c,2a+1} := \int_0^1 |||\Psi_\lambda|||^2_{c,2a+1}d\lambda,\ \Psi = \int_0^1 \Psi_\lambda d\lambda,$$

for $a \geq 0$. Then the space $\mathbb{D}_{\mathrm{c},2a+1}$ becomes a Hilbert space with the Hilbertian norm $|||\cdot|||_{\mathrm{c},2a+1}$, and the exotic Laplacian $\mathbf{\Delta}_{H,\mathrm{c},2a+1}$ defined on the linear span of $\{\Phi_{ite_{a,\lambda}}\}_{\lambda\in[0,1]}$ can be extended to a linear operator densely defined on $\mathbb{D}_{\mathrm{c},2a+1}$. We use the notation $\mathbf{\Delta}_{\mathrm{c},2a+1}$ for the extension.

8. Stochastic Process Generated by Exotic Laplacian

Let K be a self-adjoint operator densely defined on $L^2(\mathbb{R})$ by $K = I + \frac{d^2}{dt^2}$. For $p \geq 0$, we define p-norms $|\cdot|_{K,p}$ on $L^2(\mathbb{R})$ and $|||\cdot|||_{\mathrm{c},2a+1,p}$ on $\mathbb{D}_{\mathrm{c},2a+1}$ by

$$|f|_{K,p} := |K^p f|_0 = \left(\int_{\mathbb{R}} (1+t^2)^{2p} |\widehat{f}(t)|^2 dt\right)^{1/2} \in [0,\infty], \ f \in L^2(\mathbb{R}),$$

and

$$|||\Psi|||_{\mathrm{c},2a+1,p} := |||(I - \mathbf{\Delta}_{\mathrm{c},2a+1})^p \Psi|||_{\mathrm{c},2a+1} \in [0,\infty], \ \Psi \in \mathbb{D}_{\mathrm{c},2a+1},$$

respectively. Then for each $p \geq 0$, we can also define spaces $\mathbb{D}_{\mathrm{c},2a+1,p}$ by

$$\mathbb{D}_{\mathrm{c},2a+1,p} := \left\{ \int_0^1 \mathbf{I}_\lambda(f_\lambda) d\lambda \, : \, \mathbf{I}_\lambda(f_\lambda) \in \widetilde{\mathcal{E}^\lambda_{2a+1}}, \ \int_0^1 |f_\lambda|^2_{K,p} d\lambda < \infty \right\}$$

and define

$$\mathbb{D}_{\mathrm{c},2a+1,\infty} := \bigcap_{p=0}^{\infty} \mathbb{D}_{\mathrm{c},2a+1,p}$$

with the projective limit topology of $\{\mathbb{D}_{\mathrm{c},2a+1,p}\}_{p\geq 0}$.

The Gateaux differential operator D_y ($y \in H_{\mathrm{c},2a+1}$) defined on the linear span of $\{\Phi_{ite_{a,\lambda}}\}_{\lambda\in[0,1]}$ can be extended to a continuous linear operator on $\mathbb{D}_{\mathrm{c},2a+1,\infty}$, and the estimate

$$\begin{aligned}
\left|\left|\left| D_y \int_0^1 \mathbf{I}_\lambda(f_\lambda) d\lambda \right|\right|\right|^2_{\mathrm{c},2a+1,p} &= \int_0^1 |||D_y \mathbf{I}_\lambda(f_\lambda)|||^2_{\mathrm{c},2a+1,p} d\lambda \\
&= \int_0^1 |||D_y (I - \mathbf{\Delta}_{\mathrm{c},2a+1})^p \mathbf{I}_\lambda(f_\lambda)|||^2_{\mathrm{c},2a+1} d\lambda \\
&= \int_0^1 (y, e_{a,\lambda})^2_{\mathrm{c},2a+1} \int_{\mathbb{R}} t^2 (1+t^2)^{2p} |\widehat{f_\lambda}(t)|^2 dt d\lambda \\
&\leq |y|^2_{\mathrm{c},2a+1} \int_0^1 |f_\lambda|^2_{K,p+1/2} d\lambda \\
&= |y|^2_{\mathrm{c},2a+1} \left|\left|\left| \int_0^1 \mathbf{I}_\lambda(f_\lambda) d\lambda \right|\right|\right|^2_{\mathrm{c},2a+1,p+1/2}
\end{aligned}$$

holds.

