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STOCHASTIC MAGNETO-HYDRODYNAMIC SYSTEM PERTURBED BY GENERAL NOISE

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Abstract. The existence and uniqueness of solutions to the two-dimensional stochastic magneto-hydrodynamic system is established in the presence of either a multiplicative noise or an additive fractional Brownian noise. The method of monotonicity is employed when the noise is multiplicative. In the case of a fractional Brownian noise, a unique mild solution of the system is established under suitable conditions.

1. Introduction

Magneto-hydrodynamics refers to the study of motion of an electrically conductive fluid in the presence of a magnetic field. Magneto-hydrodynamic (MHD) system consists of the Navier-Stokes equations coupled in a certain manner with the Maxwell equations. The MHD system has been studied in a number of articles (see [2], [3], [17]). The stochastic MHD system was studied by us [14] using the jump-Markov process approximation scheme.

In the first part of this paper, our goal is to prove the existence and uniqueness of solutions to the two-dimensional MHD system perturbed by a multiplicative noise. Local monotonicity of the coefficients was established by Menaldi and Sritharan [8] which led to the solvability of the two-dimensional Navier-Stokes system with additive noise. The Minty-Browder argument was used by us [15] to establish the existence and uniqueness of the two-dimensional Navier-Stokes system with a small multiplicative noise. In our earlier works such as [15] and [7], the objective was to prove the Freidlin-Wentzell type large deviations result for solutions of stochastic Navier-Stokes equations and certain shell models of turbulence.

In this article, the method of monotonicity is employed to obtain a strong solution (in the sense of partial differential equations as well as stochastic analysis) of the stochastic MHD system perturbed by a suitable multiplicative force driven by a space-time Wiener process.

In the second part of this paper, the objective is to establish the solvability of the two-dimensional MHD system in the presence of an addtive fractional Brownian noise (fBn) term. Fractional Brownian motions are processes with long-range memory, and are not semimartingales. Therefore the usual notions of stochastic

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integrals, energy equality, and the ensuing martingale methods are not applicable in this context.

Tindel, Tudor and Viens [18] have studied stochastic heat equations with a fBn. Subsequently, we [6] have built the tools to establish the existence and uniqueness of mild solutions of the two-dimensional Navier-Stokes equations perturbed by a fBn. Our method is illustrated in the context of the two-dimensional MHD system.

The paper is organized as follows: The MHD system is cast as an evolution equation in section 2. The noise term driven by a Wiener process is also described in it. Local monotonicity of the coefficients, and a priori estimates for solutions are given in Section 3. Section 4 contains the proof of existence and uniqueness of strong solutions. Stochastic integration with respect to an infinite-dimensional fractional Brownian motion is discussed in section 5. A unique mild solution is shown to exist for the MHD system under suitable conditions.

2. The Stochastic MHD Problem

Suppose that $G \subset \mathbf{R}^2$ is a bounded, open, simply connected domain with a smooth boundary ∂G. The stochastic MHD system (see Temam [17]) in the non-dimensional form is given by

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v - \frac{1}{R_e}\Delta v - S(B \cdot \nabla)B + \nabla(p + \frac{S|B|^2}{2}) = \sigma_1(v, B)\,\partial W_1(t), \quad (2.1)$$

where v denotes the velocity, B, the magnetic field, p, the pressure field, and R_e, the Reynolds number. Let $S := \frac{M^2}{R_e R_m}$, where M is the Hartman number and R_m is the magnetic Reynolds number. The Maxwell equation is

$$\frac{\partial B}{\partial t} + (v \cdot \nabla)B + \frac{1}{R_m}\text{curl}(\text{curl } B) - (B \cdot \nabla)v = \sigma_2(v, B)\,\partial W_2(t), \quad (2.2)$$

where $\nabla \cdot v = 0$ and $\nabla \cdot B = 0$. The processes W_1 and W_2 are independent Wiener processes taking values in a divergence-free function space.

It is worthwhile to remember that for a vector-valued function $u = (u_1, u_2)$ defined on G,

$$\text{curl } u = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}.$$

If ϕ is a scalar-valued function defined on G,

$$\text{curl } \phi = (\frac{\partial \phi}{\partial y}, -\frac{\partial \phi}{\partial x}).$$

Equations (2.1), (2.2) are equipped with the following boundary conditions: $v = 0$ on ∂G ; $B \cdot \mathbf{n} = 0$ and curl $B = 0$ on ∂G, where $\mathbf{n}$ is the unit outer normal on ∂G. The initial conditions are $v(x, 0) = v_0(x)$ and $B(x, 0) = B_0(x)$ for $x \in G$.

The system needs to be written as an abstract evolution equation which in the integral form reads is given by

$$\mathbf{y}(t) = \mathbf{y}(0) + \int_0^t [-\mathbf{A}\mathbf{y}(s) - \mathbf{B}(\mathbf{y}(s))]\,ds + \int_0^t \sigma(\mathbf{y}(s))\,dW(s), \quad (2.3)$$

where $\mathbf{y}$ denotes the transpose of (v, B). The noise term σdW is the transpose of $\{\sigma_1 dW_1, \sigma_2 dW_2\}$. As in Sermange and Temam [12], we will begin with the choice of function spaces.

Set $H = H_1 \times H_2$ and $V = V_1 \times V_2$, where

$$\begin{aligned} H_1 &= \{\phi \in L^2(G); \nabla \cdot \phi = 0,\ \phi \cdot \mathbf{n}|_{\partial G} = 0\} \\ H_2 &= H_1 \\ V_1 &= \{\phi \in H_0^1(G) : \nabla \cdot \phi = 0\} \\ V_2 &= \{\phi \in H^1(G) : \nabla \cdot \phi = 0,\ \phi \cdot \mathbf{n}|_{\partial G} = 0\} \end{aligned}$$

where

$$\begin{aligned} H^1 &= \{u \in L^2(G) : \nabla u \in L^2(G)\} \\ H_0^1 &= \{u \in L^2(G) : \nabla u \in L^2(G) \text{ and } u|_{\partial G} = 0\} \end{aligned}$$

H_1 and H_2 are equipped with the $L^2(G)$ norm. Define the inner product on H by $[\mathbf{y}_1, \mathbf{y}_2] = (v_1, v_2)_{H_1} + S(B_1, B_2)_{H_2}$ where $\mathbf{y}_i = \{v_i, B_i\}$ The above inner product is equivalent to $(\mathbf{y}_1, \mathbf{y}_2)_H = (v_1, v_2)_{H_1} + (B_1, B_2)_{H_2}$. The norm on H is given by $||\mathbf{y}||_H = \sqrt{(y, y)_H}$. The space V_1 is endowed with the inner product $[|\phi, \psi|]_{V_1} = (\nabla\phi, \nabla\psi)_{L^2(G)}$. The norm on V_1 is given by $||\phi||_{V_1} = \sqrt{[|\phi, \phi|]_{V_1}}$. Note that the V_1 norm given here is equivalent to the usual $H^1(G)$ norm since $||\nabla\phi||_{L^2(G)} \geq C||\phi||_{L^2(G)}$ for a suitable C by the Poincaré inequality.

