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INVERSE STOCHASTIC TRANSFER PRINCIPLE

MATTHEW LINN AND ANNA AMIRDJANOVA*

Abstract. In [10] a "direct" stochastic transfer principle was introduced, which represented multiple integrals with respect to fractional Brownian motion in terms of multiple integrals with respect to standard Brownian motion. The method employed in [10] involved an operator $\Gamma_H^{(n)}$, mapping a class of functions L_H^2 to L^2. However, the operator does not map L_H^2 *onto* L^2. Hence $\Gamma_H^{(n)}$ is not invertible. The non-invertibility arises from the fact that $\Gamma_H^{(n)}$ is defined in terms of n-fold Riemann-Liouville fractional integrals and it is well known that functions $f \in L^p(a,b)$ cannot always be represented as fractional integrals of functions $\varphi \in L^p(a,b)$. This paper establishes the inverse stochastic transfer principle. Our purpose is to explicitly define an operator, which, acting on a certain class of functions, gives the transfer principle inverting that in [10]. As a result, multiple stochastic integrals with respect to standard Brownian motion are represented in terms of multiple stochastic integrals with respect to persistent fractional Brownian motion. In order to do this, we first prove a characterization of a class of functions for which there exists φ, defined on a compact set in $\mathbb{R}^n$, such that f is equal to the n-fold Riemann-Liouville fractional integral of φ. We establish a method for computing φ and conclude the paper with an example of using the inverse transfer principle in filtering.

1. Introduction

Fractional Brownian motion B^H is a Gaussian processes parameterized by Hurst parameter H. When $H = \frac{1}{2}$ fractional Brownian motion (fBm) reduces to ordinary Brownian motion. When we have Hurst parameter $H > \frac{1}{2}$ the fBm exhibits long range dependence. Such a dependence arises often in applications ranging from network communications, where internet traffic has been shown to exhibit long range dependence, to macroeconomics, where the long range dependence paradigm has been used to analyze the effects of economic shock on income cycles [12].

One can define stochastic integrals with respect to Gaussian processes in the same vein as Itô's multiple stochastic integrals with respect to Brownian motion, see for example [1], [2]. Several authors have studied multiple stochastic integrals with respect to long-range dependent fBm [6], [5], [11] and long-range dependent multiparameter fBm [8]. This method of stochastic calculus is built upon the theory of reproducing kernel Hilbert spaces and the Wick product.

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In their 2002 paper [10], Pérez-Abreu and Tudor present a stochastic transfer principle enabling one to represent the multiple fractional stochastic integrals of deterministic functions f as Itô type multiple stochastic integrals of a deterministic operator $\Gamma_{H,T}^{(n)}(f)$. In Section 2 we describe the space of functions $L_H^2(T^n)$ required in order to apply this transfer principle. The operator $\Gamma_{H,T}^{(n)} : L_H^2(T^n) \to L^2(T^n)$ is defined in terms of multivariate Riemann-Liouville fractional integrals, which will be discussed in Section 2. This very nice result allows one to take many of the known properties of multiple stochastic integrals with respect to Brownian motion and apply these properties to multiple fractional stochastic integrals.

However, the inverse relation, giving a scheme for representing multiple Wiener Itô integrals of a deterministic function in terms of a multiple *fractional* Wiener integral is not immediately clear. This stems from the fact that the Riemann-Liouville fractional integrals of functions $f \in L^p$ are invertible only for a certain subclass of L^p. As these fractional integrals play a key role in the definition of $\Gamma_{H,T}^{(n)}$, whether there exists an inverse operator depends on the inversion properties of the Riemann-Liouville fractional integrals.

From the transfer principle in [10], the form of the left inverse of the $\Gamma_{H,T}^{(n)}$ seems very intuitive. However, $\Gamma_{H,T}^{(n)} f$ may not be invertible. Obviously for functions f which can be represented as $\Gamma_{H,T}^{(n)}\phi$ for some $\phi \in L_H^2(T^n)$, $\Gamma_{H,T}^{(n)} f$ is invertible. The fact that the operator involves n-dimensional fractional integrals however, makes the invertibility non-trivial.

This paper contributes two main results. The first result gives a class of functions for which the multiparameter Riemann-Liouville fractional integral operator is invertible on a compact domain. In proving this result, we give an explicit method for identifying a function φ such that f is equal to the Riemann-Liouville fractional integral of φ, provided the function f lives in $I^\alpha(L^p)$. The second result gives an explicit class of functions $L_{\Phi,H}^2$ for which $\Gamma_{H,T}^{(n)}$ from [10] is invertible. Inclusion in this class can be verified by checking the existence of the limits specified in Section 4. For such functions we establish the Inverse Stochastic Transfer Principle (ISTP).

The ISTP is applicable in a number of settings. Firstly, in combination with the direct stochastic transfer principle, it allows to obtain general representations of multiple fractional stochastic integrals of Hurst index H_1 in terms of multiple fractional stochastic integrals of Hurst index H_2, for arbitrary $H_1, H_2 \in (0,1)$. Secondly, it is a useful tool in certain problems where fractional chaos decomposition is desired. For example, in many continuous time nonlinear filtering applications with fractional noise, the goal is to describe the optimal filter (which is the best mean-square estimate of an underlying signal of interest) in terms of the trajectory of the observation process (call it, Y). When the noise corrupting the observation is fractional white noise, then, under an appropriate reference measure, Y is a fractional Brownian motion.

In [3] and [9] it has been shown that the optimal filter can, with probability one, be represented as a ratio of infinite series of multiple stochastic integrals of various types. The latter integrals can be taken with respect to a suitably constructed ordinary Brownian motion (with an appropriately defined integrand). However, it is much more natural and computationally efficient to describe the optimal filter in terms of multiple fractional integrals with respect to observation process Y itself ([4], [3]). The inverse transfer principle allows to easily transform integral expansions of the optimal filter with respect to ordinary Brownian motion to the more natural and numerically convenient expansion involving multiple fractional integrals with respect to the observation process. Such a transformation of the integral expansions makes numerical approximations of the optimal filter possible whereas the implementation of the original filter approximation is computationally inefficient.

This paper is organized as follows. Section 2 is a self-contained introduction to multiple stochastic integration with respect to fBm and fractional calculus, laying out the tools we will need throughout our analysis. In Section 3 we give criteria that guarantee a function f belongs to $I(L^p)$ on a compact domain. The criteria also establish a way of finding such a function $\varphi \in L^p$ such that f is equal to the fractional integral of φ. In Section 4, we establish the Inverse Stochastic Transfer Principle. In Section 5 we give an example of an application of the ISTP applied to integral expansions of optimal filters for a nonlinear filtering problem with fractional noise.

2. Multiple Fractional Stochastic Integrals and Fractional Calculus

For $t, s \in \mathbb{R}_+, H \in (\frac{1}{2}, 1)$, we define the process $B^H(t)$, as the mean zero Gaussian process with covariance

$$R(t,s) := E\Big[B^H(t)B^H(s)\Big] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right). \tag{2.1}$$

We will denote the multiple stochastic integral of order n with respect to fractional Brownian motion $\mathcal{I}_n$ where we assume the integral to be taken with respect to a process with Hurst parameters $H \in (\frac{1}{2}, 1)$ unless otherwise stated.

