

6-1-2010

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### Recommended Citation

Rajput, Balram S (2010) "The asymptotic dependence behavior of Ornstein-Uhlenbeck semi-stable processes," *Communications on Stochastic Analysis*: Vol. 4 : No. 2 , Article 7.

DOI: 10.31390/cosa.4.2.07

Available at: <https://digitalcommons.lsu.edu/cosa/vol4/iss2/7>

## THE ASYMPTOTIC DEPENDENCE BEHAVIOR OF ORNSTEIN-UHLENBECK SEMI-STABLE PROCESSES

BALRAM S. RAJPUT

ABSTRACT. Let  $\mathbb{X} = \{X_t\}$  be an infinitely divisible stationary process. A good measure of the asymptotic dependence structure of  $\mathbb{X}$  is provided by the limit of  $\rho_{\mathbb{X}}(t)$  as  $t \rightarrow \infty$ , where  $\rho_{\mathbb{X}}(t)$  is equal to the joint characteristic function of  $(X_t, X_0)$  minus the product of the characteristic functions of  $X_t$  and  $X_0$ . An interesting case is when  $\rho_{\mathbb{X}}(t) \rightarrow 0$ ; which roughly says that, as time becomes large, the future of the random phenomenon (represented by  $\mathbb{X}$ ) is becoming independent of its past. In this paper, we study the rate of decay of  $\rho_{\mathbb{X}}(t)$  (as  $t \rightarrow \infty$ ) when  $\mathbb{X}$  is an Ornstein-Uhlenbeck  $(r, \alpha)$ -semi-stable process. The results obtained here generalize and complement the corresponding results for Ornstein-Uhlenbeck  $\alpha$ -stable and Ornstein-Uhlenbeck (Gaussian) processes.

### 1. Introduction

Let  $\mathbb{X} \equiv \{X_t : t \in R, \text{ the reals}\}$  be a real infinitely divisible stationary process. If  $\mathbb{X}$  is Gaussian then its covariance function describes the dependence behavior of  $\mathbb{X}$ . On the other hand, if  $\mathbb{X}$  is  $\alpha$ -stable or more generally  $(r, \alpha)$ -semi-stable then it has infinite variance and so covariance is not defined. To study the long range dependence structure of processes  $\mathbb{X}$ , in these and other similar infinitely divisible process cases, the following which is called the co-difference function of  $\mathbb{X}$  is considered in the literature

$$\rho_{\mathbb{X}}(\theta_1, \theta_2; t) = E \exp \{i(\theta_1 X_t + \theta_2 X_0)\} - E \exp \{i\theta_1 X_t\} E \exp \{i\theta_2 X_0\} \quad (1.1)$$

for  $\theta_1, \theta_2 \in R$ . Unlike the covariance function,  $\rho_{\mathbb{X}}(t) (\equiv \rho_{\mathbb{X}}(\theta_1, \theta_2; t))$  is always defined and its behavior as  $t \rightarrow \infty$  provides a good measure of the asymptotic dependence structure of  $\mathbb{X}$ . In fact, as is shown by Levy and Taqqu [1], the asymptotic behavior of  $\rho_{\mathbb{X}}$  is very useful in distinguishing between the long range dependence structure of several  $\alpha$ -stable processes (e.g., moving average, sub-Gaussian and real harmonizable). They also point out that when  $\mathbb{X}$  is Gaussian,  $\rho_{\mathbb{X}}$  is asymptotically proportional to the covariance when  $\rho_{\mathbb{X}}(t) \rightarrow 0$ .

The case of special interest (not only in the Gaussian case but also in the case of other infinitely divisible stationary processes) is when  $\rho_{\mathbb{X}}(t) \rightarrow 0$ ; this, as (1.1) suggests, roughly says that with time the future of the process is becoming independent of its past. In this paper we provide the rate of decay of  $\rho_{\mathbb{X}}(t)$  (as

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Received 2010-2-8; Communicated by the editors.

2000 *Mathematics Subject Classification*. Primary 60E02, 60G52; Secondary 60G10.

*Key words and phrases*. Ornstein-Uhlenbeck process, long range dependence, semi-stable and stable processes.

$t \rightarrow \infty$ ) when  $\mathbb{X}$  is an Ornstein-Uhlenbeck  $(r, \alpha)$ -semi-stable process. The results obtained generalize and complement the corresponding known results due to Levy and Taqqu [1] for Ornstein-Uhlenbeck  $\alpha$ -stable and Ornstein-Uhlenbeck (Gaussian) processes.

### 2. Preliminaries

As in [1], writing

$$A_{\mathbb{X}}(\theta_1, \theta_2) \equiv A_{\mathbb{X}} \equiv -\ln E \exp\{i\theta_1 X_t\} - \ln E\{i\theta_2 X_0\} \tag{2.1}$$

in (1.1) and factoring  $e^{-A_{\mathbb{X}}(\theta_1, \theta_2)}$  one writes  $\rho_{\mathbb{X}}$  as follows

$$\rho_{\mathbb{X}}(t) \equiv \rho_{\mathbb{X}}(\theta_1, \theta_2; t) \equiv e^{-A_{\mathbb{X}}(\theta_1, \theta_2)} \left[ e^{-I_{\mathbb{X}}(\theta_1, \theta_2; t)} - 1 \right], \tag{2.2}$$

where, of course,

$$I_{\mathbb{X}}(t) \equiv I_{\mathbb{X}}(\theta_1, \theta_2; t) \equiv -\ln E \exp\{i(\theta_1 X_t + \theta_2 X_0)\} + \ln E \exp\{i\theta_1 X_t\} + \ln E \exp\{i\theta_2 X_0\}. \tag{2.3}$$

Note that  $A_{\mathbb{X}}(\theta_1, \theta_2)$  does not depend on  $t$  because  $\mathbb{X}$  is stationary. We also note that the characteristic functions that arise here are exponential functions, so that  $\ln$  are just the exponents. Clearly, as is noted in [1], we have from (2.2) that  $\rho_{\mathbb{X}}(t) \rightarrow 0 \Leftrightarrow I_{\mathbb{X}}(t) \rightarrow 0$  (the limits are always as  $t \rightarrow \infty$ ), and when  $\rho_{\mathbb{X}}(t) \rightarrow 0$ ,

$$\rho_{\mathbb{X}}(t) \sim e^{-A(\theta_1, \theta_2)}(-I_{\mathbb{X}}(t)). \tag{2.4}$$

(For non-zero functions  $\phi, \psi$ , by  $\phi(t) \sim \eta\psi(t)$  we mean that  $\phi(t) \rightarrow 0, \psi(t) \rightarrow 0$  and  $\phi(t)\psi^{-1}(t)$  converges to a nonzero constant  $\eta$ ).