Let V_2 be endowed with the inner product $[|\phi, \psi|]_{V_2} = (\operatorname{curl}\phi, \operatorname{curl}\psi)_{L^2(G)}$ so that the norm on V_2 is given by $||\phi||_{V_2} = \sqrt{[|\phi, \phi|]_{V_2}}$. From Proposition 1.8 and Lemma 1.6 of [16], we can conclude that the norm given by $\{||\phi||^2_{L^2(G)} + ||\operatorname{curl}\phi||^2_{L^2(G)} + ||\phi \cdot \mathbf{n}||^2_{H^{1/2}(\partial G)}\}^{1/2}$ is equivalent to the $H^1(G)$ norm.Thus on V_2, the norm defined above is equivalent to the $H^1(G)$ norm. We thus endow the space V with the scalar product:

$$[|\mathbf{y}_1, \mathbf{y}_2|] = [|v_1, v_2|]_{V_1} + S[|B_1, B_2|]_{V_2}.$$

Having defined the spaces H and V, we have the dense, continuous and compact embedding:

$$V \hookrightarrow H = H' \hookrightarrow V'.$$

The next step consists in defining the operators $\mathbf{A}$ and $\mathbf{B}$ that appear in (2.3). $\mathbf{A}$ is defined through a bilinear coercive form, and $\mathbf{B}$ by means of a trilinear continuous form.

Define a function $a : V \times V \to \mathbf{R}$ as follows:

$$a(\mathbf{y}_1, \mathbf{y}_2) = \frac{1}{R_e}[|v_1, v_2|]_{V_1} + \frac{S}{R_m}[|B_1, B_2|]_{V_2} \tag{2.4}$$

Proposition 2.1. *The function a defined by (2.4) is continuous and coercive.*

Proof. Let R_e and R_m be equal to 1 without loss of generality. $||.||_{L^2(G)}$ will simply be denoted $||.||$ from now on. There exists a constant k such that

$$\begin{aligned} |a(\mathbf{y}_1, \mathbf{y}_2)| &= ||\nabla v_1||||\nabla v_2|| + S||\operatorname{curl} B_1||||\operatorname{curl} B_2|| \\ &\leq k(||v_1||_{H^1(G)}||v_2||_{H^1(G)} + S||B_1||_{H^1(G)}||B_2||_{H^1(G)}) \\ &\leq k(||v_1||^2_{H^1(G)} + ||B_1||^2_{H^1(G)})^{1/2}\ (||v_2||^2_{H^1(G)} + ||B_2||^2_{H^1(G)})^{1/2}. \end{aligned}$$

Hence $|a(\mathbf{y}_1,\mathbf{y}_2)| \le C||\mathbf{y}_1||_V\, ||\mathbf{y}_2||_V$ for $\mathbf{y}_1,\mathbf{y}_2 \in V$.

To establish the coercive property, consider

$$\begin{aligned} a(\mathbf{y},\mathbf{y}) &= ||\nabla v||^2 + S||\mathrm{curl}\, B||^2 \\ &\ge c(||v||^2_{H^1(G)} + ||B||^2_{H^1(G)}) \end{aligned}$$

for a constant c so that $a(\mathbf{y},\mathbf{y}) \ge c||\mathbf{y}||^2_V$. □

By the Lax-Milgram lemma, there exists an operator $\mathbf{A} : V \to V'$ such that $a(\mathbf{y},\mathbf{z}) = <\mathbf{A}\mathbf{y},\mathbf{z}>_{V'\times V}$ for all $\mathbf{y},\mathbf{z}\in V$.

Therefore $\mathbf{A} : V \to V'$ can be restricted to a self-adjoint operator $\mathbf{A} : \mathcal{D}(\mathbf{A}) \to H$.

We can write $\mathcal{D}(\mathbf{A}) = \mathcal{D}(\mathbf{A}_1)\times\mathcal{D}(\mathbf{A}_2)$ where $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathcal{D}(\mathbf{A}_1)$, and $\mathcal{D}(\mathbf{A}_2)$ are obtained as follows:

Consider the "Stokes" problem in G:

$$-\frac{1}{R_e}\Delta v + \nabla p = g \quad \text{with} \quad \nabla . v = 0, \quad \text{and} \quad v|_{\partial G} = 0.$$

This can be written as $\frac{1}{R_e}[|v,\phi|]_{V_1} = (g,\phi)$ for $\phi\in V_1$.

Define $\mathbf{A}_1 v = \frac{1}{R_e}[|v,\cdot|]_{V_1} = g \in V_1'$. By the Cattabriga regularity theorem, we can conclude that $\mathcal{D}(\mathbf{A}_1) = H^2(G)\cap V_1$ whenever $g\in H_1$.

To define $\mathbf{A}_2$, consider the elliptic problem in G:

$$\frac{1}{R_m}\mathrm{curl}\ \mathrm{curl}\, B = g \quad \text{with} \quad \nabla\cdot B = 0;\ B\cdot\mathbf{n}|_{\partial G} = 0;\ \mathrm{curl}\, B|_{\partial G} = 0.$$

so that $\mathbf{A}_2 B = \frac{1}{R_m}\mathrm{curl}\ \mathrm{curl}\, B = g$. It is shown in [12] that if $g\in H$, then

$$\mathcal{D}(\mathbf{A}_2) = \{u\in H^2(G) : \nabla\cdot u = 0, u\cdot\mathbf{n}|_{\partial G} = 0 \text{ and } \mathrm{curl}\, u|_{\partial G} = 0\}$$

.

The next step consists in defining $\mathbf{B}$ that figures in (2.3). Consider the trilinear form $b(\cdot,\cdot,\cdot) \colon V\times V\times V \to \mathbf{R}$ by

$$b(\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3) = \tilde{b}(v_1,v_2,v_3) - S\tilde{b}(B_1,B_2,v_3) + S\tilde{b}(v_1,B_2,B_3) - S\tilde{b}(B_1,v_2,B_3) \quad (2.5)$$

for all $\mathbf{y}_i = (v_i,B_i)\in V$ where $\tilde{b}(\cdot,\cdot,\cdot)\colon (H^1(G))^{\otimes 3}\to\mathbf{R}$ is defined by

$$\tilde{b}(\phi,\psi,\theta) = \sum_{i,j=1}^{2}\int_G \phi_i\frac{\partial\psi_j}{\partial x_i}\theta_j\, dx.$$

Proposition 2.2. *The function b defined by (2.5) is continuous.*

Proof. It suffices to show the continuity of $\tilde{b}$. Towards this note that

$$\begin{aligned} |\tilde{b}(\phi,\psi,\theta) &= |\sum_{i,j}\int_G \phi_i\frac{\partial\psi_j}{\partial x_i}\theta_j\, dx| \\ &\le ||\phi||_{L^4(G)}||\nabla\psi||_{L^2(G)}||\theta||_{L^4(G)} \end{aligned}$$

using the Hölder's inequality.

$H^1(G) \subset L^4(G)$ by the Sobolev embedding theorem, using which we get

$$|\tilde{b}(\phi, \psi, \theta)| \leq c\, ||\phi||_{H^1}\, ||\psi||_{H^1}\, ||\theta||_{H^1}$$

so that

$$\begin{aligned} |b(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3)| &\leq c\big(||v_1||_{H^1}\, ||v_2||_{H^1}\, ||v_3||_{H^1} + ||B_1||_{H^1}\, ||B_2||_{H^1}\, ||v_3||_{H^1} \\ &\quad + ||v_1||_{H^1}\, ||B_2||_{H^1}\, ||B_3||_{H^1} + ||B_1||_{H^1}\, ||v_2||_{H^1}\, ||B_3||_{H^1}\big) \\ &\leq c||\mathbf{y}_1||_V\, ||\mathbf{y}_2||_V\, ||\mathbf{y}_3||_V \end{aligned}$$

since $||\mathbf{y}||_V$ is equivalent to $(||v||^2_{H^1} + ||B||^2_{H^1})^{1/2}$. □

Note that for all $\phi \in H^1(G)$ and $\psi \in V_1$,

$$\begin{aligned} \tilde{b}(\phi, \psi, \psi) &= \sum_{i,j} \int_G \phi_i \frac{\partial \psi_j}{\partial x_i} \psi_j \, dx \\ &= 0 \end{aligned}$$

by integration by parts. Also, $\tilde{b}(\phi, \psi, \theta) = -\tilde{b}(\phi, \theta, \psi)$ if ψ or $\theta \in V_1$ and $\phi \in H^1(G)$. Therefore $b(\mathbf{y}, \mathbf{z}, \mathbf{z}) = 0$.