For $t, s \in \mathbb{R}_+$, define

$$\phi_H(t,s) := \frac{\partial^2}{\partial t \partial s} R(t,s) = H(2H-1)\left|t-s\right|^{2H-2}. \tag{2.2}$$

The stochastic integral with respect to fractional Brownian motion is constructed by first defining the integral for $\mathcal{S}_1$, a class of simple functions on $[0,T]$ of the form

$$f(s) = \sum_{i=1}^{n} a_i 1_{(c_i, d_i]}(s),$$

where $a_i \in \mathbb{R}$ are constants and $(c_i, d_i]$ $i = 1, ..., n$ are disjoint intervals in $[0,T]$. Equip $\mathcal{S}_1$ with the scalar product

$$\langle f, g\rangle_{L^2_H} := \int_{[0,T]} \phi_H(s,t) f(s) g(t) ds dt. \tag{2.3}$$

Let $\mathcal{H}_1^H := \overline{\text{span}}\left\{B^H(t) \ : \ t \in [0,T]\right\}$ and equip this space with the scalar product $\langle X, Y\rangle_{\mathcal{H}_1^H} = E(XY)$. For $f \in \mathcal{S}_1$, define the integral with respect to B^H as the map from $\mathcal{S}_1$ to $\mathcal{H}_1^H$ given by

$$\mathcal{I}_1^T(f) := \int_{[0,T]} f(s)dB_s^H := \sum_{i=1}^n a_i(B_{d_i}^H - B_{c_i}^H).$$

Then for all $f, g \in \mathcal{S}_1$, we have the isometry $\langle f, g\rangle_{L_H^2} = \langle \mathcal{I}_1^T(f), \mathcal{I}_1^T(g)\rangle_{\mathcal{H}_1^H}$. Define Λ_1^H to be the completion of $\mathcal{S}_1$ under the norm induced by $\langle\cdot,\cdot\rangle_{L_H^2}$, so that $\{\Lambda_1^H, \langle\cdot,\cdot\rangle_{L_H^2}\}$ is a Hilbert space and $\mathcal{I}_1^T$ can be extended to an isometric isomorphism mapping Λ_1^H to $\mathcal{H}_1^H$. We will call this extension $\mathcal{I}_1^T$ as well. For the remainder of the paper, we will drop the superscript and simply write $\mathcal{I}_1$.

Note that the elements of Λ_1^H may not be functions but rather generalized functions. For this reason, it is convenient to work with the following subspace L_H^2 of functions which is dense in Λ_1^H. Define the space

$$L_H^2 = \left\{ f : [0,T] \to \mathbb{R},\ s.t.\ \|f\|_{L_H^2} < \infty \right\}.$$

Let $\{e_i\}_{i=1}^\infty$ be a complete orthonormal system in L_H^2. Define the class of deterministic symmetric functions $\mathcal{S}_n$ of the form:

$$f(t_1, ..., t_n) = \sum_{1\le k_i\le k} a_{k_1,...,k_2} e_{k_1}(t_1)...e_{k_n}(t_n). \tag{2.4}$$

For $f \in \mathcal{S}_n$, define the map $\mathcal{I}_n$ by

$$\mathcal{I}_n := \sum_{1\le k_i\le k} a_{k_1,...,k_2}\mathcal{I}_1^{\diamond 1}(e_{k_1}) \diamond ... \diamond \mathcal{I}_1^{\diamond 1}(e_{k_n}), \tag{2.5}$$

where $\diamond$ and $(\cdot)^{\diamond 1}$ represent the Wick product and the first Wick power as defined in [14]. Then $\mathcal{I}_n$ is the nth order Wiener integral with respect to fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$. As in the case $n = 1$, the mapping for arbitrary n, $\mathcal{I}_n$ can be extended to the class of integrands

$$L_H^2([0,T]^n) := \left\{ f : [0,T]^n \to \mathbb{R} : \|f\|_{H,n} < \infty \right\} \tag{2.6}$$

where the norm $\|\cdot\|_{H,n}$ is induced by the scalar product

$$\langle f, g\rangle_{H,n} := \int_{[0,T]^n} f(t_1, ..., t_n)g(s_1, ..., s_n)\phi_H(s_1, t_1)\cdots\phi_H(s_n, t_n)ds_1dt_1\cdots ds_ndt_n \tag{2.7}$$

and for non symmetric f, $\mathcal{I}_n(f)$ is defined by $\mathcal{I}_n(\widetilde{f})$, where $\widetilde{f}$ is the symmetrization of f, defined by

$$\widetilde{f}(t_1, ..., t_n); = \frac{1}{n!}\sum_\sigma f(t_{\sigma(1)}, ..., t_{\sigma(n)}),$$

where the sum is taken over σ, the set of all permutations of $\{1, ..., n\}$.

When dealing with sets $a = (a_1, ..., a_n),\ b = (b_1, ..., b_n)$ in the n-dimensional plane we will use the notation $a \prec b$ to describe the relation $a_i < b_i\ \forall\ i = 1, ..., n$ and similarly we write $a \preceq b$ if $a_i \le b_i\ \forall\ i = 1, ..., n$.

We note that fractional and standard Brownian motion are connected through a number of fractional integral relations. For example, $B_t^H = \int_0^t K_H(t,s)dW_s$ for some standard Brownian motion W, where

$$K_H(t,s) = C_H\left(\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2})s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}du\right) \quad (2.8)$$

where

$$C_H = \left(\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}\right)^{\frac{1}{2}}.$$

The above Brownian motion W can be reconstructed from B^H by

$$W_t = \int_0^t K_H^{-1}(t,s)dB_s^H,$$

where

$$K_H^{-1}(t;s) = c'_H\left(\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{\frac{1}{2}-H} - (H-\frac{1}{2})s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{3}{2}}(u-s)^{\frac{1}{2}-H}du\right), \quad (2.9)$$

and $c'_H = \frac{1}{\Gamma(\frac{3}{2}-H)}\sqrt{\frac{\Gamma(2-2H)}{2H\Gamma(\frac{3}{2}-H)\Gamma(H+\frac{1}{2})}}$.

Theorem 2.1. ([10]) *For $H > \frac{1}{2}$, $L^2([0,T]^n)$ is a dense subclass of $L_H^2([0,T]^n)$.*

Next we discuss concepts of fractional calculus which will be used throughout the paper, namely fractional integration and differentiation of functions of many variables. As in [13] (section 24, Chapter 5), let us make use of the following extensions of the Riemann-Liouville fractional integral and derivative.

Definition 2.2. Let $[a,b] = [a_1,b_1]\times\cdots\times[a_n,b_n]$, where each $[a_i,b_i]$ are fixed intervals in $\mathbb{R}$. For the function $\varphi(x) : [a,b] \to \mathbb{R}$, $\varphi \in L^1([a,b])$, the *right-sided mixed fractional integral* of order $\alpha = (\alpha_1,...,\alpha_n)$ is defined by

$$(I_{b-}^{\alpha,n}\varphi)(x) = (I_{b_1-}^{\alpha_1}\circ ...\circ I_{b_n-}^{\alpha_n}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_{x_1}^{b_1}...\int_{x_n}^{b_n}\frac{\varphi(t)dt}{(t-x)^{1-\alpha}}, \quad 0 \prec \alpha \prec 1, \quad (2.10)$$

where $(t-x)^{1-\alpha} = (t_1-x_1)^{1-\alpha_1}\cdots(t_n-x_n)^{1-\alpha_n}$, $\Gamma(\alpha) = \Gamma(\alpha_1)\cdots\Gamma(\alpha_n)$ and $dt = dt_1...dt_n$.

