Journal of Stochastic Analysis

Volume 3 | Number 2 Article 3

June 2022

A Closed Form Formula for the Stochastic Exponential of a Matrix-Valued Semimartingale

Peter Kern
Heinrich Heine University, Düsseldorf, Germany, kern@hhu.de

Christian Müller
Heinrich Heine University, Düsseldorf, Germany, c.mueller@hhu.de

Recommended Citation
Kern, Peter and Müller, Christian (2022) "A Closed Form Formula for the Stochastic Exponential of a Matrix-Valued Semimartingale," *Journal of Stochastic Analysis*: Vol. 3: No. 2, Article 3.
DOI: 10.31390/josa.3.2.03
Available at: https://digitalcommons.lsu.edu/josa/vol3/iss2/3

Journal of Stochastic Analysis
Vol. 3, No. 2 (2022) Article 3 (19 pages)
DOI: 10.31390/josa.3.2.03

LSU | Libraries
digitalcommons.lsu.edu/josa

A CLOSED FORM FORMULA FOR THE STOCHASTIC EXPONENTIAL OF A MATRIX-VALUED SEMIMARTINGALE

PETER KERN AND CHRISTIAN MÜLLER*

Abstract. We prove a closed form formula for the stochastic exponential of a matrix-valued semimartingale under the assumption that various commutativity conditions are fulfilled. This extends a corresponding result for continuous semimartingales by Yan in [9] to semimartingales with jump parts. We give three examples of a semimartingale in a Lie group setting for which the commutativity conditions are easily verified and explicitly compute their stochastic exponential.

1. Introduction

The stochastic exponential of a real-valued semimartingale $X = (X_t)_{t\geq 0}$ is the unique strong solution of the stochastic differential equation $\mathrm{d}Y_t = Y_{t-}\,\mathrm{d}X_t$ with the initial condition $Y_0 = 1$ and can be viewed as the stochastic analog of the deterministic exponential function. We write $Y = \mathsf{Exp}(X)$ for the stochastic exponential of X. Oftentimes it is also denoted by $\mathcal{E}(X)$ and known under the name Doléans-Dade exponential. It can be written in the closed form

$$\mathsf{Exp}(X_t) = \exp\left(X_t - \frac{1}{2}[X,X]_t\right) \prod_{0<s\leq t} (1+\Delta X_s) \exp\left(-\Delta X_s + \frac{1}{2}(\Delta X_s)^2\right), \tag{1.1}$$

where $[X,X]$ denotes the quadratic variation of X and $\Delta X_t = X_t - X_{t-}$ is the jump of X at time $t>0$ with $\Delta X_0 \coloneqq 0$. See for example Theorem II.37 in [8] for a proof of this formula. Due to the non-commutativity of matrix multiplication there are two types of stochastic exponentials one can consider for matrix-valued semimartingales. The left stochastic exponential $\overleftarrow{\mathsf{Exp}}(X)$ is the unique strong solution of the stochastic differential equation $\mathrm{d}Y_t = Y_{t-}\,\mathrm{d}X_t$ and the right stochastic exponential $\overrightarrow{\mathsf{Exp}}(X)$ is the unique strong solution of the stochastic differential equation $\mathrm{d}Y_t = \mathrm{d}X_t\,Y_{t-}$, both with the initial condition $Y_0 = I$, where I denotes the identity matrix. Multivariate stochastic exponentials often appear in the solution of linear stochastic differential equations such as the multivariate generalized Ornstein-Uhlenbeck process which solves $\mathrm{d}V_t = \mathrm{d}U_t\,V_{t-} + \mathrm{d}L_t$ or $\mathrm{d}V_t = V_{t-}\,\mathrm{d}U_t + \mathrm{d}L_t$ for some bivariate Lévy process (U,L) with values in $\mathbb{R}^{n\times n}\times\mathbb{R}^n$, see for example

Received 2022-4-12; Accepted 2022-6-26; Communicated by the editors.
2020 *Mathematics Subject Classification.* 60H10, 60B99, 60G48, 60H20.
Key words and phrases. Stochastic exponential, Doléans-Dade exponential, semimartingale, linear stochastic differential equation, matrix Lie group.
* Corresponding author.

[1], [2], [6]. Thus understanding stochastic exponentials is important for studying the properties of the solution process of such equations. YAN proved in [9] that the stochastic exponential of a continuous matrix-valued semimartingale, under the assumption that various processes commute in the sense that their left and right stochastic integral with respect to each other are equal, can be written in the closed form

$$\overleftarrow{\mathsf{Exp}}(X_t) = \overrightarrow{\mathsf{Exp}}(X_t) = \exp\left(X_t - \frac{1}{2}[X,X]_t\right). \tag{1.2}$$

In Section 4 we prove that the stochastic exponential of a not necessarily continuous matrix-valued semimartingale, under even more commutativity conditions as in [9], can be written in the closed form

$$\overleftarrow{\mathsf{Exp}}(X_t) = \overrightarrow{\mathsf{Exp}}(X_t) = \exp\left(X_t - \frac{1}{2}[X,X]_t^c + \sum_{0<s\le t}\Big(\log(I+\Delta X_s) - \Delta X_s\Big)\right), \tag{1.3}$$

where $[X,X]^c$ denotes the continuous part of the quadratic variation of X and the sum is computed over the at most countably many jump times s of X up until time t. Eq. (1.3) extends both Eq. (1.1) and Eq. (1.2). To show the applicability of our main result, in section 5 we give three examples of a semimartingale with jumps in a matrix LIE group setting for which the commutativity conditions are easily verified, and compute an explicit formula for the corresponding stochastic exponential. Preliminary to our main result, in sections 2 and 3 we recall the construction of stochastic integrals with respect to matrix-valued semimartingales and some general properties of stochastic exponentials.

2. Stochastic Integrals and Covariations

Let $(\Omega, \mathcal{A}, \mathcal{F}, P)$ be a filtered and complete probability space such that the filtration $\mathcal{F} = (\mathcal{F}_t)_{t\ge 0}$ is right-continuous and $\mathcal{F}_0$ contains all null sets of $\mathcal{A}$. A real-valued $\mathcal{F}$-adapted càdlàg process $X = (X_t)_{t\ge 0}$ is called semimartingale if there exist a local martingale $M = (M_t)_{t\ge 0}$ and a process $A = (A_t)_{t\ge 0}$ with paths of finite variation on compacts such that $M_0 = A_0 = 0$ and $X_t = X_0 + M_t + A_t$. We shortly recall the construction of stochastic integrals with respect to semimartingales following chapter II in the monograph [8] of PROTTER.

A simple predictable process is a process $H = (H_t)_{t\ge 0}$ of the form

$$H_t = K_0 \mathbf{1}_{\{0\}}(t) + \sum_{i=1}^{m} K_i \mathbf{1}_{(T_i, T_{i+1}]}(t), \tag{2.1}$$

where $0 = T_1 \le \ldots \le T_{m+1}$ are a.s. finite stopping times and $K_0, \ldots, K_m$ are a.s. finite random variables such that $\sigma(K_i) \subseteq \mathcal{F}_{T_i}$ for all $i = 0, \ldots, m$. Let $\mathbf{S}_{ucp}$ be the set of all simple predictable processes, $\mathbb{D}_{ucp}$ the set of adapted processes with càdlàg paths, and $\mathbb{L}_{ucp}$ the set of adapted processes with càglàd paths, each endowed with the ucp-topology which is induced by uniform convergence on compacts in probability. More precisely, $H^m \xrightarrow{\text{ucp}} H$ for processes $(H^m)_{m\in\mathbb{N}}$ and H in $\mathbf{S}_{ucp}$, $\mathbb{D}_{ucp}$, or $\mathbb{L}_{ucp}$ if $\sup_{0\le s\le t} |H_s^m - H_s| \xrightarrow{\text{P}} 0$ for all $t > 0$. For a simple predictable process $H \in \mathbf{S}_{ucp}$ and an adapted càdlàg process $X \in \mathbb{D}_{ucp}$ the stochastic integral of H with respect to X is defined as

$$(H \cdot X)_t := \int_0^t H_s \, \mathrm{d}X_s := J_X(H)_t := K_0 X_0 + \sum_{i=1}^{m} K_i (X_{t \wedge T_{i+1}} - X_{t \wedge T_i}), \tag{2.2}$$

which induces a linear mapping $J_X : \mathbf{S}_{ucp} \longrightarrow \mathbb{D}_{ucp}$. In the case that X is a semimartingale this mapping is continuous. Since $\mathbf{S}_{ucp}$ is dense in $\mathbb{L}_{ucp}$ and $\mathbb{D}_{ucp}$ is a complete metric space, there exists a unique continuous and linear extension $J_X : \mathbb{L}_{ucp} \longrightarrow \mathbb{D}_{ucp}$ which is called the stochastic integral with respect to X.

A matrix-valued and $\mathcal{F}$-adapted càdlàg process $X = (X_t)_{t \geq 0}$ is called semimartingale if each component $X^{(i,j)}$ is a real-valued semimartingale. In the following let X, Y, Z be semimartingales in $\mathbb{R}^{n \times n}$ and let G, H be adapted processes in $\mathbb{R}^{n \times n}$ with càglàd paths.

