Journal of Stochastic Analysis

Volume 3 | Number 2 Article 2

June 2022

Induced Matrices: Recurrences and Markov Chains

Philip Feinsilver
Southern Illinois University, Carbondale, Illinois 62901, USA, phfeins@siu.edu

Journal of Stochastic Analysis
Vol. 3, No. 2 (2022) Article 2 (45 pages)
DOI: 10.31390/josa.3.2.02

INDUCED MATRICES: RECURRENCES AND MARKOV CHAINS

PHILIP FEINSILVER*

Abstract. We present an approach to transforms of recurrence sequences using induced matrices (symmetric tensor powers). Background material on recurrences and shift matrices as well as that for induced matrices is presented. Krawtchouk polynomials appear and an interesting connection with the golden ratio is found. Properties of induced matrices are shown in some detail, including a cyclic binomial identity and shift property. We study the connection between the shift operator for the recurrences and (some) subgroups of SL(2), group of matrices of determinant one, through the Lie algebra $\mathfrak{sl}(2)$. This leads to the Γ-map, a Lie algebra homomorphism, which yields recurrences for entries of the induced matrix for a general 2×2 matrix. Then we connect up with Markov chains using results on recurrences. Induced matrices lead to induced chains which are described in detail. We show that the Γ-map takes generators to generators in the context of (continuous-time) jump Markov chains.

1. Introduction

Many sequences arising in combinatorics and algebra satisfy recurrences with constant coefficients, including well-known examples such as Chebyshev polynomials, Dickson polynomials, Fibonacci and Lucas numbers. We present a linear algebra approach to discover transformations of such sequences that reveal some of their less evident properties. A matrix transform will map a set of consecutive elements, according to an arithmetic progression, into another set of consecutive elements, comprising elements obeying a different arithmetic progression, usually with some additional scaling factors.

In Section 2, we review material connecting matrices and recurrences, important for establishing notation and our approach. We include as well how the negative index sequence is formed and introduce a complementary matrix, which turns out to be the adjugate of our shift operator. Our main examples are introduced: Fibonacci numbers, and partial geometric series. In §3 the basic techniques are shown for deducing formulas for sequences satisfying a given recurrence from equations in the shift matrix. An application to powers of an arbitrary 2×2 matrix is

Received 2022-3-11; Accepted 2022-3-31; Communicated by A. Boukas

2020 *Mathematics Subject Classification.* Primary 11B37, 15A72, 60J10; Secondary 15A69, 22E60.

Key words and phrases. Krawtchouk polynomials, symmetric tensor powers, 3-term recurrences, Kravchuk matrices, induced matrices, matrix transforms, cyclic binomials, sl(2), matrix tree theorem, jump Markov chain, generator.

* Corresponding author.

found. We then, in §4, present the definition and properties of *induced matrices*. [Alternative approach to symmetric tensor powers, see [4, pp. 472–5]. Cf. [11, pp. 95–99].] §5 shows the relation of induced matrices with Krawtchouk polynomials. Matrix transforms and results for recurrences are then presented in §6. There we find a surprising connection between the [golden ratio and] Fibonacci numbers and Krawtchouk polynomials. Some connections with SL(2) are developed in §7. The associated Lie map, i.e., the induced matrix generating any particular one-parameter subgroup appears there as well. That map, the Γ-map, is explored further in §8. The connection with recurrences is shown. In particular, recurrences for the elements of an induced matrix for a general 2×2 matrix are found. In §9 we show how induced matrices lead to induced Markov chains. Adjugate matrices appear and associated Matrix Tree theorems are recalled. Connections with generators of jump Markov chains and the Γ-map are shown.

For induced matrices we will consider mainly the 2×2 case for the initial matrix, although we include a discussion for induced matrices starting from a matrix of arbitrary (square) size. Our recurrences will be mainly 2-step (i.e., 3-term) with constant coefficients.

Remark 1.1. See works of Koshy [6, 7] for a comprehensive presentation involving Fibonacci numbers and related material. Matrix-related material appears in Ch. 20 of [6], e.g.

Remark 1.2. We are working over a subfield of $\mathbb{C}$, as needed for availability of eigenvalues as well as to ensure characteristic zero.

Remark 1.3. Note that throughout the paper we will use indexing starting from 0. Thus, the standard basis for two-vectors is

$$\mathbf{e}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{e}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For example, for the induced matrix in degree N, (see §4), indices run from 0 to N, with the corresponding matrices $(N+1) \times (N+1)$.

2. Matrices and Recurrences

For a general 2×2 matrix, A, the Cayley-Hamilton theorem gives the relation

$$A^2 = aA + bI$$

with $a = \operatorname{tr} A$ and $b = -\det A$, I denoting the identity matrix. Multiplying through by A^{n-1}, for $n \geq 1$ yields

$$A^{n+1} = aA^n + bA^{n-1} \tag{2.1}$$

In other words, $x_n = A^n$ satisfies the recurrence

$$x_{n+1} = a\, x_n + b\, x_{n-1} \tag{2.2}$$

We are interested as well in numerical solutions to this recurrence. Setting

$$x_n = \mathbf{v}^\top A^n \mathbf{w} \tag{2.3}$$

(the $^\top$ denoting transpose), the sequence $\{x_n\}$ satisfies the recurrence for any vectors $\mathbf{v}$ and $\mathbf{w}$. In particular, with $\{\mathbf{e}_i\}$ denoting the standard basis, the entries of A^n, $\mathbf{e}_i^\top A^n \mathbf{e}_j$, satisfy recurrence (2.2).

We will work with the shift matrix

$$T = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$$

which satisfies

$$T^2 = aT + bI$$

noting tr $T = a$, $\det T = -b$. T shifts along solutions to the recurrence:

$$T \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} ax_n + bx_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} \ . \tag{2.4}$$

We set

$$T^n = a_n T + b_n I = \begin{pmatrix} aa_n + b_n & ba_n \\ a_n & b_n \end{pmatrix} \tag{2.5}$$

With $T^0 = I = a_0 T + b_0 I$ and $T = a_1 T + b_1 I$, observe the initial conditions

$$\begin{aligned} a_0 &= 0 \qquad & a_1 &= 1 \\ b_0 &= 1 \qquad & b_1 &= 0 \end{aligned} \tag{2.6}$$

We have

$$T^{n+1} = T^n T$$

that is

$$\begin{pmatrix} aa_{n+1} + b_{n+1} & ba_{n+1} \\ a_{n+1} & b_{n+1} \end{pmatrix} = \begin{pmatrix} aa_n + b_n & ba_n \\ a_n & b_n \end{pmatrix} \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} * & * \\ aa_n + b_n & ba_n \end{pmatrix}$$

with the stars as placeholders. We see that

$$a_{n+1} = aa_n + b_n \tag{2.7}$$

$$b_{n+1} = ba_n \tag{2.8}$$

Substituting (2.8), with the index shifted back, into (2.7) yields

$$a_{n+1} = aa_n + ba_{n-1}$$

so that the sequences $\{a_n\}$ and $\{b_n\}$ both satisfy recurrence (2.2).

Referring to (2.5), we see that T^n has the form

$$T^n = \begin{pmatrix} a_{n+1} & b_{n+1} \\ a_n & b_n \end{pmatrix}$$

From (2.4), we have

$$T^n \begin{pmatrix} x_1 \\ x_0 \end{pmatrix} = \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} * \\ a_n x_1 + b_n x_0 \end{pmatrix} \tag{2.9}$$

so that

$$x_n = x_1 a_n + x_0 b_n = \begin{pmatrix} 0 & 1 \end{pmatrix} T^n \begin{pmatrix} x_1 \\ x_0 \end{pmatrix} \tag{2.10}$$

cf. (2.4). Setting $\mathbf{x} = \begin{pmatrix} x_1 \\ x_0 \end{pmatrix}$, we have

$$x_n = \begin{pmatrix} 0 & 1 \end{pmatrix} T^n \mathbf{x} \tag{2.11}$$

Thus, the sequences $\{a_n\}$ and $\{b_n\}$ form a basis for solutions to the recurrence (2.2).

Remark 2.1. The *first fundamental solution*,$\{h_n\}$, is given by the homogeneous symmetric functions in the eigenvalues of T. It satisfies the initial conditions $h_0 = 1$, $h_1 = a = \operatorname{tr} T$. Thus we identify $h_n = a_{n+1}$. I.e., h_n appears as the entry in the upper-left corner of T^n.

For $n \geq 1$, we may thus write

$$T^n = \begin{pmatrix} h_n & bh_{n-1} \\ h_{n-1} & bh_{n-2} \end{pmatrix} \tag{2.12}$$

with $h_{-1} = 0$.

Remark 2.2. As in (2.3), a sequence of the form

$$x_n = \mathbf{v}^\top T^n \mathbf{w}$$

(the $^\top$ denoting transpose) satisfies the recurrence for any vectors $\mathbf{v}$ and $\mathbf{w}$.

Remark 2.3. Combining (2.1) with (2.11), we can calculate individual entries $(A^n)_{ij}$ in the powers of A. The relations $A^0 = I$, $A^1 = A$ give initial conditions $x_0 = \delta_{ij}$ and $x_1 = A_{ij}$.

Thus,

Proposition 2.4. *Let i and j be given. Let $\tau = \operatorname{tr} A$, $\Delta = \det A$. The (i, j) entries of powers of A satisfy the recurrence*

$$x_{n+1} = \tau x_n - \Delta x_{n-1}$$

with initial conditions $x_0 = \delta_{ij}$ and $x_1 = A_{ij}$, and hence are given by

$$(A^n)_{ij} = \begin{pmatrix} 0 & 1 \end{pmatrix} T^n \begin{pmatrix} A_{ij} \\ \delta_{ij} \end{pmatrix}$$

where T is the shift matrix

$$T = \begin{pmatrix} \tau & -\Delta \\ 1 & 0 \end{pmatrix} .$$

This is effectively the first part of eq. (2.5) written in terms of A.

Example 2.5. Consider the "sampled" sequence $y_n = x_{ln}$, for a fixed positive integer ℓ. Set

$$S = T^\ell = \begin{pmatrix} a_{\ell+1} & b_{\ell+1} \\ a_\ell & b_\ell \end{pmatrix}$$

and with $\tau = \operatorname{tr} S$, $\Delta = \det S$, we have

$$S^2 = \tau\, S - \Delta\, I = (a_{\ell+1} + b_\ell) S + (a_\ell b_{\ell+1} - a_{\ell+1} b_\ell)\, I$$

Thus, setting

$$T_\ell = \begin{pmatrix} a_{\ell+1} + b_\ell & a_\ell b_{\ell+1} - a_{\ell+1} b_\ell \\ 1 & 0 \end{pmatrix}$$

we have

$$(S^n)_{ij} = \begin{pmatrix} 0 & 1 \end{pmatrix} (T_\ell)^n \begin{pmatrix} S_{ij} \\ \delta_{ij} \end{pmatrix}$$

Note. Throughout, we will assume the non-degeneracy conditions $b \neq 0$ and that T has distinct eigenvalues λ, μ. Note that, then, the sequence T^n extends to all integers $n \in \mathbb{Z}$, with sequences $\{x_n\}$ extending accordingly.

2.1. Trace, determinant.

2.1.1. *Trace.* Note that $t_n = \operatorname{tr} T^n$ satisfies the recurrence with initial values $t_0 = 2$, $t_1 = a$. Hence, by (2.10), we have

$$\operatorname{tr} T^n = a a_n + 2 b_n$$

which checks with the trace directly observed to equal $a_{n+1} + b_n$, via (2.7).

2.1.2. *Determinant.* The determinant $|T^n|$ is readily seen to equal $(-b)^n$. Thus

$$\det(T^n) = \begin{vmatrix} a_{n+1} & b_{n+1} \\ a_n & b_n \end{vmatrix} = (-b)^n$$

which can be rewritten as an identity

$$a_{n+1} b_n - a_n b_{n+1} = (-1)^n b^n \ .$$

On the other hand,

$$T^n = \begin{pmatrix} a_{n+1} & b a_n \\ a_n & b a_{n-1} \end{pmatrix}$$

yields

$$\det(T^n) = b(a_{n+1} a_{n-1} - a_n^2) = (-b)^n$$

Thus,

Proposition 2.6. *We have the identities*

$$\begin{aligned} a_n^2 - a_{n+1} a_{n-1} &= (-b)^{n-1} \\ b_n^2 - b_{n-1} b_{n+1} &= (-b)^n = \det(T^n) \end{aligned}$$

Proof. We will derive the second line directly. First check that $b_{-1} = -a/b$ from (2.2) with $n = 0$. Now

$$\begin{pmatrix} b_n & b_{n+1} \\ b_{n-1} & b_n \end{pmatrix} = T^n \begin{pmatrix} b_0 & b_1 \\ b_{-1} & b_0 \end{pmatrix} = T^n \begin{pmatrix} 1 & 0 \\ -a/b & 1 \end{pmatrix}$$

Taking determinants of both sides yields the result. □

In particular, $b < 0$ implies the inequalities

$$a_{n-1} a_{n+1} < a_n^2 \text{ and } b_{n-1} b_{n+1} < b_n^2 \ .$$

2.2. Negative index. For $n < 0$, replace $n \to -n$ in (2.2) and rearrange:

$$-bx_{-(n+1)} = ax_{-n} - x_{-(n-1)}$$

Now make the substitution $y_n = (-b)^n x_{-n}$ which leads to:

$$y_{n+1} = ay_n + by_{n-1}$$

In other words, $\{y_n\}$ satisfies the same recurrence as $\{x_n\}$.