Definition 2.3. For φ as in definition 2.2, the *right-sided mixed Riemann-Liouville fractional derivative* of order $\alpha = (\alpha_1,...,\alpha_n)$ is defined by

$$(\mathcal{D}_{b-}^{\alpha,n}\varphi)(x) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial^n}{\partial x_1...\partial x_n}\int_{x_1}^{b_1}...\int_{x_n}^{b_n}\frac{\varphi(t)dt}{(t-x)^{\alpha}}, \quad 0 \prec \alpha \prec 1, \quad (2.11)$$

where $(t-x)^{\alpha} = (t_1-x_1)^{\alpha_1}....(t_n-x_n)^{\alpha_n}$, $\Gamma(1-\alpha) = \Gamma(1-\alpha_1)\cdots\Gamma(1-\alpha_n)$ and $dt = dt_1...dt_n$, provided φ is such that the derivative on the right side of (2.11) exists.

If a solution exists to the multidimensional Abel equation

$$\frac{1}{\Gamma(\alpha)}\int_{x_1}^{b_1}...\int_{x_n}^{b_n}\frac{f(t)dt}{(t-x)^{1-\alpha}} = \varphi(x), \quad x \prec b, \quad (2.12)$$

then clearly $f(x) = (\mathcal{D}_{b-}^{\alpha,n}\varphi)(x)$ is the unique solution. Hence $\mathcal{D}_{b-}^{\alpha,n}(I_{b-}^{\alpha,n}f)(x) = f(x)$. However, in general it is not the case that $I_{b-}^{\alpha,n}(\mathcal{D}_{b-}^{\alpha,n}f)(x) = f(x)$. If there exists a function ϕ satisfying $I_{b-}^{\alpha,n}\phi(x) = f(x)$ $\forall x \in [a,b]$, then clearly $I_{b-}^{\alpha,n}(\mathcal{D}_{b-}^{\alpha,n}f)(x) = f(x)$. In order to investigate existence of such a ϕ, we need to introduce the (mixed) finite difference operator $\Delta_h^{k,n}$ acting on functions $f : \mathbb{R}^n \to \mathbb{R}$ parameterized by order vector k and step vector h, defined by

$$(\Delta_h^{k,n}f)(x) = \sum_{0\preceq j\preceq k} (-1)^{|j|}\binom{k}{j} f(x - j \bullet h), \tag{2.13}$$

where $j = (j_1, ..., j_n), k = (k_1, ..., k_n)$ are vectors of integers in $\mathbb{R}^n$, and we use of the following notation,

$$\binom{k}{j} := \prod_{i=1}^{n}\binom{k_i}{j_i},\ j \bullet t := (j_1t_1, ..., j_nt_n),\ \text{and } |j| := \sum_{i=1}^{n} j_i.$$

For notational simplicity throughout the paper, let us define function

$$\theta_n(x; t, \alpha, k) = \frac{\prod_{i=1}^{n} \alpha_i^{k_i}}{\prod_{i=1}^{n} t_i^{(1+\alpha_i)k_i}(b_i - x_i)^{\alpha_i(1-k_i)}}, \quad x \in \mathbb{R}^n, \tag{2.14}$$

where $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_i \in (0, 1)$ $\forall i \in \{1, ..., n\}$. We also introduce the integral operator

$$\left(\Phi_{\alpha,k}^{b,n}f\right)(x) = (\prod_{i=1}^{n}(b_i-x_i)^{k_i-1}) \int_{t_1=0}^{b_1-x_1} \cdots \int_{t_n=0}^{b_n-x_n} \theta_n(x; t, \alpha, k)(\Delta_{-t}^{k,n}f)(x)dt_n...dt_1. \tag{2.15}$$

Similarly, define the operator

$$\begin{aligned}\left(\Phi_{\epsilon,\alpha,k}^{b,n}f\right)(x) &= \left(\prod_{i=1}^{n}(b_i - x_i - \epsilon_i)^{k_i-1}\right)\\ &\times \int_{t_1=\epsilon_1}^{b_1-x_1} \cdots \int_{t_n=\epsilon_n}^{b_n-x_n} \theta_n(x; t, \alpha, k)(\Delta_{-t}^{k,n}f)(x)dt_n...dt_1\end{aligned} \tag{2.16}$$

where $\epsilon = (\epsilon_1, ...\epsilon_n)$.

Definition 2.4. For function $f : [a, b] \to \mathbb{R}$, operator of the form

$$\mathbf{D}_{b-,\epsilon}^{\alpha,n}f(x) \equiv \left(\prod_{i=1}^{n}\frac{1}{\Gamma(1-\alpha_i)}\right)\sum_{0\preceq k\preceq 1}\left(\Phi_{\epsilon,\alpha,k}^{b,n}f\right)(x), \tag{2.17}$$

is called the *truncated Marchaud fractional derivative* and, when the L^p limit exists,

$$\mathbf{D}_{b-}^{\alpha,n}f(x) \equiv \lim_{\epsilon\to 0}\mathbf{D}_{b-,\epsilon}^{\alpha,n}f(x), \tag{2.18}$$

is called the *Marchaud fractional derivative*, where the sum over $0 \preceq k \preceq 1$ represents the sum over all vectors $k \in \{0, 1\}^n$.

Remark 2.5. Note that, when $n = 1$, $\mathbf{D}_{b-,\epsilon}^{\alpha,n}f(x)$ is the sum of two terms,

$$\mathbf{D}_{b-,\epsilon}^{\alpha,n}f(x) = \frac{f(x)}{\Gamma(1-\alpha)(b-x)^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)}\int_{t=\epsilon}^{b-x}\frac{f(x) - f(t+x)}{t^{1+\alpha}}dt. \tag{2.19}$$

In the case $n = 2$,

$$\Big(\prod_{i=1}^{2}\Gamma(1-\alpha_i)\Big)\mathbf{D}^{\alpha,n}_{b-,\epsilon}f(x) = \frac{f(x_1,x_2)}{(b_1-x_1)^{\alpha_1}(b_2-x_2)^{\alpha_2}}$$

$$+\alpha_1\int_{t_1=\epsilon_1}^{b_1-x_1}\frac{f(x_1,x_2)-f(t_1+x_1,x_2)}{t_1^{1+\alpha_1}(b_2-x_2)^{\alpha_2}}dt_1$$

$$+\alpha_2\int_{t_2=\epsilon_2}^{b_2-x_2}\frac{f(x_1,x_2)-f(x_1,t_2+x_2)}{t_2^{1+\alpha_2}(b_1-x_1)^{\alpha_1}}dt_2$$

$$+\int_{t_1=\epsilon_1}^{b_1-x_1}\int_{t_2=\epsilon_2}^{b_2-x_2}\frac{f(x_1,x_2)-f(t_1+x_1,x_2)-f(x_1,t_2+x_2)+f(t_1+x_1,t_2+x_2)}{t_1^{1+\alpha_1}t_2^{1+\alpha_2}\alpha_1^{-1}\alpha_2^{-1}}dt_2dt_1, \tag{2.20}$$

the sum of four terms. It is easy to see that for general n, $\mathbf{D}^{\alpha,n}_{b-,\epsilon}f(x)$ is given by the sum of 2^n such terms where the numerator of the integrand in each term is given by the variation of f along the axes with respect to which the integral is taken.

3. Representation of Deterministic Fractional Integrals

In this section we give conditions on functions $f : [a,b] \to \mathbb{R}$ which ensure that there exists a function $\phi \in L^p([a,b])$ such that $f(x) = (I^{\alpha,n}_{b-}\phi)(x)$ for all $x \in [a,b]$, where $[a,b] = [a_1,b_1] \times \cdots \times [a_n,b_n]$.

When looking at the truncated Marchaud fractional derivative, the form of $\left(\Phi^{b,n}_{\epsilon,\alpha,k}f\right)(x)$ depends on the sub-region x falls in. We adopt the very important convention of [13], and assume that the function $f : [a,b] \to \mathbb{R}$ vanishes outside the region $[a,b]$.

