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A S Üstünel

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PERSISTENCE OF INVERTIBILITY IN THE WIENER SPACE

A. S. ÜSTÜNEL

ABSTRACT. Let (W, H, μ) be the classical Wiener space, assume that $U = I_W + u$ is an adapted perturbation of identity where the perturbation u is an equivalence class w.r.to the Wiener measure. We study several necessary and sufficient conditions for the almost sure invertibility of such maps. In particular the subclass of these maps who preserve the Wiener measure are characterized in terms of the corresponding innovation processes. We give the following application: let U be invertible and let τ be stopping time. Define U^τ as $I_W + u^\tau$ where u^τ is given by

$$u^\tau(t, w) = \int_0^t 1_{[0,\tau(w)]}(s)\dot{u}_s(w)ds\,.$$

We prove that U^τ is also almost surely invertible. Note that this has immediate applications to stochastic differential equations.

1. Introduction

This paper continues the study of the characterization of invertible (and/or non-invertible) adapted perturbation of identity (API for short) on the Wiener space using the notion of the innovation process which has been developed by Gopinath Kallianpur and his co-authors, cf.[3]. In [7, 8] we have shown some results about the invertibility of the adapted perturbations of identity on the classical Wiener space. In particular, using the notion of the innovation process associated to an API, we have shown that the invertibility of such a mapping is equivalent to the equality of the energy of its perturbation to the relative entropy of the measure that it induces. The main ingredient for all this results originates from a result which was born from the question of representability of the absolutely continuous probability measures as an image, or push forward of the Wiener measure under API, which is a causal version of Monge-Kantorovitch measure transportation theory, cf. [1, 2]. To be accurate let $d\nu = Ld\mu$ be a probability on the Wiener space with $L > 0$ a.s., where μ is the Wiener measure. Then there exists an API $U = I_W + u$ such that $U\mu = \nu$ if and only if the causal estimation of u w.r. to U is equal to $v \circ U$, where v is the primitive of the uniquely defined adapted process whose Girsanov exponential is equal to L, cf. [7, 8]. Using this result we obtain also several interesting information about the existence of almost sure inverses of the maps API. The list of these results is completed by proving some more and also by proving the equivalence between them at the beginning of the paper, in particular

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we give a precise equivalence between the existence of strong solutions of some functional stochastic differential equations and the invertibility of the associated API, this result finds a nice application at the end of the present paper.. As a byproduct of these results we give a complete characterization of the API's which preserve the Wiener measure in terms of their innovation processes and show that they are closed under composition operator. In fact, even the existence of such API's are quite astonishing by itself in the sense that, due to the ergodicity of the translations in the Cameron-Martin space direction, one would expect that, at least in the adapted case, such transformations would be trivial, which is a totally erroneous intuition as we show there. We also prove the connections with the notion of Girsanov measure (cf.[10]) associated to an API and the existence of strong solutions of (functional) stochastic differential equations. Finally we prove that if an API is almost surely invertible, then, its stopped version, where the stopping occurs only in the drift, is again almost surely invertible. Translated into the language of stochastic differential equations, this result is rather astonishing since the stopping operation creates a quite singular drift. Let us note that all these results are easily extended to the infinite dimensional situations and also to the case of the abstract Wiener spaces by using the techniques developed in [9] and in [10].

2. Invertibility of API's

Assume that (W, H, μ) is the classical Wiener space, i.e., $W = C([0,1], \mathbb{R}^d)$, $d \geq 1$, μ is the standard Gauss measure and H is the Cameron-Martin space whose scalar product and norm are noted as $(h,k)_H = \int_0^1 \dot{h}_s \cdot \dot{k}_s ds$ and as $|\cdot|_H$ respectively. We note by $(\mathcal{F}_t, t \in [0,1])$ the canonical filtration of W which is completed with μ-null sets. Let now u be any H-valued random variable whose Lebesgue density is adapted (the $dt \times d\mu$-equivalence classes of such random variables is denoted by $L^0_a(\mu, H)$). We note by $\rho(\delta u)$ the Girsanov exponential defined as

$$\rho(\delta u) = \exp\left(\int_0^1 \dot{u}_s . dW_s - \frac{1}{2}\int_0^1 |\dot{u}_s|^2 ds\right)$$

In the sequel we shall denote the Itô integral on $[0,1]$, of $\dot{u}$ with respect to the Wiener process as δu where δ denotes the divergence operator defined w.r. to the Wiener measure μ (cf. for instance [6, 10]). As an abuse of notation we shall use again the same notation even if $\dot{u}$ is not square integrable w.r.to $dt \times d\mu$.