Let  $0 < r < 1$  and  $0 < \alpha < 2$ . We recall that an independently scattered random measure  $M$  is called an  $(r, \alpha)$ -semi-stable random measure (with Lebesgue control measure and (finite) spectral measure  $\sigma$  on  $\Delta \equiv \{s : r^{\frac{1}{\alpha}} < |s| \leq 1\}$ ) if, for every Borel set  $B$  of  $R$  with finite Lebesgue measure, the characteristic function  $\widehat{\mathcal{L}}_{M(B)}$  of  $M(B)$  is given by

$$\begin{aligned} \widehat{\mathcal{L}}_{M(B)}(u) &= \exp - \{ \text{Leb}(B) \int_{\Delta} k(us) \sigma(ds) \}, u \in R \\ &= \exp - \left\{ \int_{\Delta} \left( \int_R k(us I_B(x)) dx \right) \sigma(ds) \right\}, \end{aligned}$$

where

$$k(u) \equiv k_{r, \alpha}(u) \equiv \begin{cases} \sum_{n=-\infty}^{\infty} r^{-n} \{1 - \exp(ir^{n/\alpha} u)\} & \text{if } 0 < \alpha < 1, \\ \sum_{n=-\infty}^{\infty} r^{-n} \{1 - \exp(ir^{n/\alpha} u) + ir^{n/\alpha} u\} & \text{if } 1 < \alpha < 2, \\ \sum_{n=-\infty}^{\infty} r^{-n} (1 - \cos r^{n/\alpha} u) & \text{if } \alpha = 1. \end{cases}$$

If  $f$  is a complex function on  $R$  belonging to  $L_{\alpha}(dx)$ , then the stochastic integral  $\int_R f dM$  is well defined, and we have

$$\widehat{\mathcal{L}}_{\int_R f dM}(u) = \exp - \left\{ \int_{\Delta} \left( \int_R k(us f(x)) dx \right) \sigma(ds) \right\}, u \in R. \tag{2.5}$$

For these and related facts concerning semi-stable random measures and integrals relative to these measures we refer to [2, 3, 4]. Below we summarize some facts about the kernel (function)  $k$  which will be needed; some of these are straightforward, others are available in [2, 4].

**(F-1)**: The function  $k$  is continuous on  $R$ ,  $k(0) = 0$ ,  $k(u) \neq 0$ , if  $u \neq 0$ , and  $\bar{k}(u) = k(-u)$ , where  $\bar{k}$  denotes the conjugate of  $k$ ; and the following inequalities hold

$$c(r, \alpha)|u|^\alpha \leq \Re(k(u)) \leq |k(u)| \leq C(r, \alpha)|u|^\alpha, u \in R, \quad (2.6)$$

where the positive constants  $c \equiv c(r, \alpha)$  and  $C \equiv C(r, \alpha)$  depend only on  $r$  and  $\alpha$ . In particular, if  $\alpha = 1$ ,

$$c(r, \alpha) = \inf_{u \in R \setminus \{0\}} \frac{k(u)}{|u|} \text{ and } C(r, \alpha) = \sup_{u \in R \setminus \{0\}} \frac{k(u)}{|u|}. \quad (2.7)$$

**(F-2)**: From (2.6), we have

$$c|u|^{\alpha-1} \leq \left| \frac{k(u)}{u} \right| = \left| \frac{\bar{k}(u)}{u} \right| \leq C|u|^{\alpha-1}, u \in R \setminus \{0\}, \quad (2.8)$$

and from **(F-1)** and (2.8), it is clear that if  $1 < \alpha < 2$ , then  $\frac{k(u)}{u}$  and  $\frac{\bar{k}(u)}{u}$  are continuous on  $R$  (by setting  $\frac{k(u)}{u} = \frac{\bar{k}(u)}{u} = 0$ , at  $u = 0$ ), and these are continuous on  $R \setminus \{0\}$ , if  $0 < \alpha \leq 1$ .

Just as in the Gaussian case, the Ornstein-Uhlenbeck  $\alpha$ -stable and  $(r, \alpha)$ -semi-stable processes are solutions of the Langevin equation when the noise process is an  $\alpha$ -stable and an  $(r, \alpha)$ -semi-stable Lévy process, respectively. Precisely, for fixed  $0 < r < 1$ ,  $0 < \alpha < 2$  and  $\lambda > 0$ , we define the Ornstein-Uhlenbeck  $(r, \alpha)$ -semi-stable process  $\mathbb{X}$  by

$$X_t \equiv \int_{-\infty}^t e^{-\lambda(t-x)} M(dx), t \in R, \quad (2.9)$$

where  $M$  is the  $(r, \alpha)$ -semi-stable random measure noted above. Since the function

$$x \mapsto e^{-\lambda(t-x)} \chi_{(-\infty, t]}(x)$$

is in  $L_\alpha(dx)$ ,  $X_t$  is well defined from the remark above (as usual,  $\chi_B(\cdot)$  denotes the indicator function of  $B$ ). Using the characteristic functions method, (2.5) and (2.9), we have that this process  $\mathbb{X}$  is stationary. Further, using (2.3), (2.5), and (2.9), we observe that

$$\begin{aligned} I_{\mathbb{X}}(\theta_1, \theta_2; t) &= \int_{\Delta} \left( \int_R k(s(\theta_1 e^{-\lambda t} \chi_{(-\infty, t]}(x) + \theta_2 \chi_{(-\infty, 0]}(x)) e^{\lambda x}) dx \right) \sigma(ds) \\ &\quad - \int_{\Delta} \left( \int_R k(s\theta_1 e^{-\lambda t} \chi_{(-\infty, t]}(x) e^{\lambda x}) dx \right) \sigma(ds) \\ &\quad - \int_{\Delta} \left( \int_R k(s\theta_2 \chi_{(-\infty, 0]}(x) e^{\lambda x}) dx \right) \sigma(ds). \end{aligned} \quad (2.10)$$

Note that if one of  $\theta_1$  or  $\theta_2$  is zero then  $I_{\mathbb{X}}(\theta_1, \theta_2; t) = 0$ , for all  $t$ ; so in the following, throughout, we will assume that both  $\theta_1$  and  $\theta_2$  are nonzero. For  $t > 0$ , which is

the case of interest here,  $I_{\mathbb{X}}(\theta_1, \theta_2; t)$  in (2.10) can be written as

$$\begin{aligned} & \int_{\Delta} \left( \int_{-\infty}^0 k(s\theta_1 e^{-\lambda t} + \theta_2) e^{\lambda x} dx \right) \sigma(ds) \\ & \quad + \int_{\Delta} \left( \int_0^t k(s\theta_1 e^{-\lambda t} e^{\lambda x}) dx \right) \sigma(ds) \\ & \quad - \int_{\Delta} \left( \int_{-\infty}^0 k(s\theta_1 e^{-\lambda t} e^{\lambda x}) dx \right) \sigma(ds) \\ & \quad - \int_{\Delta} \left( \int_0^t k(s\theta_1 e^{-\lambda t} e^{\lambda x}) dx \right) \sigma(ds) \\ & \quad - \int_{\Delta} \left( \int_{-\infty}^0 k(s\theta_2 e^{\lambda x}) dx \right) \sigma(ds). \end{aligned}$$