We need initial conditions. First $y_0 = x_0$. Setting $n = 1$ gives $y_1 = -bx_{-1}$, while setting $n = 0$ in (2.2) yields, after rearrangement,

$$-bx_{-1} = ax_0 - x_1 \ .$$

Thus,

Proposition 2.7. *For negative index terms of the recurrence we have*

$$x_{-n} = (-b)^{-n} y_n$$

where $\{y_n\}$ satisfies the recurrence (2.2) *with initial conditions*

$$y_0 = x_0, \ y_1 = ax_0 - x_1$$

In fact,

$$y_n = x_0 a_{n+1} - x_1 a_n \ .$$

Proof. By (2.10), we have

$$y_n = (ax_0 - x_1)a_n + x_0 b_n = x_0(aa_n + b_n) - x_1 a_n$$

and the result follows by (2.7). □

Corollary 2.8. *Referring to the initial conditions for $\{a_n\}$ and $\{b_n\}$,* (2.6)*, we see that*

$$a_{-n} = -(-b)^{-n} \, a_n$$
$$b_{-n} = (-b)^{-n} \, a_{n+1}$$

2.3. Eigenvalues. Complementary matrix. Eigenvalues of T, λ and μ, satisfy the characteristic polynomial, e.g., in the form

$$\lambda^2 = a\lambda + b$$
$$\mu^2 = a\mu + b$$

We have immediately that $\{\lambda^n\}$ and $\{\mu^n\}$ satisfy recurrence (2.2). I.e.,

$$\lambda^{n+1} = a\lambda^n + b\lambda^{n-1}$$
$$\mu^{n+1} = a\mu^n + b\mu^{n-1}$$

As in the discussion for powers of T, we have

$$\lambda^n = a_n\lambda + b_n \tag{2.13}$$
$$\mu^n = a_n\mu + b_n \tag{2.14}$$

We are assuming $\lambda \neq \mu$ and that $\lambda\mu \neq 0$. They are related via the trace and determinant thus:

$$\mu = a - \lambda$$
$$\mu = -b/\lambda$$

We may diagonalize T as:

$$TW = W\Lambda = T\begin{pmatrix}\lambda & \mu \\ 1 & 1\end{pmatrix} = \begin{pmatrix}\lambda & \mu \\ 1 & 1\end{pmatrix}\begin{pmatrix}\lambda & 0 \\ 0 & \mu\end{pmatrix} \tag{2.15}$$

In other words, we have

$$T = W\Lambda W^{-1} = \frac{1}{\lambda - \mu}\begin{pmatrix}\lambda & \mu \\ 1 & 1\end{pmatrix}\begin{pmatrix}\lambda & 0 \\ 0 & \mu\end{pmatrix}\begin{pmatrix}1 & -\mu \\ -1 & \lambda\end{pmatrix}$$

Introduce the *complementary matrix*:

$$T' = aI - T$$

it has the property that an eigenvector of T with eigenvalue λ is an eigenvector of T' with eigenvalue μ and vice versa. Thus, T' satisfies the characteristic equation $(T')^2 = aT' + bI$ as well with corresponding properties as those of T. We have

Proposition 2.9. *Properties of T'*

$$T' = aI - T = -bT^{-1} \tag{2.16}$$
$$(T')^n = -a_nT + a_{n+1}I$$

Proof. The first relation follows from the characteristic equation for T. For the second,

$$(T')^n = a_nT' + b_nI = a_n(aI - T) + b_nI = -a_nT + a_{n+1}I$$

by (2.7). □

This last corresponds to the relation

$$\mu^n = -a_n\lambda + a_{n+1} \tag{2.17}$$

Remark 2.10. From (2.16), we have

$$T' = -bT^{-1} = (\det T)T^{-1}$$

thus $T' = \operatorname{adj}(T)$, the adjugate matrix.

2.4. Examples. Here are two main examples we will use for illustrating various properties throughout.

Example 2.11. Fibonacci sequence
Here

$$T = \begin{pmatrix}1 & 1 \\ 1 & 0\end{pmatrix}$$

with $a_n = F_n$, the Fibonacci sequence. With φ denoting the golden ratio, the eigenvalues of T are φ and $1 - \varphi = -\varphi^{-1}$. The recurrence has the well-known form

$$x_{n+1} = x_n + x_{n-1}$$

satisfied by the sequence of powers $\{\varphi^n\}$ as well as $\{(-1)^n\varphi^{-n}\}$. We have, e.g.,

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \; T^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \; T^3 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}, \; T^4 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}, \; \ldots$$

In Prop. 2.7, we have $y_n = -F_n$ so that

$$F_{-n} = (-1)^{n-1} F_n \ .$$

Note as well that in this case

$$\operatorname{tr} T^n = L_n$$

the Lucas numbers.

Example 2.12. Given α the matrix

$$T = \begin{pmatrix} 1+\alpha & -\alpha \\ 1 & 0 \end{pmatrix}$$

has eigenvalues 1 and α. The recurrence has the form

$$x_{n+1} = (1+\alpha)x_n - \alpha x_{n-1} \ .$$

satisfied by the constant sequence $\{1\}$ and the sequence of powers $\{\alpha^n\}$.

We have

$$T^2 = \begin{pmatrix} \alpha^2+\alpha+1 & -\alpha^2-\alpha \\ \alpha+1 & -\alpha \end{pmatrix}, \; T^3 = \begin{pmatrix} \alpha^3+\alpha^2+\alpha+1 & -\alpha^3-\alpha^2-\alpha \\ \alpha^2+\alpha+1 & -\alpha^2-\alpha \end{pmatrix}$$
$$T^4 = \begin{pmatrix} \alpha^4+\alpha^3+\alpha^2+\alpha+1 & -\alpha^4-\alpha^3-\alpha^2-\alpha \\ \alpha^3+\alpha^2+\alpha+1 & -\alpha^3-\alpha^2-\alpha \end{pmatrix}, \; \ldots$$

from which the general form is clear:

$$a_n = \frac{1-\alpha^n}{1-\alpha}\,, \quad b_n = \frac{\alpha^n-\alpha}{1-\alpha}$$

For $\alpha < 0, |\alpha| < 1$, T is a stochastic matrix and the powers of T converge:

$$\lim_{n\to\infty} T^n = \begin{pmatrix} \frac{1}{1+|\alpha|} & \frac{|\alpha|}{1+|\alpha|} \\ \\ \frac{1}{1+|\alpha|} & \frac{|\alpha|}{1+|\alpha|} \end{pmatrix}$$

For the sequence $\{\alpha^n\}$, in Prop. 2.7, for negative powers we have

$$y_n = a_{n+1} - \alpha a_n = 1$$

which checks $x_{-n} = \alpha^{-n}$. Similarly one can verify Prop. 2.7 for the constant sequence $\{1\}$.

We will use these special cases in our examples throughout for illustration.

3. Identities

The main observations for deriving identities are as follows. Assume we have a polynomial identity, an identity involving analytic functions works as well, of the form

$$p_1(T) = p_2(T) \tag{3.1}$$

say. Then

(1) Applying to an eigenvector with eigenvalue λ yields

$$p_1(\lambda) = p_2(\lambda)$$

and similarly for μ.

(2) Forming matrix elements, i.e., taking inner products with $\begin{pmatrix} 0 & 1 \end{pmatrix}$ on the left and $\mathbf{x}$ on the right yields identities for the recurrence, as each occurrence of a power of T becomes the corresponding element of the recurrence sequence, e.g.,

$$T^n \longrightarrow x_n$$

(3) Multiplying both sides of a relation such as (3.1) by T^m and then converting to the recurrence sequence gives relations involving shifted indices. E.g.,

$$T^n = a_n T + b_n I$$

becomes

$$T^{m+n} = a_n T^{m+1} + b_n T^m$$

yielding

$$x_{m+n} = a_n x_{m+1} + b_n x_m \ . \tag{3.2}$$

Example 3.1. Continuing Example 2.5, consider $\{a_{\ell n}\}_{n\geq 0}$. In particular, apply (3.2) for the sequence $\{a_n\}$ replacing $\{x_n\}$ with $m = n = \ell$ to get

$$a_{2\ell} = a_\ell\, a_{\ell+1} + b_\ell a_\ell = a_\ell \ \operatorname{tr} T^\ell$$

I.e.,

$$\frac{a_{2\ell}}{a_\ell} = \operatorname{tr} T^\ell$$

So the sequence

$$h_n = \frac{a_{(n+1)\ell}}{a_\ell}$$

has initial conditions

$$h_0 = 1 \quad \text{ and } \quad h_1 = \operatorname{tr} T^\ell$$

so is the first fundamental solution to the recurrence

$$x_{n+1} = (\operatorname{tr} T^\ell) x_n - (\det T^\ell) x_{n-1}$$

In particular, with T_ℓ as in example 2.5, we have

$$\frac{a_{(n+1)\ell}}{a_\ell} = \begin{pmatrix} 1 & 0 \end{pmatrix} (T_\ell)^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ .$$

Example 3.2. From $T^2 = aT + bI$ we have

$$T^{2n} = \sum_k \binom{n}{k} b^{n-k} a^k \, T^k$$

and the equivalent form with the summation reversed. Thus

$$x_{2n} = \sum_k \binom{n}{k} b^{n-k} a^k \, x_k$$

For our special cases, the Fibonacci numbers satisfy

$$F_{2n} = \sum_k \binom{n}{k} F_k$$

and we have as well

$$\begin{aligned} \alpha^{2n} &= \sum_k \binom{n}{k} (-\alpha)^{n-k} (1+\alpha)^k \alpha^k \\ &= \alpha^n \sum_k \binom{n}{k} (-1)^{n-k} (1+\alpha)^k \end{aligned}$$

easily checked.

The inverse relation is found directly by starting with $aT = T^2 - bI$ so that

$$a^n T^n = \sum_k \binom{n}{k} (-b)^{n-k} T^{2k}$$

hence

$$a^n x_n = \sum_k \binom{n}{k} (-b)^{n-k} x_{2k} \ .$$

E.g.,

$$F_n = \sum_k \binom{n}{k} (-1)^{n-k} F_{2k} \ . \tag{3.3}$$

Example 3.3. Raising both sides of (2.16) to the power n leads to

$$T^{-n} = (-b)^{-n} \sum_k \binom{n}{k} (-1)^k a^{n-k} \, T^k$$

and hence

$$x_{-n} = (-b)^{-n} \sum_k \binom{n}{k} (-1)^k a^{n-k} \, x_k \ .$$

For example,

$$(-1)^{n-1} F_n = F_{-n} = \sum_k \binom{n}{k} (-1)^{n-k} \, F_k \ . \tag{3.4}$$

And the inverse relation, via $T = aI + bT^{-1}$:

$$x_n = \sum_k \binom{n}{k} a^{n-k} b^k \, x_{-k} \ .$$

E.g.,

$$F_n = \sum_k \binom{n}{k} F_{-k} = -\sum_k \binom{n}{k} (-1)^k F_k$$

cf. (3.4).

4. Matrix Action on Polynomials. Induced Matrices

Remark 4.1. For this section, we refer the reader to [11, pp. 95–99], [10, Prop. 11.15], [8, pp. 122–123]. Also see [1]. The action defined here on polynomials is equivalent to the action on symmetric tensor powers, see [4, pp. 472–5].

4.1. Induced matrices from 2×2 matrices. Since we will be mainly interested in the 2×2 case, we discuss this separately. The general case is presented subsequently.

Starting with a 2×2 matrix, A, say we have an action on polynomials in two variables. With

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = A\mathbf{u} = A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{4.1}$$

we have, for a polynomial $p(u_1, u_2)$

$$(Ap)(u_1, u_2) = p(v_1, v_2)$$

For example, if $A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$, we map the monomial

$$u_1^2 u_2 \longrightarrow v_1^2 v_2 = (2u_1 + u_2)^2 (u_1 - u_2) = 4u_1^3 - 3u_1 u_2^2 - u_2^3 \ .$$

Noting that homogeneous degree is preserved, we form an induced matrix, $\bar{A}$, by extracting coefficients of the expansion of monomials of a given homogeneous degree, N, say. Rows are indexed by monomials in (v_1, v_2) with columns indexed by corresponding monomials in (u_1, u_2). So with A as above, with $N = 2$, we expand

$$\begin{aligned} v_1^2 &= 4u_1^2 + 4u_1 u_2 + u_2^2 \\ v_1 v_2 &= 2u_1^2 - u_1 u_2 - u_2^2 \\ v_2^2 &= u_1^2 - 2u_1 u_2 + u_2^2 \end{aligned}$$

so that

$$\bar{A}^{(2)} = \begin{pmatrix} 4 & 4 & 1 \\ 2 & -1 & -1 \\ 1 & -2 & 1 \end{pmatrix}$$

and, for $N = 3$, with rows labelled v_1^3, $v_1^2 v_2$, $v_1 v_2^2$, v_2^3 we have

$$\bar{A}^{(3)} = \begin{pmatrix} 8 & 12 & 6 & 1 \\ 4 & 0 & -3 & -1 \\ 2 & -3 & 0 & 1 \\ 1 & -3 & 3 & -1 \end{pmatrix}$$

In general, we use dictionary ordering with v_1 preceding v_2 and the same for the u's. We will use the superscript notation on $\bar{A}$ as necessary. If the degree N is arbitrary or fixed, we will just write $\bar{A}$.

More generally, for degree N, $\bar{A}$ is the matrix of coefficients in the expansions of the monomials

$$v_1^N, v_1^{N-1}v_2, \ldots, v_1^{N-m}v_2^m, \ldots, v_1v_2^{N-1}, v_2^N$$

in terms of monomials in the variables (u_1, u_2). In the 2×2 case, we can use single indexing so that:

$\bar{A}_{mn}$ is the coefficient of $u_1^{N-n}u_2^n$ in the expansion of $v_1^{N-m}v_2^m$.

I.e.,

$$v_1^{N-m}v_2^m = \sum_n \bar{A}_{mn}u_1^{N-n}u_2^n \tag{4.2}$$

By homogeneity, we can reduce to a single variable as well. Take

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

Write (4.2) in the form

$$(\alpha u_1 + \beta u_2)^{N-m}(\gamma u_1 + \delta u_2)^m = \sum_n \bar{A}_{mn}u_1^{N-n}u_2^n$$

Writing $t = u_2/u_1$, we have, dividing through by u_1^N,

$$(\alpha + \beta t)^{N-m}(\gamma + \delta t)^m = \sum_n \bar{A}_{mn}t^n \tag{4.3}$$

Thus:

Proposition 4.2. *For* $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $\bar{A}$ *has entries*

$$\bar{A}_{mn} = \sum_j \binom{N-m}{n-j}\binom{m}{j}\alpha^{N-m-n+j}\beta^{n-j}\gamma^{m-j}\delta^j$$

Proof. Expand the binomials in (4.3) to get

$$\begin{aligned}\bar{A}_{mn} &= \sum_{i,j} \binom{N-m}{i}\binom{m}{j}\alpha^{N-m-i}\beta^i\gamma^{m-j}\delta^j\, t^{i+j} \\ &= \sum_{i,j} \binom{N-m}{i}\binom{m}{j}\alpha^{N-m-i}\beta^i\gamma^{m-j}\delta^j\, t^{i+j}\end{aligned}$$

Setting $i + j = n$ yields the result. $\square$

Remark 4.3. Note that $\bar{A}^{(N)}$ is a $(N+1)\times(N+1)$ matrix.

