Journal of Stochastic Analysis

Volume 3 | Number 2 Article 1

June 2022

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Recommended Citation

Mueller, Carl and Neumann, Eyal (2022) "Self-Repelling Elastic Manifolds with Low Dimensional Range," *Journal of Stochastic Analysis*: Vol. 3: No. 2, Article 1.
DOI: 10.31390/josa.3.2.01
Available at: https://digitalcommons.lsu.edu/josa/vol3/iss2/1

Journal of Stochastic Analysis
Vol. 3, No. 2 (2022) Article 1 (16 pages)
DOI: 10.31390/josa.3.2.01

SELF-REPELLING ELASTIC MANIFOLDS WITH LOW DIMENSIONAL RANGE

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Abstract. We consider self-repelling elastic manifolds that take values in $\mathbb{R}^D$, with a domain $[-N,N]^d \cap \mathbb{Z}^d$. Our main result states that when the dimension of the domain is $d = 2$ and the dimension of the range is $D = 1$, the effective radius R_N of the manifold is approximately $N^{4/3}$. This verifies the conjecture of Kantor, Kardar and Nelson [7]. Our results for the case where $d \geq 3$ and $D < d$ give a lower bound on R_N of order $N^{\frac{1}{D}\left(d-\frac{2(d-D)}{D+2}\right)}$ and an upper bound proportional to $N^{\frac{d}{2}+\frac{d-D}{D+2}}$. These results imply that self-repelling elastic manifolds with a low dimensional range undergo a significantly stronger stretching than in the case where $d = D$, which was studied in [10].

1. Introduction

Self-repelling elastic manifolds were first introduced by Kantor, Kardar and Nelson in [5] as generalizations of polymer models to higher dimensions, in order to capture the behaviour of sheets of covalently bonded atoms and of polymerized lipid surfaces, among others. See [5, 7, 6, 11] and references therein for additional details. In the mathematical literature this model was first studied in [10] as a random surface with free boundary conditions is modeled by $\mathbb{R}^D$-valued discrete Gaussian free field (DGFF) over $[-N,N]^d \cap \mathbb{Z}^d$, with Neumann boundary conditions. A penalization term for self-intersections, which reflects the fact that different parts of the manifold cannot occupy the same position, is then added to the Hamiltonian of the DGFF. If the domain of DGFF is one dimensional, then we recover the well-known model of a random polymer. A typical object of study is the end-to-end distance of such a polymer, or the closely related concept of effective radius. There is a vast literature on such problems, see Bauerschmidt, Duminil-Copin, Goodman, and Slade [2] and the included citations.

Bounds on the effective radius R_N of the self-repelling manifold were derived in [10] for the case where $d = D$. It was proved that for the two dimensional case, that is $d = D = 2$, R_N is proportional to N in the upper and lower bounds, up to a logarithmic correction. The bounds on R_N in higher dimensions are not as sharp, with a lower bound proportional to N, but with an upper bound of order $N^{d/2}$.

Received 2022-2-28; Accepted 2022-3-31; Communicated by the editors.
2020 *Mathematics Subject Classification.* Primary, 60G60; Secondary, 60G15.
Key words and phrases. Gaussian free field, self-avoiding, elastic manifold.
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The results of [10] proved however that self-repelling elastic manifolds experience a substantial stretching in any dimension.

We should also mention the related paper [9] on the effective radius for moving polymers. Most methods available in the polymer literature fail in these more general settings, but we did take inspiration from work of Bolthausen [4].

In this paper, we deal with the case where $D < d$. We first prove that when the dimension of the domain is $d = 2$ and the dimension of the range is $D = 1$, the effective radius R_N of the manifold is approximately $N^{4/3}$. Our results for the case $d \geq 3$ and $D < d$ give a lower bound on R_N of order $N^{\frac{1}{D}\left(d-\frac{2(d-D)}{D+2}\right)}$ and an upper bound proportional to $N^{\frac{d}{2}+\frac{d-D}{D+2}}$. These results imply that self-repelling elastic manifolds with a low dimensional range undergo a significantly stronger stretching than in the case $d = D$, which was studied in [10].

The remaining case, $D > d$, looks to be much harder. For example, consider the case where the domain of the self-repelling DGFF is $\{0, \dots, N\}$ and it takes values in $\mathbb{R}^D$. For $D = 2, 3, 4$ the behavior of the effective radius of the self-repelling polymer as $N \to \infty$ is still unknown, although we have good information for $D = 1$ and for $D > 4$. See page 400 of Bauerschmidt, Duminil-Copin, Goodman, and Slade [2] and also Bauerschmidt, Slade, and Tomberg, and Wallace [3]. If D is large enough, then for self-avoiding walks, the lace expansion can be used. For DGFF however, there appears to be no analogue of the lace expansion.

2. Model Setup and Main Results

2.1. Setup. We briefly review some of the definitions and notation from Section 1 of [10] which are essential for our context. In the following, ordinary letters such as x, u take values in $\mathbb{R}$ or $\mathbb{Z}$, while boldface letters such as $\mathbf{x}, \mathbf{u}$ take values in $\mathbb{R}^d$ for $d \geq 2$.

Fix $d \geq 2, N \geq 1$ and define our parameter set as follows:

$$S_N^d := [-N, N]^d \cap \mathbb{Z}^d.$$

Note that

$$S_N^1 := \{-N, \dots, N\}.$$

Thus S_N^d is a cube in $\mathbb{Z}^d$ centered at the origin.

We denote by $\Delta = \Delta_{N,d,D}$ the discrete Neumann Laplacian on S_N^d (see Section 1 of [10] for the precise definition). In the case $D = 1$, since Δ is a self-adjoint operator on a finite-dimensional space, there exists a finite index set $\mathbb{I} = \mathbb{I}_{N,d}$ to be defined later, and an orthonormal basis of eigenfunctions $(\varphi_{\mathbf{k}})_{\mathbf{k}\in\mathbb{I}}$ with corresponding eigenvalues $(\lambda_{\mathbf{k}})_{\mathbf{k}\in\mathbb{I}}$. We can assume without loss of generality that there is a distinguished index $\mathbf{0}$ such that $\varphi_{\mathbf{0}}$ is constant and that $\lambda_{\mathbf{0}} = 0$.

Throughout, we fix a parameter $\beta > 0$, which has a physical interpretation as the inverse temperature. Let $(X_{\mathbf{k}}^{(i)})_{\mathbf{k}\in\mathbb{I}\setminus\{\mathbf{0}\}, i=1,\dots,D}$ be a collection of i.i.d. random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$X_{\mathbf{k}}^{(i)} \sim N(0, (2\beta\lambda_{\mathbf{k}})^{-1}).$$

For each $i = 1, \dots, D$ define

$$u^{(i)} = \sum_{\mathbf{k} \in \mathbb{I} \setminus \mathbf{0}} X^{(i)}_{\mathbf{k}} \varphi_{\mathbf{k}}, \tag{2.1}$$

and define DGFF as

$$\mathbf{u} = (u^{(1)}, \dots, u^{(D)}). \tag{2.2}$$

As explained in [10], this corresponds to the Gibbs measure with Hamiltonian $H(\mathbf{u}) = \sum_{\mathbf{x} \sim \mathbf{y}} |\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})|^2$, where $\mathbf{x} \sim \mathbf{y}$ means that $\mathbf{x}, \mathbf{y}$ should be nearest neighbors on S_N^d. In other words, the energy $H(\mathbf{u})$ depends on the stretching $|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})|$.