We now define $\mathbf{B}\colon V \times V \to V'$ as the continuous bilinear operator such that $b(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) =< \mathbf{B}(\mathbf{y}_1, \mathbf{y}_2), \mathbf{y}_3 >$ for all $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3 \in V$. The existence of such an operator is guaranteed by the Riesz representation theorem. Note that $\mathbf{B}(\mathbf{y})$ will denote $\mathbf{B}(\mathbf{y}, \mathbf{y})$.

The noise term in the stochastic MHD system in its integral form is given by $\int_0^t \sigma(r, \mathbf{y}(r))\, dW(r)$ where W is an H-valued Wiener process with a nuclear covariance form Q. In order to state the conditions on the noise coefficient σ, we develop the following:

Notation: From now on, we will use the notation $|y|$ to denote the H-norm of $\mathbf{y}$, and $||\mathbf{y}||$ to denote the V-norm of y.

Let $H_0 = Q^{1/2}H$. Then H_0 is a Hilbert space with the inner product

$$(\mathbf{y}, \mathbf{z})_0 = (Q^{-1/2}\mathbf{y}, Q^{-1/2}\mathbf{z})\ \forall\ \mathbf{y}, \mathbf{z} \in H_0 \tag{2.6}$$

Let $|\cdot|_0$ denote the norm in H_0. Clearly, the imbedding of H_0 in H is Hilbert-Schmidt since Q is a trace class operator.

Let L_Q denote the space of linear operators S such that $SQ^{1/2}$ is a Hilbert-Schmidt operator from H to H. Define the norm on the space L_Q by $|S|^2_{L_Q} = \mathrm{tr}\,(SQS^*)$. The noise coefficient $\sigma : [0, T] \times V \to L_Q(H_0; H)$ is such that it satisfies the following hypotheses :

(1) A.1. The function $\sigma \in C([0, T] \times V; L_Q(H_0; H))$
(2) A.2. For all $t \in (0, T)$, there exists a positive constant $K < 1$ such that $|\sigma(t, \mathbf{y})|^2_{L_Q} \leq K(1 + ||\mathbf{y}||^2)$.
(3) A.3. For all $t \in (0, T)$, there exists a positive constant $L < 1$ such that for all $\mathbf{y}, \mathbf{v} \in V$, $|\sigma(t, \mathbf{y}) - \sigma(t, \mathbf{v})|^2_{L_Q} \leq L||\mathbf{y} - \mathbf{v}||^2$.

Thus we are able to write the stochastic system in the form we can write the stochastic MHD system as

$$d\mathbf{y} + [\mathbf{A}\,\mathbf{y} + \mathbf{B}(\mathbf{y})]\, dt = \sigma(t, \mathbf{y})\, dW(t). \tag{2.7}$$

with $\mathbf{y}(0) \in H$.

Physically, the two-dimensional MHD system pertains to the case when the domain is a cylinder $G \times \mathbb{R}$ with G as a domain in $\mathbb{R}^2$, and all the quantities are independent of z. Thus v and B are parallel to the xy-plane.

3. A Priori Estimates

It is well-known that for any $\mathbf{y} \in V$,

$$||\mathbf{B}(\mathbf{y})||_{V'} \leq C\,|\mathbf{y}|\,||\mathbf{y}|| \tag{3.1}$$

where C is a constant that depends only on G. Since such constants do not play a crucial role in this paper, we will set $C = 1$. First, we prove a result on local monotonicity.

Proposition 3.1. *For a given $r > 0$, let B_r denote the $L^4(G)$ ball in V, i.e., $B_r = \{\mathbf{z} \in V : ||\mathbf{z}||_{L^4(G)} \leq r\}$. Define the nonlinear operator F on V by $F(\mathbf{y}) := -\mathbf{A}\mathbf{y} - \mathbf{B}(\mathbf{y})$. Then, the pair (F, σ) is monotone in B_r, i.e., for any $\mathbf{y} \in V$ and $\mathbf{z} \in B_r$, if $\mathbf{w}$ denotes $\mathbf{y} - \mathbf{z}$,*

$$\langle F(\mathbf{y}) - F(\mathbf{z}), \mathbf{w}\rangle - Cr^4\,|\mathbf{w}|^2 + |\sigma(t, \mathbf{y}) - \sigma(t, \mathbf{z})|^2_{L_Q} \leq 0 \tag{3.2}$$

for a suitable constant $C > 0$.

Proof. First, it is clear that $\langle \mathbf{A}\mathbf{w}, \mathbf{w}\rangle = ||\mathbf{w}||^2$. Using the bilinearity of the operator B, it follows that

$$\langle \mathbf{B}(\mathbf{y}), \mathbf{w}\rangle = -\langle \mathbf{B}(\mathbf{y}, \mathbf{w}), \mathbf{z}\rangle$$

Likewise, $\langle \mathbf{B}(\mathbf{z}), \mathbf{w}\rangle = -\langle \mathbf{B}(\mathbf{z}, \mathbf{w}), \mathbf{v}\rangle$. Using the two equations above, one obtains

$$\langle \mathbf{B}(\mathbf{y}) - \mathbf{B}(\mathbf{z}), \mathbf{w}\rangle = -\langle \mathbf{B}(\mathbf{w}), \mathbf{z}\rangle.$$

Using the Hölder inequality, and Sobolev embedding,

$$|\langle \mathbf{B}(\mathbf{y}) - \mathbf{B}(\mathbf{z}), \mathbf{w}\rangle| \leq \frac{\epsilon}{2}||\mathbf{w}||^2 + C_\epsilon |\mathbf{w}|^2\,||\mathbf{z}||^4_{L^4(G)}$$

for any $\epsilon > 0$. Here $C_\epsilon > 0$ is a constant that depends on ϵ. Using the definition of the operator F yields

$$\langle F(\mathbf{y}) - F(\mathbf{v}), \mathbf{w}\rangle \leq C_\epsilon r^4 |\mathbf{w}|^2 \tag{3.3}$$

since $||\mathbf{v}||_{L^4(G)} \leq r$. The proof is finished upon using condition (A.3) and choosing $\epsilon < 1 - L$. □

Next, we define the Galerkin approximations of the MHD system. Let $H_n := \text{span}\{e_1, e_2, \cdots, e_n\}$ where $\{e_j\}$ is any fixed orthonormal basis in H with each $e_j \in D(\mathbf{A})$. Let P_n denote the orthogonal projection of H to H_n. Define $W_n = P_n W$. Let $\sigma_n = P_n\sigma$. Define $\mathbf{y}_n$ as the solution of the following stochastic differential equation: For each $\mathbf{v} \in H_n$,

$$d(\mathbf{y}_n(t), \mathbf{v}) = (F(\mathbf{y}_n(t)), \mathbf{v})dt + (\sigma_n(t, \mathbf{y}_n(t))\,dW_n(t), \mathbf{v}) \tag{3.4}$$

with $\mathbf{y}_n(0) = P_n\mathbf{y}(0)$.