The left stochastic integral $G \cdot X = \left(\int_0^t G_u \, \mathrm{d}X_u \right)_{t \geq 0}$ has (i,j)-component

$$\left(\int_0^t G_u \, \mathrm{d}X_u \right)^{(i,j)} := \sum_{k=1}^{n} \int_0^t G_u^{(i,k)} \, \mathrm{d}X_u^{(k,j)} \,, \tag{2.3a}$$

the right stochastic integral $X : H = \left(\int_0^t \mathrm{d}X_u \, H_u \right)_{t \geq 0}$ has (i,j)-component

$$\left(\int_0^t \mathrm{d}X_u \, H_u \right)^{(i,j)} := \sum_{l=1}^{n} \int_0^t H_u^{(l,j)} \, \mathrm{d}X_u^{(i,l)} \,, \tag{2.3b}$$

and the two-sided stochastic integral $G \cdot X : H = \left(\int_0^t G_u \, \mathrm{d}X_u \, H_u \right)_{t \geq 0}$ has (i,j)-component

$$\left(\int_0^t G_u \, \mathrm{d}X_u \, H_u \right)^{(i,j)} := \sum_{k,l=1}^{n} \int_0^t G_u^{(i,k)} H_u^{(l,j)} \, \mathrm{d}X_u^{(k,l)} \,. \tag{2.3c}$$

The processes $G \cdot X$, $X : H$, and $G \cdot X : H$ are also semimartingales. The quadratic covariation $[X, Y] = ([X, Y]_t)_{t \geq 0}$ of X and Y has (i,j)-component

$$[X, Y]_t^{(i,j)} := \sum_{k=1}^{n} [X^{(i,k)}, Y^{(k,j)}]_t \tag{2.4}$$

with the usual quadratic covariation process for real-valued semimartingales on the right-hand side. It is again a semimartingale with paths of finite variation on compacts and satisfies $[X, Y]_0 = \mathbf{0}$ and $\Delta[X, Y] = \Delta X \Delta Y$. The quadratic covariation may also be introduced by the integration by parts formula

$$X_t Y_t = X_0 Y_0 + \int_0^t X_{u-} \, \mathrm{d}Y_u + \int_0^t \mathrm{d}X_u \, Y_{u-} + [X, Y]_t. \tag{2.5}$$

The continuous part $[X,Y]^c = ([X,Y]^c_t)_{t\geq 0}$ of $[X,Y]$ is given by

$$[X,Y]^c_t = [X,Y]_t - \sum_{0<s\leq t} \Delta X_s \Delta Y_s. \tag{2.6}$$

If X has paths of finite variation on compacts, then $[X,Z]^c = \mathbf{0}$. Especially, for arbitrary semimartingales in $\mathbb{R}^{n\times n}$ we have the identities

$$\big[[X,Y],Z\big]^c = \big[[X,Y]^c,Z\big]^c = \big[[X,Y]^c,Z\big] = \mathbf{0}, \tag{2.7a}$$

$$\big[[X,Y],Z\big]_t = \sum_{0<s\leq t} \Delta X_s \Delta Y_s \Delta Z_s. \tag{2.7b}$$

For stochastic integrals and quadratic covariations of stochastic integrals we have the following identities that are also stated in part (B) of Theorem 2.1 in [9]:

$$H\cdot(G\cdot X) = (HG)\cdot X, \tag{2.8a}$$

$$(Y:H):G = Y:(HG), \tag{2.8b}$$

$$[G\cdot X, Y:H] = G\cdot[X,Y]:H, \tag{2.8c}$$

$$[X:G, H\cdot Y] = [X:(GH),Y] = [X,(GH)\cdot Y]. \tag{2.8d}$$

It is not hard to verify the formulas in this section by using the definitions of quadratic covariation and stochastic integrals. For detailed derivations of the formulas we refer to chapter 2.2 of [6].

3. General Properties of the Stochastic Exponential

Let $X = (X_t)_{t\geq 0}$ be a semimartingale in $\mathbb{R}^{n\times n}$. The left stochastic exponential $\overleftarrow{\mathsf{Exp}}(X)$ of X is the unique $\mathcal{F}$-adapted càdlàg process that solves the stochastic differential equation

$$\mathrm{d}\overleftarrow{\mathsf{Exp}}(X_t) = \overleftarrow{\mathsf{Exp}}(X_{t-})\,\mathrm{d}X_t \tag{3.1a}$$

with the initial condition $\overleftarrow{\mathsf{Exp}}(X_0) = I$. The right stochastic exponential $\overrightarrow{\mathsf{Exp}}(X)$ of X is the unique $\mathcal{F}$-adapted càdlàg process that solves the stochastic differential equation

$$\mathrm{d}\overrightarrow{\mathsf{Exp}}(X_t) = \mathrm{d}X_t\,\overrightarrow{\mathsf{Exp}}(X_{t-}) \tag{3.1b}$$

with the initial condition $\overrightarrow{\mathsf{Exp}}(X_0) = I$. Eq. (3.1a) and Eq. (3.1b) are equivalent to the stochastic integral equations

$$\overleftarrow{\mathsf{Exp}}(X_t) = I + \int_0^t \overleftarrow{\mathsf{Exp}}(X_{u-})\,\mathrm{d}X_u \quad \text{and} \quad \overrightarrow{\mathsf{Exp}}(X_t) = I + \int_0^t \mathrm{d}X_u\,\overrightarrow{\mathsf{Exp}}(X_{u-}). \tag{3.1c}$$

By Theorem V.7 in [8] the solutions of Eq. (3.1a) and Eq. (3.1b) are unique in the strong sense and are again semimartingales. Thus $\overleftarrow{\mathsf{Exp}}(X)$ and $\overrightarrow{\mathsf{Exp}}(X)$ are semimartingales whenever X is a semimartingale.

Oftentimes a stochastic exponential needs to be inverted which means that $\overleftarrow{\mathsf{Exp}}(X)$ and $\overrightarrow{\mathsf{Exp}}(X)$ need to be processes in $\mathsf{GL}_n(\mathbb{R})$. By Theorem 1 in [5] this holds true if and only if

$$\det(I+\Delta X_t) \neq 0 \quad \text{for all } t\geq 0. \tag{3.2}$$

In this case the processes $\overleftarrow{\mathsf{Exp}}(X)^{-1}$ and $\overrightarrow{\mathsf{Exp}}(X)^{-1}$ are well-defined and we have the following properties of the stochastic exponential.

Lemma 3.1. *Let $X = (X_t)_{t\geq 0}$ be a semimartingale in $\mathbb{R}^{n\times n}$ which satisfies Eq. (3.2). Then for all $0 \leq s \leq t$*

$$\overleftarrow{\mathsf{Exp}}(X_t - X_s) = \overleftarrow{\mathsf{Exp}}(X_s)^{-1}\overleftarrow{\mathsf{Exp}}(X_t), \tag{3.3a}$$

$$\overrightarrow{\mathsf{Exp}}(X_t - X_s) = \overrightarrow{\mathsf{Exp}}(X_t)\overrightarrow{\mathsf{Exp}}(X_s)^{-1}. \tag{3.3b}$$

If furthermore $U = (U_t)_{t\geq 0}$ is defined by

$$U_t = -X_t + [X,X]^c_t + \sum_{0<s\leq t}\Big((I+\Delta X_s)^{-1} - I + \Delta X_s\Big), \tag{3.3c}$$

then for all $t \geq 0$

$$\overleftarrow{\mathsf{Exp}}(X_t)^{-1} = \overrightarrow{\mathsf{Exp}}(U_t), \tag{3.3d}$$

$$\overrightarrow{\mathsf{Exp}}(X_t)^{-1} = \overleftarrow{\mathsf{Exp}}(U_t), \tag{3.3e}$$

$$\det(I + \Delta U_t) = \frac{1}{\det(I+\Delta X_t)}. \tag{3.3f}$$

Proof. To prove Eq. (3.3a) we write $Z \coloneqq \overleftarrow{\mathsf{Exp}}(X)$, $Z_{s,t} \coloneqq Z_s^{-1}Z_t$, and $t = s+h$ with $h \geq 0$. Then

$$\begin{aligned} Z_{s,s+h} &= \left(I + \int_0^s Z_{u-}\,\mathrm{d}X_u\right)^{-1}\left(I + \int_0^s Z_{u-}\,\mathrm{d}X_u + \int_s^{s+h} Z_{u-}\,\mathrm{d}X_u\right) \\ &= I + Z_s^{-1}\int_s^{s+h} Z_{u-}\,\mathrm{d}X_u \\ &= I + \int_0^h Z_s^{-1}Z_{(s+u)-}\,\mathrm{d}X_{s+u} \\ &= I + \int_0^h Z_{s,(s+u)-}\,\mathrm{d}(X_{s+u} - X_s)\,. \end{aligned}$$

Since $Z_{s,s+0} = I$ and the left stochastic exponential is the unique solution of the first equation in Eq. (3.1c), it follows that

$$\overleftarrow{\mathsf{Exp}}(X_s)^{-1}\overleftarrow{\mathsf{Exp}}(X_t) = Z_{s,t} = Z_{s,s+h} = \overleftarrow{\mathsf{Exp}}(X_{s+h} - X_s) = \overleftarrow{\mathsf{Exp}}(X_t - X_s).$$

The proof of Eq. (3.3b) is analogous. Eq. (3.3d) has already been proven in Theorem 1 in [5]. The proof of Eq. (3.3e) is similar: By definition of U and Eq. (2.7a) we have $[U,X]^c_t = -[X,X]^c_t$ and

$$\Delta U_t = -\Delta X_t + (I+\Delta X_t)^{-1} - I + \Delta X_t = (I+\Delta X_t)^{-1} - I. \tag{3.4}$$