The second and fourth terms in this integral cancel out, and so finally we get

$$\begin{aligned} I_{\mathbb{X}}(\theta_1, \theta_2; t) &= \int_{\Delta} \left\{ \int_{-\infty}^0 \{k(s\theta_1 e^{-\lambda t} + \theta_2) e^{\lambda x} - k(s\theta_2 e^{\lambda x})\} dx \right\} \sigma(ds) \\ & \quad - \int_{\Delta} \left\{ \int_{-\infty}^0 k(s\theta_1 e^{-\lambda t} e^{\lambda x}) dx \right\} \sigma(ds). \end{aligned} \tag{2.11}$$

Further, for this  $\mathbb{X}$ , from (2.1), (2.5) and (2.9), we have

$$A_{\mathbb{X}} = \int_{\Delta} \left( \int_{-\infty}^0 k(s\theta_1 e^{\lambda x}) dx \right) \sigma(ds) + \int_{\Delta} \left( \int_{-\infty}^0 k(s\theta_2 e^{\lambda x}) dx \right) \sigma(ds). \tag{2.12}$$

### 3. Rate of Decay for Ornstein-Uhlenbeck Semi-Stable Processes

Before we state and prove our results, we introduce several notations and state some conventions. These we hope will help make our presentation clearer and simpler. As hinted above, in the following, unless stated otherwise, it is implicit that  $\theta_1, \theta_2 \in R$ ,  $\theta_1 \theta_2 \neq 0$ ,  $\lambda > 0$  and  $t > 0$ .

For the finite measure  $\sigma$  on  $\Delta$ , we set

$$|\sigma|_{\alpha} \equiv \int_{\Delta} |s|^{\alpha} \sigma(ds), \quad \nu(r, \alpha) \equiv c(r, \alpha) |\sigma|_{\alpha}, \quad \gamma(r, \alpha) \equiv C(r, \alpha) |\sigma|_{\alpha}, \tag{3.1}$$

where  $c(r, \alpha)$  and  $C(r, \alpha)$  are as in (2.6). For a fixed  $s \in \Delta$  and  $t > 0$ , we set  $J_1(s, t) \equiv J_1(\theta_1, \theta_2; s, t)$ , where

$$J_1(\theta_1, \theta_2; s, t) \equiv \begin{cases} \int_{[|\theta_2|, |\theta_2| + |\theta_1| e^{-\lambda t}]} u^{-1} k((\text{sgn } \theta_2) su) du & \text{if } \theta_1 \theta_2 > 0, \\ - \int_{[|\theta_2| - |\theta_1| e^{-\lambda t}, |\theta_2|]} u^{-1} k((\text{sgn } \theta_2) su) du & \text{if } \theta_1 \theta_2 < 0, \\ |\theta_2| > |\theta_1| e^{-\lambda t}, \end{cases} \tag{3.2}$$

and

$$L_1(s, t) \equiv L_1(\theta_1; s, t) \equiv \int_{(0, |\theta_1| e^{-\lambda t}]} u^{-1} (k(\text{sgn } \theta_1) su) du. \tag{3.3}$$

Next we put, for  $t$  fixed,

$$J(t) \equiv J(\theta_1, \theta_2; t) \equiv \int_{\Delta} J_1(\theta_1, \theta_2; s, t) \sigma(ds) \tag{3.4}$$

$$L(t) \equiv L(\theta_1; t) \equiv \int_{\Delta} L_1(\theta_1; s, t) \sigma(ds). \tag{3.5}$$

Though, some of the functions noted above are defined for all  $t \in R$  ( e.g.,  $L_1$  and  $L$ ); however, as noted above, we are interested in  $t$  when  $t > 0$ . We also emphasize that whenever we refer to the functions  $J$  and  $J_1$ , for the case  $\theta_1\theta_2 < 0$ , it will be implicitly assumed that  $t > 0$  is such that  $|\theta_2| > |\theta_1|e^{-\lambda t}$ . Finally we set

$$\xi(\theta) \equiv \int_{\Delta} \left( \int_{(0,|\theta|]} u^{-1} k((\text{sgn } \theta)su) du \right) \sigma(ds), \tag{3.6}$$

and

$$K(\theta_1, \theta_2) \equiv \left( \frac{\theta_1}{\theta_2} \right) \int_{\Delta} k(s\theta_2) \sigma(ds). \tag{3.7}$$

It is worth mentioning here that the functions  $L$  and  $J$  are non-zero, for every  $t > 0$ , and the constants  $K$  and  $\xi$  are also non-zero. Further, we point out that  $J_1$  and  $L_1$  depend on  $r, \alpha$  (through  $k$ ) and  $\lambda$ ; and  $\rho, I, J$  and  $L$  depend on  $r, \alpha, \lambda$  and  $\sigma$ ; and the constants  $\xi$  and  $K$  depend on  $r, \alpha$ , and  $\sigma$ . Also we note here that in the following, as we have done earlier, we sometimes suppress  $\theta_1, \theta_2$  in these functions and sometimes we do not in order for emphasizing their dependence on  $\theta_1, \theta_2$ .

Now we are ready to state our main results.

**Theorem 3.1.** *Let  $\mathbb{X}$  be the Ornstein-Uhlenbeck  $(r, \alpha)$ -semi-stable process (2.9), where  $0 < \alpha < 2, \alpha \neq 1$ . Then  $I_{\mathbb{X}}(\theta_1, \theta_2; t) \rightarrow 0$  and*

$$\rho_{\mathbb{X}}(\theta_1, \theta_2; t) \sim e^{-\frac{1}{\lambda}(\xi(\theta_1)+\xi(\theta_2))} \Phi(\theta_1, \theta_2; t),$$

where

$$\Phi(t) \equiv \Phi(\theta_1, \theta_2; t) = \begin{cases} -\lambda^{-1}K(\theta_1, \theta_2)e^{-\lambda t} & , \text{ if } 1 < \alpha < 2, \\ \lambda^{-1}L(\theta_1; t) & , \text{ if } 0 < \alpha < 1. \end{cases} \tag{3.8}$$

Further for all  $0 < \alpha < 2$  (including  $\alpha = 1$ ), the following inequalities hold for all  $t > 0$  (in fact, these hold for all  $t \in R$  but these are of interest only for  $t > 0$ , see Lemma 3.3),

$$\alpha^{-1}|\theta_1|^{\alpha} \nu(r, \alpha) e^{-\alpha \lambda t} \leq |L(t)| \leq \alpha^{-1}|\theta_1|^{\alpha} \gamma(r, \alpha) e^{-\alpha \lambda t}; \tag{3.9}$$

hence, for large  $t$ , the absolute rate of  $\rho_{\mathbb{X}}$  when  $0 < \alpha < 1$  lies within a region bounded (both sides) by a constant times  $e^{-\alpha \lambda t}$ .