We recall the definition the local time of the field $\mathbf{u}$ at level $\mathbf{z} \in \mathbb{R}^D$ as

$$\begin{aligned} \ell_N(\mathbf{z}) &= \#\{\mathbf{x} \in S_N^d : \mathbf{u}(\mathbf{x}) \in [\mathbf{z} - \mathbf{1/2}, \mathbf{z} + \mathbf{1/2}]\} \\ &= \sum_{\mathbf{x} \in S_N^d} \mathbf{1}_{\{\mathbf{u}(\mathbf{x}) \in [\mathbf{z} - \mathbf{1/2}, \mathbf{z} + \mathbf{1/2}]\}}, \end{aligned} \tag{2.3}$$

where $\mathbf{\frac{1}{2}} = (1/2, \dots, 1/2) \in \mathbb{R}^D$, and $[\mathbf{x}, \mathbf{y}] = \prod_i [x_i, y_i]$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^D$.

Now we define a weakly self-avoiding Gaussian free field. Throughout, we fix a parameter $\gamma > 0$. If $P_{N,d,D,\beta}$ denotes the original probability measure of $(\mathbf{u}(\mathbf{x}))_{\mathbf{x} \in S_N^d}$, we define the probability $Q_{N,d,D,\beta,\gamma}$ as follows. For ease of notation, we write E for the expectation with respect to $P_{N,d,D,\beta}$. Let

$$\begin{aligned} \mathcal{E}_{N,d,D,\gamma} &= \exp\left(-\gamma \int_{\mathbb{R}^D} \ell_N(\mathbf{y})^2 d\mathbf{y}\right), \\ Z_{N,d,D,\beta,\gamma} &= E[\mathcal{E}_{N,d,D,\gamma}] = E^{P_{N,d,D,\beta}}[\mathcal{E}_{N,d,D,\gamma}]. \end{aligned} \tag{2.4}$$

Then we define for any set $A \in \mathcal{F}$,

$$Q_{N,d,D,\beta,\gamma}(A) = \frac{1}{Z_{N,d,D,\beta,\gamma}} E\big[\mathcal{E}_{N,d,D,\gamma} \mathbf{1}_A\big]. \tag{2.5}$$

For ease of notation, we will often drop the subscripts except for N and write

$$P_N = P_{N,d,D,\beta}, \quad Q_N = Q_{N,d,D,\beta,\gamma}, \quad \mathcal{E}_N = \mathcal{E}_{N,d,D,\gamma}.$$

For $Z_{N,d,D,\beta,\gamma}$ in (2.4) we often write,

$$Z_{N,d,D} = Z_{N,d,D,\beta,\gamma}.$$

Finally, we define the effective radius of the field $\mathbf{u}$ as

$$R_{N,d,D} = \max_{\mathbf{w}, \mathbf{z} \in S_N^d} \|\mathbf{u}(\mathbf{z}) - \mathbf{u}(\mathbf{w})\|,$$

where $\| \cdot \|$ denotes the Euclidian norm.

2.2. Statement of the main result. Note that in our main theorem below, we assume that $D \leq d$.

Theorem 2.1. *Let $\mathbf{u}$ be the weakly self-avoiding DGFF on S_N^d taking values in $\mathbb{R}^D$. There are constants $\varepsilon_0, K_0 > 0$ not depending on β, γ, N such that*

(i) *For $d = 2$ and $D = 1$,*

$$\lim_{N\to\infty} Q_N\Big[\varepsilon_0 \frac{\gamma}{\beta+\gamma} N^{4/3}(\log N)^{-2/3} \leq R_{N,d,D} \leq K_0\Big(\frac{\beta+\gamma}{\beta}\Big)^{1/2} N^{4/3}(\log N)^{4/3}\Big] = 1.$$

(ii) *For $d \geq 3$ and $1 \leq D \leq d$,*

$$\lim_{N\to\infty} Q_N\Big[\varepsilon_0\Big(\frac{\gamma}{\beta+\gamma}\Big)^{1/D} N^{\frac{1}{D}\left(d-\frac{2(d-D)}{D+2}\right)} \leq R_{N,d,D} \leq K_0\Big(\frac{\beta+\gamma}{\beta}\Big)^{1/2} N^{\frac{d}{2}+\frac{d-D}{D+2}}\Big] = 1.$$

Remark 2.2. We compare the result of Theorem 2.1(i) to the result of Theorem 1.1(i) in [10], where it was proved that $R_{N,2,2} \approx N$. Note that by reducing the dimension of the range by 1, the manifold stretches, with a substantially larger radius of $R_{N,2,1} \approx N^{4/3}$. By comparing Theorem 2.1(ii) and Theorem 1.1(ii) in [10] we observe a similar phenomenon for $d \geq 3$, as the lower bound on the radius increases from $R_{N,d,d} \gtrsim N$ to $R_{N,d,D} \gtrsim N^{\frac{1}{D}\left(d-\frac{2(d-D)}{D+2}\right)}$ for $D < d$. This additional stretching for manifolds of lower dimensional range can be predicted by considering the local time expression in (2.3). Indeed for a fixed value of d and when the range has dimension $D < d$, $\ell_N(\mathbf{z})$ counts the same number of vertices $(2N+1)^d$, but if the radius R_N remains the same, there is less space to fit these vertices. Hence we would expect $\mathcal{E}_N$ to be larger. This in turn gives the repelling term in (2.4) a stronger influence on the configuration of the manifold (see e.g. (7.4) in Section 7).

Remark 2.3. Theorem 2.1 verifies the conjecture by Kantor, Kardar and Nelson in [7] for the case where $d = 2$ and $D = 1$. Although in the model that was presented in [7] the DGFF is defined on the triangular lattice, the heuristics that yields their result is based on Flory's argument which also applies for the rectangular lattice.

2.3. Outline of the proof. We describe the outline for the case $d = 2$, $D = 1$, as the proof for $d \geq 3$, $D < d$ follows similar lines. Define the following events.

$$\begin{aligned} A^{(1)}_{N,d,D} &= \Big\{R_{N,d,D} > K_0\Big(\frac{\beta+\gamma}{\beta}\Big)^{1/2} N^{4/3}(\log N)^{4/3}\Big\}, \\ A^{(2)}_{N,d,D} &= \Big\{R_{N,d,D} < \varepsilon_0 \frac{\gamma}{\beta+\gamma} N^{4/3}(\log N)^{-2/3}\Big\}. \end{aligned} \tag{2.6}$$

It suffices to show that for $i = 1, 2$ we have

$$\lim_{N\to\infty} Q_N\big(A^{(i)}_{N,d,D}\big) = 0.$$

From (2.5) we see that it is enough to find:

(1) a lower bound on $Z_{N,d,D}$, derived in Section 3,
(2) and an upper bound on $E\big[\mathcal{E}_{N,d,D}\mathbf{1}_{A^{(i)}_{N,d,D}}\big]$ for $i = 1, 2$, obtained in Sections 6 and 7, respectively.