Definition 3.1. For vectors $a = (a_1,...,a_n)$, $b = (b_1,...,b_n)$ and $\epsilon = (\epsilon_1,...,\epsilon_n)$, where $0 < \epsilon_i < b_i - a_i$ $\forall i = 1,...,n$, define

$$\mathcal{R}(\epsilon) = \mathcal{R}_{a,b}(\epsilon) = [a,b] - \big([a_1,b_1-\epsilon_1]\times\cdots\times[a_n,b_n-\epsilon_n]\big).$$

We will partition $\mathcal{R}(\epsilon)$ into $2^n - 1$ regions. Consider all rectangles of the form $I_1^{(\epsilon)} \times \cdots \times I_n^{(\epsilon)}$ where exactly k of the $I_j^{(\epsilon)}$'s are of the form $I_j^{(\epsilon)} = [b_j - \epsilon_j, b_j]$, and the remaining $n-k$ of the $I_j^{(\epsilon)}$'s are of the form $I_j^{(\epsilon)} = [a_j, b_j - \epsilon_j)$. There are $\binom{n}{k}$ such rectangles, let us call them $\mathcal{R}^k_1, ..., \mathcal{R}^k_{\binom{n}{k}}$, so that we have

$$\mathcal{R}(\epsilon) = \bigcup_{k=1}^{n}\bigcup_{r=1}^{\binom{n}{k}} R^k_r.$$

To see how these regions affect the form of Marchaud fractional derivative let us look at an example with $n = 2$. Take the last term in (2.20),

$$\int_{t_1=x_1+\epsilon_1}^{b_1}\int_{t_2=x_2+\epsilon_2}^{b_2}\frac{f(x_1,x_2)-f(t_1,x_2)-f(x_1,t_2)+f(t_1,t_2)}{(t_1-x_1)^{1+\alpha_1}(t_2-x_2)^{1+\alpha_2}}dt_2dt_1. \tag{3.1}$$

If $a_1 \le x_1 < b_1 - \epsilon_1,\ b_2 - \epsilon_2 \le x_2 \le b_2$, that is $x \in \mathcal{R}_2^1$, since f vanishes outside $[a,b]$, the last two terms in the numerator of the integrand become zero and we are left with

$$\left(\frac{1}{\epsilon_2^{\alpha_2}} - \frac{1}{(b_2-x_2)^{\alpha_2}}\right)\int_{t_1=x_1+\epsilon_1}^{b_1} \frac{f(x_1,x_2)-f(t_1,x_2)}{(t_1-x_1)^{1+\alpha_1}}dt_1$$

and if $b_1 - \epsilon_1 \le x_1 \le b_1,\ b_2 - \epsilon_2 \le x_2 \le b_2$ ($x \in \mathcal{R}_1^2$) then (3.1) becomes

$$f(x_1,x_2)\left(\frac{1}{\epsilon_1^{\alpha_1}} - \frac{1}{(b_1-x_1)^{\alpha_1}}\right)\left(\frac{1}{\epsilon_2^{\alpha_2}} - \frac{1}{(b_2-x_2)^{\alpha_2}}\right).$$

Theorem 3.2. *Let $[a,b]$ be a compact rectangle in $\mathbb{R}^n$. Let $f : [a,b] \to \mathbb{R}$ be a measurable function on $[a,b]$. Suppose $1 \le p < \infty$, and $\alpha_i \in (0, \frac{1}{p})\ \forall i$. If $f \in L^p([a,b])$ and the L^p limit*

$$\lim_{\substack{\epsilon\to 0\\ (L^p)}} \left(\Phi^{b,n}_{\epsilon,\alpha,k} f\right)(\cdot) \tag{3.2}$$

exists $\forall\ 0 \preceq k \preceq 1$ (i.e. for all vectors $k \in \{0,1\}^n$) where $\epsilon = (\epsilon_1, ..., \epsilon_n)$ and $\Phi^{b,n}_{\epsilon,\alpha,k}$ is given by (2.16), then there exists function $\varphi \in L^p([a,b])$ such that $f(x) = I^{\alpha,n}_{b-}\varphi(x)$, for all $x \in [a,b]$. Moreover, φ is equal to $\lim_{\substack{\epsilon\to 0\\ (L^p)}} \sum_{0\preceq k\preceq 1}\left(\Phi^{b,n}_{\epsilon,\alpha,k} f\right)(\cdot)$.

Proof. Let $f \in L^p([a,b])$ and consider the function

$$\varphi^{(n)}_\epsilon(x) = \left(\prod_{i=1}^n \frac{1}{\Gamma(1-\alpha_i)}\right) \sum_{0\preceq k\preceq 1} \left(\Phi^{b,n}_{\epsilon,\alpha,k} f\right)(x). \tag{3.3}$$

It is easy to check directly that $\varphi^{(n)}_\epsilon(x) \in L^p([a,b])$ and when the limit exists, $\lim_{\epsilon\to 0}\varphi^{(n)}_\epsilon(x) := \varphi^{(n)} \in L^p([a,b])$. We will prove that $f = I^{\alpha,n}_{b-}\varphi^{(n)}$. Since operator $I^{\alpha,n}_{b-}$ is continuous in L^p (see [13]) it is enough to show $f = \lim_{\epsilon\to 0} I^{\alpha,n}_{b-}\varphi^{(n)}_\epsilon(x)$, where the limit is taken in L^p.

Step 1. Let us first focus on the region where $x_i < b_i - \epsilon_i\ \forall i$. Let us show that over this region,

$$I^{\alpha,n}_{b-}\varphi^{(n)}_\epsilon(x) = \int_{t_1=0}^{\frac{b_1-x_1}{\epsilon_1}} \cdots \int_{t_n=0}^{\frac{b_n-x_n}{\epsilon_n}} \left(\prod_{i=1}^n \mathcal{K}_{\alpha_i}(t_i)\right) f(t_1\epsilon_1+x_1, ..., t_n\epsilon_n+x_n)dt_n...dt_1, \tag{3.4}$$

where $\mathcal{K}_{\alpha_i}(t_i) = \frac{1}{\Gamma(\alpha_i)\Gamma(1-\alpha_i)}\left(\frac{t_i^{\alpha_i} - (t_i-1)_+^{\alpha_i}}{t_i}\right)$, which has the property that

$$\int_{t_i=0}^{\infty} \mathcal{K}_{\alpha_i}(t_i)dt_i = 1. \tag{3.5}$$

The right hand side of (3.4) can be rewritten as

$$\frac{1}{\Gamma(\alpha_n)\Gamma(1-\alpha_n)}\int_{t_1=0}^{\frac{b_1-x_1}{\epsilon_1}} \mathcal{K}_{\alpha_1}(t_1)... \int_{t_{n_1}=0}^{\frac{b_{n-1}-x_{n-1}}{\epsilon_{n-1}}} \mathcal{K}_{\alpha_{n-1}}(t_{n-1})$$
$$\times\Bigg[\int_{t_n=0}^{\frac{b_n-x_n}{\epsilon_n}} t_n^{\alpha_n-1} f(t_1\epsilon_1+x_1, ..., t_n\epsilon_n+x_n)dt_n$$

$$-\int_{t_n=1}^{\frac{b_n-x_n}{\epsilon_n}} \frac{(t_n-1)^{\alpha_n}}{t_n} f(t_1\epsilon_1+x_1,...,t_n\epsilon_n+x_n)dt_n\Bigg] dt_{n-1}...dt_1.$$