Definition 2.1. Assume that $A, B : W \to W$ are measurable maps, we say that A is a *(almost sure) right inverse* of B if

(1) the image of μ under A, denoted as $A\mu$ is absolutely continuous w.r.to μ,
(2)
$$B \circ A(w) = w$$
μ-almost surely.

If there is another measurable map $C : W \to W$ such that B is a right inverse to C as defined above (including the absolute continuity of $B\mu$ w.r.to μ), then we say that B is almost surely invertible and in this case obviously we have $A = C$ almost surely.

Theorem 2.2. *Assume that $U = I_W + u$ is an API such that $E[\rho(-\delta u)] = 1$. Suppose that there exists a measurable map $V : W \to W$ such that $V \circ U = I_W$ μ-a.s., i.e., V is a left inverse of U. Then V is also a right inverse, $V\mu \sim \mu$ (i.e. equivalent), it is also an API, hence of the form $V = I_W + v$ with $v \in L^0_a(\mu, H)$. Moreover the associated stochastic processes $(t, w) \to U(w)(t)$ and $(t, w) \to V(w)(t)$ denoted respectively as $(U_t(w), (t, w) \in [0, 1] \times W)$ and $(V_t(w), (t, w) \in [0, 1] \times W)$ are the unique strong solutions of the following stochastic differential equations*

$$dU_t = -\dot{v}_t \circ U dt + dW_t,\ U_0 = 0\,. \tag{2.1}$$

$$dV_t = -\dot{u}_t \circ V dt + dW_t,\ V_0 = 0\,. \tag{2.2}$$

Conversely, assume that there are adapted process $(U_t(w), (t, w) \in [0, 1] \times W)$ and $(V_t(w), (t, w) \in [0, 1] \times W)$ which are adapted strong solutions of the equations (2.1) and (2.2) respectively, then $E[\rho(-\delta u)] = E[\rho(-\delta v)] = 1$ and the corresponding API's are almost sure inverses of each other.

Proof. We have, for any $f \in C_b(W)$,

$$E[f \circ V] = E[f \circ V \circ U\, \rho(-\delta u)] = E[f \rho(-\delta u)]\,,$$

hence $V\mu \sim \mu$. Let $\Omega = \{w \in W : V \circ U(w) = w\}$, since $\Omega \subset U^{-1}(U(\Omega))$[1], we have $\mu(U(\Omega)) = 1$ and evidently $U \circ V(w) = w$ for any $w \in \Omega$ and this proves the invertibility of U. It is clear also that V is of the form $V = I_W + v$. To show that V is an API, we need to prove that the Lebesgue density of v, denoted by $\dot{v}_t$ is $\mathcal{F}_t$-measurable for almost all $t \in [0, 1]$. For this, note first that $\dot{v} \circ U = \dot{u}$ $dt \times d\mu$-a.s., hence $\dot{v} \circ U$ is adapted to the Brownian filtration, then, by multiplying $\dot{v}$ by $1_{B_n} \circ \dot{v}$, where B_n denotes the ball of radius n in $\mathbb{R}^d$, we may suppose that $\dot{v}$ is bounded. Let η be an element of $L^\infty_a(\mu, H)$, denote by π the operator of optional projection, then we have

$$\begin{aligned} E[(\eta \circ U, v \circ U)_H \rho(-\delta u)] &= E[(\eta, v)_H] \\ &= E[(\eta, \pi v)_H] \\ &= E[(\eta \circ U, (\pi v) \circ U)_H \rho(-\delta u)] \end{aligned}$$

since η is arbitrary, we conclude $(\pi v) \circ U = v \circ U$ $dt \times d\mu$-a.s., since $U\mu \sim \mu$, it follows that $\pi v = v$ $dt \times d\mu$-a.s. Now the processes $(U_t, t \in [0, 1])$ and $(V_t, t \in [0, 1])$ are clearly strong solutions of (2.1) and (2.2) respectively. Conversely, any adapted strong solutions of the equations (2.1), (2.2) define API's $U = I_W + u, V = I_W + v$ with the property that $\rho(-\delta u) \circ V \rho(-\delta v) = \rho(-\delta v) \circ U \rho(-\delta u) = 1$, hence from the Girsanov theorem we get $E[\rho(-\delta u)] = E[\rho(-\delta v)] = 1$. □

Remark 2.3. The existence of strong solutions to (2.1) and (2.2) simultaneously implies the fact that $E[\rho(-\delta u)] = E[\rho(-\delta v)] = 1$. If we suppose that only one of them has a strong solution, say e.g. (2.1) (which says that V is a left inverse) the integrability condition $E[\rho(-\delta u)] = 1$ (or the condition $V\mu \sim \mu$) does not follow automatically and it should be added explicitly as it is given in the following corollary:

[1]Note that $U(\Omega)$ is a Souslin set, hence it is universally measurable

Corollary 2.4. *Let $U = I_W + u$ be an API with $E[\rho(-\delta u)] = 1$ and let $V : W \to W$ be a measurable map such that $V\mu \sim \mu$ and that $U \circ V = I_W$ μ-a.s. Then U is almost surely invertible with inverse V which is also an API and all the conclusions of Theorem 2.2 are also valid.*

Proof. Let $\Omega = \{w : U \circ V(w) = w\}$, since $\Omega \subset V^{-1}(V(\Omega))$,

$$E[1_{V(\Omega)} \circ V] = 1\,,$$

since $V\mu \sim \mu$, we have $\mu(V(\Omega)) = 1$, hence $V \circ U = I_W$ almost surely and Theorem 2.2 implies that V is also an API. □

Another version of Corollary 2.4, where we do not need to assume the fact that $E[\rho(-\delta u)] = 1$ is given as

Corollary 2.5. *Assume that $U = I_W + u$, $V = I_W + v$ are API's such that $U \circ V = I_W$ a.s. Then $E[\rho(-\delta u)] = 1$ and consequently $V \circ U = I_W$ a.s.*

Proof. Since u, v are both adapted, the hypothesis implies that

$$\rho(-\delta u) \circ V\, \rho(-\delta v) = 1$$

a.s., hence from the Girsanov theorem, we get $E[\rho(-\delta u)] = 1$ and the proof follows from Corollary 2.4. □

The above results will be used often in terms of the Lebesgue densities of the API's under question, hence we reformulate them below using their densities:

Corollary 2.6.

(1) *Assume that $U = I_W + u$ is an API such that $E[\rho(-\delta u)] = 1$ and $V = I_W + v$ with $v \in L^0(\mu, H)$ such that*

$$\dot{u}_t + \dot{v}_t \circ U = 0$$

$dt \times d\mu$-a.s., then V is also an API and it is the almost sure inverse of U.

(2) *Assume that $U = I_W + u$ is an API and $V : W \to W$ a measurable map such that $V\mu \sim \mu$ and that*

$$\dot{v}_t + \dot{u}_t \circ V = 0$$

$dt \times d\mu$-a.s., then V is also an API and it is the almost sure inverse of U.

Theorem 2.7. *Assume that $U = I_W + u$ is an API with $E[\rho(-\delta u)] = 1$, then we have*

$$E[\rho(-\delta u)|U]\, \frac{dU\mu}{d\mu} \circ U = 1$$

μ-a.s. In particular the following equation holds true

$$E[\rho(-\delta u)|U] = \rho(-\delta u)$$

if and only if U is a.s. invertible.

Proof. Let us denote by L the Radon-Nikodym density of $U\mu$ w.r.to μ. From the Girsanov theorem, we have

$$\begin{aligned} E[f \circ U\, L \circ U\, E[\rho(-\delta u)|U]] &= E[f \circ U\, L \circ U\, \rho(-\delta u)] \\ &= E[f\, L] \\ &= E[f \circ U] \end{aligned}$$

for any $f \in C_b(W)$, hence the first claim follows. If U is almost surely invertible, then the sigma algebra generated by U is equal to $\mathcal{F}_1$, hence the equality $E[\rho(-\delta u)|U] = \rho(-\delta u)$ follows. Conversely, suppose that the latter holds, we can denote the density L as $L = \rho(-\delta v)$, with $v \in L^0_a(\mu, H)$. The equality implies that $\dot{u} + \dot{v} \circ U = 0$ $dt \times d\mu$-a.s., hence $V \circ U = I_W$ μ-a.s., where $V = I_W + v$ and the proof follows from Corollary 2.6. □

The following proposition whose proof follows from the Girsanov theorem, gives a necessary and sufficient condition for a density to be the Radon-Nikodym derivative of an API denoted by U and in such a case we say that the measure (or the density) is **represented** by the mapping U:

Proposition 2.8. *Assume that $L = \rho(-\delta v)$, where $v \in L^0_a(\mu, H)$, i.e., $\dot{v}$ is adapted and $\int_0^1 |\dot{v}_s|^2 ds < \infty$ a.s. Then there exists $U = I_W + u$, with $u : W \to H$ adapted such that $U\mu = L\mu$ and $E[\rho(-\delta u)] = 1$ if and only if the following condition is satisfied:*

$$\begin{aligned} 1 &= L_t \circ U \; E\left[\rho(-\delta u^t)|\mathcal{U}_t\right] &(2.3)\\ &= L_t \circ U \; E\left[\rho(-\delta u)|\mathcal{U}_t\right] &(2.4)\end{aligned}$$

almost surely for any $t \in [0,1]$, where u^t is defined as $u^t(\tau) = \int_0^{t\wedge\tau} \dot{u}_s ds$ and $\mathcal{U}_t$ is the sigma algebra generated by $(w(\tau) + u(\tau), \tau \le t)$.