4.1.1. *Properties of the induced matrix map.*
We start with
1. The main property of the map $A \to \bar{A}$ is the *homomorphism property.*

Proposition 4.4. Homomorphism property
The map on matrices, $A \to \bar{A}$ *is a multiplicative homomorphism from* 2×2 *matrices to* $(N+1)\times(N+1)$ *matrices.*

Proof. Let A, B be 2×2 matrices. Write $\mathbf{v} = AB\mathbf{u}$, $\mathbf{w} = B\mathbf{u}$. I.e., $\mathbf{v} = A\mathbf{w}$. Then

$$\begin{aligned} v_1^{N-m} v_2^m &= \sum_k \bar{A}_{mk} w_1^{N-k} w_2^k \\ &= \sum_k \bar{A}_{mk} \bar{B}_{kn} u_1^{N-n} u_2^n \\ &= \sum_k \overline{AB}_{mn} u_1^{N-n} u_2^n \end{aligned}$$

and the result follows. □

In particular, since the identity matrix maps to the corresponding identity matrix, we have

$$\overline{A^{-1}} = \bar{A}^{\,-1}$$

2. The eigenvalues of $\bar{A}$ are the corresponding monomials in the eigenvalues of A, namely, with λ, μ the eigenvalues of A,

Proposition 4.5. *The eigenvalues of $\bar{A}^{(N)}$ are*

$$\{\lambda^N, \lambda^{N-1}\mu, \dots, \lambda^{N-m}\mu^m, \dots, \lambda\mu^{N-1}, \mu^N\}$$

Alternatively,

$$\{\lambda^N, (-b)\lambda^{N-2}, \dots, (-b)^m \lambda^{N-2m}, \dots, (-b)^{N-1}\lambda^{2-N}, (-b)^N \lambda^{-N}\}$$

Proof. Note that if A is diagonal, with $v_1 = \lambda u_1$ and $v_2 = \mu u_2$, then $\bar{A}$ is diagonal with corresponding monomials as entries. The result in general holds by diagonalization, or upper-triangularization, since similarity is preserved via the homomorphism property.

The alternative form follows via the substitution $\mu = -b/\lambda$. □

3. From #2 follows immediately

Proposition 4.6. *The trace of $\bar{A}^{(N)}$ is the N^{th} homogeneous symmetric function in the eigenvalues of A.*

Denoting by h_N the corresponding homogeneous symmetric functions, this may be reformulated as

$$\frac{1}{\det(I - tA)} = \sum_{N \ge 0} t^N \operatorname{tr} \bar{A}^{(N)} = \sum_{N \ge 0} t^N h_N \tag{4.4}$$

4. Now, from the characteristic polynomial of A,

$$\det(sI - A) = s^2 - \tau s + \Delta$$

with $\tau = \operatorname{tr} A$ and $\Delta = \det A$, we have

$$A^2 = \tau A - \Delta I$$

which has the same characteristic polynomial, hence eigenvalues, as the shift matrix

$$T = \begin{pmatrix} \tau & -\Delta \\ 1 & 0 \end{pmatrix}$$

Writing (4.4) in the form

$$\frac{1}{1-\tau t+\Delta t^2}=\sum_{N\geq 0} t^N \operatorname{tr}\bar{A}^{(N)}=\sum_{N\geq 0} t^N h_N$$

confirms $\{h_n\}$ as a solution to the recurrence $x_{n+1}=\tau x_n-\Delta x_{n-1}$ with initial conditions $h_0=1$, $h_1=\tau$, which we call the first fundamental solution. This gives another perspective to our initial discussion relating matrices and recurrences.

5. Combining Props. 4.2 and 4.6, we have

Proposition 4.7. Trace Identity

$$\sum_{n,j}\binom{N-n}{n-j}\binom{n}{j}\alpha^{N-2n}(\beta\gamma)^{n-j}(\alpha\delta)^j=\sum_n\binom{N-n}{n}(\alpha+\delta)^{N-2n}(\beta\gamma-\alpha\delta)^n$$

Proof. The left side is the trace of $\bar{A}$ via Prop. 4.2. Now we use the fact that the shift matrix

$$T=\begin{pmatrix}\tau & -\Delta\\ 1 & 0\end{pmatrix} \tag{4.5}$$

has the same eigenvalues as does A. Applying Prop. 4.2, to $\bar{T}$, noting that $\delta=0$, hence the index j vanishes, we have

$$\bar{T}_{mn}=\binom{N-m}{n}\tau^{N-m-n}(-\Delta)^n \tag{4.6}$$

and hence

$$\operatorname{tr}\bar{A}=\operatorname{tr}\bar{T}=\sum_n\binom{N-n}{n}\tau^{N-2n}(-\Delta)^n \tag{4.7}$$

which is the required result. □

With some changes of notation we have

Corollary 4.8. *The solution to recurrence $x_{n+1}=ax_n+bx_{n-1}$ with initial conditions $x_0=1$, $x_1=a$, i.e., $\{h_n\}$, the homogeneous symmetric functions in the eigenvalues of the shift matrix T, is given by*

$$h_n(a,b)=\sum_k\binom{n-k}{k}a^{n-2k}\,b^k\ .$$

Proof. The shift matrix for the recurrence (2.2) is

$$T=\begin{pmatrix}a & b\\ 1 & 0\end{pmatrix}$$

Now compare with (4.5) and (4.7). □

Recall that this sequence occurs in the upper left corner of the powers T^n. In fact, via eq. (2.12), we have the explicit form of the basic sequences $\{a_n\}$ and $\{b_n\}$. Using $h_n=a_{n+1}$, $b_n=ba_{n-1}$,

Corollary 4.9. *For* $T^n = \begin{pmatrix} a_{n+1} & b_{n+1} \\ a_n & b_n \end{pmatrix}$*, we have*

$$T^n = \begin{pmatrix} \displaystyle\sum_k \binom{n-k}{k} a^{n-2k} b^k & \displaystyle\sum_k \binom{n-1-k}{k} a^{n-1-2k} b^{k+1} \\ a_n & b_n \end{pmatrix}$$

with the bottom row derived from the top row upon replacing $n \to n-1$.

We can find the trace as well. Writing $t_n(a,b)$ for $\operatorname{tr} T^n$,

Corollary 4.10. *For* $n \geq 1$*, the trace of* T^n *is given by*

$$t_n(a,b) = \sum_k \frac{n}{n-k} \binom{n-k}{k} a^{n-2k}\, b^k \ .$$

Proof. With $b_n = \sum_k \binom{n-2-k}{k} a^{n-2-2k} b^{k+1}$, shift the index $k+1 \to k$:

$$b_n = \sum_{k \geq 1} \binom{n-1-k}{k-1} a^{n-2k} b^k = \sum_k \binom{n-k}{k} \frac{k}{n-k} a^{n-2k} b^k$$

and combining with the upper left corner yields the result. □

Note that, in the Fibonacci case these give Lucas numbers, and for $b \to -b$, Dickson polynomials.

Example 4.11. Continuing Example 3.1, we have $\tau = \operatorname{tr} T^\ell = a_{\ell+1} + b_\ell$ and $\Delta = \det T^\ell = (-b)^\ell$. Thus

$$\frac{a_{(n+1)\ell}}{a_\ell} = h_n(\tau, -\Delta) = \sum_k \binom{n-k}{k} (a_{\ell+1} + b_\ell)^{n-2k}\, (-1)^{(\ell+1)k} b^{k\ell}$$

In particular, for the Fibonacci case, $a = b = 1$, we have the ratio

$$\frac{F_{(n+1)\ell}}{F_\ell} = \sum_k \binom{n-k}{k} {L_\ell}^{n-2k}\, (-1)^{(\ell+1)k}$$

with Lucas numbers L_ℓ.

Example 4.12. Continuing Example 2.12, we have $\tau = \operatorname{tr} T^\ell = 1 + \alpha^\ell$ and $\Delta = \det T^\ell = \alpha^\ell$. Thus, recalling $a_n = (1-\alpha^n)/(1-\alpha)$,

$$\frac{1-\alpha^{(n+1)\ell}}{1-\alpha^\ell} = \sum_{0 \leq j \leq n} \alpha^{j\ell} = \sum_k \binom{n-k}{k} (1+\alpha^\ell)^{n-2k}\, (-1)^k \alpha^{k\ell}$$

which, setting $\alpha^\ell = x$, is the identity for geometric series

$$\frac{1-x^{n+1}}{1-x} = \sum_{0 \leq j \leq n} x^j = \sum_k \binom{n-k}{k} (-1)^k\, (1+x)^{n-2k}\, x^k$$

Example 4.13. We can use the homomorphism property to find the form of

$$\overline{T^\ell}^{(N)} = (\bar{T}^{(N)})^\ell$$

with T the shift matrix $\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$. First, we restate eq. (4.6):

Proposition 4.14. *The entries for the level N induced matrix for $T = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ are*

$$\bar{T}_{mn} = \binom{N-m}{n} a^{N-m-n} b^n$$

Thus

$$\begin{aligned}
\overline{T^2} &= \sum_k \binom{N-m}{k}\binom{N-k}{n} a^{N-m-k} b^k a^{N-k-n} b^n \\
&= \sum_k \binom{N-m}{k}\binom{N-k}{n} a^{2N-m-2k-n} b^{k+n}
\end{aligned}$$

and for $\ell = 3$

$$\begin{aligned}
\overline{T^3} &= \sum_{k_1,k_2} \binom{N-m}{k_1}\binom{N-k_1}{k_2}\binom{N-k_2}{n} a^{N-m-k_1} b^{k_1} a^{N-k_1-k_2} b^{k_2} a^{N-k_2-n} b^n \\
&= \sum_{k_1,k_2} \binom{N-m}{k_1}\binom{N-k_1}{k_2}\binom{N-k_2}{n} a^{3N-2k_1-2k_2-m-n} b^{k_1+k_2+n}
\end{aligned}$$

In general, iterating, we have

Proposition 4.15.

$$\begin{aligned}
\overline{T^\ell}_{mn} = &\sum_{k_1,\ldots,k_{\ell-1}} \binom{N-m}{k_1}\binom{N-k_1}{k_2}\cdots\binom{N-k_{\ell-2}}{k_{\ell-1}}\binom{N-k_{\ell-1}}{n} \\
&\times a^{N\ell-2(k_1+k_2+\cdots+k_{\ell-1})-m-n}\, b^{k_1+k_2+\cdots+k_{\ell-1}+n}
\end{aligned}$$

Following Examples 3.1 and 4.11, taking the trace, we have the *cyclic binomial identity*. Setting $m = n = k_\ell$, rearranging and summing:

Corollary 4.16. *Cyclic binomial identity* (cf. [9, Th. 4.4])

$$\begin{aligned}
\frac{a_{(N+1)\ell}}{a_\ell} = &\sum_{k_1,\ldots,k_\ell} \binom{N-k_1}{k_2}\binom{N-k_2}{k_3}\cdots\binom{N-k_{\ell-1}}{k_\ell}\binom{N-k_\ell}{k_1} \\
&\times a^{N\ell-2(k_1+k_2+\cdots+k_\ell)}\, b^{k_1+k_2+\cdots+k_\ell}
\end{aligned}$$

We can find the entries of $\overline{T^\ell}$ directly, from $T^\ell = \begin{pmatrix} a_{\ell+1} & b_{\ell+1} \\ a_\ell & b_\ell \end{pmatrix}$, and Prop. 4.2, yielding

$$\overline{T^\ell}_{mn} = \sum_j \binom{N-m}{n-j}\binom{m}{j} a_{\ell+1}^{N-m-n+j} b_{\ell+1}^{n-j} a_\ell^{m-j} b_\ell^j \tag{4.8}$$

which gives a formula for the trace as well, setting $m = n$ and summing:

$$\frac{a_{(N+1)\ell}}{a_\ell} = \sum_{n,j} \binom{N-n}{n-j} \binom{n}{j} a_{\ell+1}^{N-2n} (b_{\ell+1} a_\ell)^{n-j} (a_{\ell+1} b_\ell)^j$$

cf. Prop. 4.7.

4.2. Induced matrices, $d \times d$ case. Now we present induced matrices for the general case. We will show the basic properties after introducing multi-indices for labelling monomials.

Consider polynomials in the variables $u_1, \ldots, u_d$. We will work with the vector space whose basis elements are the homogeneous polynomials of degree N in these variables, *i.e.*, with

$$\{u_1^{n_1} \cdots u_d^{n_d} \mid n_1 + \cdots + n_d = N, \text{each } n_\ell \geq 0\},$$

this vector space has dimension $\binom{N+d-1}{N}$.

The *induced matrix* of a $d \times d$ matrix $A = (a_{\ell\ell'})$ is the matrix of the action on polynomials according to this relation:

$$u_1^{n_1} \cdots u_d^{n_d} \to v_1^{n_1} \cdots v_d^{n_d},$$

where

$$v_\ell = \sum_{\ell'} a_{\ell\ell'} u_{\ell'}$$

or, more compactly, $\mathbf{v} = A\mathbf{u}$. That is, define the matrix element $\bar{A}_{mn}$ to be the coefficient of $u_1^{n_1} \cdots u_d^{n_d}$ in $v_1^{m_1} \cdots v_d^{m_d}$. Given $(m_1, \ldots, m_d)$, we have

$$v_1^{m_1} \cdots v_d^{m_d} = \sum_{(n_1,\ldots,n_d)} \bar{A}_{mn} u_1^{n_1} \cdots u_d^{n_d}. \tag{4.9}$$

Observe that the total degree $N = |n| = \sum n_\ell = |m| = \sum m_\ell$, i.e., homogeneity of degree N is preserved. We use multi-indices: $m = (m_1, \ldots, m_d)$ and $n = (n_1, \ldots, n_d)$. Then, for a given m, (4.9) becomes

$$v^m = \sum_n \bar{A}_{mn} u^n.$$

Successive application of B then A shows that this is a homomorphism of the multiplicative semi-group of square $d \times d$ matrices into the multiplicative semi-group of square $\binom{N+d-1}{N} \times \binom{N+d-1}{N}$ matrices, namely

Proposition 4.17. *Matrix elements satisfy the homomorphism property*

$$\overline{AB}_{mn} = \sum_k \bar{A}_{mk} \bar{B}_{kn} \ .$$

Proof. Let $\mathbf{v} = (AB)\mathbf{u}$ and $\mathbf{w} = B\mathbf{u}$. Then $\mathbf{v} = A\mathbf{w}$, so

$$\begin{aligned} v^m &= \sum_n \overline{AB}_{mn} u^n \\ &= \sum_k \bar{A}_{mk} w^k \\ &= \sum_n \sum_k \bar{A}_{mk} \bar{B}_{kn} u^n. \end{aligned}$$

□

Now, we see directly that if A is diagonal or upper-triangular, with eigenvalues $\lambda_1, \ldots, \lambda_d$, then, $\bar{A}^{(N)}$ has eigenvalues the corresponding monomials of homogeneous degree N in the variables $\{\lambda_1, \ldots, \lambda_d\}$. (Recall that the superscript indicates the homogeneous degree.) Thus the trace

$$\operatorname{tr} \bar{A}^{(N)} = h_N(\lambda_1, \ldots, \lambda_d)$$

the N^{th} homogeneous symmetric function in the eigenvalues.