Making the substitution $y_n = t_n\epsilon_n + x_n$, the term inside the brackets becomes

$$\int_{x_n}^{b_n} \frac{f(t_1\epsilon_1+x_1,\cdots,y_n)}{\epsilon_n^{\alpha_n}(y_n-x_n)^{1-\alpha_n}}dy_n - \int_{x_n+\epsilon_n}^{b_n} \frac{f(t_1\epsilon_1+x_1,\cdots,y_n)(y_n-x_n-\epsilon_n)^{\alpha_n}}{\epsilon_n^{\alpha_n}(y_n-x_n)}dy_n. \tag{3.6}$$

Using the fact that

$$\frac{(y-\epsilon-x)^{\alpha}}{\epsilon^{\alpha}(y-x)} = \int_{s=x}^{y-\epsilon} \frac{1}{(s-x)^{1-\alpha}(y-s)^{1+\alpha}}ds \ (x<y),$$

which follows from the indefinite integral

$$\int \frac{1}{(s-x)^{1-\alpha}(y-s)^{1+\alpha}}ds = \frac{(y-s)^{-\alpha}(s-x)^{\alpha}}{\alpha(y-x)},$$

(3.6) becomes

$$\int_{x_n}^{b_n} \frac{f(t_1\epsilon_1+x_1,...,y_n)}{\epsilon_n^{\alpha_n}(y_n-x_n)^{1-\alpha_n}}dy_n - \alpha_n \int_{s_n=x_n}^{b_n-\epsilon_n}\int_{y_n=s_n+\epsilon}^{b_n} \frac{f(t_1\epsilon_1+x_1,...,y_n)ds_n dy_n}{(s_n-x_n)^{1-\alpha_n}(y_n-s_n)^{1+\alpha_n}}. \tag{3.7}$$

Adding and subtracting the term

$$\int_{y_n=x_n}^{b_n-\epsilon_n} \frac{f(t_1\epsilon_1+x_1,...,y_n)}{(y_n-x_n)^{1-\alpha_n}(b_n-y_n)^{\alpha_n}}dy_n - \frac{1}{\epsilon^{\alpha_n}}\int_{y_n=x_n}^{b_n-\epsilon_n} \frac{f(t_1\epsilon_1+x_1,...,y_n)}{(y_n-x_n)^{1-\alpha_n}}dy_n$$

$$+\int_{y_n=b_n-\epsilon_n}^{b_n} \frac{f(t_1\epsilon_1+x_1,...,y_n)}{(y_n-x_n)^{1-\alpha_n}(b_n-y_n)^{\alpha_n}}dy_n,$$

we see that (3.7) becomes

$$\int_{y_n=x_n}^{b_n} \frac{1}{(y_n-x_n)^{1-\alpha_n}}\Bigg\{\frac{f(t_1\epsilon_1+x_1,...,y_n)}{(b_n-y_n)^{\alpha_n}}$$

$$+\alpha_n\int_{s_n=y_n+\epsilon_n}^{b_n} \frac{f(t_1\epsilon_1+x_1,...,y_n)-f(t_1\epsilon_1+x_1,...,s_n)}{(s_n-y_n)^{1+\alpha_n}}ds_n\Bigg\}dy_n.$$

Thus the right hand side of (3.4) is equal to

$$\frac{1}{\Gamma(\alpha_n)\Gamma(1-\alpha_n)}\int_{t_1=0}^{\frac{b_1-x_1}{\epsilon_1}} \mathcal{K}_{\alpha_1}(t_1)...\int_{t_{n-1}=0}^{\frac{b_{n-1}-x_{n-1}}{\epsilon_{n-1}}} \mathcal{K}_{\alpha_{n-1}}(t_{n-1})$$

$$\times\Bigg[\int_{y_n=x_n}^{b_n} \frac{1}{(y_n-x_n)^{1-\alpha_n}}\Bigg\{\frac{f(t_1\epsilon_1+x_1,...,y_n)}{(b_n-y_n)^{\alpha_n}}$$

$$+\alpha_n\int_{s_n=y_n+\epsilon_n}^{b_n} \frac{f(t_1\epsilon_1+x_1,...,y_n)-f(t_1\epsilon_1+x_1,...,s_n)}{(s_n-y_n)^{1+\alpha_n}}ds_n\Bigg\}dy_n\Bigg]dt_{n-1}...dt_1.$$

Consider $f(t_1\epsilon_1 + x_1, ..., y_n)$ as a function of y_n only and let

$$\begin{aligned}\varphi_\epsilon^{(1)}(t_1\epsilon_1 + x_1, ..., y_n) \quad &:= \quad \frac{1}{\Gamma(1-\alpha_n)}\Bigg[\frac{f(t_1\epsilon_1 + x_1, ..., y_n)}{(b_n - y_n)^{\alpha_n}} \\ &+ \quad \alpha_n \int_{y_n+\epsilon_n}^{b_n} \frac{f(t_1\epsilon_1+x_1,...,y_n)-f(t_1\epsilon_1+x_1,...,s_n)}{(s_n - y_n)^{1+\alpha_n}} ds_n\Bigg].\end{aligned}$$

Then the right hand side of (3.4) is equal to

$$\left(\prod_{i=1}^{n-1} \frac{1}{\Gamma(\alpha_i)\Gamma(1-\alpha_i)}\right) \int_{t_1=0}^{\frac{b_1-x_1}{\epsilon_1}} \left(\frac{t_1^{\alpha_1} - (t_1-1)_+^{\alpha_1}}{t_1}\right) ...$$

$$... \int_{t_{n-1}=0}^{\frac{b_{n-1}-x_{n-1}}{\epsilon_{n-1}}} \left(\frac{t_{n-1}^{\alpha_{n-1}} - (t_{n-1}-1)_+^{\alpha_{n-1}}}{t_{n-1}}\right) I_{b-}^{\alpha,1} \varphi_\epsilon^{(1)}(t_1\epsilon_1 + x_1, ..., y_n) dt_1...dt_{n-1} dy_n, \tag{3.8}$$

where $I_{b-}^{\alpha,1}\varphi$ is the mixed fractional integral, that is the single parameter Riemann-Liouville fractional integral taken over only the nth axis. Using the fact that mixed fractional integrals of $L^p([a,b])$ functions are in $L^p([a,b])$ for $\alpha_i < \frac{1}{p_i}$ (see [13]), applying Fubini's Theorem and repeating the above arguments in each of the remaining $n-1$ coordinates gives (3.4).

In view of (3.4) and (3.5), it follows that

$$\begin{aligned} &I_{b-}^{\alpha,n}\varphi_\epsilon^{(n)}(x) - f(x) \\ &= \int_{t_1=0}^{\infty} \cdots \int_{t_n=0}^{\infty} \left(\prod_{i=1}^{n} \mathcal{K}_{\alpha_i}(t_i)\right) \Big[f(t_1\epsilon_1 + x_1, ..., t_n\epsilon_n + x_n) - f(x_1, ..., x_n)\Big] dt_n...dt_1 \end{aligned} \tag{3.9}$$

for $x \in [a_1, b_1 - \epsilon_1) \times \cdots \times [a_n, b_n - \epsilon_n)$.