Proposition 3.2. *Let $E(|\mathbf{y}(0)|^2) < \infty$. Let $\mathbf{y}_n$ denote the unique strong solution of the finite system of equations (3.4) in $C([0, T] : H_n)$. Then,with K as in condition (A.2), the following estimates hold:*

(1) *For all* $0 \le t \le T$,

$$E(|\mathbf{y}_n(t)|^2) + \int_0^t E||\mathbf{y}_n(s)||^2\, ds \le E|\mathbf{y}(0)|^2 + KT \tag{3.5}$$

and

$$E(\sup_{0\le t\le T} |\mathbf{y}_n(t)|^2 + \int_0^T ||\mathbf{y}_n(s)||^2 ds) \le C \tag{3.6}$$

where C *is a constant that depends on* $E|\mathbf{y}(0)|^2$ *and* T.

(2) *Fix any* $\delta > 0$. *Then*

$$E|\mathbf{y}_n(t)|^2\, e^{-\delta t} + \int_0^t E||\mathbf{y}_n(s)||^2 e^{-\delta s}\, ds \le E|\mathbf{y}(0)|^2 + \frac{K}{\delta} \tag{3.7}$$

(3) *If* $E|\mathbf{y}(0)|^4 < \infty$, *then for all* $0 \le t \le T$,

$$\begin{aligned} E|\mathbf{y}_n(t)|^4\, e^{-\delta t} &+ 3\int_0^t E||\mathbf{y}_n(s)||^2 |\mathbf{y}_n(s)|^2 e^{-\delta s}\, ds \\ &\le E|\mathbf{y}(0)|^4 + \frac{4K}{\delta} \sup_{0\le t\le T} E(|\mathbf{y}_n(s)|^2) \end{aligned} \tag{3.8}$$

Proof. To prove (3.5), we use the Itô Lemma to obtain

$$\begin{aligned} (d|\mathbf{y}_n(t)|^2 + 2\nu\, ||\mathbf{y}_n(t)||^2 dt) &= \mathrm{tr}\,(\sigma_n(t, \mathbf{y}_n(t)) Q \sigma_n(t, \mathbf{y}_n(t)))dt \\ &\quad + 2(\sigma_n(t, \mathbf{y}_n(t)) dW_n(t), \mathbf{y}_n(t)). \end{aligned} \tag{3.9}$$

Stopping (3.9) at $\tau_N = \inf\{t : |\mathbf{y}_n(t)|^2 + \int_0^t ||\mathbf{y}_n(s)||^2\, ds > N\}$, one can take expectation, and use the hypothesis (A.2). A Gronwall argument allows us to get (3.5) till $t \wedge \tau_N$.

The proof of (3.6) requires us to take supremum in (3.9) on both sides upto $T \wedge \tau_N$, and then take expectation using the Burkholder-Davis-Gundy inequality. It also shows that $T \wedge \tau_N$ increases to T a.s. as $N \to \infty$. Taking the limit as $N \to \infty$ gives the inequality (3.6).

By applying the Itô Lemma to each of the functions $g(t, \xi) = e^{-\delta t}|\xi|^2$ and $h(t, \xi) = e^{-\delta t}|\xi|^4$, and proceeding as above, (3.7) and (3.8) are obtained. □

Proposition 3.3. *Let* $E|\mathbf{y}(0)|^4 < \infty$. *Then following inequality holds:*

$$E\{\sup_{0\le t\le T} |\mathbf{y}_n(t)|^4\, e^{-\delta t} + \int_0^T ||\mathbf{y}_n(t)||^2\, |\mathbf{y}_n(t)|^2\, e^{-\delta t}\, dt\} \le E|\mathbf{y}_n(0)|^4 + \frac{1}{\delta}$$

Proof. The proof follows by a use of the Burkholder-Davis-Gundy inequality, and the estimates given in the previous proposition. □

4. Existence and Uniqueness of Solutions

The method of monotonicity to prove existence of strong solutions of SPDEs was initiated by Pardoux [11] (also see Metivier [9]). Existence of strong solutions for Navier-Stokes equations with additive noise was first shown using local-monotonicity method by Menaldi and Sritharan [8]. Since a multiplicative noise appears in the system studied in this paper, the proof of existence and uniqueness of solutions is given in full.

Theorem 4.1. *Let $E|\mathbf{y}_0|^4 < \infty$. Under the conditions (A.1)-(A.3) on σ, there exists a strong solution of the following stochastic Navier-Stokes equation:*

$$d\mathbf{y} + [\mathbf{A}\,\mathbf{y} + \mathbf{B}(\mathbf{y})]\,dt = \sigma(t, \mathbf{y})\,dW(t). \tag{4.1}$$

in $L^2(\Omega : C(0,T;H)) \cap L^2(\Omega \times (0,T);V)$. The solution is pathwise unique.

Proof. Let $\Omega_T := \Omega \times [0,T]$. Using the estimates given in the previous section, it follows from the Banach-Alaoglu theorem that along a subsequence, the Galerkin approximations $\{\mathbf{y}_n\}$ have the following limits:

$$\mathbf{y}_n \to \mathbf{y} \text{ weakly in } L^2(\Omega_T, V)$$

$$\mathbf{y}_n \to \mathbf{y} \text{ weak star in } L^4(\Omega; L^\infty(0,T;H))$$

and

$$\mathbf{y}_n(T) \to \eta \text{ weakly in } L^2(\Omega; H).$$

Recall that $F(\mathbf{y}) := -\mathbf{A}\mathbf{y} - \mathbf{B}(\mathbf{y})$. Since $F(\mathbf{y}_n)$ is bounded in $L^2(\Omega_T, V^{'})$,

$$F(\mathbf{y}_n) \to F_0 \text{ weakly in } L^2(\Omega_T, V^{'})$$

and likewise

$$\sigma_n(\cdot, \mathbf{y}_n) \to S \text{ weakly in } L^2(\Omega_T, L_Q).$$

The assertion of the last statement holds since σ has linear growth (condition A.2) and $\mathbf{y}_n$ is bounded in $L^2(0,T;V)$ uniformly in n by the a-priori estimates.

Extend the equation (3.4) to an open interval $(-\delta, T+\delta)$ by setting the terms equal to 0 outside of the interval $[0,T]$. Let $\phi(t)$ be a function in $H^1(-\delta, T+\delta)$ with $\phi(0) = 1$. Define for all integers $j \geq 1$, $e_j(t) = \phi(t)e_j$ where $\{e_j\}$ is the fixed orthonormal sequence for H.

Using the Itô formula for the function $(\mathbf{y}_n(t), e_j(t))$, and letting $n \to \infty$, one obtains

$$\begin{aligned} -\int_0^T (\mathbf{y}(s), e_j'(s)ds)ds &= (\mathbf{y}_0, e_j) + \int_0^T \langle F_0(s), e_j\rangle \phi(s)ds + \\ & \int_0^T \phi(s)(S(s)dW(s), e_j) - (\eta, e_j)\phi(T) \end{aligned} \tag{4.2}$$