**Theorem 3.2.** *Let  $\mathbb{X}$  be the Ornstein-Uhlenbeck  $(r, 1)$ -semi-stable process (2.9). Then  $\rho_{\mathbb{X}}(\theta_1, \theta_2; t) \rightarrow 0$ , and  $\rho_{\mathbb{X}} \sim e^{-\frac{1}{\lambda}(\xi(\theta_1)+\xi(\theta_2))} \Phi(\theta_1, \theta_2; t)$ , where in the case when  $\theta_1\theta_2 < 0$ ,*

$$\Phi(t) \equiv \Phi(\theta_1, \theta_2; t) = \lambda^{-1}(-K(\theta_1, \theta_2)e^{-\lambda t} + L(\theta_1; t))$$

and, for all large  $t$  (e.g.; when  $|\theta_2| > |\theta_1|e^{-\lambda t}$ )

$$(-K(\theta_1, \theta_2) + |\theta_1|\nu(r, 1))e^{-\lambda t} \leq \lambda\Phi(t) \leq (-K(\theta_1, \theta_2) + |\theta_1|\gamma(r, 1))e^{-\lambda t}. \tag{3.10}$$

In the case when  $\theta_1\theta_2 > 0$ ,  $\Phi(t) \equiv \Phi(\theta_1, \theta_2; t) = -I_{\mathbb{X}}(t)$ , and, for all  $t > 0$ ,

$$(c(r, 1) - C(r, 1))|\theta_1||\sigma|_1 e^{-\lambda t} \leq \lambda I_{\mathbb{X}}(t) \leq (C(r, 1) - c(r, 1))|\theta_1||\sigma|_1 e^{-\lambda t}. \quad (3.11)$$

The constants  $c(r, 1)$  and  $C(r, 1)$  in (3.10) and (3.11) are those noted in (2.7). (In this case, if  $\Phi(t) = 0$ , for some  $t$ , we use the convention that  $\frac{0}{0} = 1$ ).

We find it convenient to prove the theorems through the following three lemmas. Another reason to state these lemmas separately is that it is likely that the results stated in them may be useful elsewhere.

**Lemma 3.3.** For the Ornstein-Uhlenbeck  $(r, \alpha)$ -semi-stable process  $\mathbb{X}$  of (2.9), we have

$$A_{\mathbb{X}}(\theta_1, \theta_2) = \lambda^{-1}(\xi(\theta_1) + \xi(\theta_2)), \quad (3.12)$$

and, for  $t > 0$  with  $t$  satisfying  $|\theta_2| > |\theta_1|e^{-\lambda t}$ , if  $\theta_1\theta_2 < 0$ , we have

$$I_{\mathbb{X}}(\theta_1, \theta_2; t) = \lambda^{-1} [J(\theta_1, \theta_2; t) - L(\theta_1; t)]. \quad (3.13)$$

*Proof.* We first observe that, for fixed  $s \in \Delta$ ,  $\theta \in R$ ,  $\theta \neq 0$  and  $t \geq 0$ , we have

$$\int_{-\infty}^0 k(s\theta e^{-\lambda t} e^{\lambda x}) dx = \lambda^{-1} \left\{ \int_{(0, |\theta|e^{-\lambda t}]} u^{-1} k((\text{sgn } \theta) su) du \right\}. \quad (3.14)$$

To see this, if  $\theta > 0$ , then the transformation  $u = \theta e^{-\lambda t} e^{\lambda x}$  yields (3.14). If  $\theta < 0$ , then writing the left hand integrand in (3.14) as  $\bar{k}(s(-\theta)e^{-\lambda t} e^{\lambda x})$  (recall  $\bar{k}(-\cdot) = k(\cdot)$ ) and making the transformation  $u = (-\theta)e^{-\lambda t} e^{\lambda x}$ , the left side integral in (3.14) becomes

$$(\lambda^{-1}) \left( \int_{(0, (-\theta)e^{-\lambda t}]} u^{-1} \bar{k}(su) du \right) = (\lambda^{-1}) \left( \int_{(0, |\theta|e^{-\lambda t}]} u^{-1} k((\text{sgn } \theta) su) du \right).$$

Thus (3.14) holds. Then the proof of (3.12) is now obvious from (3.14), (2.12), and (3.6).

To prove (3.13), first note that, from (3.14), (3.3) and (3.5), the last integral in the expression of  $I_{\mathbb{X}}$  in (2.11) is clearly

$$\lambda^{-1} \left\{ \int_{\Delta} \left( \int_{(0, |\theta_1|e^{-\lambda t}]} u^{-1} k((\text{sgn } \theta_1) su) du \right) \sigma(ds) \right\} = \lambda^{-1} L(\theta_1; t).$$

Thus we need only show that the first integral in (2.11) is equal to  $\lambda^{-1} J(\theta_1, \theta_2; t)$ . Consider first the case  $\theta_1 > 0$  and  $\theta_2 > 0$ . If we make the transformation  $u = (\theta_1 e^{-\lambda t} + \theta_2) e^{\lambda x}$  in the first term, and  $u = \theta_2 e^{\lambda x}$  in the second in the inner integral of the first integral in (2.11), then this inner integral becomes

$$\begin{aligned} & \lambda^{-1} \left\{ \int_{(0, \theta_1 e^{-\lambda t} + \theta_2]} u^{-1} k(su) du - \int_{(0, \theta_2]} u^{-1} k(su) du \right\} \\ &= \lambda^{-1} \left\{ \int_{[\theta_2, \theta_1 e^{-\lambda t} + \theta_2]} u^{-1} k(su) du \right\}. \end{aligned} \quad (3.15)$$

The right side of (3.15) is, of course, equal to  $\lambda^{-1} J_1(\theta_1, \theta_2; s, t)$  (see (3.2)). Therefore the first integral in (2.11) is  $\lambda^{-1} J_1(\theta_1, \theta_2; t)$  (see (3.4)). The other cases are

treated similarly. For example, if  $\theta_1 < 0$  and  $\theta_2 > 0$ , we choose  $t > 0$  so large that  $\theta_2 + \theta_1 e^{-\lambda t} > 0$ . Then making exactly the same transformations as above the noted inner integral in (2.11) is the same as the left side of (3.15); but, since  $0 < \theta_2 + \theta_1 e^{-\lambda t} < \theta_2$ , it is equal to

$$\begin{aligned} & -(\lambda^{-1}) \left\{ \int_{[\theta_2 + \theta_1 e^{-\lambda t}, \theta_2]} u^{-1} k(su) du \right\} \\ & = -(\lambda^{-1}) \left\{ \int_{[|\theta_2| - |\theta_1| e^{-\lambda t}, \theta_2]} u^{-1} k((\operatorname{sgn} \theta_2) su) du \right\} \\ & = \lambda^{-1} J_1(\theta_1, \theta_2; s, t). \end{aligned}$$

Thus recalling (3.4), the first integral in (2.11) is  $\lambda^{-1} J(\theta_1, \theta_2; t)$ . For the other two cases (e.g.;  $\theta_1 < 0, \theta_2 < 0$ , and  $\theta_1 > 0, \theta_2 < 0$ ), we found it convenient to replace  $k(\cdot)$  in (2.11) by  $\bar{k}(\cdot)$  and then make the appropriate transformations. This finishes the proof.  $\square$

**Lemma 3.4.** *For the Ornstein-Uhlenbeck  $(r, \alpha)$ -semi-stable process  $\mathbb{X}$  of (2.9), for every  $0 < \alpha < 2$  (including  $\alpha = 1$ ), the function  $L$  satisfies the following inequalities, for all  $t \in R$ ,*

$$\nu(r, \alpha) \alpha^{-1} |\theta_1|^\alpha e^{-\alpha \lambda t} \leq \Re(L(t)) \leq |L(t)| \leq \gamma(r, \alpha) \alpha^{-1} |\theta_1|^\alpha e^{-\alpha \lambda t}, \quad (3.16)$$

and, hence,  $L(t) \rightarrow 0$ ; and, if  $1 < \alpha < 2$ , then  $e^{\lambda t} L(t) \rightarrow 0$ .