Let us calculate $E[\rho(-\delta u^t)|\mathcal{U}_t] = E[\rho(-\delta u)|\mathcal{U}_t]$ in terms of the innovation process associated to U. Recall that the term innovation, which originates from the filtering theory is defined as (cf.[3] and [10])

$$Z_t = U_t - \int_0^t E[\dot{u}_s|\mathcal{U}_s] ds$$

and it is a μ-Brownian motion with respect to the filtration $(\mathcal{U}_t, t \in [0,1])$. A similar proof as the one in [3] shows that any martingale with respect to the filtration of U can be represented as a stochastic integral with respect to Z. Hence, by the positivity assumption, $E[\rho(-\delta u)|\mathcal{U}_t]$ can be written as an exponential martingale

$$E[\rho(-\delta u)|\mathcal{U}_t] = \exp\left(-\int_0^t (\dot{\xi}_s, dZ_s) - \frac{1}{2}\int_0^t |\dot{\xi}_s|^2 ds\right).$$

Remark also that U is a Wiener process under the probability $\hat{\rho} d\mu$ where

$$\hat{\rho} = \exp\left(-\int_0^t (E[\dot{u}_s|\mathcal{U}_s], dZ_s) - \frac{1}{2}\int_0^t |E[\dot{u}_s|\mathcal{U}_s]|^2 ds\right),$$

hence a double utilization of the Girsanov theorem gives the following explicit result:

Proposition 2.9. *The following equalities hold almost surely*

$$E[\rho(-\delta u)|\mathcal{U}] = \exp\left(-\int_0^1 (E[\dot{u}_s|\mathcal{U}_s], dZ_s) - \frac{1}{2}\int_0^1 |E[\dot{u}_s|\mathcal{U}_s]|^2 ds\right), \qquad (2.5)$$

$$E[\rho(-\delta u)|\mathcal{U}_t] = \exp\left(-\int_0^t (E[\dot{u}_s|\mathcal{U}_s], dZ_s) - \frac{1}{2}\int_0^t |E[\dot{u}_s|\mathcal{U}_s]|^2 ds\right). \qquad (2.6)$$

Combining Propositions 2.8 and 2.9, we obtain

Theorem 2.10. *A necessary and sufficient condition for a density L, represented as $L = \rho(-\delta v)$, where $v \in L_a^0(\mu, H)$ to be the Radon-Nikodym density of the image of the Wiener measure μ under some API, noted as $U = I_W + u$, is that*

$$E[\dot{u}_t | \mathcal{U}_t] = -\dot{v}_t \circ U$$

$dt \times d\mu$-almost surely.

Now we state and prove a main theorem (cf. also [2] for related problems):

Theorem 2.11. *Assume that $u \in L^2(\mu, H) \cap L_a^0(\mu, H)$ with $E[\rho(-\delta u)] = 1$. Define L as*

$$L = \frac{dU\mu}{d\mu} = \rho(-\delta v)$$

where $v \in L_a^0(\mu, H)$ is given by the Itô representation theorem. The map $U = I_W + u$ is then almost surely invertible with its inverse $V = I_W + v$ if and only if

$$E[L \log L] = \frac{1}{2} E[|u|_H^2] \,.$$

In other words, U is invertible if and only if

$$H(U\mu|\mu) = \frac{1}{2} \|u\|^2_{L^2(\mu,H)} \,,$$

where $H(U\mu|\mu)$ denotes the entropy of $U\mu$ with respect to μ.

Proof. Since U represents $Ld\mu$, we have, from Theorem 2.21, $E[\dot{u}_s|\mathcal{U}_s] + \dot{v}_s \circ U = 0$ $ds \times d\mu$-almost surely. Hence, from the Jensen inequality $E[|v \circ U|_H^2] \le E[|u|_H^2]$. Moreover the Girsanov theorem gives

$$\begin{aligned} 2E[L \log L] &= E[|v|_H^2 L] \\ &= E\left[|v \circ U|_H^2\right] \\ &= E\left[\int_0^1 |E[\dot{u}_s|\mathcal{U}_s]|^2 ds\right] \,. \end{aligned}$$

Hence the hypothesis implies that

$$E[|u|_H^2] = E[\int_0^1 |E[\dot{u}_s|\mathcal{U}_s]|^2 ds] \,.$$

From which we deduce that $\dot{u}_s = E[\dot{u}_s|\mathcal{U}_s]$ $ds \times d\mu$-almost surely. Finally we get $\dot{u}_s + \dot{v}_s \circ U = 0$ $ds \times d\mu$, which is a necessary and sufficient condition for the invertibility of U from Corollary 2.6 (cf. also [7]) . The necessity is obvious. □

Corollary 2.12. *With the notations of Theorem 2.11, U is not invertible if and only if we have*

$$\frac{1}{2} E[|u|_H^2] > H(U\mu|\mu) \,.$$

Remark 2.13. This result gives an enlightenment about the celebrated counter example of Tsirelson, cf. [4].