Finally, the upper bounds divided by the lower bound should vanish as $N \to \infty$.

3. Lower Bound on the Partition Function

In this section we derive the following lower bound on $Z_{N,d,D}$.

Proposition 3.1. *Let $\beta > 0$. Then there exists a constant $C > 0$ not depending on N, β and γ such that*

(i) *for $d = 2$ and $D = 1$,*

$$\log Z_{N,d,D} \geq -C(\beta + \gamma)N^{8/3}(\log N)^{2/3}.$$

(ii) *for $d \geq 3$ and $D \leq d$,*

$$\log Z_{N,d,D} \geq -C(\beta + \gamma)N^{d+\frac{2(d-D)}{d-D+2}}.$$

In order to prove Proposition 3.1 we will introduce some additional definitions and auxiliary lemmas.

3.1. The orthonormal function basis. We first recall the orthonormal basis $\{\varphi_{\mathbf{k}}\}$ in (2.1) of eigenfunctions of $\Delta_{N,d,1}$ on $[-N, N]^d \cap \mathbb{Z}^d$ taking values in $\mathbb{R}$. Each basis function $\varphi_{\mathbf{k}}$ can be represented as a product of d functions $\phi_j : \mathbb{R}^d \to \mathbb{R}$, as follows

$$\varphi_{\mathbf{k}}(\mathbf{x}) = \phi_{k_1}(x_1) \ldots \phi_{k_d}(x_d), \tag{3.1}$$

where $\mathbf{x} = (x_1, \ldots, x_d)$ and $\mathbf{k} = (k_1, \ldots, k_d)$, $-N \leq k_i \leq N$, and $1 \leq i \leq d$. Here $\{\phi_j\}_{j=-N}^{N}$ is an orthonormal basis of eigenfunctions of $\Delta_{N,1,1}$, the Laplacian with Neumann boundary conditions on $S_N^1 = \{-N, \ldots, N\}$. Note that if $\lambda_{\mathbf{k}}$ is the eigenvalue of $\varphi_{\mathbf{k}}$ and λ_k is the eigenvalue corresponding to ϕ_k, then satisfies

$$\lambda_{\mathbf{k}} = \sum_{i=1}^{d} \lambda_{k_i}. \tag{3.2}$$

We can explicitly compute these eigenfunctions and eigenvalues (see Section 3.1 of [10]).

Our basis comprises all such combinations as in (3.1), excluding the constant eigenfunction

$$\varphi_{(0,\ldots,0)}(\mathbf{x}) = \phi_0(x_1) \ldots \phi_0(x_d).$$

We denote by $N(d)$ the number of function in our basis,

$$N(d) = |S_N^d| - 1 = (2N + 1)^d - 1. \tag{3.3}$$

3.2. Incorporating drift. Next we incorporate a linear drift into each of the components $u^{(i)}$ of $\mathbf{u}$, calling the resulting component $u_a^{(i)}$. This drift should stretch out the values of $\mathbf{u}(\cdot)$, so that $\mathbf{u}$ looks more like what we believe it would be under Q_N. Since our goal is to use Jensen's inequality, this drift should make the inequality more exact. Let

$$u_a^{(i)}(\mathbf{x}) = \sum_{k=1}^{N(d)} X_{\mathbf{k}}^{(i)} \varphi_{\mathbf{k}}(\mathbf{x}) + a x_i, \quad i = 1, \ldots, D,$$

where $a > 0$ is a constant to be determined later.

Using (2.1) and (3.1), we get

$$u_a^{(i)}(\mathbf{x}) = \sum_{(k_1,\ldots,k_d)\in S_N^d\setminus\{\mathbf{0}\}} X^{(i)}_{k_1,\ldots,k_d}\phi_{k_1}(x_1)\ldots.\phi_{k_d}(x_d) + ax_i, \tag{3.4}$$

where $\mathbf{0} = (0,\ldots,0) \in \mathbb{Z}^d$.

Regarding x_i as a function on $S_N^1 = \{-N,\ldots,N\}$ and expanding it in terms of our eigenfunctions, we see that there are coefficients $\alpha_j^{(i)}$ such that

$$x_i = (\phi_0)^{d-1} \sum_{j\in S_N^1\setminus\{0\}} \phi_j(x_i)\alpha_j^{(i)}, \tag{3.5}$$

where we have included $(\phi_0)^{(1-d)}$ for convenience in later calculations. Recall that

$$\phi_0 = \phi_0(x) = (2N+1)^{-1/2}.$$

In (3.5), we do not include $j = 0$ because x_i is orthogonal to the constant function ϕ_0.

Next we find $\alpha_j^{(i)}$ in (3.5). Since $\alpha_j^{(i)}$ are used to expand the function $f(x) = x$ for each coordinate i, we can omit the superscript i and write just α_j in what follows. Since $\{\phi_j\}_{j=-N}^N$ forms an orthonormal basis, we get

$$\alpha_j = \phi_0^{(1-d)} \sum_{n=-N}^{N} n\phi_j(n), \quad j \neq 0, \quad \text{and} \quad \alpha_0 = 0. \tag{3.6}$$

From (3.4) and (3.5) we have

$$\begin{aligned} u_a^{(i)}(\mathbf{x}) = &\sum_{(k_1,\ldots,k_d)\in S_N^d\setminus\{\mathbf{0}\}} X^{(i)}_{k_1,\ldots,k_d}\phi_{k_1}(x_1)\ldots.\phi_{k_d}(x_d) \\ &+ a\phi_0^{d-1} \sum_{j\in S_N^1\setminus\{0\}} \phi_j(x_i)\alpha_j. \end{aligned}$$

We can represent $u_a^{(i)}$ as follows:

$$\begin{aligned} u_a^{(i)}(x) = &\sum_{(k_1,\ldots,k_d)\in S_N^d\setminus\{j\mathbf{e}_i:\, j\in S_N^1\}} X^{(i)}_{k_1,\ldots,k_d}\phi_{k_1}(x_1)\cdots\phi_{k_d}(x_d) \\ &+ \phi_0^{d-1} \sum_{j\in S_N^1\setminus\{0\}} (X^{(i)}_{j\mathbf{e}_i} + a\alpha_j^{(i)})\phi_j(x_i), \end{aligned} \tag{3.7}$$

where $\{\mathbf{e}_i\}_{i=1}^d$ is the standard basis of $\mathbb{R}^d$.