Step 2. Next we show $\|I_{b-}^{\alpha,n}\varphi_\epsilon^{(n)}\|_{L^p(\mathcal{R}(\epsilon))} \to 0$ as $\epsilon \to 0$. Specifically we will examine the norm over individual regions making up $\mathcal{R}(\epsilon)$, as

$$\|I_{b-}^{\alpha,n}\varphi_\epsilon^{(n)}(x)\|_{L^p(\mathcal{R}(\epsilon))} \le \sum_{\ell=1}^{n}\sum_{r=1}^{\binom{n}{\ell}} \|I_{b-}^{\alpha,n}\varphi_\epsilon^{(n)}\|_{L^p(\mathcal{R}_r^\ell(\epsilon))}, \tag{3.10}$$

where each term in the sum satistfies

$$\|I_{b-}^{\alpha,n}\varphi_\epsilon^{(n)}(x)\|_{L^p(\mathcal{R}_r^\ell(\epsilon))} \le \Big(\prod_{i=1}^{n}\Gamma(1-\alpha_i)\Big)^{-1} \sum_{j=1}^{2^n} \|I_{b-}^{\alpha,n}\Phi_{\epsilon,\alpha,j}^{b,n} f(x)\|_{L^p(\mathcal{R}_r^\ell(\epsilon))} \tag{3.11}$$

where $\Phi_{\epsilon,\alpha,j}^{b,n} f$ are the individual terms (integrals) making up $\varphi_\epsilon^{(n)}(x)$ and each $j \in \{1, ..., 2^n\}$ represents an element of the set of vectors $0 \preceq k \preceq 1$, that is all vectors $k \in \{0,1\}^n$.

For notational simplicity we will look only at the region $\mathcal{R}_1^m(\epsilon)$, where $b_i - \epsilon_i \le x_i \le b_i$ for $i = 1, ..., m$. All other terms are handled in the same manner. The substitution $q_i = b_i + x_i - t_i$, $\forall i$ yields

$$I_{b-}^{\alpha,n}\Phi_{\epsilon,\alpha,j}^{b,n} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x_1}^{b_1} ... \int_{x_n}^{b_n} \frac{\Phi_{\epsilon,\alpha,j}^{b,n} f(b_1 + x_1 - q_1, ..., b_n + x_n - q_n)}{\prod_{i=1}^{n}(b_i - q_i)^{1-\alpha_i}} dq_n...dq_1.$$

Applying the Minkowski inequality for integrals ([7]),

$$\|I_{b-}^{\alpha,n}\Phi_{\epsilon,\alpha,j}^{b,n}f\|_{L^p(\mathcal{R}_1^m(\epsilon))}$$

$$= \frac{1}{\Gamma(\alpha)}\Bigg(\int_{x_1=b_1-\epsilon_1}^{b_1}\cdots\int_{x_m=b_m-\epsilon_m}^{b_m}\int_{x_{m+1}=0}^{b_{m+1}-\epsilon_{m+1}}\cdots\int_{x_n=0}^{b_n-\epsilon_n}$$

$$\left|\int_{q_1=x_1}^{b_1}\cdots\int_{q_n=x_n}^{b_n}\frac{\Phi_{\epsilon,\alpha,j}^{b,n}f(b_1+x_1-q_1,...,b_n+x_n-q_n)}{\prod_{i=1}^n(b_i-q_i)^{1-\alpha_i}}dq_n...dq_1\right|^p dx_n...dx_1\Bigg)^{\frac{1}{p}}$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_{q_1=b_1-\epsilon_1}^{b_1}\cdots\int_{q_m=b_m-\epsilon_m}^{b_m}\int_{q_{m+1}=0}^{b_{m+1}}\cdots\int_{q_n=0}^{b_n}\Bigg(\int_{x_1=b_1-\epsilon_1}^{q_1}\cdots\int_{x_m=b_m-\epsilon_m}^{q_m}$$

$$\int_{x_{m+1}=0}^{q_{m+1}\wedge(b_{m+1}-\epsilon_{m+1})}\cdots\int_{x_n=0}^{q_n\wedge(b_n-\epsilon_n)}\left|\frac{\Phi_{\epsilon,\alpha,j}^{b,n}f(b_1+x_1-q_1,...,b_n+x_n-q_n)}{\prod_{i=1}^n(b_i-q_i)^{1-\alpha_i}}\right|^p$$

$$dx_n\ldots dx_1\Bigg)^{1/p} dq_n\ldots dq_1$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_{q_1=b_1-\epsilon_1}^{b_1}\frac{1}{(b_1-q_1)^{1-\alpha_1}}\cdots\int_{q_n=0}^{b_n}\frac{1}{(b_n-q_n)^{1-\alpha_n}}dq_n...dq_1\|\Phi_{\epsilon,\alpha,j}^{b,n}f\|_{L^p(\mathcal{R}_1^m(\epsilon))}$$

$$= \frac{1}{\Gamma(\alpha)}\epsilon_1^{\alpha_1}\cdots\epsilon_m^{\alpha_m}b_{m+1}^{\alpha_{m+1}}\cdots b_n^{\alpha_n}\|\Phi_{\epsilon,\alpha,j}^{b,n}f\|_{L^p(\mathcal{R}_1^m(\epsilon))}$$

$$\leq \frac{1}{\Gamma(\alpha)}\epsilon_1^{\alpha_1}\cdots\epsilon_m^{\alpha_m}b_{m+1}^{\alpha_{m+1}}\cdots b_n^{\alpha_n}\|\Phi_{\epsilon,\alpha,j}^{b,n}f\|_{L^p([a,b])}\to 0 \quad \text{as } \epsilon\to 0. \tag{3.12}$$

Similar arguments show that for any $0 \preceq k \preceq 1$, $\|I_{b-}^{\alpha,n}\Phi_{\epsilon,\alpha,k}^{b,n}f\|_{L^p(\mathcal{R}_r^\ell(\epsilon))} \to 0$ for all $\mathcal{R}_r^\ell(\epsilon)$ making up $\mathcal{R}(\epsilon)$. This result, along with (3.10) and (3.11), implies that

$$\|I_{b-}^{\alpha,n}\varphi_\epsilon^{(n)}\|_{L^p(\mathcal{R}(\epsilon))}\to 0 \quad as\ \epsilon\to 0.$$

Step 3. As $f \in L^p(a,b)$,

$$\|f\|_{L^p(\mathcal{R}(\epsilon))}\to 0 \quad as\ \epsilon\to 0.$$

Upon combining this with (3.9) and using the properties of mean continuity of L^p functions and the dominated convergence theorem, we obtain that

$$\|I_{b-}^{\alpha,n}\varphi_\epsilon^{(n)}-f\|_{L^p(a,b)} \leq \|I_{b-}^{\alpha,n}\varphi_\epsilon^{(n)}-f\|_{L^p(\mathcal{R}(\epsilon))}+\int_{t_1=0}^{\infty}\cdots\int_{t_n=0}^{\infty}\left(\prod_{i=1}^n\mathcal{K}_{\alpha_i}(t_i)\right)$$
$$\times\Big\|f(t_1\epsilon_1+x_1,\ldots,t_n\epsilon_n+x_n)-f(x_1,...,x_n)\Big\|_{L^p(a,b)}dt_n\ldots dt_1 \longrightarrow 0$$

as $\epsilon\to 0$ and the desired result follows. □

4. Inverse Stochastic Transfer Principle

In this section we describe the inverse transfer principle and the operator for the class of functions $L^2_{\Phi,H}([0,T]^n)$, which allow us to represent multiple stochastic integrals with respect to Brownian motion in terms of multiple stochastic integrals with respect to fractional Brownian motion.