Choose a sequence of functions $\{\phi_k\}$ in the place of ϕ such that $\phi_k \to 1_{[0,T]}$ and the time derivative of ϕ_k converges to δ_t weakly as $k \to \infty$. Using ϕ_k in (4.2) in the place of ϕ and then letting $k \to \infty$,

$$\mathbf{y}(t) = \mathbf{y}(0) + \int_0^t F_0(s)ds + \int_0^t S(s)dW(s) \tag{4.3}$$

with $\mathbf{y}(T) = \eta$. Let $\mathbf{v} \in L^\infty(\Omega_T; H_m)$ for $m \leq n$. Define

$$r(t) = \int_0^t ||\mathbf{v}(s)||^4_{L^4(G)}ds.$$

Note that by monotonicity,

$$\begin{aligned} X_n(T) \quad := \quad & 2E\int_0^T \langle F(\mathbf{y}_n(s)) - F(\mathbf{v}(s)), \mathbf{y}_n(s) - \mathbf{v}(s)\rangle e^{-r(s)} ds \\ & +2E\int_0^T r'(s)e^{-r(s)}|\mathbf{y}_n(s) - \mathbf{v}(s)|^2 ds \\ & +E\int_0^T e^{-r(s)}|\sigma_n(s,\mathbf{y}_n(s)) - \sigma_n(s,\mathbf{v}(s))|^2_{L_Q} ds \\ & \leq 0. \end{aligned} \tag{4.4}$$

Let $X_n(T)$ be written as $Y_n + Z_n$ where

$$\begin{aligned} Y_n \quad = \quad & 2\big(E\int_0^T (\langle F(\mathbf{y}_n(s)), \mathbf{y}_n(s)\rangle + r'(s)\,|\mathbf{y}_n(s)|^2) e^{-r(s)} ds\big) \\ & +E\int_0^T e^{-r(s)}|\sigma_n(s,\mathbf{y}_n(s))|^2_{L_Q} ds \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} Z_n \quad = \quad & 2E\int_0^T (\langle F(\mathbf{v}(s)), \mathbf{v}(s) - \mathbf{y}_n(s)\rangle - \langle F(\mathbf{y}_n(s)), \mathbf{v}(s)\rangle) e^{-r(s)} ds \\ & +2E\int_0^T r'(s)e^{-r(s)}(|\mathbf{v}(s)|^2 - 2(\mathbf{y}_n(s), \mathbf{v}(s))) ds \\ & +E\int_0^T e^{-r(s)}\{|\sigma_n(s,\mathbf{v}(s))|^2_{L_Q} - 2(\sigma_n(\mathbf{y}_n(s), \sigma_n(\mathbf{v}(s)))_{L_Q}\} ds. \end{aligned} \tag{4.6}$$

By the Itô formula,

$$\begin{aligned} Y_n \quad & = \quad E(e^{-r(T)}|\mathbf{y}_n(T)|^2 - |\mathbf{y}_n(0)|^2) \\ & \geq \quad E(e^{-r(T)}|\mathbf{y}_n(T)|^2 - |\mathbf{y}_0|^2) \end{aligned} \tag{4.7}$$

Therefore,

$$\begin{aligned} \liminf_{n\to\infty} Y_n \quad \geq \quad & Ee^{-r(T)}|\mathbf{y}_n(T)|^2 - E|\mathbf{y}(0)|^2 \\ = \quad & 2E\int_0^T e^{-r(s)}\langle F_0(s), \mathbf{y}(s)\rangle ds + 2E\int_0^T r'(s)e^{-r(s)}|\mathbf{y}(s)|^2 ds \\ & +E\int_0^T e^{-r(s)}||S(s)||^2_{L_Q} ds + E\int_0^T e^{-r(s)}(S(s)dW(s), \mathbf{y}(s)) \end{aligned}$$

In Z_n, each term has a limit so that we can conclude that

$$\begin{aligned} & 2E\int_0^T e^{-r(s)}\langle F_0(s) - F(\mathbf{v}(s)), \mathbf{y}(s) - \mathbf{v}(s)\rangle ds \\ & +E\int_0^T (r'(s)|\mathbf{y}(s) - \mathbf{v}(s)|^2 + ||S(s) - \sigma(s,\mathbf{v}(s))||^2_{L_Q}) e^{-r(s)} ds \\ & \leq \liminf_{n\to\infty} X_n \\ & \leq 0. \end{aligned}$$

Take $\mathbf{v} = \mathbf{y}$ to see that $Z(s) = \sigma(s, \mathbf{y}(s))$. Take $\mathbf{v} = \mathbf{y} - \lambda \mathbf{w}$ with $\lambda > 0$. Then,

$$\lambda E \int_0^T \langle F_0(s) - F(\mathbf{y}(s) + \lambda \mathbf{w}(s)), \mathbf{w}(s) \rangle ds + \lambda^2 E \int_0^T r'(s) e^{-r(s)} |\mathbf{w}(s)|^2 ds$$

is less than or equal to 0. Dividing by λ on both sides of the inequality above, and letting $\lambda \to 0$, one obtains

$$E \int_0^T \langle F_0(s) - F(\mathbf{y}(s)), \mathbf{w}(s) \rangle ds \leq 0.$$

Since $\mathbf{w}$ is arbitrary, one can show that F_0 can be identified with $F(\mathbf{y}(s))$. Thus the existence of a strong solution has been proved.

In what follows, the proof of pathwise uniqueness is sketched. Let $\mathbf{y}_1$ and $\mathbf{y}_2$ be two solutions of the SNSE (2.7). For $i = 1, 2$, and $N > 0$, define $\tau_N^i := \inf\{t \leq T : |\mathbf{y}_i(t)| \geq N\}$. Let $\tau_N = \min\{\tau_N^1, \tau_N^2\}$.

$$P\{\tau_N < T\} \leq P\{\max\{\sup_{t \in [0,T]} |\mathbf{y}_1(t)|, \sup_{t \in [0,T]} |\mathbf{y}_2(t)|\} \geq N\} \leq \frac{C}{N^2} \tag{4.8}$$

by Chebyshev inequality. Thus $\lim_{N \to \infty} \tau_N = T$ a.s.

Let $\mathbf{w} = \mathbf{y}_1 - \mathbf{y}_2$, and $\sigma_{12} = \sigma(\mathbf{y}_1) - \sigma(\mathbf{y}_2)$. By the energy equality,

$$\begin{aligned} &|\mathbf{w}(t \wedge \tau_N)|^2 + 2 \int_0^{t \wedge \tau_N} ||\mathbf{w}(s)||^2 \, ds \\ &\leq -2 \int_0^{t \wedge \tau_N} b(\mathbf{w}(s), \mathbf{y}_1(s), \mathbf{w}(s)) \, ds + \int_0^{t \wedge \tau_N} \operatorname{tr}(\sigma_{12}(s) Q \sigma_{12}(s)) ds \\ &\quad +2 \int_0^{t \wedge \tau_N} (\mathbf{w}, \sigma_{12}(s) dW(s)). \end{aligned}$$

Let $k > 0$ be any constant. Define $\xi(t) := e^{-k \int_0^{t \wedge \tau_N}, ||\mathbf{y}_1(r)||^2 dr}$for all t. By applying the Itô formula for the function $\xi(t) \, |\mathbf{w}(t \wedge \tau_N)|^2$, and taking expectation, one obtains

$$\begin{aligned} &E e^{-k \int_0^{t \wedge \tau_N} \, ||\mathbf{y}_1(r)||^2 dr} \, |\mathbf{w}(t \wedge \tau_N)|^2 \\ &\leq L \operatorname{tr} Q \, E \int_0^{t \wedge \tau_N} e^{-k \int_0^{s \wedge \tau_N} \, ||\mathbf{y}_1(r)||^2 dr} \, |\mathbf{w}(s)|^2 \, ds \end{aligned}$$

An application of Gronwall's Lemma implies that $\mathbf{w}(t \wedge \tau_N) = 0$ a.s. Since $\tau_N \to T$ as $N \to \infty$, we get that $\mathbf{y}_1 = \mathbf{y}_2$ for all $t \in [0, T]$ a.s. □

Remark 4.2. :

(1) An additional external force $\mathbf{f}(t)dt$ can be introduced on the right side of (2.3) where $\mathbf{f} \in L^2([0, T] : H)$. The above proof carries over with very slight modifications.
(2) For the stochastic MHD system, a Freidlin-Wentzell type large deviations principle can be proved exactly as in our paper [15].

