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RISK-BASED INDIFFERENCE PRICING UNDER A STOCHASTIC VOLATILITY MODEL

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Abstract. This paper considers a risk-based approach for indifference valuation of contingent claims in the context of a continuous-time stochastic volatility model. Since the market in the model is incomplete there is more than one arbitrage-free price of an option. We adopt a risk-based approach to select a seller's and a buyer's indifference price for the option contract. A convex risk measure is used to measure risk. We formulate the valuation problems as two-person, zero-sum, stochastic differential games. Two approaches, namely, the dynamic programming principle and the maximum principle, are used to find the solutions to the games.

1. Introduction

The class of SV models is one of the major class of asset price models in finance and econometrics. It corrects some defects of GBM for modeling asset price dynamics assumed in the Black-Scholes-Merton option valuation model and incorporates some stylized facts of financial returns, such as skewness and heavy-tailedness of distribution of returns, time-varying volatility, volatility clustering and positive autocorrelation of squared returns, etc. The origin of SV models may be dated back to the early work of Clark in [5], where an information counting model for asset's returns was proposed. Tauchen and Pitts in [41] further developed the information counting model in [5] and established a link between trading volume and the price variability.

Taylor in [42], [43] first developed a practical, discrete-time, version of the stochastic volatility model. This model is also known as an autoregressive random variance process. The basic idea is to model the logarithmic volatility of a share price as a first-order autoregressive time series model. Hull and White in [29], Scott in [38] and Wiggins in [45] introduced a continuous-time version of the stochastic volatility model with a view to applying it for option valuation. Other continuous-time stochastic volatility models for option valuation were also introduced in the finance literature. Some examples include [40], [25], and others.

The main challenge of option valuation in a continuous-time SV model is that the market is in general incomplete. Consequently there is more than one equivalent martingale measure, or price kernel, and hence, more than one arbitrage-free

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price for a contingent claim. The standard Black-Scholes-Merton valuation method based on the no-arbitrage principle is not sufficient to determine an arbitrage-free price of the claim. One sequence of papers adds another option, together with the underlying instruments, to hedge the option perfectly and complete the market; see, for example, [46], [37], [27], and [6]. Another strand of the literature specifies a criterion to justify a particular choice of a martingale pricing measure. There are two popular criteria for choosing martingale pricing measures in the context of continuous-time SV models, namely, the variance-optimal martingale measures and the minimal entropy martingale measures. The variance-optimal martingale measures can be related to the quadratic utility functions. Their use was extensively investigated in [30], [4], [22], [23], among others. The minimal entropy martingale measure can be linked with the option valuation problem under an exponential utility function with constant absolute risk aversion; see, for example, [7], [36] and [26].

Hobson and Henderson in [24] considered a utility-based indifference pricing of contingent claims under stochastic volatility models. The idea of indifference pricing originates from [28] and determines a seller's, (buyer's), price so that the seller, (the buyer), is indifferent to whether the claim is sold, (bought), and, or whether it is not sold, (bought). The utility-based approach has a solid economic foundation. However, in practice, it seems not easy to elicit a utility function. This motivates the quest for an alternative objective function for indifference pricing.

Recently, some attention has been paid to the game theoretic approach for option valuation. Some works include [34], [39], [35], [1], and others. The theory of stochastic differential games is an important topic in both mathematical and economic science. Some early works on stochastic differential games include [8], [9], [10], [11], [17], [18], and others. Recent interest on the game theoretic approach for option valuation is to highlight the link between risk-based indifference pricing and stochastic differential games, (see [35] and [1]). The main focus has, so far, been on risk-based indifference pricing of contingent claims under jump-diffusion models. The risk-based indifference pricing under stochastic volatility models remains an open issue.