From this it follows that

$$\begin{aligned} &X_t + U_t + [U,X]_t \\ &= X_t - X_t + [X,X]^c_t + \sum_{0<s\leq t}\Big((I+\Delta X_s)^{-1} - I + \Delta X_s\Big) + [U,X]^c_t \\ &\quad + \sum_{0<s\leq t}(\Delta U_s)(\Delta X_s) \end{aligned}$$

$$
\begin{aligned}
&= X_t - X_t + [X,X]^c_t - [X,X]^c_t \\
&\quad + \sum_{0<s\le t} \Big((I+\Delta X_s)^{-1} - I + \Delta X_s + (I+\Delta X_s)^{-1}\Delta X_s - \Delta X_s \Big) \\
&= \sum_{0<s\le t} \Big((I+\Delta X_s)^{-1}(I+\Delta X_s) - I + \Delta X_s - \Delta X_s \Big) = \mathbf{0}.
\end{aligned}
$$

The integration by parts formula Eq. (2.5) together with Eq. (3.1a), Eq. (3.1b), and Eq. (2.8c) now yield

$$
\begin{aligned}
&\overleftarrow{\mathsf{Exp}}(U_t)\overrightarrow{\mathsf{Exp}}(X_t) \\
&= \overleftarrow{\mathsf{Exp}}(U_0)\overrightarrow{\mathsf{Exp}}(X_0) + \int_0^t \overleftarrow{\mathsf{Exp}}(U_{u-})\, \mathrm{d}\overrightarrow{\mathsf{Exp}}(X_u) + \int_0^t \mathrm{d}\overleftarrow{\mathsf{Exp}}(U_u)\, \overrightarrow{\mathsf{Exp}}(X_{u-}) \\
&\quad + \Big[\overleftarrow{\mathsf{Exp}}(U), \overrightarrow{\mathsf{Exp}}(X)\Big]_t \\
&= I + \int_0^t \overleftarrow{\mathsf{Exp}}(U_{u-})\, \mathrm{d}X_u\, \overrightarrow{\mathsf{Exp}}(X_{u-}) + \int_0^t \overleftarrow{\mathsf{Exp}}(U_{u-})\, \mathrm{d}U_u\, \overrightarrow{\mathsf{Exp}}(X_{u-}) \\
&\quad + \left[I + \int_0^t \overleftarrow{\mathsf{Exp}}(U_{u-})\, \mathrm{d}U_u\,, I + \int_0^t \mathrm{d}X_u\, \overrightarrow{\mathsf{Exp}}(X_{u-}) \right] \\
&= I + \int_0^t \overleftarrow{\mathsf{Exp}}(U_{u-})\, \mathrm{d}X_u\, \overrightarrow{\mathsf{Exp}}(X_{u-}) + \int_0^t \overleftarrow{\mathsf{Exp}}(U_{u-})\, \mathrm{d}U_u\, \overrightarrow{\mathsf{Exp}}(X_{u-}) \\
&\quad + \int_0^t \overleftarrow{\mathsf{Exp}}(U_{u-})\, \mathrm{d}[U,X]_u\, \overrightarrow{\mathsf{Exp}}(X_{u-}) \\
&= I + \int_0^t \overleftarrow{\mathsf{Exp}}(U_{u-})\, \mathrm{d}\underbrace{(X_u + U_u + [U,X]_u)}_{=\mathbf{0}} \overrightarrow{\mathsf{Exp}}(X_{u-}) = I,
\end{aligned}
$$

which shows that $\overrightarrow{\mathsf{Exp}}(X_t)^{-1} = \overleftarrow{\mathsf{Exp}}(U_t)$. Finally, Eq. (3.3f) follows from Eq. (3.4) because

$$
\det(I+\Delta U_t) = \det\Big((I+\Delta X_t)^{-1}\Big) = \frac{1}{\det(I+\Delta X_t)}.
$$

□

4. Closed Form Formula

We now prove a closed form formula for the stochastic exponential of a matrix-valued semimartingale which generalizes both the known formulas Eq. (1.1) for one-dimensional semimartingales with jump parts and Eq. (1.2) for matrix-valued continuous semimartingales. We follow the proof of Yan in [9], but the possibility of jumps now requires additional commutativity assumptions. We first specify in which sense semimartingales shall commute, which corresponds to Definition 1.1 in [9].

Definition 4.1. Let $H = (H_t)_{t\ge 0}$ and $Z = (Z_t)_{t\ge 0}$ be semimartingales in $\mathbb{R}^{n\times n}$. The pair (H, Z) is called commutative if for all $t \ge 0$

$$
\int_0^t H_{u-}\, \mathrm{d}Z_u = \int_0^t \mathrm{d}Z_u\, H_{u-}, \tag{4.1}
$$

which can also be written as $(H_- \cdot Z)_t = (Z : H_-)_t$ or $H_{t-}\,\mathrm{d}Z_t = \mathrm{d}Z_t\,H_{t-}$.

Since in [9] only continuous semimartingales are considered, we now have to integrate H_{u-} rather than H_u to ensure that we integrate a process with càglàd paths. We also recall Definition 6.1 in [9].

Definition 4.2. Let $r \in (0,\infty]$ and $z_0 \in \mathbb{C}$. Let $f : B_r(z_0) \longrightarrow \mathbb{C}$ be analytic with TAYLOR series $f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$ in $B_r(z_0)$ and let $A \in \mathbb{R}^{n\times n}$ with $\|A - z_0 I\| < r$. Then the matrix

$$f(A) \coloneqq \sum_{k=0}^{\infty} a_k (A - z_0 I)^k \tag{4.2}$$

is well-defined because the series converges absolutely.

The following proposition is a collection of the results in Theorem 4.1, 4.2, and 6.1 in [9] that lead to the closed form formula for the stochastic exponential of a matrix-valued continuous semimartingale.

Proposition 4.3. *Let $X = (X_t)_{t\geq 0}$ be a continuous semimartingale in $\mathbb{R}^{n\times n}$ such that (X, X) and $(X, [X, X])$ are commutative.*

(a) *For all integers $k > 1$*

$$\mathrm{d}X_t^k = kX_t^{k-1}\,\mathrm{d}X_t + \frac{k(k-1)}{2} X_t^{k-2}\,\mathrm{d}[X,X]_t\,, \tag{4.3a}$$

$$\mathrm{d}X_t^k = k\,\mathrm{d}X_t\,X_t^{k-1} + \frac{k(k-1)}{2}\,\mathrm{d}[X,X]_t\,X_t^{k-2}. \tag{4.3b}$$

(b) *For any analytic function $f : B_r(z_0) \longrightarrow \mathbb{C}$*

$$\mathrm{d}f(X_t) = f'(X_t)\,\mathrm{d}X_t + \frac{1}{2} f''(X_t)\,\mathrm{d}[X,X]_t\,, \tag{4.4a}$$

$$\mathrm{d}f(X_t) = \mathrm{d}X_t\,f'(X_t) + \frac{1}{2}\,\mathrm{d}[X,X]_t\,f''(X_t). \tag{4.4b}$$

(c) *If additionally $X_0 = \mathbf{0}$ and $V \coloneqq X + \frac{1}{2}[X, X]$, then $Y = \exp(X)$ is the unique solution of both the stochastic integral equations*

$$Y_t = I + \int_0^t Y_u\,\mathrm{d}V_u \quad \text{and} \quad Y_t = I + \int_0^t \mathrm{d}V_u\,Y_u. \tag{4.5}$$

(d) *If additionally $([X,X], X)$ and $([X,X],[X,X])$ are commutative as well, then*

$$\overleftarrow{\mathsf{Exp}}(X) = \overrightarrow{\mathsf{Exp}}(X) = \exp\left(X - \frac{1}{2}[X,X]\right). \tag{4.6}$$

The next four theorems generalize Theorem 4.3(a)-(d) and lead to a closed form formula for the stochastic exponential of a semimartingale in dimension $n \geq 2$.

Theorem 4.4. *Let $X = (X_t)_{t\geq 0}$ be a semimartingale in $\mathbb{R}^{n\times n}$ such that (X, X) and $(X, [X, X]^c)$ are commutative, $X_0 = \mathbf{0}$, and $X_t X_{t-} = X_{t-} X_t$ for all $t \geq 0$. Then for all integers $k > 1$*

$$X_t^k = k\int_0^t X_{u-}^{k-1}\,\mathrm{d}X_u + \frac{k(k-1)}{2}\int_0^t X_{u-}^{k-2}\,\mathrm{d}[X,X]_u^c$$

$$+ \sum_{0<s\leq t} \Big(\Delta X_s^k - k X_{s-}^{k-1} \Delta X_s\Big), \tag{4.7a}$$

$$X_t^k = k \int_0^t \mathrm{d}X_u \, X_{u-}^{k-1} + \frac{k(k-1)}{2} \int_0^t \mathrm{d}[X,X]_u^c \, X_{u-}^{k-2}$$
$$+ \sum_{0<s\leq t} \Big(\Delta X_s^k - k X_{s-}^{k-1} \Delta X_s\Big). \tag{4.7b}$$

Proof. We prove Eq. (4.7a) by induction over k. For $k = 2$, since (X, X) is commutative, the integration by parts formula Eq. (2.5) yields

$$\begin{aligned} X_t^2 &= \int_0^t X_{u-} \, \mathrm{d}X_u + \int_0^t \mathrm{d}X_u \, X_{u-} + [X,X]_t \\ &= \int_0^t X_{u-} \, \mathrm{d}X_u + \int_0^t X_{u-} \, \mathrm{d}X_u + [X,X]_t^c + \sum_{0<s\leq t} (\Delta X_s)^2 \\ &= 2\int_0^t X_{u-} \, \mathrm{d}X_u + \int_0^t \mathrm{d}[X,X]_u^c + \sum_{0<s\leq t} \Big(\Delta X_s^2 - 2X_{s-}\Delta X_s\Big). \end{aligned}$$