First, for Hurst parameter $H \in (\frac{1}{2}, 1)$ and for $(t_1, ..., t_n) \in [0,T]^n$, let us recall from [10], the following continuous, isometric operator $\Gamma^{(n)}_{H,T} : L^2_H([0,T]^n) \to L^2([0,T]^n)$ defined by

$$\begin{aligned}\left(\Gamma^{(n)}_{H,T}(f)\right)(t_1, ..., t_n) \quad &:= \quad (C^*_H)^n \left(\prod_{j=1}^n (t_j)^{\frac{1}{2}-H}\right) \\ &\times \quad \left(I^{H-\frac{1}{2},n}_{T-}\Big(\prod_{k=1}^n s_k^{H-\frac{1}{2}}\Big) f(s_1, ...s_n)\right)(t_1, ...t_n), \quad (4.1)\end{aligned}$$

where

$$C^*_H = \left(\frac{2H\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)}\right)^{\frac{1}{2}}.$$

Next, let us recall the following (direct) stochastic transfer principle of Perez-Abreu and Tudor.

Theorem 4.1. (Stochastic Transfer Principle) ([10]) *For all $f \in L^2_H([0.T]^n)$,*

$$\mathcal{I}_n(f) = \mathcal{J}_n(\Gamma^{(n)}_{H,T} f) \tag{4.2}$$

where $\mathcal{I}_n(\cdot)$ is the nth order Wiener integral with respect to B^H and $\mathcal{J}_n(\cdot)$ is the multiple stochastic integral with respect to the standard Brownian motion $W_t = \int_{[0,t]} K^{-1}_H(t,s) dB^H_s$.

In order to state the inverse stochastic transfer principle, let us introduce the operator $\Gamma^{(n)(-1)}_{H,T} : L^2([0,T]^n) \to L^2_H([0,T]^n)$,

$$\begin{aligned}\left(\Gamma^{(n)(-1)}_{H,T} f\right)(t_1, ..., t_n) \quad &:= \quad \left(\frac{1}{C^*_H}\right)^n \left(\prod_{j=1}^n t_j^{\frac{1}{2}-H}\right) \\ &\times \quad \left(\mathcal{D}^{H-\frac{1}{2},n}_{T-}\Big(\prod_{k=1}^n s_k^{H-\frac{1}{2}}\Big) f(s_1, ...s_n)\right)(t_1, ...t_n) \quad (4.3)\end{aligned}$$

We also make use of the following definition:

Definition 4.2. Define $L^2_{\Phi,H}([0,T]^n)$ to be the class of functions $f : [0,T]^n \to \mathbb{R}$ such that $f \in L^2([0,T]^n)$ and the limit

$$\lim_{\substack{\epsilon\to 0 \\ (L^2)}} \left(\prod_{i=1}^n x_i\right)^{\frac{1}{2}-H} \left(\Phi^{T,n}_{\epsilon,\underline{H}-\frac{1}{2},k} f^*_H\right)(x) \tag{4.4}$$

exists $\forall\ 0 \preceq k \preceq 1$, where Φ is given by (2.16). Here $\underline{H}$ denotes the vector $\underline{H} = (H, ..., H)$ so that $\underline{H} - \frac{1}{2} = (H - \frac{1}{2}, ..., H - \frac{1}{2})$, $\epsilon = (\epsilon_1, ..., \epsilon_n)$ and $f_H^*(x_1, ..., x_n) := (x_1 \cdots x_n)^{H-\frac{1}{2}} f(x_1, ..., x_n)$.

Remark 4.3. A trivial example of a function f such that limit (4.4) exists is given by $f(x_1, ..., x_n) = C(x_1 \cdots x_n)^{\frac{1}{2}-H}$, where $C \in \mathbb{R}$.

Theorem 4.4. **(Inverse Stochastic Transfer Principle)** *For functions* $f \in L^2_{\Phi,H}([0,T]^n)$*, the following equality holds:*

$$\mathcal{J}_n(f) = \mathcal{I}_n(\Gamma_{H,T}^{(n)(-1)} f), \tag{4.5}$$

where $\mathcal{J}_n(\cdot)$ *and* $\mathcal{I}_n(\cdot)$ *are as described in Theorem 4.1.*

Proof. First note that $f \in L^2([0,T]^n)$ implies $f_H^* \in L^2([0,T]^n)$ since $x_i^{H-\frac{1}{2}}$ is bounded on the compact interval $[0,T]$ for each i. The existence of (4.4) and boundedness of $x_i^{H-\frac{1}{2}}$ on $[0,T]$ imply the existence of

$$\lim_{\substack{\epsilon\to 0\\(L^2)}} \left(\Phi^{T,n}_{\epsilon,\underline{H}-\frac{1}{2},k} f_H^*\right)(x),$$

for all $0 \preceq k \preceq 1$. Let us therefore define

$$\varphi^{(n)}_{*,\epsilon}(x) := \left(\frac{1}{\Gamma(\frac{3}{2}-H)}\right)^n \sum_{0\preceq k\preceq 1} \left(\Phi^{T,n}_{\epsilon,\underline{H}-\frac{1}{2},k} f_H^*\right)(x).$$

From Theorem (3.2) it follows that $f_H^* = \lim_{\epsilon\to 0} I_{T-}^{H-\frac{1}{2},n} \varphi^{(n)}_{*.\epsilon}$ in (L^2). By continuity of the fractional integral in L^2 it follows that $f_H^* = I_{T-}^{H-\frac{1}{2},n} \lim_{\epsilon\to 0} \varphi^{(n)}_{*.\epsilon}$ in (L^2).

Next we define the term

$$\phi^{(n)}_{*,\epsilon,H,T}(x) := \left(\frac{1}{\Gamma(\frac{3}{2}-H)}\right)^n \sum_{0\preceq k\preceq 1} \Big(\prod_{i=1}^n x_i\Big)^{\frac{1}{2}-H} \left(\Phi^{T,n}_{\epsilon\underline{H}-\frac{1}{2},k} f_H^*\right)(x),\ \forall x \in [0,T]^n. \tag{4.6}$$

Note that the L^2 limit of $\phi^{(n)}_{*,\epsilon,H,T}(x)$ exists as $\epsilon \to 0$, as a consequence of (4.4). We can write f_H^* in terms of $\phi^{(n)}_{*,\epsilon} := \phi^{(n)}_{*,\epsilon,H,b}$ as

$$\begin{aligned} f_H^*(x) &= \left(I_{T-}^{H-\frac{1}{2},n} \lim_{\substack{\epsilon\to 0\\(L^2)}} \Big(\prod_{i=1}^n u_i\Big)^{H-\frac{1}{2}} \phi^{(n)}_{*.\epsilon}(u)\right)(x) \\ &= \left(I_{T-}^{H-\frac{1}{2},n} \Big(\prod_{i=1}^n u_i\Big)^{H-\frac{1}{2}} \lim_{\substack{\epsilon\to 0\\(L^2)}} \phi^{(n)}_{*.\epsilon}(u)\right)(x). \end{aligned}$$

And hence

$$\begin{aligned} f(x) &= (x_1 \cdots x_n)^{\frac{1}{2}-H} f_H^*(x_1, ..., x_n) \\ &= \Big(\prod_{i=1}^n x_i\Big)^{\frac{1}{2}-H} \left(I_{T-}^{H-\frac{1}{2},n} \Big(\prod_{i=1}^n u_i\Big)^{H-\frac{1}{2}} \lim_{\substack{\epsilon\to 0\\(L^2)}} \phi^{(n)}_{*.\epsilon}(u)\right)(x) \end{aligned}$$

$$= (C_H^*)^n \Big(\prod_{i=1}^n x_i\Big)^{\frac{1}{2}-H} \left(I_{T-}^{H-\frac{1}{2},n} \Big(\prod_{i=1}^n u_i\Big)^{H-\frac{1}{2}} \Big(\lim_{\substack{\epsilon\to 0 \\ (L^2)}} (C_H^*)^{-n} \phi_{*.\epsilon}^{(n)} \Big)(u) \right)(x). \quad (4.7)$$