5. Fractional Brownian Motion

In this section, a quick introduction to Wiener integrals with respect to fractional Brownian motions is presented. A space-time fractional Brownian integral is built in order to study the MHD system perturbed by such a term. The symbol H that appears as a superscript in the following definition should not be confused with the Hilbert space H introduced earlier.

Definition 5.1. A centered Gaussian process $\{\beta_t^H\}$ is called a fractional Brownian motion with Hurst parameter $H \in (0,1)$ if its covariance function is given by

$$E\left[\left(\beta_t^H \beta_s^H\right)\right] = R_H(t,s) := \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right). \tag{5.1}$$

Let S be the set of all step functions on $[0,T]$. If $\phi = \sum_{j=0}^{n-1} a_j 1_{(t_j,t_{j+1}]}$, we define $\int_0^T \phi(s) d\beta_s^H := \sum_{j=0}^{n-1} a_j(\beta_{t_{j+1}}^H - \beta_{t_j}^H)$. Let $\mathcal{H}$ be the closure of S with respect to the inner product $\left\langle 1_{[0,t]}, 1_{[0,s]}\right\rangle_{\mathcal{H}} := R_H(t,s)$. Then $1_{[0,t]} \to \beta_t^H$ extends to an isometry between $\mathcal{H}$ and the $L^2(\Omega)$-closure of the linear span of $\{\beta_t^H : t \in [0,T]\}$. This extension is called the Wiener integral with respect to β^H, and is denoted by $\mathcal{H}\phi := \int_0^T \phi\,(s)\, d\beta^H\,(s) \in L^2\,(\Omega)$.

It should be noted that the Wiener integral of any function $\phi \in \mathcal{H}$ w.r.t. β^H is a centered Gaussian random variable, and that for $\phi, \psi \in \mathcal{H}$ we have that $\int_0^T \phi d\beta^H$ and $\int_0^T \psi d\beta^H$ are jointly Gaussian with covariance equal to $\langle \phi, \psi\rangle_{\mathcal{H}}$, thereby extending the Wiener integral for standard Brownian motion.

There is a connection between the standard Wiener process and fractional Brownian motions. One begins by noting that R_H is, by definition, a non-negative definite kernel, which means that there exists a kernel function K_H such that $R_H(t,s) = \int_0^{t\wedge s} K_H(t,u)K_H(s,u)du$. In fact, its expression is explicit (see [10]):

$$K_H(t,s) = c_H\left(\frac{t}{s}\right)(t-s)^{H-1/2} + s^{1/2-H}F\left(\frac{t}{s}\right)$$

where $F(z) = c_H(1/2-H)\int_0^{z-1} r^{H-3/2}\left(1-(1+r)^{H-1/2}\right)dr$. Using these facts, one proves that there exists a standard Brownian motion W such that

$$\beta_t^H = \int_0^t K_H(t,s)dW_s.$$

For $s < T$, if we define the adjoint operator K^* on a possible subset of $L^2([0,T])$ by

$$(K_T^*\phi)(s) = K(T,s)\phi(s) + \int_s^T (\phi(r)-\phi(s))\frac{\partial K}{\partial r}(r,s)dr,$$

a result of Alos, Mazet, Nualart [1] then guarantees that K_T^* is an isometry between $\mathcal{H}$ and $L^2[0,T]$, and that the Wiener integral w.r.t β^H can be represented in the following convenient way: for all $\phi \in \mathcal{H}$, $K_T^*\phi \in L^2[0,T]$ and

$$\int_0^T \phi(s)d\beta^H(s) = \int_0^T (K_T^*\phi)(s)dW_s$$

where the last integral is a Wiener integral w.r.t. standard Brownian motion. It is easy to check that $K_T^*[\phi\, 1_{[0,t]}] = K_t^*[\phi] 1_{[0,t]}$. Therefore,

$$\int_0^t \phi(s) d\beta^H(s) = \int_0^t (K_t^* \phi)(s) dW_s. \tag{5.2}$$

We will now define integrals of the form $\int_0^t \phi(s) d\mathbf{W}^H(s)$ where $\mathbf{W}^H$ is a cylindrical H-valued fBm. This $\mathbf{W}^H$ is an infinite-dimensional stochastic process with a fBm behavior in time, taking values in the Hilbert space H, with equal weights on all directions of H.

The operators $\mathbf{A}_i$, $i = 1, 2$, introduced in Section 2 have inverses from H_i into itself. The inverses are compact, self-adjoint with discrete eigenvalues, each with finite multiplicity and can accumulate only at infinity. The eigenvalues λ_n grow like n. Also, the set of eigenfunctions is complete. Let $\{e_n\}$ denote the complete orthonormal basis in H, formed by the eigenfunctions of the operator $\mathbf{A}$ on G. Define

$$\mathbf{W}^H(t) = \sum_{j=1}^{\infty} e_n\, \beta_n^H(t),$$

where $\left\{\beta_n^H\right\}_n$ is a family of IID scalar fBm's. Strictly speaking, $\mathbf{W}^H(t)$ is not a member of $L^2(\Omega, H)$, since its norm is infinite, but it will be easy to guarantee that an integral w.r.t. $\mathbf{W}^H$ will be.

Indeed, let $\{\phi(s) : s \in [0, T]\}$ be a deterministic measurable function such that for every s, $\phi(s) \in H$. So we can write $\phi(s)\, e_n = \sum_m (\phi(s)\, e_n, e_m)\, e_m$. We may now define

$$\begin{aligned}\int_0^t \phi(s)\, d\mathbf{W}^H(s) &:= \sum_{n=1}^{\infty} \int_0^t \phi(s)\, e_n\, d\beta_n^H(s) \\ &= \sum_{n=1}^{\infty} \int_0^t \left(K^*(\phi(\cdot)\, e_n)\right)(s)\; dW_n(s)\end{aligned}$$

which follows from the representation (5.2), in which W_n is the standard Brownian motion that represents β_n^H.

Since all the terms in the last expression above are independent centered Gaussian r.v.'s, we can immediately give a necessary and sufficient condition for the above integral to exist: it is a Gaussian random variable in $L^2(\Omega)$ if and only if

$$E\left[\left\|\int_0^t \phi(s)\, d\mathbf{W}^H(s)\right\|^2\right] = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \int_0^t |[K^*((\phi(\cdot)\, e_n, e_j))](s)|^2\, ds \tag{5.3}$$

is finite.

6. Mild Solutions

Consider the stochastic MHD equation

$$d\mathbf{y}(t) + [\mathbf{A}\,\mathbf{y}(t) + \mathbf{B}(\mathbf{y}(t))]\, dt = \Phi\, d\mathbf{W}^H(t) \tag{6.1}$$

which in the integral form reads as

$$\mathbf{y}(t) = \mathbf{y}(0) - \nu \int_0^t \mathbf{A}\,\mathbf{y}(s)\,ds - \int_0^t \mathbf{B}(\mathbf{y}(s))ds + \int_0^t \Phi d\mathbf{W}^H(s).$$

To write this in its evolution form, we will need $S(t)$ the semigroup generated by $\mathbf{A}$. In mild form,

$$\mathbf{y}(t) = S(t)\,\mathbf{y}(0) - \int_0^t S(t-s)\,\mathbf{B}(\mathbf{y}(s))ds + \int_0^t [S(t-s)\,\Phi]\,d\mathbf{W}^H(s). \tag{6.2}$$

The existence and uniqueness of mild solutions of stochastic evolution systems have been studied by a number of authors (cf. Da Prato and Zabczyk [4], Sohr [13], Temam [17]). Solvability of the stochastic MHD system consists in breaking up the system (6.2) into a linear stochastic system and a nonlinear partial differential equation.