*Proof.* From (2.6), we have, for  $u \neq 0$ ,

$$\begin{aligned} c(r, \alpha) |s|^\alpha |u|^{\alpha-1} & \leq \Re \left( \frac{k((\operatorname{sgn} \theta_1) su)}{|u|} \right) \\ & \leq \left| \left( \frac{k((\operatorname{sgn} \theta_1) su)}{u} \right) \right| \\ & \leq C(r, \alpha) |s|^\alpha |u|^{\alpha-1}. \end{aligned} \quad (3.17)$$

Now recalling that (see (3.3) and (3.5))

$$\begin{aligned} \Re(L(t)) & = \int_{\Delta} \left( \int_{(0, |\theta_1| e^{-\lambda t}]} \Re(u^{-1} k((\operatorname{sgn} \theta_1) su)) du \right) \sigma(ds), \\ \text{and } L(t) & = \int_{\Delta} \left( \int_{(0, |\theta_1| e^{-\lambda t}]} u^{-1} k((\operatorname{sgn} \theta_1) su) du \right) \sigma(ds), \end{aligned}$$

and using the obvious fact that

$$\Re(L(t)) \leq |L(t)| \leq \int_{\Delta} \left( \int_{(0, |\theta_1| e^{-\lambda t}]} |u^{-1} k((\operatorname{sgn} \theta_1) su)| du \right) \sigma(ds),$$

(3.16) follows from (3.17) by integrating the expression in (3.17) first on  $[0, |\theta_1| e^{-\lambda t}]$  and then on  $\Delta$  (recall that  $\nu(r, \alpha) = c(r, \alpha) \int_{\Delta} |s|^\alpha \sigma(ds)$  and that  $\gamma(r, \alpha) = C(r, \alpha) \int_{\Delta} |s|^\alpha \sigma(ds)$ , (see (3.1)). The proof of  $e^{\lambda t} L(t) \rightarrow 0$ , when  $1 < \alpha < 2$ , is, of course, clear from (3.16).  $\square$



**Lemma 3.5.** For the Ornstein-Uhlenbeck  $(r, \alpha)$ -semi-stable process  $\mathbb{X}$  of (2.9), for every  $0 < \alpha < 2$ , we have

$$e^{\lambda t} J(\theta_1, \theta_2; t) \rightarrow K(\theta_1, \theta_2) \quad (\text{and, hence, } J(\theta_1, \theta_2; t) \rightarrow 0); \quad (3.18)$$

further, if  $0 < \alpha < 1$ , then

$$(L(\theta_1, \theta_2; t))^{-1} J(\theta_1, \theta_2; t) \rightarrow 0. \quad (3.19)$$

*Proof.* First assume  $\theta_1 \theta_2 > 0$ , and fix  $s \in \Delta$ , then using continuity of  $\frac{k(u)}{u}$  on  $[|\theta_2|, |\theta_2| + |\theta_1|e^{-\lambda t}]$  (see **(F-2)**) and recalling (3.2), we can find  $u_{t,s}$  in this interval such that

$$J_1(s, t) = u_{t,s}^{-1} k((\text{sgn } \theta_1) s u_{t,s}) |\theta_1| e^{-\lambda t}. \quad (3.20)$$

Now since, by **(F-2)**,

$$u_{t,s}^{-1} |k((\text{sgn } \theta_2) s u_{t,s})| \leq |s|^\alpha u_{t,s}^{\alpha-1} C(r, s) \leq |s|^\alpha (|\theta_1| + |\theta_2|)^{\alpha-1} C(r, s), \quad (3.21)$$

and (again by the continuity of  $\frac{k(u)}{u}$ )

$$\frac{k((\text{sgn } \theta_2) s u_{t,s})}{u_{t,s}} \rightarrow \frac{k((\text{sgn } \theta_2) s |\theta_2|)}{|\theta_2|} = \frac{k(\theta_2 s)}{|\theta_2|},$$

we can use the Dominated Convergence Theorem to conclude that

$$\int_{\Delta} \frac{k((\text{sgn } \theta_2) s u_{t,s})}{u_{t,s}} \sigma(ds) \rightarrow \int_{\Delta} \frac{k(\theta_2 s)}{|\theta_2|} \sigma(ds)$$

(the measurability of  $\frac{k((\text{sgn } \theta_2) s u_{t,s})}{u_{t,s}}$  in  $s$  is a consequence of (3.20) and Fubini's Theorem). Hence, using (3.20) and noting  $\frac{|\theta_1|}{|\theta_2|} = \frac{\theta_1}{\theta_2}$ , we have

$$e^{\lambda t} J(t) \rightarrow \frac{|\theta_1|}{|\theta_2|} \int_{\Delta} k(\theta_2 s) \sigma(ds) = K(\theta_1, \theta_2).$$

Thus, the proof of (3.18) is complete if  $\theta_1 \theta_2 > 0$ . If  $\theta_1 \theta_2 < 0$ , then exactly the same proof as above shows

$$e^{\lambda t} J(t) \rightarrow -\frac{|\theta_1|}{|\theta_2|} \int_{\Delta} k(\theta_2 s) \sigma(ds).$$

But, as  $-\frac{|\theta_1|}{|\theta_2|} = \frac{\theta_1}{\theta_2}$  (if  $\theta_1 \theta_2 < 0$ ), we are done in this case also.

Now let  $0 < \alpha < 1$ ; then, by (3.20) and (3.21),

$$\begin{aligned} |J(\theta_1, \theta_2; t)| &\leq (C(r, s) \int_{\Delta} |s|^\alpha \sigma(ds)) (|\theta_1| (|\theta_1| + |\theta_2|)^{\alpha-1}) e^{-\lambda t} \\ &= \gamma(r, \alpha) |\theta_1| (|\theta_1| + |\theta_2|)^{\alpha-1} e^{-\lambda t}; \end{aligned}$$

and, hence, using (3.16),

$$|(L(t))^{-1} J(t)| \leq (\alpha |\theta_1|^{1-\alpha} (|\theta_1| + |\theta_2|)^{\alpha-1}) \left( \frac{\gamma(r, \alpha)}{\nu(r, \alpha)} \right) e^{-\lambda t(1-\alpha)}. \quad (3.22)$$

The right side of (3.22) (and hence  $(L(t))^{-1} J(t)$ ) clearly converges to zero because  $0 < \alpha < 1$ , proving (3.19); and we are done.  $\square$

*Proof of Theorem 3.1.* Let  $0 < \alpha < 2$ . From Lemma 3.4,  $L(t) \rightarrow 0$ , and from Lemma 3.5,  $J(t) \rightarrow 0$ . Hence recalling (3.13) (Lemma 3.3), we have  $I_{\mathbb{X}}(t) \rightarrow 0$ , and hence in view of the remark preceding (2.4),  $\rho_{\mathbb{X}}(t) \rightarrow 0$ .