Remark 4.18. One can check as well

$$\det \bar{A}^{(N)} = (\det A)^{\binom{N+d-1}{d}} .$$

The homomorphism property, Proposition 4.17, shows that

$$\operatorname{tr} \overline{AB}^{(N)} = \operatorname{tr} \overline{BA}^{(N)}$$

and that similar matrices have the same trace. Again by the homomorphism property, if two $d \times d$ matrices are similar, $A = MBM^{-1}$, then that relation extends to their respective induced matrices in every degree. Recall that any matrix is similar to an upper-triangular one with the same eigenvalues. Thus,

Theorem 4.19. Trace Theorem ([10, Prop. 11.15],[12]). *We have*

$$\frac{1}{\det(I - tA)} = \sum_{N=0}^{\infty} t^N \operatorname{tr} \bar{A}^{(N)}$$

Proof. With λ_ℓ denoting the eigenvalues of A,

$$\begin{aligned} \frac{1}{\det(I - tA)} &= \prod_\ell \frac{1}{1 - t\lambda_\ell} \\ &= \sum_{N \geq 0} t^N h_N(\lambda_1, \ldots, \lambda_d) \\ &= \sum_{N \geq 0} t^N \operatorname{tr} \bar{A}^{(N)}. \end{aligned}$$

□

Example 4.20. A simple application of this Theorem recovers the dimension of the space in degree N. Apply the Trace Theorem with $A = I$, thus $\operatorname{tr} \bar{A} = \operatorname{tr} \bar{I} =$

no. of rows of $\bar{A}$. We have

$$\begin{aligned}\sum_{N\geq 0} t^N \operatorname{tr} \bar{I}^{(N)} &= \frac{1}{\det(I-tI)} \\ &= (1-t)^{-d} = \sum_{N\geq 0} \frac{d(d+1)\cdots(d+N-1)}{N!}\, t^N \\ &= \sum_{N\geq 0} \binom{N+d-1}{N} t^N\end{aligned}$$

as expected.

4.3. $d\times d$ case: shift matrix associated to a given matrix. For a general $d\times d$ matrix, A, we associate a shift matrix, $T(A)$, as follows:

1. Write the characteristic polynomial of A in the form

$$\lambda^d = \alpha_1\lambda^{d-1} + \alpha_2\lambda^{d-2} + \cdots + \alpha_d$$

set

$$T(A) = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_d \\ 1 & 0 & \cdots & \vdots \\ 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \cdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

i.e., with 1's on the subdiagonal, zeros elsewhere after the first row.

Remark 4.21. Note this is a type of *companion matrix.*

$T(A)$ has the same characteristic polynomial as does A, thus the same eigenvalues as well.

As with T above, $T(A)$ will generate solutions to the recurrence

$$x_n = \alpha_1 x_{n-1} + \alpha_2 x_{n-2} + \cdots + \alpha_d x_{n-d}$$

with the upper left corner entries of $T(A)^n$ giving the first fundamental solution $\{h_n\}$, equal in turn to the homogeneous symmetric functions in the eigenvalues. E.g., $h_1 = \alpha_1 = \operatorname{tr} A$, etc.

Results analogous to Prop. 2.4 hold as well, with d initial conditions of the form $x_k = (A^k)_{ij}$ for $k = d-1, d-2, \ldots, 0$.

5. Krawtchouk Polynomials and Kravchuk Matrices

M. Kravchuk (French spelling: Krawtchouk) developed orthogonal polynomials with respect to binomial distributions. We present the connection with induced matrices. See [13] for a compilation related to the work of Kravchuk.

Example 5.1. Taking $A = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$ we have, for $N = 3$,

$$\begin{aligned}(1+2t)^{3-m}(1-t)^m \\ &= 1 + t\,(6-3m) + t^2\left(\frac{9m^2}{2} - \frac{33m}{2} + 12\right) + t^3\left(-\frac{9m^3}{2} + \frac{45m^2}{2} - 30m + 8\right)\end{aligned}$$

with m running from 0 to 3. Now,

$$\bar{A}^{(3)} = \begin{pmatrix} 1 & 6 & 12 & 8 \\ 1 & 3 & 0 & -4 \\ 1 & 0 & -3 & 2 \\ 1 & -3 & 3 & -1 \end{pmatrix}$$

The entries in the (induced) matrix are the values of the polynomials, coefficients of powers of t, evaluated for $m = 0, 1, 2, 3$.

More generally, the entries of $\bar{A}^{(N)}$ are the values of the polynomials in m, coefficients of powers of t, evaluated for $m = 0, 1, \ldots, N$. In general, we can write the entries of $\bar{A}$, for a fixed N, as functions of m

$$\phi_n(m) = \bar{A}_{mn}$$

These are orthogonal functions with respect to a binomial distribution $\{\binom{N}{m} p^{N-m} q^m\}_{0 \le m \le N}$ if

$$\sum_m \phi_{n'}(m)\phi_n(m)\binom{N}{m} p^{N-m} q^m = \delta_{nn'}\sigma_n \tag{5.1}$$

with σ_n the norm-squared of ϕ_n.

Introduce the *binomial matrix*, B, the diagonal matrix with binomial coefficients as nonzero entries:

$$B_{mn} = \binom{N}{m}\delta_{mn}$$

Starting with the matrix $\mathrm{p} = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$, with $p, q \ge 0$, $p + q = 1$, the induced matrix in degree N is the diagonal matrix, $\bar{\mathrm{p}}$, with entries

$$\bar{\mathrm{p}}_{mn} = p^{N-m} q^m \delta_{mn}$$

Then (5.1) can be formulated using matrices as (indicating N explicitly)

$$(\bar{A}^{(N)})^\top B^{(N)} \bar{\mathrm{p}}^{(N)} \bar{A}^{(N)} = \sigma^{(N)} \tag{5.2}$$

where $\sigma^{(N)}$ is the diagonal matrix of squared norms. Typically when N is fixed, we drop the superscript notations.

We use Lemma 11.1, the Transpose Lemma, relating the induced matrix of the transpose of A, $A^\top$, with that of $\bar{A}^\top$. The relation is

$$B\overline{A^\top} = \bar{A}^\top B$$

(See the Appendix for the proof.) Note that for the 2×2 case, the nonzero entries of B are the corresponding binomial coefficients as indicated above. Also, note that we will use the symbol B for general N, indicating the matrix of appropriate size understood.

Start with

$$A^\top \mathrm{p} A = \sigma^{(1)}$$

diagonal, which is orthogonality for $N = 1$. Then by the homomorphism property

$$\overline{A^\top}\bar{\mathrm{p}}\bar{A} = \bar{\sigma}^{(N)}$$

and the Transpose Lemma yields

$$\bar{A}^\top B\bar{\mathrm{p}}\bar{A} = B\bar{\sigma}^{(N)} = \sigma$$

which is (5.2).

Remark 5.2. Note that $\mathrm{tr}\, B\bar{\mathrm{p}} = (\mathrm{tr}\,\mathrm{p})^N$ and will have nonnegative/positive entries if p does. Thus, $B\bar{\mathrm{p}}$ will be a probability matrix (diagonal entries forming a probability distribution) if p is.

Theorem 5.3. *1. For a matrix of the form*

$$A = \begin{pmatrix} 1 & r \\ 1 & -s \end{pmatrix}, \qquad r, s > 0$$

the functions $\{\phi_n(m)\}_{0\le n\le N}$ are polynomials in m, orthogonal with respect to a binomial distribution with parameters $p = s/(r+s), q = r/(r+s)$.

2. For general $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, the functions $\{\phi_n\}$ will be orthogonal with respect to a (signed) binomial measure with parameters

$$p = -\gamma\delta/(\alpha\beta - \gamma\delta), \quad q = \alpha\beta/(\alpha\beta - \gamma\delta)$$

.

Proof. For each case, set the off-diagonal entries of $A^\top \mathrm{p} A$ equal to zero and solve for p and $q = 1 - p$. We have

$$\begin{pmatrix} 1 & 1 \\ r & -s \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\begin{pmatrix} 1 & r \\ 1 & -s \end{pmatrix} = \begin{pmatrix} p+q & pr - qs \\ pr - qs & pr^2 + qs^2 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha^2 p + \gamma^2 q & \alpha\beta p + \delta\gamma q \\ \alpha\beta p + \delta\gamma q & \beta^2 p + \delta^2 q \end{pmatrix}$$

and the result follows. □

For the case of Krawtchouk polynomials, cf. [5, 18.23], formula 18.23.3, we have

$$(1 + rt)^{N-m}(1 - st)^m = \sum_n t^n \phi_n(m)$$

with

$$\phi_n(m) = \sum_j \binom{N-m}{n-j}\binom{m}{j}(-1)^j r^{n-j} s^j$$

From the above proof, we see that the squared norms are

$$\|\phi_n\|^2 = \binom{N}{n}(pr^2 + qs^2)^n \ .$$

Example 5.4. Continuing Example 5.1, with $A = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$ above, for $N = 3$, we have the polynomials

$$\phi_0(m) = 1, \quad \phi_1(m) = 6 - 3m, \quad \phi_2(m) = \frac{9m^2}{2} - \frac{33m}{2} + 12$$
$$\phi_3(m) = -\frac{9m^3}{2} + \frac{45m^2}{2} - 30m + 8$$

orthogonal with respect to the binomial distribution with parameters $N = 3$, $p = 1/3$, $q = 2/3$. In fact, we have the matrix relation

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 6 & 3 & 0 & -3 \\ 12 & 0 & -3 & 3 \\ 8 & -4 & 2 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{27} & 0 & 0 & 0 \\ 0 & \frac{2}{9} & 0 & 0 \\ 0 & 0 & \frac{4}{9} & 0 \\ 0 & 0 & 0 & \frac{8}{27} \end{pmatrix} \begin{pmatrix} 1 & 6 & 12 & 8 \\ 1 & 3 & 0 & -4 \\ 1 & 0 & -3 & 2 \\ 1 & -3 & 3 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

which shows the squared norms $\|\phi_n\|^2 = \binom{3}{n} 2^n$ with $pr^2 + qs^2 = 2$.

Definition 5.5. The *standard Kravchuk matrix*, denoted Φ or $\Phi^{(N)}$ (as necessary for clarity), is the transpose of $\bar{A}$ with

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{5.3}$$

By the above Theorem, this is the symmetric case, $p = q = 1/2$. Thus we have, in degree N,

$$\Phi B \Phi^\top = 2^N B \tag{5.4}$$

Explicitly

$$(1+t)^{N-m}(1-t)^m = \sum_n t^n \Phi_{nm} \ . \tag{5.5}$$

Note that the rows label the polynomials with the entries along a given row the values of the corresponding polynomial taken at the points $m = 0, 1, \ldots, N$.

Example 5.6. As an example of the homomorphism property, observe that $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ satisfies

$$A^2 = 2I \quad \Longrightarrow \quad \bar{A}^2 = 2^N I \quad \Longrightarrow \quad \Phi^2 = 2^N I$$

Thus, multiplying both sides of (5.4) by Φ and cancelling the common factor of 2^N yields

$$B\Phi^\top = \Phi B$$

i.e., the matrix ΦB is symmetric.

6. Matrices and Transforms

Referring to the form of the eigenvalues of $\bar{A}$, it is interesting to look at monomials in λ and μ, namely $\{\lambda^{N-m}\mu^m\}$, cf. 4.5. We can convert this to a form in terms of T by recalling that on an eigenvector $\mathbf{v}$, of T with eigenvalue λ, $T'\mathbf{v} = \mu\,\mathbf{v}$. Thus

$$T^{N-m}(T')^m\mathbf{v} = \lambda^{N-m}\mu^m\,\mathbf{v}$$

Now use the relations $T' = aI - T = -bT^{-1}$ to get

$$T^{N-m}(aI - T)^m = (-b)^m T^{N-2m}$$

Once expressed solely in terms of T, identities will yield directly identities for seqences $\{x_n\}$ satisfying the recurrence (2.2). In fact we will look at identities somewhat more general. Start with

$$(T^r)^{N-m}((T')^s)^m$$

for positive integers r, s. One way, we convert to

$$(T^r)^{N-m}(-bT^{-1})^{sm} = (-b)^{sm}\,T^{rN-(r+s)m}$$

and on the other, use the relations $T^r = a_rT + b_r$, and similarly for T' to get

$$(T^r)^{N-m}((T')^s)^m = (b_rI + a_rT)^{N-m}(b_sI + a_s(aI - T))^m$$

in turn giving the relation

$$(T^r)^{N-m}((T')^s)^m = (b_rI + a_rT)^{N-m}(a_{s+1}I - a_sT))^m \tag{6.1}$$

via $aa_s + b_s = a_{s+1}$, cf. Prop. 2.9.

Now we have the form we encountered when looking at induced matrices. I.e., the expansion of (6.1), has coefficients $\bar{A}$ with

$$A = \begin{pmatrix} b_r & a_r \\ a_{s+1} & -a_s \end{pmatrix}$$

And the form of the coefficients follows via Prop. 4.2.

Let's look at some examples.

1. Consider $T^{N-m}(T')^m$. We get

$$T^{N-m}(-b)^mT^{-m} = (-b)^m\,T^{N-2m} = T^{N-m}(aI - T)^m$$

Expand the right-hand side to get

$$(-b)^mT^{N-2m} = \sum_j \binom{m}{j}(-1)^{m-j}a^jT^{N-j}$$

Substituting $N - j \to n$ yields

$$(-b)^m\,T^{N-2m} = \sum_n \binom{m}{N-n}(-1)^{m+n-N}a^{N-n}\,T^n$$

and, cancelling $(-1)^m$ on both sides, we have the relation for any sequence $\{x_n\}$ satisfying (2.2)

$$b^m\,x_{N-2m} = \sum_n \binom{m}{N-n}(-a)^{N-n}\,x_n$$

One can multiply through by T^ℓ and deduce the shifted relation as well, namely

$$b^m\, x_{\ell+N-2m} = \sum_j \binom{m}{N-n} (-a)^{N-n}\, x_{n+\ell}\ .$$

Here we are looking at the induced matrices $\bar{A}$ for

$$A = \begin{pmatrix} 0 & 1 \\ a & -1 \end{pmatrix}$$

Example 6.1. For our example with $T = \begin{pmatrix} 1+\alpha & -\alpha \\ 1 & 0 \end{pmatrix}$, we have for $N = 4$ the result

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \alpha+1 & -1 \\ 0 & 0 & \alpha^2+2\alpha+1 & -2\alpha-2 & 1 \\ 0 & \alpha^3+3\alpha^2+3\alpha+1 & -3\alpha^2-6\alpha-3 & 3\alpha+3 & -1 \\ \begin{matrix}\alpha^4+4\alpha^3+6\alpha^2 \\ +4\alpha+1\end{matrix} & -4\alpha^3-12\alpha^2-12\alpha-4 & 6\alpha^2+12\alpha+6 & -4\alpha-4 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 \\ \alpha+1 \\ \alpha^2+\alpha+1 \\ \alpha^3+\alpha^2+\alpha+1 \\ \alpha^4+\alpha^3+\alpha^2+\alpha+1 \end{pmatrix} = \begin{pmatrix} \alpha^4+\alpha^3+\alpha^2+\alpha+1 \\ \alpha^3+\alpha^2+\alpha \\ \alpha^2 \\ -\alpha^2 \\ -\alpha^3-\alpha^2-\alpha \end{pmatrix}$$