Corollary 2.14. *Assume that* $(U^n = I_W + u^n,\ n \geq 1)$ *is a sequence of adapted and almost surely invertible perturbations of identity such that* $E[\rho(-\delta u^n)] = 1$ *for any* $n \geq 1$. *Suppose that* $(U^n,\ n \geq 1)$ *converges to some* $U = I_W + u$ *in* $L^0(\mu, W)$ *with* $u \in L^0_a(\mu, H)$ *with* $E[\rho(-\delta u)] = 1$. *If*

$$\lim_n H(U^n\mu|\mu) = H(U\mu|\mu)\,,$$

then U *is almost surely invertible.*

Proof. From the hypothesis, it follows that $U\mu \sim \mu$. Let

$$L = \frac{dU\mu}{d\mu} = \rho(-\delta v)$$

where $v \in L^0_a(\mu, H)$ is uniquely defined from the Itô representation theorem. We have, from Theorem 2.10,

$$E[\dot{u}_t|\mathcal{U}_t] + \dot{v}_t \circ U = 0$$

almost surely. Moreover, from the Fatou lemma and from the lower semi continuity of the Cameron-Martin norm with respect to the Banach norm of W,

$$\begin{aligned}\frac{1}{2}\,E[|u|^2_H] &\leq \frac{1}{2}\,E[\liminf_n |u_n|^2_H] \\ &\leq \frac{1}{2}\,\liminf_n E[|u_n|^2_H] \\ &= H(U\mu|\mu) = \frac{1}{2}\,E[|\hat{u}|^2_H]\end{aligned}$$

where $\hat{u}(t) = \int_0^t E[\dot{u}_s|\mathcal{U}_s]ds$. Since $\hat{u}$ is an orthogonal projection of u, it follows that $\hat{u} = u$ a.s., hence, due to Theorem 2.11, U is almost surely invertible with inverse $V = I_W + v$. □

Theorem 2.15. *Assume that* $(U_n = I_W + u_n,\ n \geq 1)$ *is a sequence of a.s. invertible sequence of identities with* $L_n = dU_n\mu/d\mu$ *satisfying the following properties:*

(1) $\lim_n L_n = L$ *weakly in* $L^1(\mu)$.

(2) *There exists a measurable map* $U : W \to W$ *such that*

$$\lim_n E[f \circ U_n] = E[f \circ U]$$

for any $f \in C_b(W)$.

(3)
$$\lim_n E[L_n \log L_n] = E[L \log L]\,.$$

Then $\frac{dU\mu}{d\mu} = L$ *and* U *is a.s. invertible.*

Proof. By writing $u = U - I_W$, from the lower semicontinuity of the Cameron-Martin norm on W, we see that u is an H-valued map, besides, it follows from the hypothesis that it is adapted to the canonical filtration. Evidently

$$\frac{dU\mu}{d\mu} = L\,.$$

Again from the lower semicontinuity and from the last hypothesis

$$\begin{aligned} E[L\log L] &= \liminf_E [L_n \log L_n] \\ &= \liminf \frac{1}{2} E[|u_n|_H^2] \\ &\geq \frac{1}{2} E[|u|_H^2] \\ &\geq E[L\log L] \end{aligned}$$

hence we get

$$E[L\log L] = \frac{1}{2} E[|u|_H^2]$$

which is a sufficient condition for the invertibility of U. □

Theorem 2.16. *Let $U = I_W + u$, $u \in L_a^2(\mu, H)$ and let us denote by L the Radon-Nikodym derivative $dU\mu/d\mu$. Assume that*

$$H(U\mu|\mu) = \frac{1}{2} \|u\|^2_{L^2(\mu,H)}$$

and that

$$E[L\log L] + E[-\log L] < \infty\,.$$

Then U is almost surely invertible.