For $i = 1,\ldots,D$ let $\mathbf{x}^{(i)} \in \mathbf{V}$ and define

$$F(\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(D)}) = \sum_{i=1}^{D} \sum_{(k_1,\ldots,k_d)\in S_N^d\setminus\{\mathbf{0}\}} \frac{(x^{(i)}_{k_1,\ldots,k_d})^2}{2(2\beta\lambda_{k_1,\ldots,k_d})^{-1}}.$$

We rewrite $Z_{N,d,D}$ in (2.4) as follows. We should emphasize that the local time ℓ_N is random and hence a function of the random variables $(X^{(i)}_{\mathbf{k}})$, so we can write

$$\ell_N(\mathbf{y}) = \ell_N\left((X^{(i)}_{\mathbf{k}}), \mathbf{y}\right),$$

where for readability we have omitted the specification that $i = 1, \ldots, D$ and $\mathbf{k} \in S_N^d \setminus \{\mathbf{0}\}$. Then we have

$$\begin{aligned} Z_{N,d,D} = \int_{\mathbb{R}^{D\cdot N(d)}} \exp\Big(-F(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(D)}) - \gamma \int_{\mathbb{R}^D} \ell_N^2\big((\mathbf{x}^{(i)}), \mathbf{y}\big) d\mathbf{y}\Big) \\ \times \prod_{i=1}^{D} \prod_{(k_1,\ldots,k_d)\in S_N^d\setminus\{\mathbf{0}\}} dx^{(i)}_{k_1,\ldots,k_d}. \end{aligned} \tag{3.8}$$

In order to find the Radon-Nikodym derivative corresponding to the drift in (3.7) we note that,

$$\begin{aligned} &\sum_{i=1}^{D} \sum_{(k_1,\ldots,k_d)\in S_N^d\setminus\{\mathbf{0}\}} \frac{(x^{(i)}_{k_1,\ldots,k_d})^2}{2(2\beta\lambda_{k_1,\ldots,k_d})^{-1}} \\ &= \sum_{i=1}^{D} \Bigg(\sum_{(k_1,\ldots,k_d)\in S_N^d\setminus\{j\mathbf{e}_i:\, j\in S_N^1\}} \frac{(x^{(i)}_{k_1,\ldots,k_d})^2}{2(2\beta\lambda_{k_1,\ldots,k_d})^{-1}} \\ &\qquad + \sum_{j\in S_N^1\setminus\{0\}} \frac{(x^{(i)}_{j\mathbf{e}_i} + a\alpha_j)^2}{2(2\beta\lambda_{j\mathbf{e}_i})^{-1}} - \sum_{j\in S_N^1\setminus\{0\}} \frac{2a\alpha_j x^{(i)}_{j\mathbf{e}_i} + (a\alpha_j)^2}{2(2\beta\lambda_{j\mathbf{e}_i})^{-1}} \Bigg). \end{aligned} \tag{3.9}$$

We therefore define $\hat{P}^{(a)}$ (resp. $\hat{E}^{(a)}$) to be the probability measure (expectation) under which u is shifted as in (3.7). Then (3.8) and (3.9) imply

$$\frac{d\hat{P}^{(a)}}{dP} = \exp\Bigg(-\sum_{i=1}^{D} \sum_{j\in S_N^1\setminus\{0\}} \frac{2a\alpha_j x^{(i)}_{j\mathbf{e}_i} + (a\alpha_j)^2}{2(2\beta\lambda_{j\mathbf{e}_i})^{-1}} \Bigg). \tag{3.10}$$

We can therefore rewrite $Z_{N,d,D}$ in (3.8) as follows,

$$\begin{aligned} Z_{N,d,D} = \hat{E}^{(a)}\Bigg[\exp\Bigg(\sum_{i=1}^{D} \sum_{j\in S_N^1\setminus\{0\}} \frac{2a\alpha_j X^{(i)}_{j\mathbf{e}_i} + (a\alpha_j)^2}{2(2\beta\lambda_{j\mathbf{e}_i})^{-1}} \\ - \gamma \int_{\mathbb{R}^D} \ell_N^2(\mathbf{y}) d\mathbf{y} \Bigg) \Bigg]. \end{aligned} \tag{3.11}$$

We define

$$Y^{(i)}_{j\mathbf{e}_i} = \frac{2a\alpha_j X^{(i)}_{j\mathbf{e}_i} + (a\alpha_j)^2}{2(2\beta\lambda_{j\mathbf{e}_i})^{-1}}. \tag{3.12}$$

Using Jensen's inequality, we get that

$$\begin{aligned} \log Z_{N,d,D} &\geq \hat{E}^{(a)}\Big[-\gamma \int_{\mathbb{R}^D} \ell_N^2(\mathbf{y}) d\mathbf{y} \Big] - \hat{E}^{(a)}\Bigg[-\sum_{i=1}^{D} \sum_{j\in S_N^1\setminus\{0\}} Y^{(i)}_{j\mathbf{e}_i} \Bigg] \\ &=: -(I_{1,d,D} + I_{2,d,D}). \end{aligned} \tag{3.13}$$

The following proposition gives some essential bounds on $I_{i,d,D}$, $i = 1, 2$.

Proposition 3.2. *Let $\beta, \gamma > 0$. Then there exists a constant $C > 0$ not depending on N, β, γ such that*

(i) *for $d = 2$ and $D = 1$,*

$$I_{1,2,1} \le C\gamma N^2\big((\beta^{-1/2}a^{-1}N\log N) \vee 1\big),$$

(ii) *for $d \ge 3$ and $D \le d$,*

$$I_{1,d,D} \le C\gamma N^d\big(1 \vee (\beta^{-D/2}N^{d-D}a^{-D})\big).$$

(iii) *for any $d \ge 2$,*

$$I_{2,d,D} \le C\beta a^2 N^d.$$

The proof of Proposition (3.2)(i) and (ii) is postponed to Section 4. The proof of Proposition (3.2)(iii) is given in Section 5.

3.3. Proof of Proposition 3.1.

Proof of Proposition 3.1. From (3.13) and Proposition 3.2(i) and (iii) it follows that for $d = 2$ and $D = 1$,

$$\begin{aligned} \log \hat{Z}_{N,2,1} &\ge -(I_{1,2,1} + I_{2,2,1}) \\ &\ge -C\left[\gamma N^2\big((\beta^{-1/2}a^{-1}N\log N) \vee 1\big) + \beta N^2 a^2\right]. \end{aligned} \tag{3.14}$$