Therefore, in view of Lemma 3.3 (namely (3.12)), (2.4) and (3.8) (recall also (3.7)), the proof of the Theorem, for the case  $1 < \alpha < 2$ , will be complete if we can show that  $e^{\lambda t} I_{\mathbb{X}}(\theta_1, \theta_2; t) \rightarrow \lambda^{-1} K(\theta_1, \theta_2)$ ; and it will be complete, for the case  $0 < \alpha < 1$ , if we can show  $(L(t))^{-1} I_{\mathbb{X}}(\theta_1, \theta_2; t) \rightarrow -(\frac{1}{\lambda})$ . But if  $1 < \alpha < 2$ , then  $e^{\lambda t} J(\theta_1, \theta_2; t) \rightarrow K(\theta_1, \theta_2)$ ; this, in fact, is true for all  $0 < \alpha < 2$ , by Lemma 3.5 (see (3.18)); and  $e^{\lambda t} L(t) \rightarrow 0$ , by Lemma 3.4. Hence, by (3.13),  $e^{\lambda t} I_{\mathbb{X}}(t) \rightarrow \lambda^{-1} K(\theta_1, \theta_2)$ . When  $0 < \alpha < 1$ , we have  $(L(t))^{-1} J(t) \rightarrow 0$ , by Lemma 3.5 (see (3.19)), and, hence, by (3.13) again,  $(L(t))^{-1} I_{\mathbb{X}}(t) \rightarrow (-\frac{1}{\lambda})$ . That  $L(t)$  satisfies the inequalities in (3.9) is also proved in Lemma 3.4.  $\square$

*Proof of Theorem 3.2.* In view of what we noted in the beginning of the proof of Theorem 3.1, we need only show that  $\Phi(t)$  is as noted in the Theorem and prove (3.10) and (3.11). First consider the case  $\theta_1 \theta_2 < 0$ ; using (3.13) we write  $I_{\mathbb{X}}$  as follows:

$$I_{\mathbb{X}}(\theta_1, \theta_2; t) = -\lambda^{-1} [-J(\theta_1, \theta_2; t) + L(\theta_1; t)].$$

Since by Lemma 3.4,  $L(\theta_1, t) \rightarrow 0$ , we have  $[-K(\theta_1, \theta_2)e^{-\lambda t} + L(\theta_1; t)] \rightarrow 0$ . Therefore in order to prove the first part of the theorem, we need to show

$$\frac{-J(\theta_1, \theta_2; t) + L(\theta_1; t)}{-K(\theta_1, \theta_2)e^{-\lambda t} + L(\theta_1; t)} \rightarrow 1. \quad (3.23)$$

(At this point, it is important to point out that  $-J(t)$ ,  $L(t)$ , and  $-K(\theta_1, \theta_2)$  are all positive, see (3.2), (3.3), (3.7), and note that  $\frac{k(u)}{|u|} > 0$ ,  $u \neq 0$ , when  $\alpha = 1$ , and recall  $\theta_1 \theta_2 < 0$ ). Now we know from Lemma 3.5 that

$$(-K(\theta_1, \theta_2)e^{-\lambda t})^{-1} (-J(\theta_1, \theta_2; t)) \rightarrow (-K(\theta_1, \theta_2))^{-1} (-K(\theta_1, \theta_2)) = 1.$$

Using this, the proof of (3.23) follows from the simple observation that if  $a_n > 0$ ,  $b_n > 0$  and  $c_n > 0$  are such that  $b_n^{-1} a_n \rightarrow 1$ , then  $\frac{a_n + c_n}{b_n + c_n} \rightarrow 1$  (as  $n \rightarrow \infty$ ). The inequalities in (3.10) are a consequence of (3.16) and the remark in **(F-1)**.

Now let  $\theta_1 \theta_2 > 0$ ; in this case we need only prove the inequalities in (3.11). These follow from (3.16), (3.13), the fact that Real  $L(t) = L(t)$  and the following inequalities which are obtained using (2.8) and (3.2),

$$c(r, \alpha) |\theta_1| e^{-\lambda t} |\sigma|_1 \leq J(t) \leq C(r, \alpha) |\theta_1| e^{-\lambda t} |\sigma|_1.$$

$\square$

Now we shall show that our theorems recover the Levy-Taquq [1] results describing the rate of  $\rho_{\mathbb{X}}$  when  $\mathbb{X}$  is the Ornstein-Uhlenbeck  $\alpha$ -stable process for  $0 < \alpha < 2$ ,  $\alpha \neq 1$ , and for  $\alpha = 1$  when  $\beta = 0$  (see below).

We recall that a random measure  $M$  is called an  $\alpha$ -stable random measure with Lebesgue control measure, if for every Borel set  $A$  of finite Lebesgue measure,

$$\widehat{\mathcal{L}}_{M(A)}(u) = \exp - \{ \text{Leb}(A) h_{\alpha, \beta}(u) \}, u \in R,$$

where

$$h(u) \equiv h_{\alpha, \beta}(u) \equiv |u|^\alpha g_\alpha(u), u \in R,$$

and

$$g(u) \equiv g_{\alpha,\beta}(u) \equiv \left\{ 1 - i\beta(\operatorname{sgn} u) \tan \frac{\pi\alpha}{2} \right\}, u \in R,$$

and  $-1 \leq \beta \leq 1$ ; if  $\alpha = 1$ , we take  $\beta = 0$ . The process (2.9) with  $M$  replaced by this  $\alpha$ -stable random measure is called an Ornstein-Uhlenbeck  $\alpha$ -stable process. The function  $h$  satisfies all those properties of  $k$  noted in (F-1) and (F-2) and are used to prove our theorems; further, the spectral measure  $\sigma$  is absent. However, if one wishes to put the above and the following in exactly the same setting as in the semi-stable case, one can take  $\sigma = \delta_{\{1\}}$ . In that case for any (complex)  $\sigma$ -integrable function on  $R$ , and  $\theta \in R$ , one has

$$\int_{\Delta} \int_R f(su) du \sigma(ds) = \int_R f(u) du, \int_{\Delta} f(\theta s) \sigma(ds) = f(\theta).$$