Proof. Since $-\log L$ is integrable, L is a.s. strictly positive, hence it can be represented as $L = \rho(-\delta v)$, where $v \in L_a^0(\mu, H)$. We have $L \circ U\, E[\rho(-\delta u)|\mathcal{U}] \leq 1$ from the Girsanov theorem. Using the Jensen and above inequalities we get

$$\begin{aligned} E[L\log L] &= E[\log L \circ U] \\ &\leq -E[\log E[\rho(-\delta u)|\mathcal{U}]] \\ &\leq E[\delta u + \frac{1}{2}\, |u|_H^2] \\ &= E[L\log L] \end{aligned}$$

therefore

$$-E[\log E[\rho(-\delta u)|\mathcal{U}]] = E[\log L \circ U]$$

and this relation implies that

$$L \circ U E[\rho(-\delta u)|\mathcal{U}] = 1\,,$$

hence $E[\rho(-\delta u)] = 1$ and the proof follows from Theorem 2.11. □

We shall give below another application of Theorem 2.21 which is about the measure preserving adapted perturbations of identity:

Theorem 2.17. *Assume that $a \in L_a^2(\mu, H)$ with $E[\rho(-\delta a)] = 1$. Define $A = I_W + a$, then A preserves the Wiener measure, i.e., $A\mu = \mu$, if and only if we have*

$$E[\dot{a}_t|\mathcal{A}_t] = 0$$

$dt \times d\mu$-almost surely, where $(\mathcal{A}_t, t \in [0,1])$ denotes the filtration of A. In particular A is equal to its innovation process.

Proof. From the Girsanov theorem, for any $f \in C_b(W)$, we have

$$\begin{aligned} E[f \circ A] &= E[f] \\ &= [f \circ A\, \rho(-\delta a)] \\ &= [f \circ A\, E[\rho(-\delta a)|\mathcal{A}]] \,. \end{aligned}$$

Hence

$$E[\rho(-\delta a)|\mathcal{A}] = 1 \,,$$

and from Theorem 2.10, $E[\dot{a}_t|\mathcal{A}_t] = 0$ a.s. Letting $Z = (Z_t)$ be the innovation process associated to A, we get

$$\begin{aligned} Z_t &= A_t - \int_0^t E[\dot{a}_s|\mathcal{A}_s] ds \\ &= A_t \,. \end{aligned}$$

□

One can construct measure preserving API's as explained in the following example:

Example 2.18. Let $U = I_W + u$, where $\dot{u}$ is the shift given by B. Tsirelson, c.f. [5] or [4], then, as it is well-known, the API U is not invertible. On the other hand, since u is bounded, $U\mu \sim \mu$. Let

$$L = \frac{dU\mu}{d\mu} = \rho(-\delta v)$$

where $v \in L^0_a(\mu, H)$ is uniquely defined due to the Itô representation theorem. Define V as $V = I_W + v$ and let $A = V \circ U$. From the Girsanov theorem, we have

$$E[f \circ A] = E[f \circ V \circ U] = E[f \circ V \rho(-\delta v)] = E[f] \,,$$

for any $f \in C_b(W)$ and $A = I_W + a = I_W + u + v \circ U$ with $\dot{a}$ adapted. Note that this subset of API's is closed with respect to the composition operation.

Let us recall the notion of Girsanov measure which has been already described in [10]:

Definition 2.19. Let $(\Omega, \mathcal{F}, \rho)$ be a probability space on which is given a measurable map $T : \Omega \to \Omega$. A measure ν on $(\Omega, \mathcal{F})$ is called a *Girsanov measure* for (or associated to) (ρ, T) if $T\nu = \rho$; in other words, if

$$\int_\Omega f \circ T \, d\nu = \int_\Omega f \, d\rho$$

for any measurable, positive function on Ω.

Theorem 2.20. *Assume that $u \in L^0_a(\mu, H)$ such that $E[\rho(-\delta u)] = 1$. Let $U = I_W + u$, then there exists a unique absolutely continuous (w.r.to μ) Girsanov measure associated to (U, μ) if and only if U is almost surely invertible.*

Proof. To show the necessity note that $\rho(-\delta u)d\mu$ and $E[\rho(-\delta u)|\mathcal{U}]d\mu$ are two absolutely continuous Girsanov measures. The uniqueness implies that $\rho(-\delta u) = E[\rho(-\delta u)|\mathcal{U}]$ almost surely. It follows from Theorem 2.21 that $\dot{u}_t + \dot{v}_t \circ U = 0$ $dt \times d\mu$-a.s., where $\dot{v}$ is defined as $\rho(-\delta v) = dU\mu/d\mu$, hence U and $V = I_W + v$

are a.s. inverse to each other from Theorem 2.2. To show the sufficiency, let $d\nu = \Gamma d\mu$ be any Girsanov measure for (μ, U) where $\Gamma \in L^1_+(\mu)$. Then, by the a.s. invertibility of U we have $\Gamma = \rho(-\delta u)$ almost surely. $\square$

The next theorem summarizes the most notable results of this section about the invertibility of the API's:

Theorem 2.21. *Assume that $E[\rho(-\delta u)] = 1$ and denote by U the mapping $I_W + u$. The following properties are then equivalent*