Jump-diffusion models and stochastic volatility models are two popular, but fundamentally different, classes of asset price models. They focus on modeling different empirical features of asset returns. The former concerns modeling sudden jumps in asset returns caused by some extra-ordinary market news, or events, while the latter aims at modeling changes in variances of returns attributed to variations in the level of activity at a market. These variations in the level of activity may be caused by either minor events or extra-ordinary events. Despite the fundamental difference between jump-diffusion models and SV models, both of them can incorporate an important empirical feature of asset returns, namely, the heavy-tailedness of return distributions. From an economic perspective, the option valuation problem under jump-diffusion models is different from that under SV models in the sense that the former concerns the impact of jump risk on option prices while the latter focuses on the impact of time-varying volatility on option prices. They may have different implications for explaining some empirical behavior of option prices, such as implied volatility smile, or smirk, and term structure of implied volatility. Mathematically, the two option valuation problems have different structures. For example, due to the presence of stochastic volatility, a partial differential equation governing option price under a SV model has one more state variable than that under a jump-diffusion model. Based on some existing valuation approaches, many interesting mathematical results have been developed for approximating option prices under SV models, (see, for example, [15]). It seems not unreasonable to expect that the risk-based indifference approach may also open up many research opportunities for developing novel mathematical results to approximate option prices under SV models.

In this paper, we consider risk-based indifference pricing of a contingent claim in the context of a continuous-time stochastic volatility model. A version of the continuous-time stochastic volatility model is considered which includes the log-normal process of Hull and White in [29], the Ornstein-Uhlenbeck process of Stein and Stein in [40], the Cox-Ingersoll-Ross volatility process of Heston in [25], and others. A general contingent claim is considered which includes share options and volatility derivatives. Here we assume that the volatility process is observable. The more complicated situation where volatility is a latent, or unobservable, process involves some filtering techniques and will not be addressed here.

We employ a risk-based approach to select a seller's, (buyer's), indifference price of the the claim in the stochastic volatility environment. More specifically, a risk-based seller's, (buyer's), indifference price is determined so that the minimal risk faced by the seller, (the buyer), is indifferent to whether the contract is sold, (bought), or whether it is not sold, (bought). A convex risk measure is used to measure risk. With the representation of a convex risk measure, the valuation problems are formulated as two-person, zero-sum, stochastic differential games. Two approaches will be considered to find the solutions of the games, namely, the HJB dynamic programming approach and the stochastic maximum principle. The solutions of the games are then used to derive the risk-based seller's and buyer's indifference prices.

The rest of this paper is organized as follows. The next section presents a general, continuous-time, stochastic volatility model. In section 3, we first give a brief discussion for the notion of convex risk measures and then present the risk-based valuation approach for determining the indifference seller's and buyer's prices of a contingent claim. In section 4, we formulate the risk-based approach for indifference prices as stochastic differential games. Section 5 gives the HJB dynamic programming solutions to the valuation games. Section 6 considers the use of the stochastic maximum principle for solving the games. In Section 7, we give the seller's and buyer's indifference prices in terms of the solutions of the three stochastic differential games for valuation. The final section summarizes the main results in this paper.

2. The Model Dynamics

Consider a continuous-time economy consisting of two primitive securities, a bond and a share. They are traded continuously over time in a finite time horizon $\mathcal{T} := [0, T]$, where $T \in (0, \infty)$. Fix a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is a real-world probability measure. The probability space is rich enough to incorporate two sources of uncertainty attributed to the two risk factors, namely, the asset price risk and the risk due to the presence of non-traded factor, or in particular volatility risk. As mentioned in the introduction, we consider the situation that volatility is observable. This assumption may not be unreasonable. Indeed, some works, (see, for example, [41]), attempt to link volatility with some observable economic variables, such as trading volume, or to use these variables as proxies or instrumental variables to explain time-varying volatility.

Let $B := \{B(t)|t \in \mathcal{T}\}$ be the price process of the bond. Suppose r is the constant continuously compounded rate of interest of the bond, where $r > 0$. Then the price of the bond evolves over time as:

$$B(t) = e^{rt} \ , \quad B(0) = 0 \ , \quad t \in \mathcal{T} \ . \tag{2.1}$$

Suppose $S := \{S(t)|t \in \mathcal{T}\}$ and $\{Y(t)|t \in \mathcal{T}\}$ denote the price process of the share and the process of an non-traded risk factor which modulates both the appreciation rate and the volatility of the share, respectively. The extensions to the cases where the rate of interest is a deterministic function of time, say $r(t)$, and where it is stochastic, say $r(t, S(t), Y(t))$, are immediate. For each $t \in \mathcal{T}$, let $\mu(t, Y(t))$ and $\sigma(t, Y(t))$ be the appreciation rate and the volatility of the share at time t; let $\alpha(t, Y(t))$ and $\beta(t, Y(t))$ be the drift and diffusion coefficients of the non-traded risk factor. Here $\mu(t, Y(t)) \in \Re$, $\sigma(t, Y(t)) > 0$, $\alpha(t, Y(t)) \in \Re$ and $\beta(t, Y(t)) > 0$. We suppose that $\mu(t, Y(t))$, $\sigma(t, Y(t))$, $\alpha(t, Y(t))$ and $\beta(t, Y(t))$ satisfy the following regularity and growth conditions:

(1) for each $t \in \mathcal{T}$ and $y_1, y_2 \in \Re$, there exists a real constant K_1 such that

$$\begin{aligned}&|\mu(t, y_1) - \mu(t, y_2)| + |\alpha(t, y_1) - \alpha(t, y_2)| + |\sigma(t, y_1) - \sigma(t, y_2)| \\ &+|\beta(t, y_1) - \beta(t, y_2)| \leq K_1|y_1 - y_2| \ ;\end{aligned}$$

(2) for each $y \in \Re$, there exists a real constant K_2 such that

$$|\mu(t, y)| + |\alpha(t, y)| + |\sigma(t, y)| + |\beta(t, y)| \leq K_2(1 + |y|) \ . \tag{2.2}$$

The first and second conditions are the uniformly Lipschitz condition and the linear growth condition in y for the coefficients, respectively.

Let $W_1 := \{W_1(t)|t \in \mathcal{T}\}$ and $W := \{W(t)|t \in \mathcal{T}\}$ be two correlated standard Brownian motions on $(\Omega, \mathcal{F}, \mathcal{P})$ with respect to their own right-continuous, $\mathcal{P}$-completed, filtrations, where $Cov(W_1(t), W(t)) = \rho t$ and ρ is the constant correlation coefficient between W^1 and W. We assume that $|\rho| < 1$. Then under $\mathcal{P}$ the price process of the share S and the process of the non-traded risk factor Y evolve over time according to the following diffusion processes:

$$\begin{aligned} dS(t) &= S(t)(\mu(t, Y(t))dt + \sigma(t, Y(t))dW_1(t)) \ , \\ S(u) &= s \ , \\ dY(t) &= \alpha(t, Y(t))dt + \beta(t, Y(t))dW(t) \ , \\ Y(u) &= y \ , \quad u, t \in \mathcal{T} \ , \quad u \leq t \ . \end{aligned} \tag{2.3}$$

Under the uniformly Lipschitz condition and the linear growth condition for the coefficients, the above diffusion processes S and Y admit unique strong solutions.

Let $W_2 := \{W_2(t)|t \in \mathcal{T}\}$ be another standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$ so that W_2 and W_1 are stochastically independent under $\mathcal{P}$. Then we can write:

$$W(t) = \rho W_1(t) + \sqrt{1-\rho^2} W_2(t) \ , \quad t \in \mathcal{T} \ .$$

Consequently, the price process S and the process of the non-traded risk factor Y can be expressed in terms of W_1 and W_2 as follows:

$$\begin{aligned} dS(t) &= S(t)(\mu(t, Y(t)) + \sigma(t, Y(t)) dW_1(t)) \ , \\ dY(t) &= \alpha(t, Y(t))dt + \beta(t, Y(t))(\rho dW_1(t) + \sqrt{1-\rho^2} dW_2(t)) \ . \end{aligned} \tag{2.4}$$

Some particular cases of the above model are presented in the following examples.

Example 2.1. (*Hull-White Stochastic Volatility Model*). Suppose

$$\begin{aligned} &\mu(t, Y(t)) = \mu \ ; \quad \sigma(t, Y(t)) = \sqrt{Y(t)} \ ; \\ &\alpha(t, Y(t)) = \alpha Y(t) \ ; \quad \beta(t, Y(t)) = \beta Y(t) \ . \end{aligned}$$

Here we interpret the non-traded risk factor Y as the variance process of the risky asset. Then this results in the following Hull-White stochastic volatility model:

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sqrt{Y(t)} S(t) dW_1(t) \ , \\ dY(t) &= \alpha Y(t)dt + \beta Y(t) dW(t) \ . \end{aligned}$$

Under this model, the variance process is a lognormal process, or a GBM, and it is always positive. However, in general, the Hull-White model cannot incorporate the mean-reverting feature of variance processes exhibited by many financial time series. The parameter β is positive and is interpreted as the volatility of volatility.