Taking $a = \beta^{-1/2}(N\log N)^{1/3}$ in (3.14) gives,

$$\log \hat{Z}_{N,2,1} \ge -C(\beta+\gamma)N^{8/3}(\log N)^{2/3}.$$

The proof for $d \ge 3$ and $1 \le D < d$ follows the same lines with the only modification that we are using Proposition 3.2(ii) and choosing $a = \beta^{-1/2}N^{\frac{d-D}{D+2}}$ to get

$$\log \hat{Z}_{N,d,D} \ge -C(\beta+\gamma)N^{d+\frac{2(d-D)}{D+2}}.$$

□

4. Proof of Proposition (3.2)(i) and (ii)

Proof of Proposition 3.2(i) and (ii). We can write

$$\begin{aligned} \tilde{I}_{1,d,D} &:= \hat{E}^{(a)}\left[\int_{\mathbb{R}^D} \ell_N(\mathbf{y})^2 d\mathbf{y}\right] \\ &= \hat{E}^{(a)}\left[\int_{\mathbb{R}^D}\Big(\sum_{\mathbf{z}\in S_N^d} \mathbf{1}_{[\mathbf{y-1/2},\mathbf{y+1/2}]}(\mathbf{u}(\mathbf{z}))\Big)^2 d\mathbf{y}\right] \\ &= \sum_{\mathbf{z}\in S_N^d} \hat{E}^{(a)}\left[\int_{\mathbb{R}^D} \mathbf{1}_{[\mathbf{y-1/2},\mathbf{y+1/2}]}(\mathbf{u}(\mathbf{z}))d\mathbf{y}\right] \\ &\quad + \sum_{\mathbf{z},\mathbf{w}\in S_N^d,\,\mathbf{z}\ne\mathbf{w}} \hat{E}^{(a)}\left[\int_{\mathbb{R}^D} \mathbf{1}_{\mathbf{u}(\mathbf{z}),\mathbf{u}(\mathbf{w})\in[\mathbf{y-1/2},\mathbf{y+1/2}]} d\mathbf{y}\right] \\ &= (2N+1)^d + \sum_{\mathbf{z},\mathbf{w}\in S_N^d,\,\mathbf{z}\ne\mathbf{w}} \int_{\|\mathbf{y}\|\le 1} \hat{p}_{\mathbf{z},\mathbf{w}}(\mathbf{y})d\mathbf{y}, \end{aligned} \tag{4.1}$$

where $\hat{p}_{\mathbf{z},\mathbf{w}}$ is the density of $\mathbf{u}(\mathbf{z}) - \mathbf{u}(\mathbf{w})$ under $\hat{P}^{(a)}$.

We recall Proposition 3.1 from [10].

Proposition 4.1. *There exist constants $C_1, C_2 > 0$ such that,*

(i) *for $d = 2$, for all $\mathbf{w}, \mathbf{z} \in S_N^2$ with $\mathbf{w} \neq \mathbf{z}$, and for $i = 1, 2$ we have*

$$C_1\beta^{-1} \leq Var\Big(u^{(i)}(\mathbf{z}) - u^{(i)}(\mathbf{w})\Big) \leq C_2\beta^{-1}(\log N)^2,$$

(ii) *for $d \geq 3$, for all $\mathbf{w}, \mathbf{z} \in S_N^d$ with $\mathbf{w} \neq \mathbf{z}$, and for $i = 1, \dots, D$ we have*

$$C_1\beta^{-1} \leq Var\Big(u^{(i)}(\mathbf{z}) - u^{(i)}(\mathbf{w})\Big) \leq C_2\beta^{-1}.$$

Note that from (3.4) we have

$$\hat{E}^{(a)}[u^{(i)}(\mathbf{z}) - u^{(i)}(\mathbf{w})] = a(z_i - w_i), \quad \text{for } i = 1, \dots, D.$$

Since $(u^{(i)})_{i=1,\dots,D}$ are independent, we have for any $\mathbf{y} \in \mathbb{R}^D$

$$\hat{p}_{\mathbf{z},\mathbf{w}}(\mathbf{y}) := \prod_{i=1}^{D} \hat{p}^{(i)}_{\mathbf{z},\mathbf{w}}(y_i) \tag{4.2}$$

and therefore

$$\int_{\|\mathbf{y}\|\leq 1} \hat{p}_{\mathbf{z},\mathbf{w}}(\mathbf{y})d\mathbf{y} \leq \prod_{i=1}^{D} \int_{-1}^{1} \hat{p}^{(i)}_{\mathbf{z},\mathbf{w}}(y_i)dy_i, \tag{4.3}$$

where $\hat{p}^{(i)}_{\mathbf{z},\mathbf{w}}$ is the density of $u^{(i)}(\mathbf{z}) - u^{(i)}(\mathbf{w})$ under $\hat{P}^{(a)}$.

We distinguish between the following two cases.

Case 1: $d \geq 3$ and $D \leq d$. From Proposition 4.1(ii) we have

$$\hat{p}^{(i)}_{\mathbf{z},\mathbf{w}}(y_i) \leq \frac{1}{\sqrt{2\pi C_1\beta^{-1}}} \exp\Big(-\frac{a^2(z_i - w_i - y_i)^2}{2C_2\beta^{-1}}\Big), \quad \text{i=1,...,D.} \tag{4.4}$$

From (4.3) and (4.4) we therefore get

$$\begin{aligned}\sum_{\mathbf{z},\mathbf{w}\in S_N^d,\, \mathbf{z}\neq\mathbf{w}} \int_{\|\mathbf{y}\|\leq 1} \hat{p}_{\mathbf{z},\mathbf{w}}(\mathbf{y}) &\leq C \sum_{\mathbf{z},\mathbf{w}\in S_N^d,\, \mathbf{z}\neq\mathbf{w}} \int_{\mathbf{y}\in[-1,1]^D} \frac{1}{(2\pi C_1\beta^{-1})^{d/2}} \\ &\qquad \times \exp\Big(-\frac{a^2\sum_{i=1}^{D}(z_i - w_i - y_i)^2}{2C_2\beta^{-1}}\Big)d\mathbf{y} \\ &\leq \int_{\mathbf{y}\in[-1,1]} J(\mathbf{y})d\mathbf{y}.\end{aligned}$$

where

$$J(\mathbf{y}) := \sum_{\mathbf{z},\mathbf{w}\in S_N^d} \frac{1}{2\pi C_1\beta^{-1}} \exp\Big(-\frac{a^2\sum_{i=1}^{D}(z_i - w_i - y_i)^2}{2C_2\beta^{-1}}\Big).$$

The following lemma follows immediately from the proof of Lemma 3.2 from [10].

Lemma 4.2. *Let $\kappa > 0$. Then for all $y \in [-1,1]$ and $w \in S_N^1$ we have*

$$\sum_{z\in S_N^1} \exp\Big(-\kappa(z-w-y)^2\Big) \le \sum_{z=w-1}^{w+1} \exp\Big(-\kappa(z-w-y)^2\Big) + \int_{-\infty}^{\infty} \exp\Big(-\kappa(z-w-y)^2\Big)dz.$$