1. *U is almost surely invertible and its inverse V is of the form $V = I_W + v$ with $v \in L^0_a(\mu, H)$,*
2. *The following stochastic differential equation*
$$\begin{aligned} dV_t &= -\dot{u}_t \circ V\, dt + dW_t \\ V_0 &= 0 \end{aligned}$$
has a unique strong solution.
3. *The following relation holds true*
$$\frac{1}{2}\int_W |u|_H^2 d\mu = \int_W \frac{dU\mu}{d\mu} \log \frac{dU\mu}{d\mu} d\mu \,.$$
4. *We have the following identity*
$$\frac{dU\mu}{d\mu} \circ U\, \rho(-\delta u) = 1$$
almost surely.
5. *U has a unique absolutely continuous Girsanov measure.*

Proof. The equivalence between (1) and (2) follows from Theorem 2.2 and the one between (1) and (3) is proved in Theorem 2.11. The equivalence between (1) and (4) is given by Theorem 2.7 and finally the one between (1) and (5) is given by Theorem 2.20. $\square$

3. Invertibility is Preserved Under Stopping of the Adapted Shifts

Assume that $U = I_W + u$ is an invertible adapted perturbation of identity, whose inverse is given by $V = I_W + v$. If $a \in [0, 1]$ is any fixed number, define U^a as $U^a = I_W + u^a$, where u^a is defined as
$$u(t) = \int_0^t 1_{[0,a]}(s)\dot{u}_s ds \,.$$
Then it is easy to see that U^a is invertible and its inverse is given explicitly as $V^a = I_W + v^a$, where v^a is defined as u^a above. A natural question is whether this property persists if we replace the constant a with a stopping time τ. The next theorem answers this question positively:

Theorem 3.1. *Let $u, v \in L^0_a(\mu, H)$ s.t. $E[\rho(-\delta u)] = 1$, define $U = I_W + u$, $V = I_W + v$. Assume that U and V are a.s. inverse to each other, i.e.,*
$$U \circ V = V \circ U = I_W$$

μ-almost surely. Let τ be a stopping time w.r.to the filtration of the canonical Brownian motion, with values in $[0,1]$. Define U^τ as I_W+u^τ, where

$$u^\tau(t)=\int_0^t 1_{[0,\tau]}(s)\dot{u}_s ds\,.$$

Then U^τ has also a both sided inverse S of the form $S=I_W+\alpha$, where $\alpha\in L_a^0(\mu,H)$ satisfies the following identity:

$$\dot{\alpha}_t=\dot{v}_t\,1_{[0,\tau\circ S]}(t)$$

$dt\times d\mu$-almost surely. In particular we have

$$\begin{aligned}\frac{dU^\tau\mu}{d\mu} &= E\left[\frac{dU\mu}{d\mu}|\mathcal{F}_{\tau\circ S}\right]\\ &= E\left[\rho(-\delta v)|\mathcal{F}_{\tau\circ S}\right]\,.\end{aligned}$$

Proof. It suffices to prove the existence of some $\alpha\in L_a^0(\mu,H)$ such that

$$\dot{u}_t^\tau+\dot{\alpha}_t\circ U^\tau=0$$

$dt\times d\mu$-a.s. From the hypothesis $U^\tau\mu$ is equivalent to μ, let L^τ be the corresponding Radon-Nikodym density. From the Itô representation theorem, there exists some $\alpha\in L_a^0(\mu,H)$ such that

$$L^\tau=\rho(-\delta\alpha)\,.$$

From [7], we have

$$E[\dot{u}_t^\tau|\mathcal{U}_t^\tau]+\dot{\alpha}_t\circ U^\tau=0$$

a.s., where $(\mathcal{U}_t^\tau,t\in[0,1])$ is the filtration of the map U^τ, i.e.,

$$\mathcal{U}_t^\tau=\sigma(U^\tau(s),s\le t)$$

and

$$U^\tau(s)(w)=W_s(w)+\int_0^s 1_{[0,\tau(w)]}(r)\dot{u}_r(w)dr\,.$$

We claim that

$$E[\dot{u}_t^\tau|\mathcal{U}_t^\tau]=E[\dot{u}_t^\tau|\mathcal{U}_t] \tag{3.1}$$

a.s., where $(\mathcal{U}_t,t\in[0,1])$ is the filtration of U. To prove the relation (3.1), let A be in $L^\infty(\mu)$, then

$$\begin{aligned}E\left[A\,E[\dot{u}_t^\tau|\mathcal{U}_t^\tau]\right] &= E\left[E[A|\mathcal{U}_t^\tau]\dot{u}_t^\tau\right]\\ &= E\left[E[A|\mathcal{U}_t^\tau]\,1_{\{\tau>t\}}\dot{u}_t\right]\\ &= E\left[E[A|\mathcal{U}_t]\,1_{\{\tau>t\}}\dot{u}_t\right]\\ &= E\left[A\,E[\dot{u}_t^\tau|\mathcal{U}_t]\right]\,,\end{aligned}$$