With these observations in mind, we compute the constants and functions that appear in the statements of the theorems for the Ornstein-Uhlenbeck  $\alpha$ -stable case (of course, we use (3.5), (3.6), and (3.7) with  $k$  replaced by  $h$ ):

$$\begin{aligned} \xi(\theta_i) &= \int_{(0,|\theta_i|]} u^{-1} h((\operatorname{sgn} \theta_i)u) du = \alpha^{-1} |\theta_i|^\alpha g(\theta_i), \quad i = 1, 2, \\ K(\theta_1, \theta_2) &= \left( \frac{\theta_1}{\theta_2} \right) h(\theta_2) = \left( \frac{\theta_1}{\theta_2} \right) |\theta_2|^\alpha g(\theta_2), \\ L(\theta_1; t) &= \int_{(0,|\theta_1|e^{-\lambda t}]} u^{-1} h((\operatorname{sgn} \theta_1)u) du = \alpha^{-1} |\theta_1|^\alpha g(\theta_1) e^{-\alpha\lambda t}; \end{aligned}$$

and, if  $\alpha = 1$  (so  $\beta = 0$ ) and  $\theta_1\theta_2 < 0$ ,

$$-K(\theta_1, \theta_2) e^{-\lambda t} + L(\theta_1; t) = - \left( \frac{\theta_1}{\theta_2} \right) |\theta_2| e^{-\lambda t} + |\theta_1| e^{-\lambda t} = 2|\theta_1| e^{-\lambda t},$$

and, since when  $\alpha = 1$ ,  $\frac{h(u)}{|u|} = 1$ ,  $u \neq 0$ , we have  $C(r, 1) = c(r, 1) = 1$ , so  $C(r, 1) - c(r, 1) = 0$ . Therefore as noted above, our theorems recover the Levy-Taquq Theorem [1] describing the rate of  $\rho_{\mathbb{X}}$  when  $\mathbb{X}$  is the Ornstein-Uhlenbeck  $\alpha$ -stable process. We state it below for completeness.

**Corollary 3.6.** [Levy-Taquq [1]] *Let  $\mathbb{X}$  be an Ornstein-Uhlenbeck  $\alpha$ -stable process,  $0 < \alpha < 2$ . Then*

$$\rho_{\mathbb{X}}(\theta_1, \theta_2; t) \sim e^{-\frac{1}{\lambda\alpha} (|\theta_1|^\alpha g(\theta_1) + |\theta_2|^\alpha g(\theta_2))} \eta(\theta_1, \theta_2) e^{-\lambda(\alpha\wedge 1)t}$$

where the constant

$$\eta(\theta_1, \theta_2) = \begin{cases} -\lambda^{-1} \left( \frac{\theta_1}{\theta_2} \right) |\theta_2|^\alpha g(\theta_2) & , \text{if } 1 < \alpha < 2 \\ (\alpha\lambda)^{-1} |\theta_1|^\alpha g(\theta_1) & , \text{if } 0 < \alpha < 1. \end{cases}$$

If  $\alpha = 1$  (hence  $\beta = 0$ ) and  $\theta_1\theta_2 > 0$ ,  $\rho_{\mathbb{X}}(\theta_1, \theta_2; t) \equiv 0$ , and if  $\theta_1\theta_2 < 0$ , then  $\rho_{\mathbb{X}}(\theta_1, \theta_2; t) \sim e^{-\frac{1}{\lambda} (|\theta_1|g(\theta_1) + |\theta_2|g(\theta_2))} 2|\theta_1| e^{-\lambda t}$ .

As noted in Theorem 3.1, when  $0 < \alpha < 1$ , for the Ornstein-Uhlenbeck  $(r, \alpha)$ -semi-stable process  $\mathbb{X}$ , the rate of  $\rho_{\mathbb{X}}(t)$  is described by  $L(t)$  whose absolute value lies within a region bounded (both sides) by a constant times  $e^{-\alpha\lambda t}$ ; on the other hand, in the  $\alpha$ -stable case, as noted in the above corollary,  $\rho_{\mathbb{X}}(t) \sim$  (constant).  $e^{-\alpha\lambda t}$ . One may ask if in the semi-stable case the rate of  $\rho_{\mathbb{X}}(t)$  is also

(constant).  $e^{-\alpha\lambda t}$  (rather than (constant),  $L(t)$ ); similarly, one can ask if in Theorem 3.2,  $L(t)$  appearing in  $\Phi$  can be replaced by (constant),  $e^{-\lambda t}$ . We now show that the answers to both of these queries are in negative. That is, we prove that if  $0 < \alpha \leq 1$ , then  $e^{\alpha\lambda t}L(\theta_1; t)$  does not converge to a nonzero constant. In fact, we prove slightly more in the following result.

**Proposition 3.7.** *Let  $0 < \alpha < 2$ . If  $\mathbb{X}$  is an Ornstein-Uhlenbeck  $\alpha$ -stable process, then  $e^{\alpha\lambda t}L(\theta_1; t)$  converges (in fact, it is a constant for each  $t$ ); and if  $\mathbb{X}$  is a proper Ornstein-Uhlenbeck  $(r, \alpha)$ -semi-stable process (i.e., one which is not an Ornstein-Uhlenbeck  $\alpha$ -stable), then  $e^{\alpha\lambda t}L(\theta_1; t)$  does not converge.*

*Proof.* If  $\mathbb{X}$  is an Ornstein-Uhlenbeck  $\alpha$ -stable process, then we have shown above that  $L(\theta_1; t) = \alpha^{-1}|\theta_1|^\alpha g(\theta_1)e^{-\alpha\lambda t}$  or  $e^{\alpha\lambda t}L(\theta_1; t) = \alpha^{-1}|\theta_1|^\alpha g(\theta_1)$ , a constant. Now let  $\mathbb{X}$  be a proper Ornstein-Uhlenbeck semi-stable process and assume that  $e^{\lambda t}L_{\mathbb{X}}(\theta_1, t)$  converges; we will show that this leads to a contradiction. For each  $0 \leq \delta \leq 1$ , we set

$$v_n(\delta) = r^{\frac{n-\delta}{\alpha}}, \quad n = 1, 2, \dots$$

and then we put  $t_n(\delta) = -\lambda^{-1} \ln \left( \frac{v_n(\delta)}{|\theta_1|} \right)$ , so  $v_n(\delta) = |\theta_1|e^{-\lambda t_n(\delta)}$ . Now we shall compute  $\Re(L(\theta_1, t_n(\delta)))$ . For fixed  $0 \leq \delta \leq 1$ , writing  $v_n$  for  $v_n(\delta)$  and  $t_n$  for  $t_n(\delta)$ , we have, from (3.3) and (3.5),

$$\begin{aligned} \Re(L(\theta_1; t_n)) &= \int_{\Delta} \left( \int_0^{|\theta_1|e^{-\lambda t_n}} u^{-1}k'(su)du \right) \sigma(ds) \\ &= \int_{\Delta} \left( \int_0^{v_n} u^{-1}k'(su)du \right) \sigma(ds), \end{aligned}$$

where  $k'(u) = \Re(k(u)) = \sum_{-\infty}^{\infty} r^n(1 - \cos(r^{-\frac{n}{\alpha}}u))$ . Fix  $s \in \Delta$ , then