Using Lemma 4.2 and integrating over the Gaussian density gives,

$$\begin{aligned} J(\mathbf{y}) &= \frac{(2N+1)^{2(d-D)}}{2\pi C_1\beta^{-1}} \prod_{i=1}^{D} \left(\sum_{z_i\in S_N^1} \sum_{w_i\in S_N^1} \exp\Big(-\frac{a^2(z_i-w_i-y_i)^2}{2C_2\beta^{-1}}\Big)\right) \\ &\le CN^{2(d-D)} \prod_{i=1}^{D} \sum_{w_i\in S_N^1} \left(\sum_{z_i=w_i-1}^{w_i+1} \exp\Big(-\frac{a^2(z_i-w_i-y_i)^2}{2C_2\beta^{-1}}\Big) \right. \\ &\quad \left. + \beta^{-1/2}a^{-1}\frac{1}{\sqrt{2\pi C_2\beta^{-1}a^{-2}}} \int_{-\infty}^{\infty} \exp\Big(-\frac{(z_i-w_i-y_i)^2}{2C_2\beta^{-1}a^{-2}}\Big)dz_i \right) \\ &\le CN^{2(d-D)} \prod_{i=1}^{D} \sum_{w_i\in S_N^1} \left(\sum_{z_i=w_i-1}^{w_i+1} \exp\Big(-\frac{a^2(z_i-w_i-y_i)^2}{2C_2\beta^{-1}}\Big) + \beta^{-1/2}a^{-1} \right). \end{aligned} \tag{4.5}$$

It follows that

$$\begin{aligned} &\int_{\mathbf{y}\in[-1,1]} J(\mathbf{y})d\mathbf{y} \\ &\le CN^{2(d-D)} \left(\int_{-1}^{1} \sum_{w\in S_N^1} \left(\sum_{z=w-1}^{w+1} \exp\Big(-\frac{a^2(z-w-y)^2}{2C_2\beta^{-1}}\Big) + \beta^{-1/2}a^{-1} \right) dy \right)^D \\ &\le CN^{2(d-D)}N^D \left(\int_{-1}^{1} \left(\sum_{k=-1}^{1} \exp\Big(-\frac{a^2(k-y)^2}{2C_2\beta^{-1}}\Big) + \beta^{-1/2}a^{-1} \right) dy \right)^D. \end{aligned} \tag{4.6}$$

Using again integration over the Gaussian density gives for any $M > 0$ and $k = -1, 0, 1$,

$$\int_{-1}^{1} e^{-M(k+x)^2}dx \le \int_{-\infty}^{\infty} e^{-M(k+x)^2}dx \le CM^{-1/2}.$$

Plugging in these bounds to (4.6) gives

$$\int_{\mathbf{y}\in[-1,1]} J(\mathbf{y})d\mathbf{y} \le CN^{2(d-D)}N^D(\beta^{-1/2}a^{-1})^D.$$

Together with (4.1) this leads to,

$$\tilde{I}_1 \le \tilde{C}N^d\big(1 \vee (\beta^{-D/2}N^{d-D}a^{-D})\big).$$

Case 2: $d = 2$ and $D = 1$. Then from Proposition 4.1(i) we have

$$\hat{p}^{(1)}_{\mathbf{z},\mathbf{w}}(y) \leq \frac{1}{\sqrt{2\pi C_1 \beta^{-1}}} \exp\Big(- \frac{a^2(y-(z_1-w_1))^2}{2\beta^{-1}C_2(\log N)^2}\Big). \tag{4.7}$$

Then following similar steps as in Case 1 we arrive to

$$\begin{aligned} \tilde{I}_{1,2,1} &\leq (2N+1)^2 + C\beta^{-1/2}a^{-1}N^3\log N \\ &\leq \tilde{C}\big((\beta^{-1/2}a^{-1}N\log N) \vee 1\big)N^2. \end{aligned} \tag{4.8}$$

Since from (3.13) and (4.1) we have that

$$I_{1,d,D} = \gamma \tilde{I}_{1,d,D},$$

this completes the proof of Proposition 3.2 parts (i) and (ii). □

5. Proof of Proposition (3.2) (iii)

Proof of Proposition (3.2)*(iii).* Recall that $I_{2,d,D}$ was defined in (3.13),

$$I_{2,d,D} = -\sum_{i=1}^{D} \sum_{j\in S_N^1\setminus\{0\}} \hat{E}^{(a)}\left[Y^{(i)}_{j\mathbf{e}_i}\right], \tag{5.1}$$

where $Y^{(i)}_{j\mathbf{e}_i}$ was defined in (3.12). Recall also that $N(d)$ was defined in (3.3). Since the expectation on the right-hand side of (5.1) is taken over a Gaussian measure, we define the following normalizing constant

$$\begin{aligned} C_{N,\beta,d,D} &= \int_{\mathbb{R}^{D\cdot N(d)}} \exp\left(-\sum_{i=1}^{D}\sum_{k=1}^{N(d)} \frac{(x_k^{(i)})^2}{2(2\beta\lambda_k)^{-1}} \right) \prod_{i=1}^{D}\prod_{k=1}^{N(d)} dx_k^{(i)} \\ &= \frac{1}{(2\beta)^{DN(d)/2}} \prod_{k=1}^{N(d)} \frac{1}{\lambda_k^{D/2}}. \end{aligned} \tag{5.2}$$

We refer to Section 1.2 in [10] for additional in formation about the setup of the GFF measure.

We further introduce the following notation:

$$z^{(i)}_{k_1,\dots,k_d} = \frac{(x^{(i)}_{k_1,\dots,k_d})^2}{2(2\beta\lambda_{k_1,\dots,k_d})^{-1}}, \quad w^{(i)}_{j\mathbf{e}_i} = \frac{(x^{(i)}_{j\mathbf{e}_i} + a\alpha_j)^2}{2(2\beta\lambda_{j\mathbf{e}_i})^{-1}},$$

and

$$y^{(i)}_{j\mathbf{e}_i} = \frac{2a\alpha_j x^{(i)}_{j\mathbf{e}_i} + (a\alpha_j)^2}{2(2\beta\lambda_{j\mathbf{e}_i})^{-1}}.$$

Then from (3.9) and (3.10) we have

$$\begin{aligned}
&\hat{E}^{(a)}\left[Y^{(i)}_{j\mathbf{e}_i}\right] \\
&= \frac{1}{C_{N,\beta,d,D}}\int y^{(i)}_{j\mathbf{e}_i}\exp\Big(-\sum_{i=1}^{D}\Big(\sum_{(k_1,\dots,k_d)\in S_N^d\setminus\{j\mathbf{e}_i:\, j\in S_N^1\}} z^{(i)}_{k_1,\dots,k_d} \\
&\quad+\sum_{l\in S_N^1\setminus\{0\}} w^{(i)}_{l\mathbf{e}_i}\Big)\Big)\prod_{i=1}^{D}\prod_{l=1}^{N(d)} dx^{(i)}_l,
\end{aligned} \tag{5.3}$$

where $C_{N,\beta,d,D}$ was defined in (5.2).