In the last step we used the additional assumption $X_s X_{s-} = X_{s-} X_s$ for all $s \geq 0$ to obtain

$$\begin{aligned} (\Delta X_s)^2 &= (X_s - X_{s-})^2 \\ &= X_s^2 - X_s X_{s-} - X_{s-} X_s + X_{s-}^2 \\ &= X_s^2 - X_{s-}^2 + 2X_{s-}^2 - 2X_s X_{s-} \\ &= \Delta X_s^2 - 2X_{s-} \Delta X_s. \end{aligned}$$

For the induction step we assume that Eq. (4.7a) holds for some $k > 1$. Then

$$\begin{aligned} [X^k, X]_t^c &= \left[k \int_0^t X_{u-}^{k-1} \, \mathrm{d}X_u + \frac{k(k-1)}{2} \int_0^t X_{u-}^{k-2} \, \mathrm{d}[X,X]_u^c \, , X_t \right]^c \\ &\quad + \underbrace{\left[\sum_{0<s\leq t} \Big(\Delta X_s^k - k X_{s-}^{k-1} \Delta X_s\Big), X_t \right]^c}_{=\mathbf{0}} \\ &= k \int_0^t X_{u-}^{k-1} \, \mathrm{d}[X,X]_u^c + \frac{k(k-1)}{2} \int_0^t X_{u-}^{k-2} \underbrace{\mathrm{d}\Big[[X,X]^c, X\Big]_u^c}_{=\mathbf{0}} \\ &= k \int_0^t X_{u-}^{k-1} \, \mathrm{d}[X,X]_u^c \, . \end{aligned}$$

The integration by parts formula Eq. (2.5) yields

$$\begin{aligned} &X_t^{k+1} = X_t^k X_t \\ &= \int_0^t X_{u-}^k \, \mathrm{d}X_u + \int_0^t \mathrm{d}X_u^k \, X_{u-} + [X^k, X]_t \\ &= \int_0^t X_{u-}^k \, \mathrm{d}X_u + \int_0^t \left(k X_{u-}^{k-1} \, \mathrm{d}X_u + \frac{k(k-1)}{2} X_{u-}^{k-2} \, \mathrm{d}[X,X]_u^c \right) X_{u-} \end{aligned}$$

$$
\begin{aligned}
&+ \int_0^t \mathrm{d}\Bigg(\sum_{0<s\le u} \Big(\Delta X_s^k - k X_{s-}^{k-1} \Delta X_s \Big) \Bigg) X_{u-} + [X^k, X]_t^c \\
&+ \sum_{0<s\le t} (\Delta X_s^k)(\Delta X_s) \\
&= \int_0^t X_{u-}^k \,\mathrm{d}X_u + k \int_0^t X_{u-}^k \,\mathrm{d}X_u + \frac{k(k-1)}{2} \int_0^t X_{u-}^{k-1} \,\mathrm{d}[X,X]_u^c \\
&+ \sum_{0<s\le t} \Big(\Delta X_s^k - k X_{s-}^{k-1} \Delta X_s \Big) X_{s-} + k \int_0^t X_{u-}^{k-1} \,\mathrm{d}[X,X]_u^c \\
&+ \sum_{0<s\le t} (\Delta X_s^k)(\Delta X_s) \\
&= (k+1) \int_0^t X_{u-}^k \,\mathrm{d}X_u + \frac{k(k+1)}{2} \int_0^t X_{u-}^{k-1} \,\mathrm{d}[X,X]_u^c \\
&+ \sum_{0<s\le t} \Big((\Delta X_s^k - k X_{s-}^{k-1} \Delta X_s) X_{s-} + (\Delta X_s^k)(\Delta X_s) \Big) \\
&= (k+1) \int_0^t X_{u-}^k \,\mathrm{d}X_u + \frac{k(k+1)}{2} \int_0^t X_{u-}^{k-1} \,\mathrm{d}[X,X]_u^c \\
&+ \sum_{0<s\le t} \Big(\Delta X_s^{k+1} - (k+1) X_{s-}^k \Delta X_s \Big).
\end{aligned}
$$

Here the last step follows from

$$
\begin{aligned}
&(\Delta X_s^k - k X_{s-}^{k-1} \Delta X_s) X_{s-} + (\Delta X_s^k)(\Delta X_s) \\
&= \Big(X_s^k - X_{s-}^k - k X_{s-}^{k-1} (X_s - X_{s-}) \Big) X_{s-} + (X_s^k - X_{s-}^k)(X_s - X_{s-}) \\
&= X_s^k X_{s-} - X_{s-}^{k+1} - k X_{s-}^{k-1} X_s X_{s-} + k X_{s-}^{k+1} \\
&\quad + X_s^{k+1} - X_s^k X_{s-} - X_{s-}^k X_s + X_{s-}^{k+1} \\
&= -k X_{s-}^{k-1} X_s X_{s-} + k X_{s-}^{k+1} + X_s^{k+1} - X_{s-}^k X_s \\
&= X_s^{k+1} - X_{s-}^{k+1} + (k+1) X_{s-}^{k+1} - k X_{s-}^{k-1} X_s X_{s-} - X_{s-}^k X_s \\
&= X_s^{k+1} - X_{s-}^{k+1} + (k+1) X_{s-}^{k+1} - (k+1) X_{s-}^k X_s \\
&= X_s^{k+1} - X_{s-}^{k+1} - (k+1) X_{s-}^k (X_s - X_{s-}) \\
&= \Delta X_s^{k+1} - (k+1) X_{s-}^k \Delta X_s.
\end{aligned}
$$

The proof of Eq. (4.7b) is analogous. □

Theorem 4.5. *Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be analytic and let $Z = (Z_t)_{t\ge 0}$ be a semimartingale in $\mathbb{R}^{n\times n}$ such that (Z, Z) and $(Z, [Z, Z]^c)$ are commutative, $Z_0 = z_0 I$, and $Z_t Z_{t-} = Z_{t-} Z_t$ for all $t \ge 0$. Then*

$$
\begin{aligned}
f(Z_t) = f(Z_0) &+ \int_0^t f'(Z_{u-}) \,\mathrm{d}Z_u + \frac{1}{2} \int_0^t f''(Z_{u-}) \,\mathrm{d}[Z,Z]_u^c \\
&+ \sum_{0<s\le t} \Big(\Delta f(Z_s) - f'(Z_{s-}) \Delta Z_s \Big),
\end{aligned} \tag{4.8a}
$$

$$\begin{aligned} f(Z_t) = f(Z_0) + \int_0^t \mathrm{d}Z_u\, f'(Z_{u-}) + \frac{1}{2}\int_0^t \mathrm{d}[Z,Z]_u^c\, f''(Z_{u-}) \\ + \sum_{0<s\le t} \Big(\Delta f(Z_s) - f'(Z_{s-})\Delta Z_s\Big). \end{aligned} \tag{4.8b}$$

Proof. Let $f(z) = \sum_{k=0}^\infty a_k(z-z_0)^k$ be the TAYLOR series of f and let $X_t \coloneqq Z_t - z_0 I$. Then $\mathrm{d}X_t = \mathrm{d}Z_t$, $\mathrm{d}[X,X]_t^c = \mathrm{d}[Z,Z]_t^c$, and $\Delta X_t = \Delta Z_t$. X satisfies the assumptions of Theorem 4.4 because

$$\begin{aligned} X_0 &= Z_0 - z_0 I = z_0 I - z_0 I = \mathbf{0}, \\ X_t X_{t-} &= Z_t Z_{t-} - z_0 Z_t - z_0 Z_{t-} + z_0^2 I \\ &= Z_{t-} Z_t - z_0 Z_{t-} - z_0 Z_t + z_0^2 I = X_{t-} X_t, \\ X_{t-}\,\mathrm{d}X_t &= Z_{t-}\,\mathrm{d}Z_t - z_0 I\,\mathrm{d}Z_t \\ &= \mathrm{d}Z_t\, Z_{t-} - \mathrm{d}Z_t\, z_0 I = \mathrm{d}X_t\, X_{t-}, \\ X_{t-}\,\mathrm{d}[X,X]_t^c &= Z_{t-}\,\mathrm{d}[Z,Z]_t^c - z_0 I\,\mathrm{d}[Z,Z]_t^c \\ &= \mathrm{d}[Z,Z]_t^c\, Z_{t-} - \mathrm{d}[Z,Z]_t^c\, z_0 I = \mathrm{d}[X,X]_t^c\, X_{t-}. \end{aligned}$$

We can thus apply Eq. (4.7a), which obviously also holds true for $k \in \{0,1\}$, to every power $(Z_t - z_0 I)^k = X_t^k$ in the TAYLOR series of $f(Z_t)$. This results in

$$\begin{aligned} f(Z_t) &= \sum_{k=0}^\infty a_k (Z_t - z_0 I)^k = \sum_{k=0}^\infty a_k X_t^k \\ &= \sum_{k=0}^\infty a_k \Bigg(k \int_0^t X_{u-}^{k-1}\,\mathrm{d}X_u + \frac{k(k-1)}{2}\int_0^t X_{u-}^{k-2}\,\mathrm{d}[X,X]_u^c \\ &\quad + \sum_{0<s\le t}\Big(\Delta X_s^k - k X_{s-}^{k-1}\Delta X_s\Big)\Bigg) \\ &= \int_0^t \left(\sum_{k=1}^\infty k a_k X_{u-}^{k-1}\right)\mathrm{d}X_u + \frac{1}{2}\int_0^t \left(\sum_{k=2}^\infty k(k-1) a_k X_{u-}^{k-2}\right)\mathrm{d}[X,X]_u^c \\ &\quad + \sum_{0<s\le t}\left(\sum_{k=0}^\infty a_k X_s^k - \sum_{k=0}^\infty a_k X_{s-}^k - \left(\sum_{k=1}^\infty k a_k X_{s-}^{k-1}\right)\Delta X_s\right) \\ &= \int_0^t \left(\sum_{k=1}^\infty k a_k (Z_{u-} - z_0 I)^{k-1}\right)\mathrm{d}Z_u \\ &\quad + \frac{1}{2}\int_0^t \left(\sum_{k=2}^\infty k(k-1) a_k (Z_{u-} - z_0 I)^{k-2}\right)\mathrm{d}[Z,Z]_u^c \\ &\quad + \sum_{0<s\le t}\Bigg(\sum_{k=0}^\infty a_k (Z_s - z_0 I)^k - \sum_{k=0}^\infty a_k (Z_{s-} - z_0 I)^k \\ &\quad - \left(\sum_{k=1}^\infty k a_k (Z_{s-} - z_0 I)^{k-1}\right)\Delta Z_s\Bigg) \end{aligned}$$

$$= \int_0^t f'(Z_{u-})\,\mathrm{d}Z_u + \frac{1}{2}\int_0^t f''(Z_{u-})\,\mathrm{d}[Z,Z]^c_u + \sum_{0<s\le t}\Big(\Delta f(Z_s) - f'(Z_{s-})\Delta Z_s\Big).$$