In order to prove the main theorem, we need the following Lemmas to ensure that the convolution space-time stochastic integral that appears in (6.2) takes values in $L^4([0,T]\times G)$ a.s. The first Lemma is taken from [6], and is an improvement of a result in [18].

Lemma 6.1. *For any $\lambda, t \geq 0$, there is a constant $c_{t,H}$ such that*

$$\mathbf{E}\left[\left(\int_0^t e^{-\lambda(t-s)} d\beta^H(s)\right)^2\right] = |\mathbf{1}_{[0,t]} e^{-\lambda(t-\cdot)}|^2_{\mathcal{H}} \leq c_{t,H}\lambda^{-2H}.$$

In fact, there exists a constant $C(H)$ depending only on H such that $c_{t,H}$ is given as follows:

$$\begin{cases} c_{t,H} \leq C(H) \text{ for all } H > 1/2, \\ c_{t,H} \leq C(H)(1+t^{2H-1}) \text{ for all } H < 1/2. \end{cases}.$$

Proof. *Case 1:* $H > 1/2$. Using the notation in [18] (see equation (23) therein), we have for any $\lambda \geq 0$,

$$\begin{aligned} A(\lambda,t) := |\mathbf{1}_{[0,t]} e^{-\lambda(t-\cdot)}|^2_{\mathcal{H}} &= \int_0^{\lambda t} v^{2H-2} e^{-v}[1-e^{-2(t\lambda - v)}]dv \\ &\leq \int_0^\infty v^{2H-2}e^{-v}dv =: C_0(H). \end{aligned}$$

Case 2: $H < 1/2$. Using the notation in [18] (see the calculation immediately preceding equation (26) therein), we have for all $\lambda \geq 0$,

$$A(\lambda,t) := |\mathbf{1}_{[0,t]} e^{-\lambda(t-\cdot)}|^2_{\mathcal{H}} \leq B_1(\lambda,t) + B_2(\lambda,t)$$

where

$$B_1(\lambda,t) := \lambda^{-2H}c(H)\int_0^{2\lambda t} e^{-v}v^{2H-1}(t - v/(2\lambda))^{2H-1}dv$$

with the well-known constant $c(H)$ defined for instance in [5], Theorem 3.2, and

$$B_2(\lambda,t) := C(H)\int_0^t e^{-2\lambda s}(\int_0^s (e^{\lambda r}-1)r^{H-3/2}dr)^2 ds$$

where $C(H) := c(H)(H-1/2)$.

By a linear change of variables, and then using Lemma 2 in [18] with the constant K_A defined therein, we get

$$
\begin{aligned}
B_2(\lambda,t) &= C(H)t^{2H}\int_0^1 e^{-2\lambda ts}(\int_0^s (e^{\lambda tr}-1)r^{H-3/2}) \\
&\le C(H)K_{H-1/2}\, t^{2H}(\lambda t)^{-2H} \\
&=: C_2(H)\lambda^{-2H}.
\end{aligned}
$$

Now for the term B_1, splitting the integral up at the midpoint λt, and changing the variable for the second half of the interval, we can write,

$$
\begin{aligned}
B_1(t,\lambda) &\le c(H)\lambda^{-2H}[(t/2)^{2H-1}\int_0^\infty e^{-v}v^{2H-1}dv \\
&\quad + (\lambda t)^{2H-1}\int_0^{\lambda t} e^{-(2\lambda t-v)}(v/(2\lambda))^{2H-1}dv] \\
&\le c(H)\lambda^{-2H}(t/2)^{2H-1}[\int_0^\infty e^{-v}v^{2H-1}dv + e^{-\lambda t}(\lambda t)^{2H}/(2H)].
\end{aligned}
$$

The function $x \mapsto e^{-x}x^{2H}$ attains its maximum value of $e^{-2H}(2H)^{2H}$ on $\mathbf{R}_+$ at the point $x = 2H$. Therefore we can write

$$B_1(t,\lambda) \le \lambda^{-2H}c(H)(t/2)^{2H-1}d(H)$$

where $d(H) := \int_0^\infty e^{-v}v^{2H-1}dv + e^{-2H}(2H)^{2H}$. We now have for all $\lambda, t \ge 0$, when $H < 1/2$, with $c(H)$, $d(H)$, and $C_2(H)$ defined above,

$$A(\lambda,t) \le \lambda^{-2H}(2^{1-2H}c(H)d(H)t^{2H-1} + C_2(H)).$$

Gathering our results, the lemma now follows, with

$$C(H) = \max(C_0(H), 2^{1-2H}c(H)d(H), C_2(H)).$$

□

Using the above Lemma, one can establish the following Lemmas. The proofs are long and hence omitted. The interested reader can find full proofs of these results in [6].

Lemma 6.2. *Assume $H > 1/4$. The process $\{\int_0^t S(t-s)\Phi d\mathbf{W}_s^H \,:\, t \in [0,T]\}$, takes values in $L^4(\Omega \times [0,T] \times G)$ provided that*

$$\sum_n (\textstyle\sum_j \lambda_j^{-H+1/4}(\Phi e_n, e_j))^2 < \infty, \tag{6.3}$$

where $(\lambda_j, e_j)_j$ are the eigen-elements of $\mathbf{A}$.

Lemma 6.3. *Assume $H \in (1/8; 1/4]$. The process $\{\int_0^t S(t-s)\Phi d\mathbf{W}_s^H\,;\, t \in [0,T]\}$, takes values in $L^4(\Omega \times [0,T] \times G)$ provided that*

$$\sum_n \left(\sum_i |(\Phi e_n, e_i)|\,\lambda_i^{1/4}\right)^2 < \infty.$$

In Lemma 6.2, it is understood that the Hurst parameter H is not equal to $1/2$. The main theorem on existence and uniqueness of solutions to the MHD system perturbed by a fractional Brownian noise is given. Full details of the proof is given in [6].

Theorem 6.4. *Let $\{e_n : n \in \mathbf{N}\}$ be the orthonormal basis in the Hilbert space H of eigenfunctions of the operator $\mathbf{A}$. Under the following two conditions, there exists a unique mild solution of the stochastic MHD system, i.e. $\mathbf{P}$-almost surely, there is a unique solution in $L^4([0,T] \times G)$ to equation (6.2) driven by the infinite-dimensional fractional Brownian noise $\Phi \mathbf{W}^H$:*

1. *$H > 1/4$, and $\sum_n \left(\sum_j j^{1/4-H} |(\Phi e_n, e_j)|\right)^2 < \infty$, or*
2. *$1/8 < H \leq 1/4$, and $\sum_n \left(\sum_j j^{1/4} |(\Phi e_n, e_j)|\right)^2 < \infty$.*

Proof. Step 1: Consider the system

$$d\mathbf{y} + [\nu \mathbf{A}\, \mathbf{y} + \mathbf{B}(\mathbf{y})]\, dt = \Phi d\mathbf{W}_t^H.$$