Here $A = \begin{pmatrix} 0 & 1 \\ 1+\alpha & -1 \end{pmatrix}$ with $x_n = h_n$ the upper left corner of powers of T. The inverse transform is given by the induced matrix of

$$A^{-1} = \begin{pmatrix} \frac{1}{\alpha+1} & \frac{1}{\alpha+1} \\ 1 & 0 \end{pmatrix}$$

That is, $(A^{-1})^{(4)} = \dfrac{1}{(1+\alpha)^4} \times$

$$\begin{pmatrix} 1 & 4 & 6 & 4 & 1 \\ \alpha+1 & 3\alpha+3 & 3\alpha+3 & \alpha+1 & 0 \\ \alpha^2+2\alpha+1 & 2\alpha^2+4\alpha+2 & \alpha^2+2\alpha+1 & 0 & 0 \\ \alpha^3+3\alpha^2+3\alpha+1 & \alpha^3+3\alpha^2+3\alpha+1 & 0 & 0 & 0 \\ \alpha^4+4\alpha^3+6\alpha^2+4\alpha+1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For the induced matrices from A^{-1}, we have

$$(1+\alpha)^{-N}(1+t)^{N-m}(1+\alpha)^m = (1+\alpha)^{-N} \sum_n \binom{N-m}{n} (1+\alpha)^m\, t^n\ .$$

Example 6.2. For the next example, consider

$$(T^2)^{N-m}(T')^m = (bI+aT)^{N-m}(aI-T)^m = (-b)^m\, T^{2N-3m}$$

We are looking at the induced matrix in degree N for

$$A = \begin{pmatrix} b & a \\ a & -1 \end{pmatrix} .$$

From Prop. 4.2, we have

$$(-b)^m\, T^{2N-3m} = \sum_n \bar{A}_{mn}\, T^n$$

with

$$\bar{A}_{mn} = \sum_j \binom{N-m}{n-j}\binom{m}{j}(-1)^j b^{N-m-n+j} a^{n+m-2j}$$

with the corresponding identity for solutions to the recurrence:

$$(-b)^m\, x_{2N-3m} = \sum_n \bar{A}_{mn}\, x_n$$

For the Fibonacci numbers, we have $a = b = 1$ as $T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, and hence we are in the context of Krawtchouk polynomials, cf. (5.3), $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Here we take the eigenvalue $\lambda = \varphi = (1+\sqrt{5})/2$, the golden ratio, with $\mu = 1 - \varphi = -\varphi^{-1}$. We have

$$(-1)^m \varphi^{2N-3m} = (1+\varphi)^{N-m}(1-\varphi)^m = \sum_n \varphi^n \Phi_{nm}$$

cf. (5.5), where Φ is the Kravchuk matrix, the transpose of $\bar{A}$ with

$$\bar{A}_{mn} = \sum_j \binom{N-m}{n-j}\binom{m}{j}(-1)^j$$

For example, with $N = 4$ we have

$$\begin{pmatrix} 1 & 4 & 6 & 4 & 1 \\ 1 & 2 & 0 & -2 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \varphi \\ \varphi^2 \\ \varphi^3 \\ \varphi^4 \end{pmatrix} = \begin{pmatrix} \varphi^8 \\ -\varphi^5 \\ \varphi^2 \\ -\varphi^{-1} \\ \varphi^{-4} \end{pmatrix}$$

And similarly, the Fibonacci numbers satisfy

$$(-1)^m F_{2N-3m} = \sum_n F_n \Phi_{nm}$$

for $m = 0, 1, \ldots, N$. which holds as well with an arbitrary shift ℓ, say. E.g., for $N = 4$ with a shift of $\ell = 4$, we have

$$\begin{pmatrix} 1 & 4 & 6 & 4 & 1 \\ 1 & 2 & 0 & -2 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 8 \\ 13 \\ 21 \end{pmatrix} = \begin{pmatrix} 144 \\ -34 \\ 8 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{12} \\ -F_9 \\ F_6 \\ -F_3 \\ F_0 \end{pmatrix}$$

this last corresponding to $(-1)^m F_{\ell+2N-3m}$.

Example 6.3. For our next example, consider

$$(\lambda^2)^{N-m}\mu^{2m} = (b+a\lambda)^{N-m}(a^2+b-a\lambda)^m = b^{2m}\lambda^{2N-4m}$$

as $a_3 = a^2+b$. These are the eigenvalues of $\overline{T^2}$, with $T^2 = \begin{pmatrix} a^2+b & ab \\ a & b \end{pmatrix}$. In terms of T, we have

$$b^{2m}T^{2N-4m} = (b+aT)^{N-m}(a^2+b-aT)^m = \sum_n \bar{A}_{mn}\, T^n$$

where $A = \begin{pmatrix} b & a \\ a^2+b & -a \end{pmatrix}$. So

$$\bar{A}_{mn} = a^n \sum_j \binom{N-m}{n-j}\binom{m}{j}(-1)^j\, b^{N-m-n+j}(a^2+b)^{m-j}$$

and

$$b^{2m}x_{2N-4m} = \sum_n \bar{A}_{mn}\, x_n$$

for sequences satisfying recurrence (2.2).

Example 6.4. For this example, we will use only powers of T. Consider

$$(T^2)^{N-m}\, T^m = (b+aT)^{N-m}\,(0+1T)^m = T^{2N-m}$$

Including a shift by ℓ, we have

$$x_{\ell+2N-m} = \sum_n \bar{A}_{mn} x_{\ell+n} \tag{6.2}$$

where $A = \begin{pmatrix} b & a \\ 0 & 1 \end{pmatrix}$. Notice that

$$A = T\mathfrak{J}$$

where $\mathfrak{J}$ is the *reversal matrix*, reversing the order of the columns of the matrix it postmultiplies, equivalently, reversing the order of the coordinates when acting on a vector.

$$\mathfrak{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Note that the entries of $\mathfrak{J}$ satisfy

$$\mathfrak{J}_{mn} = \delta_{1,m+n}$$

with $0 \le m, n \le 1$, i.e., indexing runs starting from 0.

Proposition 6.5. *The matrix $\bar{\mathfrak{J}}$ satisfies*

$$\bar{\mathfrak{J}}_{mn} = \delta_{N,m+n}$$

i.e., $\bar{\mathfrak{J}}$ is the $(N+1)\times(N+1)$ reversal matrix.

Proof. We have

$$\mathfrak{J}\begin{pmatrix}1\\t\end{pmatrix}=\begin{pmatrix}t\\1\end{pmatrix}$$

so

$$t^{N-m}=\sum_n \bar{\mathfrak{J}}_{mn}t^n \quad\Longrightarrow\quad \bar{\mathfrak{J}}_{mn}=1 \text{ if } N=m+n,\ 0 \text{ otherwise}$$

as required. □

In our example, $A=T\mathfrak{J}$ implies

$$\bar{A}=\bar{T}\,\bar{\mathfrak{J}}$$

which is just $\bar{T}$ with reversed columns. Equation (6.2) thus reads, reversing the coordinates on the right-hand side,

$$x_{\ell+2N-m}=\sum_n \bar{T}_{mn}\,x_{\ell+N-n} \tag{6.3}$$

which shows the action of induced powers of T on segments of sequences. The matrix T, the case $N=1$, shifts a pair of elements of the sequence up by 1. Reading the above equation for $N\geq 1$, comparing with eqs. (2.4), (2.9), we have

Theorem 6.6. *The induced matrix $\bar{T}^{(N)}$ shifts the indices of a segment of $N+1$ consecutive elements of the sequence $\{x_n\}$, satisfying the recurrence* (2.2)*, up by amount N:*

$$\bar{T}^{(N)}\begin{pmatrix}x_{n+N}\\x_{n+N-1}\\\vdots\\x_n\end{pmatrix}=\begin{pmatrix}x_{n+2N}\\x_{n+2N-1}\\\vdots\\x_{n+N}\end{pmatrix}$$

Recalling Prop. 4.14

Corollary 6.7. *We have*

$$x_{\ell+2N-m}=\sum_n\binom{N-m}{n}a^{N-m-n}b^n\,x_{\ell+N-n}$$

Here is an example for the Fibonacci case. Taking $N=3$, starting from F_5 to F_8, we have, with $T=\begin{pmatrix}1&1\\1&0\end{pmatrix}$,

$$\bar{T}^{(3)}=\begin{pmatrix}1&3&3&1\\1&2&1&0\\1&1&0&0\\1&0&0&0\end{pmatrix}$$

and

$$\bar{T}^{(3)}\begin{pmatrix}F_8\\F_7\\F_6\\F_5\end{pmatrix}=\begin{pmatrix}1&3&3&1\\1&2&1&0\\1&1&0&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}21\\13\\8\\5\end{pmatrix}=\begin{pmatrix}89\\55\\34\\21\end{pmatrix}=\begin{pmatrix}F_{11}\\F_{10}\\F_9\\F_8\end{pmatrix}$$

Here, cf. eq. (4.7), $\bar{T}$ has matrix elements

$$\bar{T}_{mn} = \binom{N-m}{n}$$

yielding

$$F_{\ell+2N-m} = \sum_n \binom{N-m}{n} F_{\ell+N-n}$$

extending the basic recurrence to N steps.

Example 6.8. Continuing the previous example, consider

$$(T^{r+1})^{N-m} (T^r)^m = (b_{r+1} + a_{r+1}T)^{N-m} (b_r + a_r T)^m = T^{(r+1)N-m}$$

Including a shift by ℓ, we have

$$x_{\ell+(r+1)N-m} = \sum_n \bar{A}_{mn} x_{\ell+n} \tag{6.4}$$

where $A = \begin{pmatrix} b_{r+1} & a_{r+1} \\ b_r & a_r \end{pmatrix}$. That is,

$$A = T^r \mathfrak{J}$$

thus

$$\bar{A} = \overline{T^r}\, \bar{\mathfrak{J}} = \bar{T}^r \bar{\mathfrak{J}}$$

We have

$$x_{\ell+(r+1)N-m} = \sum_n \bar{T}^r_{mn} x_{\ell+N-n}$$

corresponding to iterating the action of Theorem 6.6, equivalently, relation (6.3), r times. This can be expressed in various ways, recalling the matrix entries of $\overline{T^r}$, as in Example 4.13, Props. 4.14, 4.15, and eq. (4.8).

Corollary 6.9. *The induced matrix* $\overline{T^r}^{(N)}$ *shifts the indices of a segment of* $N+1$ *consecutive elements of the sequence* $\{x_n\}$*, satisfying the recurrence* (2.2)*, up by amount* rN*:*

$$\overline{T^r}^{(N)} \begin{pmatrix} x_{n+N} \\ x_{n+N-1} \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_{n+(r+1)N} \\ x_{n+(r+1)N-1} \\ \vdots \\ x_{n+rN} \end{pmatrix}$$

7. Relation with SL(2)

7.1. Group elements. By $\mathfrak{sl}(2)$ we mean the Lie algebra of matrices of trace zero. And SL(2) denotes the group of 2×2 matrices of determinant equal to one. For the Lie algebra $\mathfrak{sl}(2)$, we have the *raising operator*, R, and the *lowering operator*, L, with

$$R = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } L = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

so-called by the action on the basis:

$$\begin{aligned} R\mathbf{e}_0 &= \mathbf{e}_1 \quad \text{with } R\mathbf{e}_1 = 0 \\ L\mathbf{e}_1 &= \mathbf{e}_0 \quad \text{with } L\mathbf{e}_0 = 0 \end{aligned}$$

R on $\mathbf{e}_1$ takes you "out" of the space, so to the zero vector, while L lowers $\mathbf{e}_0$ to zero. We will find realizations of the Lie algebra acting in higher dimensional spaces where these operators act similarly. The third element of the basis we denote by

$$\rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so that these satisfy the commutation relations

$$[L, R] = \rho\,, \quad [R, \rho] = 2R\,, \quad [\rho, L] = 2L$$

We consider group elements of the form

$$g(s,t,u) = \exp(tL)\, u^\rho\, \exp(sR) = \begin{pmatrix} \frac{st}{u} + u & \frac{t}{u} \\ \frac{s}{u} & \frac{1}{u} \end{pmatrix} \tag{7.1}$$

where

$$\begin{aligned} \exp(tL) = e^{tL} &= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \\ u^\rho = e^{\log(u)\rho} &= \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \\ \exp(sR) = e^{sR} &= \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \end{aligned}$$

with the identity $g(0, 0, 1)$.

Given a matrix (of the above form), checking for determinant one, we find the coordinates as follows:

Proposition 7.1. *Let* $g = \begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix}$ *be an element of* SL(2) *of the form* $g(s,t,u)$. *Then we recover the coordinates by the mapping*

$$s = g_{10}/g_{11}\,, \quad t = g_{01}/g_{11}\,, \quad u = 1/g_{11}$$

Proof. Verify via eq. (7.1). □

First, note that $\det(b^{-1}T^2) = 1$. Now for our key observation.

Proposition 7.2. *Let* $T = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ *be the shift matrix for recurrence* (2.2)*. Then*

$$\frac{1}{b} T^2 = g(a/b, a, 1) = g(s, t, 1)$$

with the correspondences

$$\begin{aligned} (a, b) &\longleftrightarrow (t, t/s) \\ (s, t) &\longleftrightarrow (a/b, a) \end{aligned} \tag{7.2}$$

Proof. Compare

$$\tfrac{1}{b}\,T^2 = \frac{1}{b}\begin{pmatrix} a^2+b & ab \\ a & b \end{pmatrix} = \begin{pmatrix} 1+a^2/b & a \\ a/b & 1 \end{pmatrix}$$

Now compare with eq. (7.1) and Prop. 7.1. □

In other words, we can write

$$g = g(s,t,1) = g(a/b,a,1) = \frac{1}{b}T^2 = e^{aL}e^{(a/b)R}\ . \tag{7.3}$$

Now we can use our results to compute powers of the group element $g(s,t,1)$. We rewrite in terms of h_n functions, Cor. 4.9,

$$b^{-n}T^{2n} = b^{-n}\begin{pmatrix} h_{2n} & bh_{2n-1} \\ h_{2n-1} & bh_{2n-2} \end{pmatrix}$$

Thus, via the correspondence (7.2),

$$\begin{aligned}
g^n_{00} &= \sum_k \binom{2n-k}{k}(st)^{n-k} \\
g^n_{01} &= \sum_k \binom{2n-1-k}{k} t^{n-k}s^{n-k-1} \\
g^n_{10} &= \sum_k \binom{2n-1-k}{k} s^{n-k}t^{n-k-1} \\
g^n_{11} &= \sum_k \binom{2(n-1)-k}{k}(st)^{n-1-k}
\end{aligned}$$

Noting g^n_{10} and g^n_{01} are related by switching $s \leftrightarrow t$, while $g^n_{11} = g^{n-1}_{00}$.