Lemma 8.1. *Let $y \in H_{\mathrm{c},2a+1}$ be given. Then for any $\Psi \in \mathbb{D}_{\mathrm{c},2a+1,\infty}$, the series*

$$T_y \Psi := \sum_{k=0}^{\infty} \frac{1}{k!} D_y^k \Psi$$

converges in $\mathbb{D}_{\mathrm{c},2a+1,\infty}$. Furthermore, the operator T_y, called the translation operator, on $\mathbb{D}_{\mathrm{c},2a+1,\infty}$ is isometric.

Proof. We first note that for any $x \in H_{\mathrm{c},2a+1}$, the series

$$T_y\Phi_x := \sum_{n=0}^{\infty} \frac{1}{n!} D_y^n \Phi_x = \sum_{n=0}^{\infty} \frac{1}{n!} (x,y)_{\mathrm{c},2a+1}^n \Phi_x = e^{(x,y)_{\mathrm{c},2a+1}} \Phi_x$$

converges in $\mathbb{D}_{\mathrm{c},2a+1,\infty}$. Therefore, for any $p \geq 0$, we obtain that

$$\begin{aligned}
&|||T_y \mathbf{I}_\lambda(f)|||_{\mathrm{c},2a+1,p}^2 \\
&= E\left[\left\| \int_{\mathbb{R}} e^{-\frac{t^2}{2}} e^{it(y,e_{a,\lambda})_{\mathrm{c},2a+1}} (1+t^2)^p \widehat{f_\lambda}(t) \Phi_{ite_{a,\lambda}} dB(t) \right\|_{\mathrm{c},2a+1}^2 \right] \\
&= \int_{\mathbb{R}} (1+t^2)^{2p} |\widehat{f_\lambda}(t)|^2 dt = |f_\lambda|_{K,p}^2,
\end{aligned}$$

from which we see that for any $\Psi \in \mathbb{D}_{\mathrm{c},2a+1,\infty}$,

$$|||T_y \Psi|||_{\mathrm{c},2a+1,p} = |||\Psi|||_{\mathrm{c},2a+1,p}$$

for any $p \geq 0$. □

Define an operator $\mathbf{\Delta}_{\mathrm{G},e_a}$ on $\mathbb{D}_{\mathrm{c},2a+1,\infty}$ by

$$\mathbf{\Delta}_{\mathrm{G},e_a} \Psi := \int_0^1 D_{e_{a,\lambda}}^2 \mathbf{I}_\lambda(f) d\lambda$$

for $\Psi = \int_0^1 \mathbf{I}_\lambda(f) d\lambda \in \mathbb{D}_{\mathrm{c},2a+1,\infty}$. Then $\mathbf{\Delta}_{\mathrm{G},e_a}$ is an extension of $\mathbf{\Delta}_{\mathrm{c},2a+1}$ defined on $\widetilde{\mathcal{E}}_{2a+1}^\lambda$ to the continuous linear operator on $\mathbb{D}_{\mathrm{c},2a+1,\infty}$.

Lemma 8.2. *The operator $\mathbf{\Delta}_{\mathrm{G},e_a}$ is a continuous linear operator on $\mathbb{D}_{\mathrm{c},2a+1,\infty}$ into itself.*

Proof. The proof is straightforward. □

Let $w(\cdot,\cdot)$ be a two-dimensional white noise. For each $t \geq 0$ and $a \geq 0$, put

$$B_u(t) := \int_0^t w(u,s) ds, \quad \mathbf{B}_a(t) := \int_0^1 B_u(t) e_{a,u} du.$$

Then for any $\xi \in H_{\mathrm{c},2a+1}$, we have

$$\begin{aligned}
\langle \mathbf{B}_a(t), \xi \rangle &= \int_0^1 B_u(t) \langle e_{a,u}, \xi \rangle du = \frac{\sqrt{2a+1}}{\pi^a} \int_0^1 B_u(t) \langle \delta_u^{(a)}, \xi \rangle du \\
&= \frac{\sqrt{2a+1}}{\pi^a} \sum_{k=1}^{\infty} \left(\int_0^t \int_0^1 e_k(u) w(u,s) du ds \right) \langle e_k^{(a)}, \xi \rangle.
\end{aligned} \tag{8.1}$$