In order to find the solution $\mathbf{y}$, we will use the previous theorems, which tell us how to find the unique evolution (mild) solution $\mathbf{z}(t)$ of

$$d\mathbf{z}\,(t) + \mathbf{Azdt} = \Phi d\mathbf{W}_t^H,$$

with $\mathbf{z}(0) = 0$. If $\mathbf{y}$ existed, say in a strong sense, we would denote $\mathbf{v} := \mathbf{y} - \mathbf{z}$, and notice that

$$\begin{aligned}
\frac{\partial \mathbf{v}}{\partial t} &= \frac{\partial \mathbf{y}}{\partial t} - \frac{\partial \mathbf{z}}{\partial t} \\
&= (-\mathbf{A}\mathbf{y} - \mathbf{B}(\mathbf{y}) + \Phi \frac{d\mathbf{W}^H}{dt}) - (-\mathbf{A}\mathbf{z} + \Phi \frac{d\mathbf{W}^H}{dt}) \\
&= -\mathbf{A}(\mathbf{y} - \mathbf{z}) - \mathbf{B}(\mathbf{y}) = -\mathbf{A}\mathbf{v} - \mathbf{B}(\mathbf{v} + \mathbf{z})
\end{aligned}$$

Therefore, with $\mathbf{z}$ given, solving for $\mathbf{y}$ in (6.2) would be equivalent to solving for $\mathbf{v}$ in

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{A}\mathbf{v} + \mathbf{B}(\mathbf{v} + \mathbf{z}) = 0 \tag{6.4}$$

with initial data $\mathbf{v}(0) = \mathbf{y}_0 \in H$.

It can be shown that the hypotheses on Φ guarantee the existence (and uniqueness) in $L^4(\Omega \times [0,T] \times G)$ of $\mathbf{z}$ as a mild solution of

$$\mathbf{z}\,(t) - \mathbf{z}\,(0) - \int_0^t \mathbf{A}\mathbf{z}\,(s)\, ds = \Phi \mathbf{W}^H\,(t) \tag{6.5}$$

which is given by the formula

$$\mathbf{z}(t) := \int_0^t S(t-s) \Phi d\mathbf{W}_s^H \tag{6.6}$$

Therefore the evolution equation (6.2) has a unique solution mild in that same space (starting from $\mathbf{y}_0$) if the evolution (mild) version of equation (6.4) admits a solution in $L^4(\Omega \times [0,T] \times G)$ as well. This evolution solution $\mathbf{v}$, when it exists in that space, satisfies

$$\mathbf{v}(t) = S(t)\mathbf{y}_0 - \int_0^t S(t-s)\mathbf{B}(\mathbf{v}(s) + \mathbf{z}(s)) ds \tag{6.7}$$

where $S(t) = e^{-t\mathbf{A}}$ is the semigroup generated by the operator $\mathbf{A}$. Let us introduce notation meant to signify that equation (6.7) is a fixed point problem:

$$\Lambda(\mathbf{w}) := S(t)\mathbf{y}_0 - \int_0^t S(t-s)\mathbf{B}(\mathbf{w}(s)+\mathbf{z}(s))ds.$$

Studying the properties of this operator Λ is the main object of the next step.

Step 2: Let $\mathbf{w} \in L^4([0,T]\times G)\cap V$. It easily follows that $\mathbf{B}(\mathbf{w}+\mathbf{z}) \in L^2(0,T;V')$.

Define the space $Y := L^\infty(0,T;H)\cap L^2(0,T;V)$. Using the Sobolev embedding theorem and the interpolation theorem, one obtains $\Lambda(\mathbf{w}) \in Y$, and in fact,

$$||\Lambda(\mathbf{w})||_Y \le ||\mathbf{B}(\mathbf{w}+\mathbf{z})||_{L^2(0,T;V')}.$$

Let L^4 denote $L^4([0,T]\times G) = L^4(0,T;L^4(G))$. We claim that for any $\mathbf{w}_1, \mathbf{w}_2 \in L^4([0,T]\times G)\cap V$, we have

$$|\Lambda(\mathbf{w}_1) - \Lambda(\mathbf{w}_2)|_{L^4} \le C\,|\mathbf{w}_1-\mathbf{w}_2|_{L^4}\left(|\mathbf{w}_1+\mathbf{z}|_{L^4} + |\mathbf{w}_2+\mathbf{z}|_{L^4}\right)$$

for a suitable constant C.

Indeed, by the basic estimates on $\mathbf{B}$, and the Jensen inequality, one gets

$$|\mathbf{B}(\mathbf{y}_1)-\mathbf{B}(\mathbf{y}_2)|_{L^2(0,T;V')} \le C|\mathbf{y}_1-\mathbf{y}_2|_{L^4}(|\mathbf{y}_1|_{L^4}+|\mathbf{y}_2|_{L^4})$$

and thus

$$\begin{aligned}&|\mathbf{B}(\Lambda(\mathbf{w}_1)+\mathbf{z})-\mathbf{B}(\Lambda(\mathbf{w}_2)+\mathbf{z})|_{L^2(0,T;V')}\\ &\le C|\Lambda(\mathbf{w}_1)-\Lambda(\mathbf{w}_2)|_{L^4}(|\Lambda(\mathbf{w}_1)+\mathbf{z}|_{L^4}+|\Lambda(\mathbf{w}_2)+\mathbf{z}|_{L^4}).\end{aligned}$$

Let $\mathbf{y}_j = \Lambda(\mathbf{w}_j)+\mathbf{z}$ for $j=1,2$. Using again the Sobolev embedding of $L^4(G)$ in $W^{1/2,2}$, we get

$$\begin{aligned}|\Lambda(\mathbf{w}_1)-\Lambda(\mathbf{w}_2)|_{L^4} \le& C|\mathbf{y}_1-\mathbf{y}_2|_{L^4}(|\mathbf{y}_1|_{L^4}+|\mathbf{y}_2|_{L^4})\\ =& C|\mathbf{w}_1-\mathbf{w}_2|_{L^4}(|\mathbf{w}_1+\mathbf{z}|_{L^4}+|\mathbf{w}_2+\mathbf{z}|_{L^4}).\end{aligned}$$

Step 3: From the previous step, we get that the operator

$$\Lambda : \begin{cases} L^4 \mapsto L^4 \\ \mathbf{w} \mapsto \Lambda(\mathbf{w}) := S(\cdot)\mathbf{y}_0 - \int_0^\cdot S(\cdot - s)\mathbf{B}(\mathbf{w}(s)+\mathbf{z}(s))ds \end{cases}$$

is well-defined as mapping L^4 to itself. By the hypotheses of the theorem, $\mathbf{z}$ is in $L^4(\Omega\times[0,T]\times G)$, which implies that $\mathbf{z}\in L^4$ almost surely. Fix any ω in this almost sure set. Note that

$$\Lambda(0) = S(\cdot)\mathbf{y}_0 - \int_0^\cdot S(\cdot - s)\mathbf{B}(\mathbf{z}(s))ds$$

is then a fixed function on $[0,T]\times G$, and a member of L^4. The fixed point theorem can be applied to $\{\Lambda^n(0) : n \ge 1\}$ in a small interval $[0,T_1]$. The unique fixed point of the map Λ is the unique solution in L of equation (6.7) restricted to $[0,T_1]$. The solution can be continued to the entire inerval $[0,T]$ by a standard argument for mild solutions. Therefore, for almost every ω, the unique solution exists on the entire time interval $[0,T]$, and belongs to L^4. □

Remark 6.5. In Equation (6.1), an additive force $\sigma(t)dW_t$ where W is an infinite-dimensional Wiener process can be introduced. If σ satisfies the condition that the process $\int_0^t S(t-s)\sigma(s)dW_s$ belongs to $L^4(\Omega \times [0,T] \times G)$, the existence and uniqueness of a mild solution follows from the above proof. To keep the discussion simple, we have avoided the introduction of such extra forces.

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