From another point of view, write

$$T^{2n} = \begin{pmatrix} a_{2n+1} & b_{2n+1} \\ a_{2n} & b_{2n} \end{pmatrix}\Bigg|_{a=t,b=t/s}$$

and thus express g^n in coordinates:

$$g(s,t,1)^n = g(a_{2n}/b_{2n}, b_{2n+1}/b_{2n}, b^n/b_{2n})\big|_{a=t,b=t/s}$$

7.2. Induced matrices from group elements. Gamma map. Including the factor of b via the matrix factor bI, rewrite eq. (7.3) as

$$T^2 = e^{aL}(bI)e^{aR/b} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a/b & 1 \end{pmatrix}$$

This is a UDL decomposition, upper-triangular $\times$ diagonal $\times$ lower-triangular factorization. Forming induced matrices preserves this form in every degree. That is, for a given degree N:

$$\bar{T}^2 = \overline{e^{aL}}(b^N I)\overline{e^{(a/b)R}} = b^N\ \overline{e^{aL}}\ \overline{e^{(a/b)R}}$$

The exponential factors are of the generic form e^{vX}, forming one-parameter groups. Via the homomorphism property,

$$\overline{e^{vX}} = e^{v\Gamma(X)}$$

for some matrix $\Gamma(X)$, we call the map $X \to \Gamma(X)$, the Γ-map. Note that, explicitly indicating the degree,

$$\Gamma^{(N)}(X) = \frac{\mathrm{d}}{\mathrm{d}v}\bigg|_{v=0} e^{v\Gamma^{(N)}(X)}$$

which is the directional derivative of the induced-matrix map at the identity. Thus we can compute as well

$$\Gamma^{(N)}(X) = \frac{\mathrm{d}}{\mathrm{d}v}\bigg|_{v=0} \overline{(I + vX)}^{(N)}$$

In the Appendix, are summarized the main properties of the Γ-map, including: (1) linearity and (2) preservation of Lie brackets.

Definition 7.3. We define projection operators according to:

$$P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Together with R and L, these form a basis for 2×2 matrices.

Some examples will illustrate the patterns that arise. For $N = 4$, we have

$$\Gamma(P_0) = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \Gamma(P_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

while

$$\Gamma(L) = \begin{pmatrix} 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \Gamma(R) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix}$$

By linearity this leads to $\Gamma(X)$ for general 2×2 matrix X.

From the pattern we see that for general N,

$$\Gamma^{(N)}_{mn}(P_0) = \begin{cases} N - m, & \text{if } m = n \\ 0, & \text{otherwise} \end{cases} \qquad \Gamma^{(N)}_{mn}(P_1) = \begin{cases} m, & \text{if } m = n \\ 0, & \text{otherwise} \end{cases}$$

while for R and L,

$$\Gamma^{(N)}_{mn}(L) = \begin{cases} N - m, & \text{if } m = n - 1 \\ 0, & \text{otherwise} \end{cases} \qquad \Gamma^{(N)}_{mn}(R) = \begin{cases} m, & \text{if } m = n + 1 \\ 0, & \text{otherwise} \end{cases}$$

As operators acting on vectors, we have, in degree N,

$$\begin{gathered} \Gamma(R)\mathbf{e}_n = (n+1)\mathbf{e}_{n+1} \quad \text{and} \quad \Gamma(L)\mathbf{e}_n = (N - (n-1))\mathbf{e}_{n-1} \\ \Gamma(P_0)\mathbf{e}_n = (N - n)\mathbf{e}_n \quad \text{and} \quad \Gamma(P_1)\mathbf{e}_n = n\mathbf{e}_n \end{gathered} \tag{7.4}$$

with $\Gamma(R)\mathbf{e}_N = \Gamma(L)\mathbf{e}_0 = 0$.

Remark 7.4. We see that $\Gamma(P_1)$ acts as a "number operator," multiplying $\mathbf{e}_n$ by n.

Note that for any 2×2 matrix, X, $\Gamma(X)$ is tridiagonal.

8. Gamma Map and Spectral Relations. Recurrences.

We look at the eigenvalues of $\Gamma(X)$. Then consider some interesting spectral relations. In particular, some recurrences with non-constant coefficients will arise.

8.1. Eigenvalues of $\Gamma(X)$. For eigenvalues, consider the diagonal matrix $\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = \lambda P_0 + \mu P_1$. We have $\Gamma(\Lambda)_{mm} = (N-m)\lambda + m\mu$, i.e.,

Proposition 8.1. *For eigenvalues $\{\lambda, \mu\}$ of X, the eigenvalues of $\Gamma^{(N)}(X)$ are*

$$\{N\lambda, (N-1)\lambda + \mu, \ldots, (N-m)\lambda + m\mu, \ldots, \lambda + (N-1)\mu, N\mu\} \ .$$

Alternatively,

$$\{N\lambda, (N-2)\lambda + a, \ldots, (N-2m)\lambda + ma, \ldots, (2-N)\lambda + (N-1)a, -N\lambda + Na\} \ .$$

This alternative form follows via the substitution $\mu = a - \lambda$.

Here are some examples tying up the induced matrix and the Γ-matrix.

Example 8.2. For $T = \begin{pmatrix} 1+\alpha & -\alpha \\ 1 & 0 \end{pmatrix}$, we have eigenvalues 1 and α. We find, for $N = 4$,
$\bar{T} =$

$$\begin{pmatrix} \alpha^4 + 4\alpha^3 + 6\alpha^2 + 4\alpha + 1 & -4\alpha^4 - 12\alpha^3 - 12\alpha^2 - 4\alpha & 6\alpha^4 + 12\alpha^3 + 6\alpha^2 & -4\alpha^4 - 4\alpha^3 & \alpha^4 \\ \alpha^3 + 3\alpha^2 + 3\alpha + 1 & -3\alpha^3 - 6\alpha^2 - 3\alpha & 3\alpha^3 + 3\alpha^2 & -\alpha^3 & 0 \\ \alpha^2 + 2\alpha + 1 & -2\alpha^2 - 2\alpha & \alpha^2 & 0 & 0 \\ \alpha + 1 & -\alpha & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with eigenvalues $\{1, \alpha, \alpha^2, \alpha^3, \alpha^4\}$. While

$$\Gamma(T) = \begin{pmatrix} 4\alpha + 4 & -4\alpha & 0 & 0 & 0 \\ 1 & 3\alpha + 3 & -3\alpha & 0 & 0 \\ 0 & 2 & 2\alpha + 2 & -2\alpha & 0 \\ 0 & 0 & 3 & \alpha + 1 & -\alpha \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix}$$

with eigenvalues $\{4, 3 + \alpha, 2 + 2\alpha, 1 + 3\alpha, 4\alpha\}$.

Example 8.3. The Fibonacci case leads to some interesting patterned matrices. For $T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, we have, for $N = 4$,

$$\bar{T} = \begin{pmatrix} 1 & 4 & 6 & 4 & 1 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with eigenvalues $\{\phi^{4-m}(-\phi)^{-m}\}_{0\le m\le 4}$ simplifying to

$$\{2+3\phi, -1-\phi, 1, -2+\phi, 5-3\phi\} \ .$$

While

$$\Gamma(T) = \begin{pmatrix} 4 & 4 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix}$$

with eigenvalues $\{(N-m)\phi - m\phi^{-1}\}_{0\le m\le 4}$ which, via $-\phi^{-1} = 1-\phi$, simplifies to

$$\{4\phi, 1+2\phi, 2, 3-2\phi, 4-4\phi\}$$

Similar binomial matrices to $\bar{T}$ and patterned matrices as that of $\Gamma(T)$ will have corresponding spectra expressible in terms of the golden ratio.

We see that

Proposition 8.4. *Assume X is diagonalizable, with $XW = W\Lambda$, where the columns of W are eigenvectors of X with corresponding eigenvalues along the diagonal of Λ. Then, for a given degree N,*

1. *We have*
$$\Gamma(X)\bar{W} = \bar{W}\Gamma(\Lambda)$$
2. *$\bar{X}$ and $\Gamma(X)$ have common eigenvectors given by the columns of $\bar{W}$.*
3. *The eigenvalues of $\bar{X}$ form the diagonal of $\bar{\Lambda}$ and the eigenvalues of $\Gamma(X)$ are given by the diagonal matrix $\Gamma(\Lambda)$.*

Proof. The relation $XW = W\Lambda$ implies $\bar{X}\bar{W} = \bar{W}\bar{\Lambda}$, which gives the results for $\bar{X}$ Now multiply both sides by v and add W to get

$$(I+vX)W = W(I+v\Lambda) \ .$$

Forming induced matrices yields

$$\overline{(I+vX)}\bar{W} = \bar{W}\overline{(I+v\Lambda)}$$

Now differentiate with respect to v and set v equal to zero to convert to the Γ-map, and the required results follow. □

Corollary 8.5. *The relation $WX = \Lambda W$ entails $\bar{W}\,\Gamma(X) = \Gamma(\Lambda)\bar{W}$.*

Some further properties of the Γ-map.

Proposition 8.6. *Let $VW = 0$, with V, W both $n \times n$ matrices. Then for each level N,*

$$\bar{V}\,\Gamma(W) = \Gamma(V)\bar{W} = 0$$

Proof. We have $V(I + vW) = V$ so that $\bar{V}\,\overline{I + vW} = \bar{V}$. Differentiating with respect to v at $v = 0$ yields $\bar{V}\,\Gamma(W) = 0$. The other way is similar. □

In particular

Corollary 8.7. *Let A be an $n \times n$ matrix of rank $n - 1$. Then, for each level N,*

$$\Gamma(\operatorname{adj} A)\bar{A} = \bar{A}\,\Gamma(\operatorname{adj} A) = \Gamma(A)\,\overline{\operatorname{adj} A} = \overline{\operatorname{adj} A}\,\Gamma(A) = 0$$

Proof. Start with $A(\operatorname{adj} A) = (\operatorname{adj} A)A = 0$ and apply the above Proposition. □

8.2. Gamma map and recurrences. Let's look at some Krawtchouk systems, cf. #1 of Theorem 5.3 .

First, we want to think of $\bar{A}$ as an array of the values of functions $\phi_n(m)$. For polynomials, we would like ϕ_n to be of degree n. Taking ϕ_0 to be constant, we normalize to $\phi_0 = 1$, i.e., the first column of $\bar{A}$ should consist of all 1's. ϕ_1 should be linear in m, and we will take the values $\phi_1(m)$ to be the x variable in a 3-term recursion such as $x\phi_n = \alpha_n\phi_{n+1} + \beta_n\phi_n + \gamma_n\phi_{n-1}$. Then higher-order functions ϕ_n can be expressed either in terms of x or in terms of m. In any case, in degree N, m and n take values in $\{0, 1, \ldots, N\}$.

First, we make some observations.

Proposition 8.8. *Let* $A = \begin{pmatrix} 1 & r \\ 1 & -s \end{pmatrix}$. *Then*

1. *For every $N > 0$, the first column, $n = 0$, of $\bar{A}^{(N)}$ consists of all 1's.*
2. *The second column, $n = 1$, of $\bar{A}^{(N)}$ is the same as the diagonal entries of $\Gamma^{(N)}(\Lambda)$ where Λ is the diagonal matrix formed from the second column of A,* $\Lambda = \begin{pmatrix} r & 0 \\ 0 & -s \end{pmatrix}$.

Proof. Start with

$$(1 + rt)^{N-m}(1 - st)^m = \sum_n \bar{A}_{mn}t^n \tag{8.1}$$

We are looking on the right-hand side for the constant term and the term first-order in t. Setting $t = 0$ immediately gives $\bar{A}_{m0} = 1$ for all m. Next, we have

$$(1 + rt)^{N-m}(1 - st)^m = 1 + ((N - m)r - ms)t + \cdots$$

Comparing the coefficient of t with the statement of Prop. 8.1 yields the result. □

Remark 8.9. These properties extend to the $d \times d$ case. Cf., [1, §3].

We form the matrix

$$X = A^{-1}\Lambda A$$

where Λ is the diagonal matrix formed from the second column of A. (Note that using the first column just results in an identity matrix for X.) That is,

$$AX = \Lambda A$$

As in Corollary 8.5, we have

$$\bar{A}\Gamma(X) = \Gamma(\Lambda)\bar{A}$$

We will work through some examples, illustrating the signifance of this relation.

Example 8.10. For the case $A = \begin{pmatrix} 1 & r \\ 1 & -s \end{pmatrix}$, we have $\Lambda = \begin{pmatrix} r & 0 \\ 0 & -s \end{pmatrix}$ and we find

$$X = \begin{pmatrix} 0 & rs \\ 1 & r-s \end{pmatrix}$$

For $N = 3$, we have

$$\Gamma(X) = \begin{pmatrix} 0 & 3rs & 0 & 0 \\ 1 & r-s & 2rs & 0 \\ 0 & 2 & 2r-2s & rs \\ 0 & 0 & 3 & 3r-3s \end{pmatrix} \text{ and } \bar{\Lambda} = \begin{pmatrix} 3r & 0 & 0 & 0 \\ 0 & 2r-s & 0 & 0 \\ 0 & 0 & r-2s & 0 \\ 0 & 0 & 0 & -3s \end{pmatrix}$$

with

$$\bar{A} = \begin{pmatrix} 1 & 3r & 3r^2 & r^3 \\ 1 & 2r-s & r^2-2rs & -r^2s \\ 1 & r-2s & -2rs+s^2 & rs^2 \\ 1 & -3s & 3s^2 & -s^3 \end{pmatrix}$$

Write

$$X = R + (r-s)P_1 + rsL$$

and applying Γ yields

$$\Gamma(X) = \Gamma(R) + (r-s)\Gamma(P_1) + rs\Gamma(L)$$

Now act on $\mathbf{e}_n$, using eqs. (7.4), to get

$$\Gamma(X)\mathbf{e}_n = (n+1)\mathbf{e}_{n+1} + (r-s)n\mathbf{e}_n + rs(N-n+1)e_{n-1}$$

Applying $\bar{A}$ to both sides, using $\bar{A}\mathbf{e}_n = \sum_m \bar{A}_{mn}\mathbf{e}_m = \sum_m \phi_n(m)\mathbf{e}_m$, we get

$$\bar{A}\Gamma(X) = \sum_m \left((n+1)\phi_{n+1}(m) + (r-s)n\phi_n(m) + rs(N+n-1)\phi_{n-1}(m)\right)\mathbf{e}_m$$

On the other hand, we apply $\Gamma(\Lambda)$ to $\bar{A}\mathbf{e}_n$ to get

$$\Gamma(\Lambda)\sum_m \phi_n(m)\mathbf{e}_m = \sum_m (r(N-m)-sm)\phi_n(m)\mathbf{e}_m$$

And we have

Proposition 8.11. *The functions $\phi_n(m)$ satisfy the recurrence relation*

$$(r(N-m)-sm)\phi_n(m) = (n+1)\phi_{n+1}(m) + (r-s)n\phi_n(m) + rs(N+n-1)\phi_{n-1}(m)$$

for $n \geq 0$, with initial conditions $\phi_{-1}(m) = 0$, $\phi_0(m) = 1$ and boundary condition $\phi_{N+1}(m) = 0$, $0 \leq m \leq N$.