The proof of Eq. (4.8b) is analogous. □

Theorem 4.6. *Let $Z = (Z_t)_{t\ge 0}$ be a semimartingale in $\mathbb{R}^{n\times n}$ such that (Z, Z), $(Z, [Z, Z]^c)$ are commutative, $Z_0 = \mathbf{0}$, and $Z_t Z_{t-} = Z_{t-} Z_t$ for all $t \ge 0$. Let*

$$V_t := Z_t + \frac{1}{2}[Z,Z]^c_t + \sum_{0<s\le t}\Big(\exp(\Delta Z_s) - (I + \Delta Z_s)\Big). \tag{4.9}$$

Then $Y = \exp(Z)$ is the unique solution of both of the stochastic integral equations

$$Y_t = I + \int_0^t Y_{u-}\,\mathrm{d}V_u \quad \text{and} \quad Y_t = I + \int_0^t \mathrm{d}V_u\, Y_{u-}. \tag{4.10}$$

Proof. We use Theorem 4.5 with the analytic function $f : \mathbb{C} \longrightarrow \mathbb{C}$ given by $f(z) := \exp(z)$ to show that $Y = \exp(Z) = f(Z)$ is a solution of the first equation in Eq. (4.10). An application of Eq. (4.8a) to $f(Z_t)$ yields

$$\begin{aligned}
\exp(Z_t) = f(Z_t) &= f(Z_0) + \int_0^t f'(Z_{u-})\,\mathrm{d}Z_u + \frac{1}{2}\int_0^t f''(Z_{u-})\,\mathrm{d}[Z,Z]^c_u \\
&\quad + \sum_{0<s\le t}\Big(\Delta f(Z_s) - f'(Z_{s-})\Delta Z_s\Big) \\
&= I + \int_0^t \exp(Z_{u-})\,\mathrm{d}Z_u + \frac{1}{2}\int_0^t \exp(Z_{u-})\,\mathrm{d}[Z,Z]^c_u \\
&\quad + \sum_{0<s\le t}\Big(\Delta \exp(Z_s) - \exp(Z_{s-})\Delta Z_s\Big) \\
&= I + \int_0^t \exp(Z_{u-})\,\mathrm{d}\left(Z_u + \frac{1}{2}[Z,Z]^c_u\right) \\
&\quad + \sum_{0<s\le t}\Big(\exp(Z_s) - \exp(Z_{s-})(I + \Delta Z_s)\Big).
\end{aligned}$$

We now rewrite the remaining sum as a right stochastic integral of $\exp(Z_{u-})$ with respect to $\sum_{0<s\le u} M_s$ with $M_s := \exp(\Delta Z_s) - (I + \Delta Z_s)$. Note that $M_s = \mathbf{0}$ if $\Delta Z_s = \mathbf{0}$. Since $\Delta \sum_{0<s\le u} M_s = M_u$, we have

$$\begin{aligned}
&\sum_{0<s\le t}\Big(\exp(Z_s) - \exp(Z_{s-})(I + \Delta Z_s)\Big) \\
&= \sum_{0<s\le t} \exp(Z_{s-})\exp(Z_{s-})^{-1}\Big(\exp(Z_s) - \exp(Z_{s-})(I + \Delta Z_s)\Big) \\
&= \sum_{0<s\le t} \exp(Z_{s-})\Big(\exp(Z_{s-})^{-1}\exp(Z_s) - (I + \Delta Z_s)\Big) \\
&= \sum_{0<s\le t} \exp(Z_{s-})\Big(\exp(\Delta Z_s) - (I + \Delta Z_s)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{0<s\le t} \exp(Z_{s-}) M_s \\
&= \sum_{0<s\le t} \exp(Z_{s-}) \Delta\Bigg(\sum_{0<r\le s} M_r \Bigg) \\
&= \int_0^t \exp(Z_{u-})\, \mathrm{d}\Bigg(\sum_{0<s\le u} M_s \Bigg) .
\end{aligned}
$$

Combining both terms into one integral then results in

$$
\begin{aligned}
\exp(Z_t) &= I + \int_0^t \exp(Z_{u-})\, \mathrm{d}\left(Z_u + \frac{1}{2}[Z,Z]^c_u \right) \\
&\quad + \sum_{0<s\le t} \Big(\exp(Z_s) - \exp(Z_{s-})(I+\Delta Z_s) \Big) \\
&= I + \int_0^t \exp(Z_{u-})\, \mathrm{d}\left(Z_u + \frac{1}{2}[Z,Z]^c_u \right) + \int_0^t \exp(Z_{u-})\, \mathrm{d}\Bigg(\sum_{0<s\le u} M_s \Bigg) \\
&= I + \int_0^t \exp(Z_{u-})\, \mathrm{d}\Bigg(Z_u + \frac{1}{2}[Z,Z]^c_u + \sum_{0<s\le u} M_s \Bigg) \\
&= I + \int_0^t \exp(Z_{u-})\, \mathrm{d}V_u .
\end{aligned}
$$

The proof of the second equation in Eq. (4.10) is analogous. □

We are now ready to state our main result which generalizes Theorem 4.2 in [9] from continuous semimartingales to semimartingales with jump parts.

Theorem 4.7. *Let $X = (X_t)_{t\ge 0}$ be a semimartingale in $\mathbb{R}^{n\times n}$ such that $X_0 = \mathbf{0}$ and $\|\Delta X_t\| < 1$ for all $t > 0$. Let $L^X = (L^X_t)_{t\ge 0}$ be given by*

$$
L^X_t := \sum_{0<s\le t} \Big(\log(I+\Delta X_s) - \Delta X_s \Big). \tag{4.11}
$$

Assume that for all $s,t \ge 0$ we have

$$
X_t X_{t-} = X_{t-} X_t \quad , \quad \Delta X_t \Delta X_s = \Delta X_s \Delta X_t \quad , \quad [X,X]^c_t \Delta X_t = \Delta X_t [X,X]^c_t
$$

and that the pairs

$$
(X,X), (X,[X,X]^c), ([X,X]^c,X), ([X,X]^c,[X,X]^c), (L^X,X), (L^X,[X,X]^c)
$$

are commutative. Then the left and right stochastic exponential of X are equal and given by

$$
\overleftarrow{\mathsf{Exp}}(X_t) = \overrightarrow{\mathsf{Exp}}(X_t) = \exp\left(X_t - \frac{1}{2}[X,X]^c_t + L^X_t \right). \tag{4.12}
$$

Remark 4.8. For a continuous semimartingale X it holds that $X_{t-} = X_t$, $\Delta X_t = 0$, and $[X,X]^c_t = [X,X]_t$ for all $t \ge 0$. Our commutativity assumptions in Theorem 4.7 then reduce to the assumptions given in Theorem 4.2 of [9] and Eq. (4.12) reduces to Eq. (1.2). Due to $[X,X]^c_t = [X,X]_t - \sum_{0<s\le t}(\Delta X_s)^2$ we may also rewrite the right-hand side in Eq. (4.12) as

$$\exp\left(X_t - \frac{1}{2}[X,X]_t + \sum_{0<s\le t}\Big(\log(I+\Delta X_s) - \Delta X_s + \frac{1}{2}(\Delta X_s)^2\Big)\right).$$

In the one-dimensional case $n=1$ this simplifies to

$$\exp\left(X_t - \frac{1}{2}[X,X]_t\right)\prod_{0<s\le t}(1+\Delta X_s)\exp\left(-\Delta X_s + \frac{1}{2}(\Delta X_s)^2\right),$$

which is the well-known formula Eq. (1.1) for the one-dimensional stochastic exponential as in Theorem II.37 in [8].