Since the expected value in (5.3) is symmetric with respect to i, we can use $i=1$ in what follows in order to ease the notation. We therefore consider

$$\begin{aligned}
\hat{E}^{(a)}\left[Y^{(1)}_{j\mathbf{e}_1}\right] &= \frac{1}{C_{N,\beta,d,D}}\int y^{(1)}_{j\mathbf{e}_1}\exp(-w^{(1)}_{j\mathbf{e}_1}) \\
&\quad\times\int\exp\Big(-\Big(\sum_{i=1}^{D}\sum_{(k_1,\dots,k_d)\in S_N^d\setminus\{l\mathbf{e}_i:\, l\in S_N^1\}} z^{(i)}_{k_1,\dots,k_d} \\
&\quad+\sum_{i=2}^{D}\sum_{l\in\{-N,\dots,N\}\setminus\{0\}} w^{(i)}_{l\mathbf{e}_i}+\sum_{l\in S_N^1\setminus\{0,j\}} w^{(1)}_{l\mathbf{e}_1}\Big)\Big)\prod_{i=1}^{D}\prod_{l=1}^{N(d)} dx^{(i)}_l.
\end{aligned} \tag{5.4}$$

We notice that we have three types of integrals above, which can be evaluated as follows. We have $D((2N+1)^d-(2N+1))$ integrals of the form

$$\begin{aligned}
\int_{\mathbb{R}}\exp\big(-z^{(i)}_{k_1,\dots,k_d}\big)dx^{(i)}_{k_1,\dots,k_d} &= \int_{\mathbb{R}}\exp\left(-\frac{(x^{(i)}_{k_1,\dots,k_d})^2}{2(2\beta\lambda_{k_1,\dots,k_d})^{-1}}\right)dx^{(i)}_{k_1,\dots,k_d} \\
&= \sqrt{2\pi}(2\beta\lambda_{k_1,\dots,k_d})^{-1/2}.
\end{aligned}$$

We have $2N(D-1)+2N-1$ integrals of the form

$$\begin{aligned}
\int_{\mathbb{R}}\exp\big(-w^{(i)}_{l\mathbf{e}_i}\big)dx^{(i)}_{l\mathbf{e}_i} &= \int_{\mathbb{R}}\exp\left(-\frac{(x^{(i)}_{\mathbf{0}_i(l)}+a\alpha_l)^2}{2(2\beta\lambda_{l\mathbf{e}_i})^{-1}}\right)dx^{(i)}_{l\mathbf{e}_i} \\
&= \sqrt{2\pi}(2\beta\lambda_{l\mathbf{e}_i})^{-1/2},
\end{aligned}$$

and one integral as follows

$$\begin{aligned}
&\int y_{j\mathbf{e}_1}\exp(-w^{(1)}_{j\mathbf{e}_1})dx^{(1)}_{j\mathbf{e}_1} \\
&= \int_{\mathbb{R}}\frac{2a\alpha_j x^{(1)}_{j\mathbf{e}_1}+(a\alpha_j)^2}{2(2\beta\lambda_{j\mathbf{e}_1})^{-1}}\exp\left(-\frac{(x^{(1)}_{j\mathbf{e}_1}+a\alpha_j)^2}{2(2\beta\lambda_{j\mathbf{e}_1})^{-1}}\right)dx^{(1)}_{j\mathbf{e}_1} \\
&= -\sqrt{2\pi}\frac{(a\alpha_j)^2}{2(2\beta\lambda_{j\mathbf{e}_1})^{-1/2}}.
\end{aligned}$$

Plugging in all the above integrals into (5.4) gives

$$\begin{aligned}
&\hat{E}^{(a)}\left[Y^{(1)}_{j\mathbf{e}_1}\right] \\
&= -\frac{1}{C_{N,\beta,d,D}}\sqrt{2\pi}\frac{(a\alpha_j)^2}{2(2\beta\lambda_{j\mathbf{e}_1})^{-1/2}} \\
&\quad\times\prod_{i=1}^{D}\prod_{(k_1,\dots,k_d)\in S_N^d\setminus\{l\mathbf{e}_i:\, l\in S_N^1\}}\sqrt{2\pi}(2\beta\lambda_{k_1,\dots,k_d})^{-1/2} \\
&\quad\times\prod_{i=2}^{D}\prod_{l\in S_N^1\setminus\{0\}}\sqrt{2\pi}(2\beta\lambda_{l\mathbf{e}_i})^{-1/2}\prod_{l\in S_N^1\setminus\{0,k\}}\sqrt{2\pi}(2\beta\lambda_{l\mathbf{e}_1})^{-1/2} \\
&= -\frac{1}{C_{N,\beta,d,D}}\frac{(2\pi)^{((2N+1)^d-1)D/2}(a\alpha_j)^2\lambda_{j\mathbf{e}_1}}{2(2\beta)^{((2N+1)^d-1)D/2-1}}\prod_{i=1}^{D}\prod_{(k_1,\dots,k_d)\in S_N^d\setminus\{\mathbf{0}\}}\frac{1}{\lambda^{1/2}_{k_1,\dots,k_d}}.
\end{aligned}$$

Together with (5.2) we get

$$\hat{E}^{(a)}\left[Y^{(1)}_{j\mathbf{e}_1}\right] = -\beta(a\alpha_j)^2\lambda_{j\mathbf{e}_1}. \tag{5.5}$$

Plugging (5.5) into (5.1) gives

$$\begin{aligned}
I_{2,d,D} &= \beta\sum_{i=1}^{D}\sum_{j\in S_N^1\setminus\{0\}}(a\alpha_j)^2\lambda_{j\mathbf{e}_i} \\
&= D\beta a^2\sum_{j\in S_N^1\setminus\{0\}}\alpha_j^2\lambda_{j\mathbf{e}_1},
\end{aligned} \tag{5.6}$$

where we used the fact that $\lambda_{j\mathbf{e}_i} = \lambda_{j\mathbf{e}_1}$, by the symmetry of the eigenvalues (see (3.2)).

In order to complete the proof we recall Lemma 4.1 from [10].

Lemma 5.1. *There exists a constant $C > 0$ not depending on N and β such that,*

$$\sum_{j\in S_N^1\setminus\{0\}}\alpha_j^2\lambda_{j\mathbf{e}_1} \le CN^d.$$

From Lemma 5.1 and (5.6) we conclude that

$$I_{2,d,D} \le C\beta a^2 N^d, \tag{5.7}$$

which completes the proof of Proposition 3.2 part (iii). □

6. Large Distance Tail Estimates

Assume first that $d = 2$ and $d = 1$. Let $\alpha > 0$, then from (2.5) we have

$$\begin{aligned}
&\log Q_N(R_{N,2,1} > \alpha N^{4/3}(\log N)^{4/3}) \\
&\le \log P(R_{N,2,1} > \alpha N^{4/3}(\log N)^{4/3}) - \log Z_{N,2,1}.
\end{aligned} \tag{6.1}$$

From the proof in Section 5 of [10], which uses standard Gaussian estimates, it follows that for any constant $\kappa(N) > 0$ we have

$$P(R_{N,2,1} > \alpha\beta^{-1/2}\kappa(N)) \leq C_1 \exp\left\{-c_2\kappa(N)^2(\log N)^{-2}\right\}.$$

where $C_1, c_2 > 0$ are constants not depending on N.