Proof. We have to check consistency with the known forms of ϕ_0 and ϕ_1:

$$(r(N-m)-sm)\phi_0(m) = \phi_1(m)$$

which checks. For integer m from 0 to N inclusive the form eq. (8.1) shows that the coefficient of t^{N+1} is zero. □

Example 8.12. As a variation of the Fibonacci case, let's take $s = 0$. We have $A = \begin{pmatrix} 1 & r \\ 1 & 0 \end{pmatrix}$, and $\Lambda = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}$, leading to

$$X = \begin{pmatrix} 0 & 0 \\ 1 & r \end{pmatrix} = R + rP_1$$

For $N = 3$, we have

$$\Gamma(X) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & r & 0 & 0 \\ 0 & 2 & 2r & 0 \\ 0 & 0 & 3 & 3r \end{pmatrix} \text{ and } \bar{\Lambda} = \begin{pmatrix} 3r & 0 & 0 & 0 \\ 0 & 2r & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

while

$$\bar{A} = \begin{pmatrix} 1 & 3r & 3r^2 & r^3 \\ 1 & 2r & r^2 & 0 \\ 1 & r & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

The recurrence reads

$$r(N - m)\phi_n(m) = (n + 1)\phi_{n+1}(m) + rn\phi_n(m)$$

Solving for ϕ_{n+1}, we have

$$\phi_{n+1}(m) = \frac{r(N - m - n)}{n + 1}\, \phi_n(m)$$

all in agreement with

$$\phi_n(m) = r^n \binom{N - m}{n} \; .$$

Example 8.13. For general $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ writing

$$(a + bt)^{N-m}(c + dt)^m = a^{N-m}c^m(1 + (b/a)t)^{N-m}(1 + (d/c)t)^m \tag{8.2}$$

shows that without restricting $a = c = 1$, functions exponential in m arise. We will follow the approach above to derive recurrence relations for the functions $\phi_n(m)$. Here we will be interested in both X_1 and X_2. We have

$$X_1 = A^{-1}\Lambda_1 A \quad \text{and} \quad X_2 = A^{-1}\Lambda_2 A$$

with $\Lambda_1 = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$, $\Lambda_2 = \begin{pmatrix} b & 0 \\ 0 & d \end{pmatrix}$.

For convenience avoiding denominators, we will use $\operatorname{adj} A$. With $\Delta = \det A$, from $AX = \Lambda A$, we have $A(\Delta X) = \Delta\Lambda A$, with $\Delta X = (\operatorname{adj} A)\Lambda A$. That is, for $i = 1, 2$,

$$\bar{A}\Gamma(\Delta X_i) = \Delta\Gamma(\Lambda_i)\bar{A}$$

by linearity of Γ. For $N = 3$,

$$\bar{A} = \begin{pmatrix} a^3 & 3a^2b & 3ab^2 & b^3 \\ a^2c & a^2d + 2abc & 2abd + b^2c & b^2d \\ ac^2 & 2acd + bc^2 & ad^2 + 2bcd & bd^2 \\ c^3 & 3c^2d & 3cd^2 & d^3 \end{pmatrix}$$

1. First,

$$\Delta X_1 = \begin{pmatrix} a^2 d - bc^2 & bd\,(a-c) \\ -ac\,(a-c) & -ac\,(b-d) \end{pmatrix}$$

$\Gamma(\Delta X_1) =$

$$\begin{pmatrix} 3\left(a^2 d - bc^2\right) & 3bd\,(a-c) & 0 & 0 \\ -ac\,(a-c) & 2a^2 d - abc + acd - 2bc^2 & 2bd\,(a-c) & 0 \\ 0 & -2ac\,(a-c) & \begin{matrix} a^2 d - 2abc \\ \quad + 2acd - bc^2 \end{matrix} & bd\,(a-c) \\ 0 & 0 & -3ac\,(a-c) & -3ac\,(b-d) \end{pmatrix}$$

and

$$\bar{\Lambda}_1 = \begin{pmatrix} 3a & 0 & 0 & 0 \\ 0 & 2a+c & 0 & 0 \\ 0 & 0 & a+2c & 0 \\ 0 & 0 & 0 & 3c \end{pmatrix}$$

The form

$\Gamma(\Delta X_1) = -ac\,(a-c)\,\Gamma(R) + (a^2 d - bc^2)\Gamma(P_0) - ac(b-d)\Gamma(P_1) + bd(a-c)\Gamma(L)$

yields one side of the recurrence as above. The other side comes from $\Delta\bar{\Lambda}_1$. We find

$$\begin{aligned} \Delta((N-m)a + mc)\phi_n = {} & -ac(a-c)(n+1)\phi_{n+1} \\ & + ((a^2 d - bc^2)(N-n) - ac(b-d)n)\phi_n \\ & + bd(a-c)(N-n+1)\phi_{n-1} \end{aligned}$$

Going back to eq. (8.2), we have initial conditions

$$\phi_{-1}(m) = 0 \quad \text{and} \quad \phi_0(m) = a^{N-m} c^m$$

Differentiating with respect to t at $t = 0$ yields

$$\phi_1(m) = a^{N-m} c^m \big((N-m)(b/a) + m(d/c)\big)$$

which checks consistently with the recurrence stated.

2. Second,

$$\Delta X_2 = \begin{pmatrix} bd\,(a-c) & bd\,(b-d) \\ -ac\,(b-d) & ad^2 - b^2 c \end{pmatrix}$$

with $\Gamma(\Delta X_2)$ and $\bar{\Lambda}_2$ similar to those associated with X_1. We find

$\Gamma(\Delta X_2) = -ac(b-d)\Gamma(R) + bd(a-c)\Gamma(P_0) + (ad^2 - b^2 c)\Gamma(P_1) + bd(b-d)\Gamma(L)$

and the recurrence

$$\begin{aligned} \Delta((N-m)b + md)\phi_n = {} & -ac(b-d)(n+1)\phi_{n+1} \\ & + (bd(a-c)(N-n) + (ad^2 - b^2 c)n)\phi_n \\ & + bd(b-d)(N-n+1)\phi_{n-1}\ . \end{aligned}$$

with the same initial conditions as above.

Example 8.14. Working from the other side, consider the diagonalization of T, eq. (2.15). Here we will get a difference equation in the variable m. Starting with $TW = W\Lambda$, where

$$T = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}, W = \begin{pmatrix} \lambda & \mu \\ 1 & 1 \end{pmatrix}, \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

we have $\Gamma(T)\bar{W} = \bar{W}\Gamma(\Lambda)$ with

$$\Gamma(T) = \Gamma(R) + a\Gamma(P_0) + b\Gamma(L)$$

Thus, with

$$\bar{W}\mathbf{e}_n = \sum_m \phi_n(m)\mathbf{e}_m \quad \text{and} \quad \Gamma(\Lambda)\mathbf{e}_n = \big((N-n)\lambda + n\mu\big)\mathbf{e}_n$$

we get

$$\sum_m \phi_n(m)\big[(m+1)\mathbf{e}_{m+1} + a(N-m)\mathbf{e}_m + b(N-m+1)\mathbf{e}_{m-1}\big]$$
$$= \sum_m \big((N-n)\lambda + n\mu\big)\phi_n(m)\mathbf{e}_m$$

With zero boundary conditions for negative indices, matching $\mathbf{e}_m$ components, we find the recurrence, difference equation,

$$\begin{aligned} m\phi_n(m-1) + a(N-m)\phi_n(m) + &b(N-m)\phi_n(m+1) \\ &= \big((N-n)\lambda + n\mu\big)\phi_n(m) \end{aligned}$$

And expressing this in terms of λ and μ:

$$\begin{aligned} m\phi_n(m-1) + (\lambda+\mu)(N-m)\phi_n(m) - &\lambda\mu(N-m)\phi_n(m+1) \\ &= \big((N-n)\lambda + n\mu\big)\phi_n(m) \end{aligned}$$

For $m = 0$, we have $(\lambda + \mu t)^N = \sum_n \phi_n(0)\, t^n$, i.e.,

$$\phi_n(0) = \binom{N}{n} \lambda^{N-n}\mu^n \ .$$

9. Two-state Markov Chains. Convergence and Induced Chains.

We apply our approach to the case of 2-state Markov chains. Start with

$$P = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} \quad \text{such that} \quad P\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

with $p_{ij} \geq 0$. Take one eigenvalue $\lambda = 1$. Then

$$\mu = \operatorname{tr} P - 1 = p_{00} - p_{10}$$

Since $|\mu| \leq 1$, we have $|\mu| = 1$ in the cases $P = I$ or $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the flip-flop chain. These cases aside, assume that $|\mu| < 1$.

9.1. Convergence. Referring to Example 2.12, here with μ replacing α, we have

$$P^n = \frac{1-\mu^n}{1-\mu}P + \frac{\mu^n-\mu}{1-\mu}I = \frac{1}{1-\mu}\left[P - \mu I + \mu^n(I-P)\right]$$

For $|\mu| < 1$, we have convergence to a stationary distribution given by either row of Π, where

$$\Pi = \lim_{n\to\infty} P^n = \frac{1}{1-\mu}(P-\mu I) = \frac{1}{p_{10}+p_{01}}\begin{pmatrix} p_{10} & p_{01} \\ p_{10} & p_{01}\end{pmatrix}$$

9.1.1. *Adjugates and Matrix Tree Theorem.* We have $\operatorname{rank}(I-P) = 1$, thus Π is proportional to $\operatorname{adj}(I-P)$ as $\operatorname{adj}(I-P)(I-P) = 0 = \Pi(I-P)$ with both Π and $\operatorname{adj}(I-P)$ being rank one with identical rows. The main property of $I-P$ is that its rows sum to zero. We recall, informally, the

Markov Chain/Matrix Tree Theorem:

1. The adjugate of a matrix with zero row sums is a rank one matrix with identical rows. In particular, if the matrix has rank one less than its size (no. of rows, columns), the adjugate will be nonzero with any one row a left eigenvector with eigenvalue zero.

2. Entries in a given row of the adjugate can be computed by drawing the transition graph without loops with weights on the edges equal to the corresponding entries of the given matrix. Then the entry in column n of the adjugate equals the sum of the weights of all directed trees ending at vertex n with the weight of a tree equal to the product of the weights along its edges. (Depending on the dimension, there may be an overall minus sign.)

Note: We will consider matrices with zero row sums with negative entries along the diagonal and nonnegative off-diagonal entries, such as $P-I$ for a stochastic matrix (of any size) P.

Example 9.1. With $P = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11}\end{pmatrix}$, we have $P - I = \begin{pmatrix} -p_{01} & p_{01} \\ p_{10} & -p_{10}\end{pmatrix}$. We have

$$0 \;\underset{p_{10}}{\overset{p_{01}}{\rightleftarrows}}\; 1$$

and $\operatorname{adj}(P-I) = -\begin{pmatrix} p_{10} & p_{01} \\ p_{10} & p_{01}\end{pmatrix}$.

A more interesting example is provided by the 3×3 case. We have

$$P - I = \begin{pmatrix} -p_{01}-p_{02} & p_{01} & p_{02} \\ p_{10} & -p_{10}-p_{12} & p_{12} \\ p_{20} & p_{21} & -p_{20}-p_{21}\end{pmatrix}$$

And $\operatorname{adj}(P-I) =$

$$\begin{pmatrix} \begin{array}{l} p_{10}p_{20} \\ \quad + p_{10}p_{21} + p_{12}p_{20} \end{array} & \begin{array}{l} p_{01}p_{20} \\ \quad + p_{01}p_{21} + p_{02}p_{21} \end{array} & \begin{array}{l} p_{01}p_{12} \\ \quad + p_{02}p_{10} + p_{02}p_{12} \end{array} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

with the second and third rows identical to the first. The first row is readily checked graphically. E.g., the term $p_{10}p_{20}$ comes via the tree

$$1 \xrightarrow{p_{10}} 0 \xleftarrow{p_{20}} 2$$

and similarly for the other terms.

See [2] for details and further examples.

9.2. Induced chains. First we look at row sums of induced matrices. Then we will look at chains at each level N starting with a stochastic matrix. Then we look at generators of one-parameter stochastic semigroups corresponding to jump Markov processes with finite state space.

9.2.1. *Row sums.*

Proposition 9.2.

1. Let Q be a matrix with zero row sums. Then $\bar{Q}^{(N)}$ has zero row sums for every $N \geq 1$. Furthermore, $\Gamma(Q)$ has zero row sums at every level N.

2. Let P be a stochastic matrix. Then $\bar{P}^{(N)}$ is stochastic for every $N \geq 1$.

Proof. Recall the diagonal matrix B, in level N, with $B_{nn} = \binom{N}{n}$. Let U denote the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and U_N the $(N+1) \times (N+1)$ matrix of all ones. We see directly that $\bar{U} = U_N B$, i.e., the n^{th} column of $\bar{U}$ is a column of 1's times the corresponding binomial coefficient. Now a zero row sum matrix Q means $QU = 0$, hence $\bar{Q}\bar{U} = \bar{Q}U_N B = 0$ so that $\bar{Q}U_N = 0$, and $\bar{Q}$ has zero row sums. Applying Prop. 8.6 in the case $QU = 0$ yields zero row sums for $\Gamma(Q)$. Similarly, $PU = U$ implies $\bar{P}U_N B = U_N B$ and cancelling B shows that rows of $\bar{P}$ sum to 1. The definition of $\bar{P}$ shows directly that positivity (nonnegativity) of entries of P implies positivity (nonnegativity) of entries of $\bar{P}$. ☐

Remark 9.3. Note that the above extends to matrices of any size, $n \times n$, with B replaced by the appropriate diagonal matrix of multinomial coefficients.