since A is arbitrary this proves the relation (3.1). Since u is adapted and U is invertible $\mathcal{U}_t=\mathcal{F}_t$, where $(\mathcal{F}_t)$ is the filtration of the Wiener process, we get

$$\dot{u}_t^\tau+\dot{\alpha}_t\circ U^\tau=0 \tag{3.2}$$

$dt \times d\mu$-a.s., which implies that $I_W + \alpha$ is the two-sided inverse of U^τ almost surely. To complete the proof it suffices to verify that α given as in the claim satisfies the relation (3.2):

$$\begin{aligned}\left(\dot{v}_t\, 1_{[0,\tau\circ S]}(t)\right)\circ U^\tau &= \dot{v}_t \circ U^\tau\, 1_{[0,\tau\circ S\circ U^\tau]}(t)\\ &= \dot{v}_t \circ U^\tau\, 1_{[0,\tau]}(t)\\ &= \dot{v}_t \circ U\, 1_{[0,\tau]}(t)\\ &= -\dot{u}_t\, 1_{[0,\tau]}(t)\\ &= -\dot{u}_t^\tau\end{aligned}$$

and this completes the proof. □

Example 3.2. A typical and elementary example is obtained if we take u to be constant vector field $h \in H$ and if τ is any stopping time: then the mapping $T = I_W + h^\tau$, where $h^\tau(t,w) = \int_0^{t\wedge\tau(w)} \dot{h}_s ds$, is almost surely invertible.

We can interpret this result in the language of the stochastic differential equations (SDE) as

Corollary 3.3. *Assume that* $u \in L^2_a(\mu, H)$ *with* $E[\rho(-\delta u)] = 1$. *If the SDE*

$$\begin{aligned}dV_t &= -\dot{u}_t \circ V dt + dW_t\\ V_0 &= 0\end{aligned}$$

has a unique strong solution, then so does also the following SDE

$$\begin{aligned}dS_t &= -(\dot{u}_t 1_{[0,\tau]}) \circ S dt + dW_t\\ V_0 &= 0\end{aligned}$$

for any stopping time τ. *In particular we have*

$$\frac{dU^\tau\mu}{d\mu} = E\left[\frac{dU\mu}{d\mu}|\mathcal{F}_{\tau\circ S}\right].$$

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References

1. Feyel, D. and Üstünel, A. S.: Monge-Kantorovitch measure transportation and Monge-Ampère equation on Wiener space, *Probab. Theor. Relat. Fields* **128** (2004) 347–385.
2. Feyel, D., Üstünel, A. S., and Zakai, M.: Realization of positive random variables via absolutely continuous transformations of measure on Wiener space, *Probability Surveys* (electronic) **3** (2006) 170–205.
3. Fujisaki, M., Kallianpur, G., and Kunita, H.: Stochastic differential equations for the non linear filtering problem, *Osaka J. Math.* **9** (1972) 19–40.
4. Ikeda, I. and Watanabe, S.: *Stochastic Differential Equations and Diffusion Processes*, North Holland, Amsterdam (Kodansha Ltd., Tokyo), 1981.
5. Tsirelson, B. S.: An example of stochastic differential equation having no strong solution, *Theor. Prob. Appl.* **20** (1975) 416–418.
6. Üstünel, A. S.: *Introduction to Analysis on Wiener Space*. Lecture Notes in Math. Vol. 1610. Springer, 1995.
7. Üstünel, A. S.: A necessary and sufficient condition for the invertibility of adapted perturbations of identity on the Wiener space. *C.R. Acad. Sci. Paris, Ser. I*, **346** (2008) 897–900.

8. Üstünel, A. S.: Entropy, invertibility and variational calculus of adapted shifts on Wiener space, *Preprint* (to appear in Journal of Functional Analysis).
9. Üstünel, A. S. and Zakai, M.: The construction of filtrations on abstract Wiener space, *J. Funct. Anal.* **143** (1997) 10–32.
10. Üstünel, A. S. and Zakai, M.: *Transformation of Measure on Wiener Space*, Springer Verlag, 1999.
11. Üstünel, A. S. and Zakai, M.: Sufficient conditions for the invertibility of adapted perturbations of identity on the Wiener space, *Probab. Theory Relat. Fields* **139** (2007) 207–234.

A. S. ÜSTÜNEL: INSTITUT TELECOM, TELECOM- PARISTECH, LTCI CNRS DÉPT. INFRES, 46, RUE BARRAULT, 75013, PARIS, FRANCE.
E-mail address: ustunel@enst.fr