Proof of Theorem 4.7. Since $\|\Delta X_t\| < 1$ for all $t \ge 0$, the matrix logarithm $\log(I+\Delta X_t)$ is defined for all $t\ge 0$ by Theorem 2.8 in [3], and we can define

$$Z_t \coloneqq X_t - \frac{1}{2}[X,X]^c_t + L^X_t = Z^{(1)}_t - \frac{1}{2}Z^{(2)}_t + Z^{(3)}_t,$$

where $Z^{(1)}_t = X_t$, $Z^{(2)}_t = [X,X]^c_t$, and

$$Z^{(3)}_t = L^X_t = \sum_{0<s\le t}\Big(\log(I+\Delta X_s) - \Delta X_s\Big) = \sum_{0<s\le t}\sum_{k=2}^{\infty}\frac{(-1)^{k+1}}{k}(\Delta X_s)^k \qquad (4.13)$$

with increments $\Delta Z^{(1)}_t = \Delta X_t$, $\Delta Z^{(2)}_t = \mathbf{0}$, and

$$\Delta Z^{(3)}_t = \log(I+\Delta X_t) - \Delta X_t = \sum_{k=2}^{\infty}\frac{(-1)^{k+1}}{k}(\Delta X_t)^k.$$

We verify that $Z = (Z_t)_{t\ge 0}$ fulfills the assumptions of Theorem 4.6. First, $Z_0 = \mathbf{0}$ is clear by definition. Second, the pair (Z,Z) is commutative because

$$\begin{aligned}
\big(Z^{(1)}_- \cdot Z^{(1)}\big)_t &= (X_- \cdot X)_t = (X : X_-)_t = \big(Z^{(1)} : Z^{(1)}_-\big)_t,\\
\big(Z^{(1)}_- \cdot Z^{(2)}\big)_t &= (X_- \cdot [X,X]^c)_t = ([X,X]^c : X_-)_t = \big(Z^{(1)} : Z^{(1)}_-\big)_t,\\
\big(Z^{(1)}_- \cdot Z^{(3)}\big)_t &= \sum_{0<s\le t} X_{s-}\Big(\log(I+\Delta X_s) - \Delta X_s\Big)\\
&= \sum_{0<s\le t}\sum_{k=2}^{\infty}\frac{(-1)^{k+1}}{k}X_{s-}(\Delta X_s)^k\\
&= \sum_{0<s\le t}\sum_{k=2}^{\infty}\frac{(-1)^{k+1}}{k}(\Delta X_s)^k X_{s-}\\
&= \sum_{0<s\le t}\Big(\log(I+\Delta X_s) - \Delta X_s\Big)X_{s-}\\
&= \big(Z^{(3)} : Z^{(1)}_-\big)_t,\\
\big(Z^{(2)}_- \cdot Z^{(1)}\big)_t &= ([X,X]^c \cdot X)_t = (X : [X,X]^c)_t = \big(Z^{(1)} : Z^{(2)}_-\big)_t,\\
\big(Z^{(2)}_- \cdot Z^{(2)}\big)_t &= ([X,X]^c \cdot [X,X]^c)_t = ([X,X]^c : [X,X]^c)_t = \big(Z^{(2)} : Z^{(2)}_-\big)_t,
\end{aligned}$$

$$\begin{aligned}
\big(Z^{(2)}_- \cdot Z^{(3)}\big)_t &= \sum_{0<s\le t} [X,X]^c_s \Big(\log(I+\Delta X_s) - \Delta X_s \Big) \\
&= \sum_{0<s\le t} \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} [X,X]^c_s (\Delta X_s)^k \\
&= \sum_{0<s\le t} \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} (\Delta X_s)^k [X,X]^c_s \\
&= \sum_{0<s\le t} \Big(\log(I+\Delta X_s) - \Delta X_s \Big) [X,X]^c_s \\
&= \big(Z^{(3)} : Z^{(2)}_-\big)_t, \\
\big(Z^{(3)}_- \cdot Z^{(1)}\big)_t &= (L^X_- \cdot X)_t = (X : L^X_-)_t = \big(Z^{(1)} : Z^{(3)}_-\big)_t, \\
\big(Z^{(3)}_- \cdot Z^{(2)}\big)_t &= (L^X_- \cdot [X,X]^c)_t = ([X,X]^c : L^X_-)_t = \big(Z^{(2)} : Z^{(3)}_-\big)_t, \\
\big(Z^{(3)}_- \cdot Z^{(3)}\big)_t &= \sum_{0<s\le t} \sum_{0<r\le s-} \Big(\log(I+\Delta X_r) - \Delta X_r \Big) \Big(\log(I+\Delta X_s) - \Delta X_s \Big) \\
&= \big(Z^{(3)} : Z^{(3)}_-\big)_t,
\end{aligned}$$

using the power series representation Eq. (4.13), and thus we get $(Z_- \cdot Z)_t = (Z : Z_-)_t$ for all $t \ge 0$. Third, the pair $(Z, [Z,Z]^c)$ is commutative because $[Z,Z]^c_t = [X,X]^c_t = Z^{(2)}_t$ for all $t \ge 0$ and thus

$$\begin{aligned}
(Z_- \cdot [Z,Z]^c)_t &= \big(Z^{(1)}_- \cdot Z^{(2)}\big)_t - \frac{1}{2}\big(Z^{(2)}_- \cdot Z^{(2)}\big)_t + \big(Z^{(3)}_- \cdot Z^{(2)}\big)_t \\
&= \big(Z^{(2)} : Z^{(1)}_-\big)_t - \frac{1}{2}\big(Z^{(2)} : Z^{(2)}_-\big)_t + \big(Z^{(2)} : Z^{(3)}_-\big)_t = ([Z,Z]^c : Z_-)_t.
\end{aligned}$$

Fourth, $Z_t Z_{t-} = Z_{t-} Z_t$ for all $t \ge 0$ because

$$\begin{aligned}
Z^{(1)}_t Z^{(1)}_{t-} &= X_t X_{t-} = X_{t-} X_t = Z^{(1)}_{t-} Z^{(1)}_t, \\
Z^{(2)}_t Z^{(2)}_{t-} &= [X,X]^c_t [X,X]^c_{t-} = [X,X]^c_t [X,X]^c_t = [X,X]^c_{t-} [X,X]^c_t = Z^{(2)}_{t-} Z^{(2)}_t, \\
Z^{(3)}_t Z^{(3)}_{t-} &= \sum_{0<s\le t} \sum_{0<r\le t-} \Big(\log(I+\Delta X_s) - \Delta X_s \Big) \Big(\log(I+\Delta X_r) - \Delta X_r \Big) \\
&= Z^{(3)}_{t-} Z^{(3)}_t,
\end{aligned}$$

again using the power series representation Eq. (4.13), and for the mixed terms we have

$$\begin{aligned}
& Z^{(1)}_t Z^{(2)}_{t-} + Z^{(2)}_t Z^{(1)}_{t-} \\
&= Z^{(2)}_{t-} Z^{(1)}_t + Z^{(1)}_{t-} Z^{(2)}_t + \Delta Z^{(1)}_t Z^{(2)}_{t-} - Z^{(2)}_{t-} \Delta Z^{(1)}_t + \Delta Z^{(2)}_t Z^{(1)}_{t-} - Z^{(1)}_{t-} \Delta Z^{(2)}_t \\
&= Z^{(2)}_{t-} Z^{(1)}_t + Z^{(1)}_{t-} Z^{(2)}_t + \underbrace{\Big(\Delta X_t [X,X]^c_t - [X,X]^c_t \Delta X_t \Big)}_{=\mathbf{0}} \\
&\quad + \underbrace{\Delta [X,X]^c_t Z^{(1)}_{t-}}_{=\mathbf{0}} - \underbrace{Z^{(1)}_{t-} \Delta [X,X]^c_t}_{=\mathbf{0}} \\
&= Z^{(2)}_{t-} Z^{(1)}_t + Z^{(1)}_{t-} Z^{(2)}_t
\end{aligned}$$

and

$$\begin{aligned}
&Z_t^{(1)} Z_{t-}^{(3)} + Z_t^{(3)} Z_{t-}^{(1)} \\
&= Z_{t-}^{(3)} Z_t^{(1)} + Z_{t-}^{(1)} Z_t^{(3)} + \Delta Z_t^{(1)} Z_{t-}^{(3)} - Z_{t-}^{(3)} \Delta Z_t^{(1)} + \Delta Z_t^{(3)} Z_{t-}^{(1)} - Z_{t-}^{(1)} \Delta Z_t^{(3)} \\
&= Z_{t-}^{(3)} Z_t^{(1)} + Z_{t-}^{(1)} Z_t^{(3)} + \sum_{0<s\le t-} \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} \underbrace{\Big((\Delta X_t)(\Delta X_s)^k - (\Delta X_s)^k (\Delta X_t) \Big)}_{=\mathbf{0}} \\
&\quad + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} \underbrace{\Big((\Delta X_t)^k X_{t-} - X_{t-} (\Delta X_t)^k \Big)}_{=\mathbf{0}} \\
&= Z_{t-}^{(3)} Z_t^{(1)} + Z_{t-}^{(1)} Z_t^{(3)}
\end{aligned}$$

and

$$\begin{aligned}
&Z_t^{(2)} Z_{t-}^{(3)} + Z_t^{(3)} Z_{t-}^{(2)} \\
&= Z_{t-}^{(3)} Z_t^{(2)} + Z_{t-}^{(2)} Z_t^{(3)} + \Delta Z_t^{(2)} Z_{t-}^{(3)} - Z_{t-}^{(3)} \Delta Z_t^{(2)} + \Delta Z_t^{(3)} Z_{t-}^{(2)} - Z_{t-}^{(2)} \Delta Z_t^{(3)} \\
&= Z_{t-}^{(3)} Z_t^{(2)} + Z_{t-}^{(2)} Z_t^{(3)} + \underbrace{\Delta [X,X]_t^c Z_{t-}^{(3)}}_{=\mathbf{0}} - \underbrace{Z_{t-}^{(3)} \Delta [X,X]_t^c}_{=\mathbf{0}} \\
&\quad + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} \underbrace{\Big((\Delta X_t)^k [X,X]_t^c - [X,X]_t^c (\Delta X_t)^k \Big)}_{=\mathbf{0}} \\
&= Z_{t-}^{(3)} Z_t^{(2)} + Z_{t-}^{(2)} Z_t^{(3)}
\end{aligned}$$

and thus

$$\begin{aligned}
Z_t Z_{t-} &= Z_t^{(1)} Z_{t-}^{(1)} + \frac{1}{4} Z_t^{(2)} Z_{t-}^{(2)} + Z_t^{(3)} Z_{t-}^{(3)} + \Big(Z_t^{(1)} Z_{t-}^{(3)} + Z_t^{(3)} Z_{t-}^{(1)} \Big) \\
&\quad - \frac{1}{2} \Big(Z_t^{(1)} Z_{t-}^{(2)} + Z_t^{(2)} Z_{t-}^{(1)} \Big) - \frac{1}{2} \Big(Z_t^{(2)} Z_{t-}^{(3)} + Z_t^{(3)} Z_{t-}^{(2)} \Big) \\
&= Z_{t-}^{(1)} Z_t^{(1)} + \frac{1}{4} Z_{t-}^{(2)} Z_t^{(2)} + Z_{t-}^{(3)} Z_t^{(3)} + \Big(Z_{t-}^{(3)} Z_t^{(1)} + Z_{t-}^{(1)} Z_t^{(3)} \Big) \\
&\quad - \frac{1}{2} \Big(Z_{t-}^{(2)} Z_t^{(1)} + Z_{t-}^{(1)} Z_t^{(2)} \Big) - \frac{1}{2} \Big(Z_{t-}^{(3)} Z_t^{(2)} + Z_{t-}^{(2)} Z_t^{(3)} \Big) \\
&= Z_{t-} Z_t .
\end{aligned}$$