So the existence of $\lim_{\substack{\epsilon\to 0 \\ (L^2)}} \phi_{*,\epsilon}^{(n)}(x)$ implies that for any $f \in L^2([0,T]^n)$, there exists $\phi \in L^2([0,T]^n)$ such that

$$f(x) = (C_H^*)^n \Big(\prod_{i=1}^n x_i\Big)^{\frac{1}{2}-H} \left(I_{T-}^{H-\frac{1}{2},n} \Big(\prod_{i=1}^n u_i\Big)^{H-\frac{1}{2}} \phi(u) \right)(x). \quad (4.8)$$

It follows that $\Gamma_{H,T}^{(n)(-1)} f(x) = \phi(x)$ and $f(x) = \Gamma_{H,T}^{(n)}\phi(x)$ $\forall x \in [0,T]^n$. Using the fact that

$$\widetilde{\Gamma_H^{(n)}\phi(x)} = \Gamma_H^{(n)}\widetilde{\phi(x)}, \quad (4.9)$$

we obtain that

$$\mathcal{J}_n(f) = \mathcal{J}_n(\Gamma_{H,T}^{(n)}\phi) = \mathcal{J}_n(\widetilde{\Gamma_{H,T}^{(n)}\phi}) = \mathcal{J}_n(\Gamma_{H,T}^{(n)}\widetilde{\phi}) = \mathcal{I}_n(\widetilde{\phi}) = \mathcal{I}_n(\widetilde{\Gamma_{H,T}^{(n)(-1)}f}), \quad (4.10)$$

where the third equality follows from (4.9) and the fourth follows from direct application of (4.2). Since $\phi \in L^2([0,T]^n)$, by theorem 2.1, $\phi \in L_H^2([0,T]^n)$ so that $\mathcal{I}_n(\widetilde{\phi})$ and $\mathcal{J}_n(\widetilde{\Gamma_{H,T}^{(n)}\phi})$ are well defined. □

5. An Example in the Context of Nonlinear Filtering

As an application of Theorem 4.4, we look at the nonlinear filtering problem described in [9], [3] and [4]. Consider the following nonlinear filtering problem for a random process $X = (X_t, t \in [0,T])$, where $T < \infty$, and X is not observed directly. Suppose that a noisy process Y is observed instead and that the observation model, which is of interest in many engineering and economics applications, is given by:

$$Y_t = \int_0^t h(X_s)ds + B_t^H, \quad t \in [0,T] \quad (5.1)$$

where h is a suitably regular and integrable nonlinear function, $B^H = (B_t^H)$ is a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ and the signal process X is assumed to be independent of the observation noise B^H. The goal is to characterize the optimal filter, or, in other words, to study the best mean-square estimate of the signal, conditioning on $\mathcal{F}_t^Y := \sigma\{Y_s : 0 \le s \le t\}$, the σ-field generated by the observation process.

Fix a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, where the signal and observation processes are adapted to the filtration $(\mathcal{F}_t)$. Next define the following random fields:

$$W_t^Y := \int_0^t K_H^{-1}(t;s)dY_s, \quad t \in [0,T], \quad (5.2)$$

where the kernel K_H^{-1} is as defined in (2.9). It can be shown (see [4]) that under an appropriate change of measure, $(W_t^Y, t \in [0,T])$ is a standard Brownian motion.

Under certain technical assumptions on the signal process X and the function h discussed in depth in [4] and [9], we define for almost all $\omega \in \Omega$, the function $(\delta_t(X), t \in [0,T])$ by

$$\delta_t(X)(\omega) := \frac{1}{c_H^*}\big(t^{H-\frac{1}{2}}\big)\big(\mathcal{D}_{0+}^{H-\frac{1}{2}}\hat{h}_\cdot(X)(\omega)\big)(t), \quad t \in [0,T], \tag{5.3}$$

where $c_H^* = \Gamma(H+\frac{1}{2})\sqrt{\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}}$, and $\hat{h}_t(X)(\omega) := t^{\frac{1}{2}-H}h(X_t(\omega))$.

It has been shown in [9] that for any $f \in C_b$, the optimal filter of $f(X_t)$, conditioning on $\mathcal{F}_t^Y$, can be represented as a ratio of infinite sums of multiple stochastic fractional integrals of the Itô type. In particular, the following representaion of the optimal filter holds:

$$\mathbb{E}\big[f(X_t)|\mathcal{F}_t^Y\big] \overset{a.s.}{=} \frac{\sum_{p=0}^{\infty}\frac{1}{p!}I_p^{W^Y,t}\left(\mathbb{E}\left[f(X_t)(\delta_\cdot(X))^{\otimes p}\right]\right)}{\sum_{p=0}^{\infty}\frac{1}{p!}I_p^{W^Y,t}\left(\mathbb{E}\left[(\delta_\cdot(X))^{\otimes p}\right]\right)}, \tag{5.4}$$

where the series converge in L^2, $(\cdot)^{\otimes p}$ denotes the pth order tensor product and $I_p^{W^Y,t}$ denotes the pth order multiple stochastic fractional integral of the Itô type with respect to the Brownian motion W^Y, with the integral taken over $[0,t]$.

By truncating the sum in (5.4) it is possible to approximate the optimal filter by

$$\mathbb{E}\big[f(X_t)|\mathcal{F}_t^Y\big] \approx \frac{\sum_{p=0}^{N}\frac{1}{p!}I_p^{W^Y,t}\left(\mathbb{E}\left[f(X_t)(\delta_\cdot(X))^{\otimes p}\right]\right)}{\sum_{p=0}^{N}\frac{1}{p!}I_p^{W^Y,t}\left(\mathbb{E}\left[(\delta_\cdot(X))^{\otimes p}\right]\right)}. \tag{5.5}$$

From a practical point of view however, this representation is not easy to implement, since it requires that we compute the trajectory of $(W_t^Y,\ t \in [0.T])$ using the kernel K_H^{-1}, whose form changes at each point t in the parameter space $[0,T]$.

We can use the Inverse Stochastic Transfer Principle to represent the integrals in (5.5) as multiple fractional stochastic integrals with respect to observation process Y. Namely, provided

$$\mathbb{E}\left(f(X_t)(\delta_\cdot(X))^{\otimes p}\right)(\cdot) \in L^2_{\Phi,H}([0,T]^p) \quad \forall p = 1,...,N,\ \forall t \in [0,T]$$

we can approximate (5.4) by

$$\mathbb{E}\big[f(X_t)|\mathcal{F}_t^Y\big] \approx \frac{\sum_{p=0}^{N}\frac{1}{p!}I_p^{Y,t}\left(\Gamma_{H,T}^{(p)(-1)}\mathbb{E}\left[f(X_t)(\delta_\cdot(X))^{\otimes p}\right]\right)}{\sum_{p=0}^{N}\frac{1}{p!}I_p^{Y,t}\left(\Gamma_{H,T}^{(p)(-1)}\mathbb{E}\left[(\delta_\cdot(X))^{\otimes p}\right]\right)}, \tag{5.6}$$

where $I_p^{Y,t}$ denotes the pth order multiple stochastic integral of the Itô type with respect to the observation process Y, with the integral taken over $[0,t]$.

Remark 5.1. Approximation (5.6) allows for numerical implementation of the representations given by (5.4). The Inverse Stochastic Transfer Principle gives multiple integrals with respect to the observation process so we can directly use the observation process to evaluate the integrals as opposed to those in (5.4) which require continuously updating kernel K_H^{-1} to obtain the process W^Y.

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