For each $k \in \mathbb{N}$ and $t \geq 0$, put

$$b_k(t) = \int_0^t \int_0^1 e_k(u) w(u,s) du ds.$$

Then $\{\{b_n(t)\}_{t \geq 0}\}_{n=1}^{\infty}$ becomes a sequence of independent one dimensional Brownian motions on $((E_{\mathbb{R}}^*)^{\otimes 2}, \mu_a \otimes \mu_a)$. Then $\mathbf{B}_a(t)$ can be expressed in the form:

$$\mathbf{B}_a(t) = \frac{\sqrt{2a+1}}{\pi^a} \sum_{k=1}^{\infty} b_k(t) e_k^{(a)}.$$

Remark 8.3. Let $a \in \mathbb{N}$. For any $\xi \in E$ with $\xi^{(k)}(1) = \xi^{(k)}(0) = 0$ for all $k = 1, 2, \cdots, a-1$, by the integration by parts formula, we obtain that

$$\int_0^1 B_u(t)\langle \delta_u^{(a)}, \xi\rangle du = \int_0^1 B_u(t)\xi^{(a)}(u)du = (-1)^a \int_0^1 \left(\left(\frac{\partial}{\partial u}\right)^a B_u(t)\right)\xi(u)du.$$

Therefore, from (8.1), $\mathbf{B}_a(t)$ can be considered as time derivative of white noise:

$$\mathbf{B}_a(t) = \frac{\sqrt{2a+1}}{\pi^a}(-1)^a \left(\frac{\partial}{\partial u}\right)^a B_u(t).$$

Lemma 8.4. *For $t \in \mathbb{R}$, $\mathbf{B}_a(t)$ is in $H_{\mathrm{c},2a+1}$.*

Proof. For any sequence $\{\{B_n(t)\}_{t\geq 0}\}_{n=1}^\infty$ of independent one-dimensional Brownian motions $\{B_n(t)\}_{t\geq 0}$, by the law of large numbers, we see that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N B_n(t)^2 = E[B_n(t)^2] = t \text{ (a.e.)}.$$

Therefore, by Theorem 6.6, we see that

$$|\mathbf{B}_a(t)|_{\mathrm{c},2a+1}^2 = \lim_{N\to\infty} \frac{1}{N^{2a+1}}\sum_{n=1}^N \langle e_n, \mathbf{B}_a(t)\rangle^2$$

converges. This implies the assertion. □

Then we have the following:

Theorem 8.5. *The process $\{\mathbf{B}_a(t) \ : \ t \in \mathbb{R}\}$ is generated by $\frac{1}{2}\mathbf{\Delta}_{\mathrm{G},e_a}$, i.e., it holds that*

$$E[T_{\mathbf{B}_a(t)}\Psi] = e^{\frac{t}{2}\mathbf{\Delta}_{\mathrm{G},e_a}}\Psi = e^{\frac{t}{2}\mathbf{\Delta}_{\mathrm{c},2a+1}}\Psi, \ \Psi \in \mathbb{D}_{\mathrm{c},2a+1,\infty}.$$

For any $a \in \mathbb{N}_0$, the operator $\Gamma((\partial^\alpha)^*)$ can be extended to a continuous linear operator on $\mathbb{D}_{\mathrm{c},2a+1}$. Then we have the following:

Lemma 8.6. *For any $a \in \mathbb{N}_0$ and $\alpha > 0$, we have $\Gamma((\partial^\alpha)^*)\mathbb{D}_{\mathrm{c},2a+1} = \mathbb{D}_{\mathrm{c},2(a+\alpha)+1}$.*

Corollary 8.7. *For any $a \in \mathbb{N}_0$, $t \geq 0$ and $\Psi \in \mathbb{D}_{\mathrm{c},1,\infty}$, the equality*

$$E[T_{\mathbf{B}_a(t)}\Gamma((\partial^\alpha)^*)\Psi] = e^{t\frac{\pi^a}{2(2a+1)}\mathbf{\Delta}_{\mathrm{L},a}}\Gamma((\partial^\alpha)^*)\Psi$$

holds.