Therefore Z fulfills all the assumptions of Theorem 4.6 and $Y = \exp(Z)$ is the unique solution of both of the stochastic integral equations in Eq. (4.10) with V given by Eq. (4.9). The equations $[Z,Z]_t^c = [X,X]_t^c$ and

$$\Delta Z_t = \Delta Z_t^{(1)} + \Delta Z_t^{(2)} + \Delta Z_t^{(3)} = \Delta X_t + \log(I + \Delta X_t) - \Delta X_t = \log(I + \Delta X_t)$$

show that

$$\begin{aligned}
V_t &= Z_t + \frac{1}{2}[Z,Z]_t^c + \sum_{0<s\le t} \Big(\exp(\Delta Z_s) - (I + \Delta Z_s) \Big) \\
&= X_t - \frac{1}{2}[X,X]_t^c + \sum_{0<s\le t} \Big(\log(I + \Delta X_s) - \Delta X_s \Big)
\end{aligned}$$

$$+\frac{1}{2}[X,X]_t^c + \sum_{0<s\le t}\Big(\exp\big(\log(I+\Delta X_s)\big) - \big(I+\log(I+\Delta X_s)\big)\Big)$$
$$= X_t + \sum_{0<s\le t}\Big(\log(I+\Delta X_s) - \Delta X_s + I + \Delta X_s - I - \log(I+\Delta X_s)\Big)$$
$$= X_t.$$

We conclude that $\overleftarrow{\mathsf{Exp}}(X_t) = \overleftarrow{\mathsf{Exp}}(V_t) = Y_t = \exp(Z_t)$ and $\overrightarrow{\mathsf{Exp}}(X_t) = \overrightarrow{\mathsf{Exp}}(V_t) = Y_t = \exp(Z_t)$. Plugging in the definition of Z yields Eq. (4.12). □

5. Examples

We give three examples of a semimartingale X in a matrix Lie group setting for which the assumptions of Theorem 4.7 are fulfilled and the stochastic exponential can be computed with Eq. (4.12). The explicit representations of these stochastic exponentials may be useful in interest rate modeling with jumps in the spirit of [7] or in the study of well-posedness of linear backward stochastic differential equations (BSDEs) in the context of [4]. For details on Lie groups and Lie algebras we refer to the monograph [3] of Hall.

Example 5.1. Let $M=(M_t)_{t\ge0}$ be a continuous real-valued semimartingale with $M_0=0$ and let $Y=(Y_t)_{t\ge0}$ be a real-valued semimartingale of pure jump type $Y_t=\sum_{0<s\le t}\Delta Y_s$ with $|\Delta Y_t|<1$ for all $t>0$ and with paths of finite variation on compacts. For example, we can take $M=B$ to be a standard Brownian motion and $Y=aN$ for some $a\in(-1,1)$ and a standard Poisson process N. Let $X=(X_t)_{t\ge0}$ be defined by

$$X_t \coloneqq \begin{pmatrix} Y_t & M_t \\ -M_t & Y_t \end{pmatrix} = Y_t I + M_t \boldsymbol{E},$$

where

$$\boldsymbol{E} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is the generator of the Lie algebra $\mathfrak{so}(2)$ of the special orthogonal group $\mathsf{SO}(2)$. We verify that X fulfills the assumptions of Theorem 4.7 and compute $\overleftarrow{\mathsf{Exp}}(X)$ and $\overrightarrow{\mathsf{Exp}}(X)$ with Eq. (4.12). For this purpose we observe that $\Delta X_t = \Delta Y_t I$,

$$[X,X]_t^c = \begin{pmatrix} [Y,Y]_t^c - [M,M]_t^c & 2[Y,M]_t^c \\ -2[Y,M]_t^c & [Y,Y]_t^c - [M,M]_t^c \end{pmatrix} = -[M,M]_t I,$$

and

$$L_t^X = \sum_{0<s\le t}\Big(\log(I+\Delta Y_s I) - \Delta Y_s I\Big) = \left(\sum_{0<s\le t}\log(1+\Delta Y_s) - Y_t\right) I$$

are multiples of the identity matrix. Hence for all $s,t\ge0$ we have

$$X_t X_{t-} = X_{t-}X_t \quad , \quad \Delta X_t \Delta X_s = \Delta X_s \Delta X_t \quad , \quad [X,X]_t^c \Delta X_t = \Delta X_t [X,X]_t^c$$

and the pairs $(X,[X,X]^c)$, $([X,X]^c,X)$, $([X,X]^c,[X,X]^c)$, (L^X,X), $(L^X,[X,X]^c)$ are easily shown to be commutative. Since we have $X_0=\mathbf{0}$ and $\|\Delta X_t\| = |\Delta Y_t| < 1$ by definition and the pair (X,X) is commutative because multiples of $\mathsf{SO}(2)$

elements commute, the assumptions of Theorem 4.7 are fulfilled and by Eq. (4.12) we get

$$\overleftarrow{\mathsf{Exp}}(X_t) = \overrightarrow{\mathsf{Exp}}(X_t) = \exp\left(X_t + \frac{1}{2}[M,M]_t I + \left(\sum_{0<s\leq t} \log(1+\Delta Y_s) - Y_t\right) I\right)$$
$$= \prod_{0<s\leq t}(1+\Delta Y_s) e^{\frac{1}{2}[M,M]_t} \exp(M_t \boldsymbol{E})$$
$$= \prod_{0<s\leq t}(1+\Delta Y_s) e^{\frac{1}{2}[M,M]_t} \begin{pmatrix} \cos(M_t) & \sin(M_t) \\ -\sin(M_t) & \cos(M_t) \end{pmatrix}.$$

In this example we see that $\overleftarrow{\mathsf{Exp}}(X_t) = \overleftarrow{\mathsf{Exp}}(Y_t I + M_t \boldsymbol{E}) = \mathsf{Exp}(Y_t)\overleftarrow{\mathsf{Exp}}(M_t \boldsymbol{E})$ as a combination of the one-dimensional formula Eq. (1.1) applied to Y and the formula Eq. (1.2) applied to the continuous semimartingale $M\boldsymbol{E}$, but it is not immediately clear from the definition of the stochastic exponential that $\overleftarrow{\mathsf{Exp}}(X_t) = \overleftarrow{\mathsf{Exp}}(Y_t I + M_t \boldsymbol{E})$ decomposes into $\overleftarrow{\mathsf{Exp}}(Y_t I) = \mathsf{Exp}(Y_t) I$ and $\overleftarrow{\mathsf{Exp}}(M_t \boldsymbol{E})$.

Example 5.2. Let $M^{(1)} = (M_t^{(1)})_{t\geq 0}$ and $M^{(2)} = (M_t^{(2)})_{t\geq 0}$ be two continuous real-valued semimartingales with $M_0^{(1)} = M_0^{(2)} = 0$ and let $Y = (Y_t)_{t\geq 0}$ be as in Theorem 5.1. Let $X = (X_t)_{t\geq 0}$ be defined by

$$X_t := \begin{pmatrix} Y_t & M_t^{(1)} & M_t^{(2)} \\ 0 & Y_t & M_t^{(1)} \\ 0 & 0 & Y_t \end{pmatrix} = Y_t I + M_t^{(1)}(\boldsymbol{E}_1 + \boldsymbol{E}_2) + M_t^{(2)} \boldsymbol{E}_3,$$

where

$$\boldsymbol{E}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \boldsymbol{E}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \boldsymbol{E}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are the generators of the LIE algebra $\mathfrak{h}$ of the HEISENBERG group H. In contrast to $\mathsf{SO}(2)$ in Theorem 5.1, H is a non-ABELian group, but since H is nilpotent it is still possible to explicitely calculate exponentials. We verify that X fulfills the assumptions of Theorem 4.7 and compute $\overleftarrow{\mathsf{Exp}}(X)$ and $\overrightarrow{\mathsf{Exp}}(X)$ with Eq. (4.12). As in Theorem 5.1 we get $\Delta X_t = \Delta Y_t I$,

$$[X,X]_t^c = \begin{pmatrix} [Y,Y]_t^c & 2[Y,M^{(1)}]_t^c & 2[Y,M^{(2)}]_t^c + [M^{(1)},M^{(1)}]_t^c \\ 0 & [Y,Y]_t^c & 2[Y,M^{(1)}]_t^c \\ 0 & 0 & [Y,Y]_t^c \end{pmatrix}$$
$$= [M^{(1)},M^{(1)}]_t \boldsymbol{E}_3,$$

and

$$L_t^X = \sum_{0<s\leq t}\Big(\log(I+\Delta Y_s I) - \Delta Y_s I\Big) = \left(\sum_{0<s\leq t}\log(1+\Delta Y_s) - Y_t\right) I.$$