We therefore get,

$$P(R_{N,2,1} > \alpha\beta^{-1/2}N^{4/3}(\log N)^{4/3}) \leq C_1 \exp\left\{-c_2\alpha^2 N^{8/3}(\log N)^{2/3}\right\}.$$

Using this bound together with Proposition 3.1(i) and (6.1) we get for all $\alpha \geq 1$,

$$\begin{aligned}
&\log Q_N(R_{N,2,1} > \alpha\beta^{-1/2}(\beta+\gamma)^{1/2}N^{4/3}(\log N)^{4/3}) \\
&\leq \log P(R_{N,2,1} > \alpha\beta^{-1/2}(\beta+\gamma)^{1/2}N^{4/3}(\log N)^{4/3}) - \log Z_{N,2,1} \\
&\leq -(\beta+\gamma)N^{8/3}(\log N)^{2/3}\big(c_3\alpha^2 - c_4\big).
\end{aligned}$$

We then can choose α to be large enough to get the large distance tail estimate in Theorem 2.1.

The proof for $d \geq 3$ and $1 \leq D \leq d$ follows similar lines, only now we use the bound

$$P(R_{N,d,D} > \alpha\beta^{-1/2}\kappa(N)) \leq C_1 \exp\left\{-c_2\kappa(N)^2\right\}.$$

Note that the log-correction does not appear on the right-hand side due to Proposition 4.1(ii).

Together with Proposition 3.1(ii) we get

$$\begin{aligned}
&\log Q_N(R_{N,d,D} > \alpha\beta^{-1/2}(\beta+\gamma)^{1/2}N^{\frac{d}{2}+\frac{d-D}{D+2}}) \\
&\leq \log P(R_{N,d,D} > \alpha\beta^{-1/2}(\beta+\gamma)^{1/2}N^{\frac{d}{2}+\frac{d-D}{D+2}}) - \log Z_N \\
&\leq -(\beta+\gamma)N^{d+\frac{2(d-D)}{D+2}}\big(c_3\alpha^2 - c_4\big).
\end{aligned}$$

We then can choose α to be large enough to get the large distance tail estimate in Theorem 2.1.

7. Small Distance Tail Estimates

Let $\varepsilon > 0$, and $\kappa(N) > 0$ to be specified later. Then from (2.5) we have the following:

$$\begin{aligned}
&\log Q_N(R_{N,d,D} < \varepsilon\kappa(N)) \\
&\quad \leq \log E\left[\exp\left\{-\gamma\int_{\mathbb{R}^D}\ell_N(\mathbf{y})^2 d\mathbf{y}\right\}\mathbf{1}_{\{R_{N,d,D}<\varepsilon\kappa(N)\}}\right] - \log Z_{N,d,D}.
\end{aligned} \tag{7.1}$$

Let

$$\tilde{J}_{N,d,D} := E\left[\exp\left\{-\gamma\int_{\mathbb{R}^D}\ell_N(\mathbf{y})^2 d\mathbf{y}\mathbf{1}_{\{R_{N,d,D}<\varepsilon\kappa(N)\}}\right\}\right]. \tag{7.2}$$

Note that on $\{R_{N,d,D} < \varepsilon\kappa(N)\}$ we have

$$\begin{aligned}\int_{\mathbb{R}^D} \ell_N(\mathbf{y})^2 d\mathbf{y} &= 2^d \kappa(N)^D \varepsilon^D \int_{[-\varepsilon\kappa(N),\varepsilon\kappa(N)]^D} \ell_N(\mathbf{y})^2 \frac{1}{2^D\kappa(N)^D\varepsilon^D} d\mathbf{y} \\ &\geq 2^D \kappa(N)^D \varepsilon^D \left(\int_{[-\varepsilon\kappa(N),\varepsilon\kappa(N)]^D} \ell_N(\mathbf{y}) \frac{1}{2^D\kappa(N)^D\varepsilon^D} d\mathbf{y} \right)^2 \\ &= \frac{1}{2^D\kappa(N)^D\varepsilon^D} \left(\int_{[-\varepsilon\kappa(N),\varepsilon\kappa(N)]^D} \ell_N(\mathbf{y}) d\mathbf{y} \right)^2,\end{aligned} \tag{7.3}$$

where we used Jensen's inequality. Since on $\{R_{N,d,D} < \varepsilon\kappa(N)\}$ we have

$$\int_{[-\varepsilon\kappa(N),\varepsilon\kappa(N)]^D} \ell_N(\mathbf{y}) d\mathbf{y} = |S_N| = (2N+1)^d,$$

together with (7.3) we get that

$$\int_{\mathbb{R}^D} \ell_N(\mathbf{y})^2 d\mathbf{y} \geq \frac{2^{2d-D} N^{2d} \kappa(N)^{-D}}{\varepsilon^D}. \tag{7.4}$$

From (7.2) and (7.4) we have

$$\tilde{J} \leq \exp\left(-\gamma \frac{2^{2d-D} N^{2d} \kappa(N)^{-D}}{\varepsilon^D} \right). \tag{7.5}$$

From (7.1), (7.2), (7.5) and Proposition 3.1(i) we get for $d = 2$ and D=1 (taking $\kappa(N) = \gamma(\beta+\gamma)^{-1} N^{4/3} (\log N)^{-2/3}$):

$$\begin{aligned}&\log Q_N(R_{N,2,1} < \varepsilon\gamma(\beta+\gamma)^{-1} N^{4/3} (\log N)^{-2/3}) \\ &\leq -(\beta+\gamma) \left(\frac{8N^{8/3}(\log N)^{2/3}}{\varepsilon} - CN^{8/3}(\log N)^{2/3} \right).\end{aligned}$$

By choosing $\varepsilon > 0$ small enough it follows that

$$\lim_{N\to\infty} \log Q_N(R_{N,2,1} < \varepsilon\gamma(\beta+\gamma)^{-1} N^{4/3} (\log N)^{-2/3}) = 0.$$

Repeating the same steps as in the case where $d = 2$ and $D = 1$, plugging in

$$\kappa(N) = \gamma^{1/D}(\beta+\gamma)^{-1/D} N^{\frac{1}{D}\left(d - \frac{2(d-D)}{d-D+2}\right)}$$

to (7.5) we get,

$$\tilde{J} \leq e^{-(\beta+\gamma) \frac{2^{2d-D} N^{d + \frac{2(d-D)}{d-D+2}}}{\varepsilon^D}}.$$

Together with Proposition 3.1(ii) this give the following bound for $d \geq 3$ and $1 \leq D \leq d$,

$$\begin{aligned}&\log Q_N \left(R_{N,d,D} < \varepsilon\gamma^{1/D}(\beta+\gamma)^{-1/D} N^{\frac{1}{D}\left(d - \frac{2(d-D)}{D+2}\right)} \right) \\ &\leq -(\beta+\gamma) \left(\frac{N^{d+\frac{2(d-D)}{D+2}}}{\varepsilon^D} - CN^{d+\frac{2(d-D)}{D+2}} \right).\end{aligned}$$

Then choosing $\varepsilon > 0$ sufficiently small and taking the limit where $N \to \infty$ completes the proof of Theorem 2.1.

Acknowledgment. Research of Carl Mueller is partially supported by Simons Foundation grant 513424.

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