$$\begin{aligned} \int_0^{v_n} u^{-1}k'(su)du &= \int_0^{r^{\frac{n-\delta}{\alpha}}} u^{-1}k'(su)du \\ &= \sum_{m=n}^{\infty} \left( \int_{r^{\frac{m+1}{\alpha}}}^{r^{\frac{m}{\alpha}}} u^{-1}k'(su)du \right) + \int_{r^{\frac{n}{\alpha}}}^{r^{\frac{n-\delta}{\alpha}}} u^{-1}k'(su)du \\ &= \sum_{m=n}^{\infty} \left( \int_{r^{\frac{1}{\alpha}}}^1 u^{-1}k'(sr^{\frac{m}{\alpha}}u)du \right) + \int_{r^{\frac{1}{\alpha}}}^{r^{\frac{1-\delta}{\alpha}}} u^{-1}k'(sr^{\frac{n-1}{\alpha}}u)du \\ &= \sum_{m=n}^{\infty} r^m \left( \int_{r^{\frac{1}{\alpha}}}^1 u^{-1}k'(su)du \right) + (r^{n-1}) \int_{r^{\frac{1}{\alpha}}}^{r^{\frac{1-\delta}{\alpha}}} u^{-1}k'(su)du \\ &= \left( \frac{r^n}{1-r} \right) \left( \int_{r^{\frac{1}{\alpha}}}^1 u^{-1}k'(su)du \right) + (r^{n-1}) \int_{r^{\frac{1}{\alpha}}}^{r^{\frac{1-\delta}{\alpha}}} u^{-1}k'(su)du \\ &= \left( \frac{r^n}{1-r} \right) Q(s) + (r^{n-1}) \int_{r^{\frac{1}{\alpha}}}^{r^{\frac{1-\delta}{\alpha}}} u^{-1}k'(su)du \end{aligned}$$

(using  $k'(r^{\frac{m}{\alpha}}su) = r^m k'(su)$ ,  $m = 1, 2, \dots$ ), where  $Q(s) = \int_{r^{\frac{1}{\alpha}}}^1 u^{-1} k'(su) du$ . Set

$$Q \equiv \int_{\Delta} Q(s) ds, \quad \phi(\delta, s) \equiv \int_{r^{\frac{1}{\alpha}}}^{r^{\frac{1-\delta}{\alpha}}} u^{-1} k'(su) du.$$

Then we have

$$\Re(L(\theta_1); t_n(\delta)) = \left( \frac{r^n}{1-r} \right) Q + r^{n-1} \left( \int_{\Delta} \phi(\delta, s) \sigma(ds) \right);$$

and, hence,

$$\begin{aligned} \Psi_n(\delta) &\equiv e^{\alpha \lambda t_n(\delta)} \Re(L(\theta_1); t_n(\delta)) \\ &= \left( \frac{|\theta_1|^\alpha}{v_n(\delta)^\alpha} \right) \left\{ \frac{r^n}{1-r} Q + r^{n-1} \left( \int_{\Delta} \phi(\delta, s) \sigma(ds) \right) \right\} \\ &= \left( \frac{|\theta_1|^\alpha}{r^{n-\delta}} \right) \left\{ \frac{r^n}{1-r} Q + r^{n-1} \left( \int_{\Delta} \phi(\delta, s) \sigma(ds) \right) \right\} \\ &= |\theta_1|^\alpha r^\delta \left\{ \frac{1}{1-r} Q + \frac{1}{r} \left( \int_{\Delta} \phi(\delta, s) \sigma(ds) \right) \right\}. \end{aligned}$$

Now note that  $\Psi_n(\equiv \Psi)$  does not depend on  $n$  and  $\Psi(0) = \Psi(1) = |\theta_1|^\alpha \left( \frac{Q}{1-r} \right)$ . If we assume that  $e^{\alpha \lambda t} L(\theta_1; t)$  converges, then so will  $e^{\alpha \lambda t} \Re(L(\theta_1; t))$ ; therefore, we must have  $\Psi(\delta) = |\theta_1|^\alpha \left( \frac{Q}{1-r} \right)$ , for all  $0 \leq \delta \leq 1$ , equivalently,

$$\begin{aligned} |\theta_1|^\alpha \left( \frac{Q}{1-r} \right) &= (|\theta_1|^\alpha r^\delta) \left\{ \frac{1}{1-r} Q + \frac{1}{r} \left( \int_{\Delta} \phi(\delta, s) \sigma(ds) \right) \right\} \\ \text{equivalently, } \frac{rQ}{1-r} (r^{-\delta} - 1) &= \int_{\Delta} \phi(\delta, s) \sigma(ds). \end{aligned} \quad (3.24)$$

Now taking derivatives of both sides of (3.24) and simplifying we get

$$\frac{\alpha r Q}{1-r} r^{-\delta} = \int_{\Delta} k'(sr^{\frac{1-\delta}{\alpha}}) \sigma(ds), \quad 0 \leq \delta \leq 1. \quad (3.25)$$

(For this one uses the continuity of  $\frac{k'(u)}{u}$ , and (2.8) to justify the interchange in the order of differentiation and integration). If we put  $u = r^{\frac{1-\delta}{\alpha}}$ , then  $u^\alpha = r^{1-\delta}$ ; therefore, since  $0 \leq \delta \leq 1$ , we have  $r^{\frac{1}{\alpha}} \leq u \leq 1$ , thus (3.25) becomes  $\frac{\alpha Q}{1-r} u^\alpha = \int_{\Delta} k'(su) \sigma(ds)$ , for all  $r^{\frac{1}{\alpha}} \leq u \leq 1$ . Now using  $k'(sr^{\frac{1}{\alpha}}u) = r k'(su)$  and the fact that  $k'(-u) = k'(u)$ , we have that

$$\frac{\alpha Q}{1-r} |u|^\alpha = \int_{\Delta} k'(su) du, \quad \text{for all } u \in R.$$

But this is a contradiction, using the uniqueness of the exponents in the characteristic functions of stable and semi-stable random variables.  $\square$

We conclude by pointing out that the analog of Corollary 3.6 for non-symmetric 1-stable case (when  $\beta$  is a non-zero constant) is also obtained in [1]; however, at this time it is not clear to us the way to treat the analog of Theorem 3.2 in the non-symmetric  $(r, 1)$ -semi-stable case.

**Acknowledgments.** The author thanks his friends and colleagues John Long and Carl Sundberg for their comments regarding the result in Proposition 3.7. The main basic research ideas this paper is based on were formulated a couple of years ago, however, the paper's completion and writing was delayed because of a sad and horrifying accident in the author's family. The author thanks Professor Hui-Hsiung Kuo for encouraging him (the author) to get back in to research and for inviting him to contribute a paper to this issue of the Journal COSA honoring Professor Gopinath Kallianpur on his 85th birthday. Eventually the paper was completed in its final form when the author was visiting the Indian Statistical Institute, Bangalore and New Delhi, he also thanks these Institutes for the support and gracious hospitality he received.

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