9. Concluding Remark

Let $\mathbf{I}_{n,\lambda}(f)$ be an n-ple $\Gamma(H_{c,2a+1,\mathbb{C}})$-Wiener integral given by

$$\mathbf{I}_{n,\lambda}(f) := \int_{\mathbb{R}^n} e^{-\sum_{j=1}^n \frac{t_j^2}{2}}\hat{f}(t_1,\ldots,t_n)\prod_{j=1}^n \Phi_{it_je_{a,\lambda}}dB(t_1)\cdots dB(t_n)$$

for $n \in \mathbb{N}_0, \lambda \in [0,1]$ and $f \in L^2(\mathbb{R})^{\hat{\otimes}n}$. For each $n \in \mathbb{N}_0, \lambda \in [0,1]$ and $a \geq 0$, we define $\widetilde{\mathcal{E}_{2a+1}^{n,\lambda}}$ by

$$\widetilde{\mathcal{E}_{2a+1}^{n,\lambda}} := \{\mathbf{I}_{n,\lambda}(f) \ : \ f \in L^2(\mathbb{R})^{\hat{\otimes}n}\} \subset L^2(\Omega, P; \ \Gamma(H_{\mathrm{c},2a+1,\mathbb{C}}))$$

with norm $|||\cdot|||_{c,2a+1}$ given by

$$|||\mathbf{I}_\lambda(f)|||_{c,2a+1}^2 := E[||\mathbf{I}_\lambda(f)||_{c,2a+1}^2],\ \lambda\in[0,1],\ f\in L^2(\mathbb{R})^{\hat{\otimes}n}.$$

Then, similarly we see that $\widetilde{\mathcal{E}_{2a+1}^{n,\lambda}}$ is a closed subspace of $L^2(\Omega,P;\ \Gamma(H_{c,2a+1,\mathbb{C}}))$.

We also define the space $\mathbb{D}^n_{c,2a+1}$ by

$$\mathbb{D}^n_{c,2a+1} := \left\{\int_0^1 \Psi_\lambda d\lambda\ :\ \Psi_\lambda\in\widetilde{\mathcal{E}_{2a+1}^{n,\lambda}},\ \int_0^1 |||\Psi_\lambda|||_{c,2a+1}^2 d\lambda<\infty\right\}$$

with the norm $|||\cdot|||_{c,2a+1}$ for $a\geq 0$ and $n\in\mathbb{N}_0$. Then we can define the space $[\mathbb{D}_{c,2a+1}]$ by

$$[\mathbb{D}_{c,2a+1}] := \bigoplus_{n=0}^{\infty}\mathbb{D}^n_{c,2a+1}.$$

For each $n\in\mathbb{N}_0$ and $p\geq 0$, we can also define spaces $\mathbb{D}_{c,2a+1,p}$ by

$$\mathbb{D}^n_{c,2a+1,p} := \left\{\int_0^1 \mathbf{I}_{n,\lambda}(f_\lambda)d\lambda\ :\ \mathbf{I}_{n,\lambda}(f_\lambda)\in\widetilde{\mathcal{E}_{2a+1}^{n,\lambda}},\ \int_0^1 |f_\lambda|_{K,p}^2 d\lambda<\infty\right\}$$

where $|f_\lambda|_{K,p} := |(K^{\otimes n})^p f_\lambda|_0$, and

$$[\mathbb{D}_{c,2a+1,\infty}] := \bigcap_{p=0}^{\infty}\bigoplus_{n=0}^{\infty}\mathbb{D}^n_{c,2a+1,p}.$$

We can obtain similar results on the space $[\mathbb{D}_{c,2a+1,\infty}]$ parallelly in previous section.

Let $W(x) := \Phi_{ix}\diamond$ (the Wick multiplication operator) for $x\in H_{c,2a+1}$ with $a\geq 0$. Then $W(x)$ is the Weyl operator acting on $LS\{\Phi_{ix};\ x\in H_{c,2a+1}\}$. The exotic Laplacian $\mathbf{\Delta}_{c,2a+1}$ can be extended to a continuous linear operator on $LS\{W(x);\ x\in H_{c,2a+1}\}$ with operator norm. We can also obtain extensions of results in this paper based on the Weyl operator. Those studies are now in progress and will appear in a separate paper.

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