9.2.2. *Systems with N nodes.* Starting with a 2×2 stochastic matrix, P, we determine a Markov chain where the states are *configurations* of N *nodes*. The chain with transition matrix P is the *underlying chain*. Each node is either in state 0 or state 1. A configuration of the N nodes is a specification of m_0 nodes in state 0 of the underlying chain and $m_1 = N - m_0$ nodes in state 1 of the underlying chain. A transition from (m_0, m_1) to (n_0, n_1), with $n_0 + n_1 = N$, has probability

$$\bar{P}_{(m_0,m_1),(n_0,n_1)} = \sum_{\sum_j n_{ij} = m_i, \sum_i n_{ij} = n_j} \frac{m_0!}{n_{00}!n_{01}!} \frac{m_1!}{n_{10}!n_{11}!} p_{00}^{n_{00}} p_{01}^{n_{01}} p_{10}^{n_{10}} p_{11}^{n_{11}}$$

with $\sum_{i,j} n_{ij} = N$. Here n_{00} are the number of nodes in state 0 remaining in state 0, similarly n_{ij} is the number of nodes in state i transitioning to state j, noting that j may equal i. The probabilities are determined according to binomial choices. The following array summarizes the conditions on the numbers n_{ij}:

$$\begin{array}{cc|c} n_{00} & n_{01} & m_0 \\ n_{10} & n_{11} & m_1 \\ \hline n_0 & n_1 & N \end{array}$$

The right margin are the row sums, the bottom margin are the column sums. All such choices of n_{ij} resulting in column sums (n_0, n_1) are added to yield the transition probability from (m_0, m_1) to (n_0, n_1). For reference here is $\bar{P}^{(3)}$:

$$\begin{pmatrix} p_{00}^3 & 3p_{00}^2 p_{01} & 3p_{00}p_{01}^2 & p_{01}^3 \\ p_{00}^2 p_{10} & p_{00}^2 p_{11} + 2p_{00}p_{01}p_{10} & 2p_{00}p_{01}p_{11} + p_{01}^2 p_{10} & p_{01}^2 p_{11} \\ p_{00}p_{10}^2 & 2p_{00}p_{10}p_{11} + p_{01}p_{10}^2 & p_{00}p_{11}^2 + 2p_{01}p_{10}p_{11} & p_{01}p_{11}^2 \\ p_{10}^3 & 3p_{10}^2 p_{11} & 3p_{10}p_{11}^2 & p_{11}^3 \end{pmatrix}$$

For $N = 4$, e.g., we have $\bar{P}_{(3,1),(2,2)} = 3p_{00}^2 p_{01} p_{11} + 3p_{00}p_{01}^2 p_{10}$.

Remark 9.4. The induced chains for finite-state underlying chains with more states are described similarly, with multinomial probabilities extending the binomial probabilities of the 2-state case.

9.2.3. *Convergence.* Given a stochastic matrix P, with $P^n \to \Pi$, we have $\overline{P^n} = (\bar{P})^n$, with $\bar{P}$ stochastic. We expect $\lim_{n\to\infty} (\bar{P})^n \to \bar{\Pi}$. Let's verify this.

First, let $P = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ be stochastic and consider eq. (4.3) with t replaced by $e^{i\theta}$, $\theta \in \mathbb{R}$:

$$G(\theta) = (p + qe^{i\theta})^{N-m}(r + se^{i\theta})^m = \sum_n \bar{P}_{mn} e^{in\theta}$$

so that

$$\bar{P}_{mn} = \frac{1}{2\pi} \int_0^{2\pi} G(\theta) e^{-in\theta}\, d\theta$$

For two stochastic matrices P_1 and P_2, we have, with corresponding functions G_1, G_2:

$$|(\bar{P}_1)_{mn} - (\bar{P}_2)_{mn}| \le \frac{1}{2\pi} \int_0^{2\pi} |G_1(\theta) - G_2(\theta)|\, d\theta \tag{9.1}$$

Observe that each of the N factors in $G(\theta)$ are at most 1 in absolute value. We recall the identity for any sets of numbers $\{u_i\}_{1\le i\le n}$, $\{v_i\}_{1\le i\le n}$

$$\prod_{1\le i\le n} u_i - \prod_{1\le i\le n} v_i = \sum_{k=1}^{n} \Big(\prod_{j<k} u_j\Big)(u_k - v_k)\Big(\prod_{i>k} v_i\Big)$$

(empty products equal to 1) observing that this is a telescoping sum. If in turn $|u_i| \le 1$ and $|v_i| \le 1$ for all $1 \le i \le n$, then we have the inequality

$$\Big|\prod_{1\le i\le n} u_i - \prod_{1\le i\le n} v_i\Big| \le \sum_{k=1}^{n} |u_k - v_k|$$

cf. [3, p. 258]

Proposition 9.5. *Let P_1 and P_2 be stochastic matrices. Then for fixed $N \ge 1$ the induced matrices satisfy*

$$|(\bar{P}_1^{(N)})_{mn} - (\bar{P}_2^{(N)})_{mn}| \le N \sum_{0\le i,j\le 1} \big|(P_1)_{ij} - (P_2)_{ij}\big|$$

Proof. Each factor in a product of the form $G(\theta)$ has absolute value less or equal to 1 as the corresponding matrix is stochastic. Thus, bounding the right-hand side of eq. (9.1), we have

$$\begin{aligned}
&|G_1(\theta) - G_2(\theta)| \\
&\qquad = |(p_1 + q_1 e^{i\theta})^{N-m}(r_1 + s_1 e^{i\theta})^m - (p_2 + q_2 e^{i\theta})^{N-m}(r_2 + s_2 e^{i\theta})^m| \\
&\qquad \le (N-m)(|p_1 - p_2| + |q_1 - q_2|) + m(|r_1 - r_2| + |s_1 - s_2|) \\
&\qquad \le N(|p_1 - p_2| + |q_1 - q_2| + |r_1 - r_2| + |s_1 - s_2|)
\end{aligned}$$

as required. □

Corollary 9.6. *Given a stochastic matrix P such that $P^n \to \Pi$, then, for each $N \ge 1$, the induced matrices satisfy $\bar{P}^n \to \bar{\Pi}$.*

Thus for an ergodic chain, the induced chains at every level are ergodic, converging to the corresponding stationary distributions.

Remark 9.7. Again, note that analogous results hold for any finite-state Markov chain. Also note that one could study induced chains in the non-homogeneous case as well.

9.2.4. *Generators of jump processes.* Begin by considering a finite-state jump process with d states, $d > 1$. Then the process corresponds to a one-parameter stochastic semigroup $P(t) = e^{tQ}$, where $Q = (q_{ij})$ satisfies the conditions:
(1) zero row sums, $\sum_j q_{ij} = 0,\ \forall i$ (2) $q_{ii} < 0,\ \forall i$ (3) $q_{ij} \ge 0,\ \forall i \ne j$.

Theorem 9.8. *If Q is the generator for a jump Markov process, then so is $\Gamma(Q)$.*

Proof. This is essentially the definition of $\Gamma(Q)$. We have $P(t)$ stochastic and thus

$$\bar{P}(t) = \overline{e^{tQ}} = e^{t\Gamma(Q)}$$

is stochastic so that $\Gamma(Q)$ is a generator and has the same descriptive properties as Q. □

In the irreducible case, we have a stationary distribution. If Π is the limiting matrix, with equal rows each equal to the stationary distribution, then, as $\operatorname{adj} Q$ has equal rows and satisfies $(\operatorname{adj} Q)\, Q = 0$, we have $\operatorname{adj} Q$ proportional to Π. At level N, we have, cf. Cor. 8.7, both $\overline{\operatorname{adj} Q}\,\Gamma(Q) = 0$ and $(\operatorname{adj}\Gamma(Q))\Gamma(Q) = 0$, with $\Gamma(Q)$ the generator of a stochastic semigroup. We see that $\operatorname{adj} Q$ is rank one with identical rows with the same for $\overline{\operatorname{adj} Q}$ and $\operatorname{adj}\Gamma(Q)$. We have the following relation.

Proposition 9.9. *Let Q be a generator for an irreducible continuous-time jump process. Then, for the 2×2 case, at level N,*

$$\operatorname{adj}\Gamma(Q) = N!\,\overline{\operatorname{adj} Q}$$

Proof. Let $Q = \begin{pmatrix} -r & r \\ s & -s \end{pmatrix}$, with $r, s > 0$. Then $\operatorname{adj} Q = -\begin{pmatrix} s & r \\ s & r \end{pmatrix}$ and

$$\Gamma(Q) = s\Gamma(R) + r\Gamma(L) - r\Gamma(P_0) - s\Gamma(P_1)$$

First, we consider the overall sign(s). $\overline{\operatorname{adj} Q}$ will pick up a sign of $(-1)^N$, while $\Gamma(Q)$ is $(N+1)\times(N+1)$ so $\operatorname{adj}\Gamma(Q)$ will pick up a sign of $(-1)^{(N+1-1)}$, which checks accordingly. Henceforth, we will ignore overall signs.

From $\operatorname{adj} Q$, we see that $\overline{\operatorname{adj} Q}$ has equal rows with entries $\binom{N}{n}s^{N-n}r^n$. For $\Gamma(Q)$, we have seen the form of $\Gamma(L)$ and $\Gamma(R)$, which are the relevant entries as we will use the Tree Theorem to determine $\operatorname{adj}\Gamma(Q)$. For example, for $N=5$, we have

$$\Gamma(L) = \begin{bmatrix} 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \Gamma(R) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \end{bmatrix}$$

which leads to the directed graph

$$0 \; \overset{1s}{\underset{5r}{\leftrightarrows}} \; 1 \; \overset{2s}{\underset{4r}{\leftrightarrows}} \; 2 \; \overset{3s}{\underset{3r}{\leftrightarrows}} \; 3 \; \overset{4s}{\underset{2r}{\leftrightarrows}} \; 4 \; \overset{5s}{\underset{1r}{\leftrightarrows}} \; 5$$

The entry in column n of a row in $\operatorname{adj}\Gamma(Q)$ is given by the product of the weights along the sequences of edges terminating at n. In our example, for $n=2$, we find

$$(5s)(4s)(3s)(5r)(4r) = \frac{5!}{2!}\frac{5!}{3!}s^3r^2 = N!\binom{N}{n}s^{N-n}r^n$$

To see this form in general, moving from node N to node n, the weight from j to $j-1$ is js, while moving from node 0 to node n, the weight from j to $j+1$ is $(N-j)r$. Thus the weight of the tree is

$$N(N-1)\cdots(n+1)\cdot N(N-1)\cdots(N-n+1)\cdot s^{N-n}r^n = (N!/n!)(N!/(N-n)!)s^{N-n}r^n$$

as required. □

10. Conclusion

We have shown that induced matrices lead to interesting extensions of known properties of certain systems based on matrices. A family of induced matrices produces a hierarchy of systems that lend insight to a given system as well as being of interest in their own right. In this work, we have focused on recurrences and then Markov chains. Our approach indicates as well how to consider systems larger than 2×2 which look to be promising for further research.

11. Appendix

11.1. Proof of the transpose lemma.

Note. Here B denotes the diagonal matrix with nonzero entries multinomial coefficients of the appropriate order. It may be defined alternatively as the diagonal of $\bar{J}$ where J is the all-ones matrix.

We show the relation between $\overline{A^\top}$ and $\bar{A}^\top$. The idea is to consider the bilinear form $\sum_{i,j} x_i A_{ij} y_j$. The multinomial expansion yields, using multi-index notation,

$$(\sum_{i,j} x_i A_{ij} y_j)^N = \sum_m x^m \binom{N}{m} (Ay)^m = \sum_m x^m \binom{N}{m} \bar{A}_{mn} y^n \tag{11.1}$$

for degree N.

Replacing A by $A^\top$ yields

$$(\sum_{i,j} x_i A_{ji} y_j)^N = \sum_{m,n} x^m \binom{N}{m} \overline{A^\top}_{mn} y^n$$

which, interchanging indices i and j, equals

$$\begin{aligned}(\sum_{i,j} y_i A_{ij} x_j)^N &= \sum_{m,n} y^m \binom{N}{m} \overline{A}_{mn} x^n = \sum_{m,n} x^m \binom{N}{n} \overline{A}_{nm} y^n \\ &= \sum_{m,n} x^m \binom{N}{n} \bar{A}^\top_{mn} y^n\end{aligned}$$

exchanging indices m and n. Comparing shows that

$$\overline{A^\top}_{mn} = \binom{N}{m}^{-1} \binom{N}{n} \bar{A}^\top_{mn}$$

In terms of B, we see that

Lemma 11.1. Transpose Lemma*The induced matrices satisfy*

$$\overline{A^\top} = B^{-1} \bar{A}^\top B$$

11.2. Properties of the gamma map. Basic properties of the Γ-map for induced matrices (symmetric representation):

The Γ-map is defined by the relation

$$\overline{e^{tX}} = e^{t\Gamma(X)}$$

with the homogeneous degree N understood.

The first property shows that Γ is a linear map. These properties combined show that it is a Lie homomorphism.

1. Linearity:
For scalar λ, $\Gamma(\lambda X + Y) = \lambda\Gamma(X) + \Gamma(Y)$.
2. The Lie bracket is preserved:
With $[X, Y] = XY - YX$ denoting the commutator of X and Y, we have

$$\Gamma([X, Y]) = [\Gamma(X), \Gamma(Y)] \ .$$

3. Zero row sums are preserved. A generator maps to a generator.
(See Appendix of [1] for proofs of #1 and #2.)

References

1. Feinsilver, Philip: Krawtchouk-Griffiths systems I: matrix approach. *Commun. Stoch. Anal.*, 10(3):Article 3, 297–320, 2016.
2. Feinsilver, Philip: Matrices with zero row sums, tree theorems and a Markov chain on trees. In *Probability on algebraic and geometric structures*, volume 668 of *Contemp. Math.*, pages 67–79. Amer. Math. Soc., Providence, RI, 2016.
3. Feller, William: *An introduction to probability theory and its applications. Vol. II.* John Wiley & Sons, Inc., New York-London-Sydney, 1966.
4. Fulton, W. and Harris, J.: Representation Theory, A First Course, Graduate Texts in Mathematics, 129, Springer-Verlag, 1991.
5. Koornwinder, T., et al. : URL: `http://dlmf.nist.gov/18`.
6. Koshy, Thomas: *Fibonacci and Lucas numbers with applications. Vol. 1.* Pure and Applied Mathematics (Hoboken). John Wiley & Sons, Inc., Hoboken, NJ, 2018. Second edition of [MR1855020].
7. Koshy, Thomas: *Fibonacci and Lucas numbers with applications. Vol. 2.* Pure and Applied Mathematics (Hoboken). John Wiley & Sons, Inc., Hoboken, NJ, 2019.
8. Littlewood, D. E.: *A university algebra.* Dover Publications, Inc., New York, 1970. An introduction to classic and modern algebra, Republication of the second (1958) edition.
9. McSorley, John P. and Feinsilver, Philip: Multivariate matching polynomials of cyclically labelled graphs. *Discrete Math.*, 309(10):3205–3218, 2009.
10. Neusel, Mara D.: *Invariant theory*, volume 36 of *Student Mathematical Library.* American Mathematical Society, Providence, RI, 2007.
11. Sloane, N. J. A.: Error-correcting codes and invariant theory: new applications of a nineteenth-century technique. *Amer. Math. Monthly*, 84(2):82–107, 1977.
12. Springer, T.: Invariant Theory, Lecture Notes in Mathematics, v. 585, 1977.
13. Virchenko, N., et al., eds. Development of the Mathematical Ideas of Mykhailo Kravchuk (Krawtchouk), Shevchenko Scientific Society, Kyiv-New York, 2004.

Philip Feinsilver: Department of Mathematics, Southern Illinois University, Carbondale, Illinois 62901, USA

Email address: `phfeins@siu.edu`

URL: `http://chanoir.math.siu.edu`