Because of $\boldsymbol{E}_1\boldsymbol{E}_2 = \boldsymbol{E}_3$ and $\boldsymbol{E}_i\boldsymbol{E}_j = \boldsymbol{0}$ for $(i,j) \neq (1,2)$ we then have for all $s,t \geq 0$

$$X_t X_{t-} = X_{t-} X_t \quad , \quad \Delta X_t \Delta X_s = \Delta X_s \Delta X_t \quad , \quad [X,X]_t^c \Delta X_t = \Delta X_t [X,X]_t^c$$

and the pairs $(X,[X,X]^c)$, $([X,X]^c,X)$, $([X,X]^c,[X,X]^c)$, (L^X,X), $(L^X,[X,X]^c)$ are easily shown to be commutative. Since $X_0=\mathbf{0}$ and $\|\Delta X_t\|=|\Delta Y_t|<1$ by definition and the pair (X,X) is commutative because

$$
\begin{aligned}
X_-\cdot X &= \begin{pmatrix} Y_-\cdot Y & Y_-\cdot M^{(1)}+M^{(1)}\cdot Y & Y_-\cdot M^{(2)}+M^{(2)}\cdot Y\\ 0 & Y_-\cdot Y & Y_-\cdot M^{(1)}+M^{(1)}\cdot Y\\ 0 & 0 & Y_-\cdot Y\end{pmatrix}\\
&= \begin{pmatrix} Y:Y_- & Y:M^{(1)}+M^{(1)}:Y_- & Y:M^{(2)}+M^{(2)}:Y_-\\ 0 & Y:Y_- & Y:M^{(1)}+M^{(1)}:Y_-\\ 0 & 0 & Y:Y_-\end{pmatrix} = X:X_-,
\end{aligned}
$$

the assumptions of Theorem 4.7 are fulfilled and by Eq. (4.12) we get

$$
\begin{aligned}
&\overleftarrow{\mathsf{Exp}}(X_t)=\overrightarrow{\mathsf{Exp}}(X_t)\\
&=\exp\left(X_t-\frac12[M^{(1)},M^{(1)}]_t\boldsymbol{E}_3+\left(\sum_{0<s\le t}\log(1+\Delta Y_s)-Y_t\right)I\right)\\
&=\prod_{0<s\le t}(1+\Delta Y_s)\exp\left(M_t^{(1)}(\boldsymbol{E}_1+\boldsymbol{E}_2)+\left(M_t^{(2)}-\frac12[M^{(1)},M^{(1)}]_t\right)\boldsymbol{E}_3\right)\\
&=\prod_{0<s\le t}(1+\Delta Y_s)\begin{pmatrix}1 & M_t^{(1)} & M_t^{(2)}-\frac12[M^{(1)},M^{(1)}]_t+\frac12(M_t^{(1)})^2\\ 0 & 1 & M_t^{(1)}\\ 0 & 0 & 1\end{pmatrix}.
\end{aligned}
$$

Example 5.3. Let $B^{(1)}=(B_t^{(1)})_{t\ge0}$ and $B^{(2)}=(B_t^{(2)})_{t\ge0}$ be two independent one-dimensional BROWNian motions and let $Y=(Y_t)_{t\ge0}$ be as in Theorem 5.1. Let $X=(X_t)_{t\ge0}$ be defined by

$$X_t:=\begin{pmatrix}Y_t+B_t^{(1)} & B_t^{(2)}\\ 0 & Y_t+B_t^{(1)}\end{pmatrix}=(Y_t+B_t^{(1)})I+B_t^{(2)}\boldsymbol{E}_2,$$

where

$$\boldsymbol{E}_1=\begin{pmatrix}1&0\\0&0\end{pmatrix}\quad,\quad\boldsymbol{E}_2=\begin{pmatrix}0&1\\0&0\end{pmatrix}$$

are the generators of the LIE algebra $\mathfrak{aff}(2)$ of the affine group $\mathsf{Aff}(2)$. We again verify that X fulfills the assumptions of Theorem 4.7 and compute $\overleftarrow{\mathsf{Exp}}(X)$ and $\overrightarrow{\mathsf{Exp}}(X)$ with Eq. (4.12). As in Theorem 5.1 we see that $\Delta X_t=\Delta Y_t I$,

$$[X,X]_t^c=\begin{pmatrix}[Y+B^{(1)},Y+B^{(1)}]_t^c & 2[Y+B^{(1)},B^{(2)}]_t^c\\ 0 & [Y+B^{(1)},Y+B^{(1)}]_t^c\end{pmatrix}=[B^{(1)},B^{(1)}]_tI=tI,$$

since $[B^{(1)},B^{(2)}]_t=0$ for the independent BROWNian motions, and

$$L_t^X=\sum_{0<s\le t}\Big(\log(I+\Delta Y_sI)-\Delta Y_sI\Big)=\left(\sum_{0<s\le t}\log(1+\Delta Y_s)-Y_t\right)I$$

are multiples of the identity matrix. Hence for all $s,t\ge0$ we have

$$X_tX_{t-}=X_{t-}X_t\quad,\quad\Delta X_t\Delta X_s=\Delta X_s\Delta X_t\quad,\quad[X,X]_t^c\Delta X_t=\Delta X_t[X,X]_t^c$$

and the pairs $(X,[X,X]^c)$, $([X,X]^c,X)$, $([X,X]^c,[X,X]^c)$, (L^X,X), $(L^X,[X,X]^c)$ are easily shown to be commutative. Since $X_0=\mathbf{0}$ and $\|\Delta X_t\|=|\Delta Y_t|<1$ by definition and the pair (X,X) is commutative because

$$\begin{aligned} X_- \cdot X &= \begin{pmatrix} (Y_-+B^{(1)})\cdot(Y+B^{(1)}) & (Y_-+B^{(1)})\cdot B^{(2)}+B^{(2)}\cdot(Y+B^{(1)}) \\ 0 & (Y_-+B^{(1)})\cdot(Y+B^{(1)}) \end{pmatrix} \\ &= \begin{pmatrix} (Y+B^{(1)}):(Y_-+B^{(1)}) & (Y+B^{(1)}):B^{(2)}+B^{(2)}:(Y_-+B^{(1)}) \\ 0 & (Y+B^{(1)}):(Y_-+B^{(1)}) \end{pmatrix} \\ &= X:X_-, \end{aligned}$$

the assumptions of Theorem 4.7 are fulfilled and by Eq. (4.12) we get

$$\begin{aligned} \overleftarrow{\mathsf{Exp}}(X_t)=\overrightarrow{\mathsf{Exp}}(X_t) &= \exp\left(X_t-\frac{t}{2}I+\left(\sum_{0<s\le t}\log(1+\Delta Y_s)-Y_t\right)I\right) \\ &= e^{-\frac{t}{2}}\prod_{0<s\le t}(1+\Delta Y_s)\exp\Big(B_t^{(1)}\boldsymbol{E}_1+B_t^{(2)}\boldsymbol{E}_2\Big) \\ &= e^{-\frac{t}{2}}\prod_{0<s\le t}(1+\Delta Y_s)\begin{pmatrix} e^{B_t^{(1)}} & \frac{B_t^{(2)}}{B_t^{(1)}}(e^{B_t^{(1)}}-1) \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

References

1. Behme, A.: *Generalized Ornstein-Uhlenbeck Processes and Extensions*, Doc. Disser., Univ. of Braunschweig, 2011, available at: `http://www.digibib.tu-bs.de/?docid=00040009`
2. Behme, A.; Lindner, A.: Multivariate generalized Ornstein-Uhlenbeck processes, *Stoch. Process. Appl.* **122** (2012) 1487–1518.
3. Hall, B.: *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*, Second Edition, Springer Cham, 2015.
4. Jackson, J.: The reverse Hölder inequality for matrix-valued stochastic exponentials and applications to quadratic BSDE systems, arXiv preprint, 2022, available at: `https://doi.org/10.48550/arXiv.2202.13886`
5. Karandikar, R. L.: Multiplicative decomposition of nonsingular matrix valued semimartingales, *Séminaire de Probabilités de Strasbourg* **25** (1991) 262–269.
6. Müller, C.: *Multivariate generalized Ornstein-Uhlenbeck processes and connections to semi-selfsimilarity properties*, Doctoral Dissertation, Heinrich Heine University Düsseldorf, 2020, available at: `https://docserv.uni-duesseldorf.de/servlets/DocumentServlet?id=52460`
7. Park, F. C.; Chun, C. M.; Han, C. W.; Webber, N.: Interest rate models on Lie groups. *Quantitative Finance* **11**(4) (2011) 559–572.
8. Protter, P. E.: *Stochastic Integration and Differential Equations*, Second edition, Springer Berlin, 2005.
9. Yan, L.: Right and left matrix-valued stochastic exponentials and explicit solutions to systems of SDEs, *Stoch. Anal. Appl.* **30** (2012) 160–173.

Peter Kern: Mathematical Institute, Heinrich Heine University Düsseldorf, Universitätsstr. 1, D-40225 Düsseldorf, Germany
Email address: `kern@hhu.de`

Christian Müller: Mathematical Institute, Heinrich Heine University Düsseldorf, Universitätsstr. 1, D-40225 Düsseldorf, Germany
Email address: `c.mueller@hhu.de`