Example 2.2. (*Stein-Stein Stochastic Volatility Model*). In this case, we assume that

$$\begin{aligned} &\mu(t, Y(t)) = \mu \ ; \quad \sigma(t, Y(t)) = Y(t) \ ; \\ &\alpha(t, Y(t)) = \alpha(m - Y(t)) \ ; \quad \beta(t, Y(t)) = \beta \ . \end{aligned}$$

Now, $Y(t)$ is interpreted as the volatility, or standard deviation, process of the risky asset. Then the following Stein-Stein stochastic volatility model results:

$$\begin{aligned} dS(t) &= \mu S(t)dt + Y(t)S(t)dW_1(t) \ , \\ dY(t) &= \alpha(m - Y(t))dt + \beta dW(t) \ . \end{aligned}$$

Here the volatility process is modeled by an Ornstein-Uhlenbeck process. It incorporates the mean-reverting feature of the volatility. α and m are the speed of mean reversion and the mean-reversion level, (or the long-run mean of the volatility), respectively. The main defect of the Stein-Stein stochastic volatility is that there is a positive, (though small), probability that the variance process $Y(t)$ may become negative.

Example 2.3. (*Heston Stochastic Volatility Model*). Suppose the coefficients of the model are the same as those in the Stein-Stein stochastic volatility model, except that $\beta(t, Y(t)) = \beta\sqrt{Y(t)}$. Then the Heston stochastic volatility model is given by:

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sqrt{Y(t)}S(t)dW_1(t) , \\ dY(t) &= \alpha(m - Y(t))dt + \beta\sqrt{Y(t)}dW(t) . \end{aligned}$$

The variance process Y under the Heston model is modeled as the Cox-Ingersoll-Ross process. Again, m and α are the mean-reversion level and the speed of mean reversion of the variance process, respectively. The necessary and sufficient condition for the variance process staying positive is:

$$\alpha m > \frac{1}{2}\beta .$$

In what follows, we define an investment portfolio consisting of the bond and the share, and derive its corresponding wealth process. For each $t \in \mathcal{T}$, let $\pi(t)$ be the amount invested in the share at time t. Write $\pi := \{\pi(t)|t \in \mathcal{T}\}$ for the corresponding portfolio process. Let $V^\pi := \{V^\pi(t)|t \in \mathcal{T}\}$ be the total wealth process of the investor with endowment, or initial wealth, $v > 0$ if the investor holds the portfolio process π. We suppose that the portfolio process π is self-financing, (i.e. there is no net inflow or outflow of capital). Consequently, the amount invested in the bond at time t is $V^\pi(t) - \pi(t)$. The evolution of the wealth process V^π over time is governed by the following stochastic differential equation:

$$\begin{aligned} dV^\pi(t) &= [rV^\pi(t) + \pi(t)(\mu(t, Y(t)) - r)]dt + \sigma(t, Y(t))\pi(t)dW(t) , \\ V^\pi(u) &= v , \quad u, t \in \mathcal{T} , \quad u \le t . \end{aligned} \tag{2.5}$$

Let $F^S := \{F^S(t)|t \in \mathcal{T}\}$ and $F^Y := \{F^Y(t)|t \in \mathcal{T}\}$ be the $\mathcal{P}$-augmentation of the right-continuous, $\mathcal{P}$-completed, natural filtration generated by the histories of the share process and the non-traded factor process, respectively. For each $t \in \mathcal{T}$, let $G(t) := F^S(t) \vee F^Y(t)$, the minimal σ-field containing $F^S(t)$ and $F^Y(t)$. Write $G := \{G(t)|t \in \mathcal{T}\}$. Then a portfolio process π is said to be admissible if it satisfies the following conditions:

(1) π is F^Y-predictable;
(2) the stochastic differential equation for the wealth process has a unique strong solution;
(3)
$$\int_0^T \Big(|rV^\pi(t) + (\mu(t, Y(t)) - r)\pi(t)| + \sigma^2(t, Y(t))\pi^2(t)\Big)dt < \infty , \quad \mathcal{P}\text{-a.s.};$$
(4) π is self-financing.

We write $\mathcal{A}$ for the set of all admissible portfolio processes. The first condition means that the investor decides the amount invested in the share based only on the observable market information generated by the return process of the share. The second and third conditions are technical conditions.

3. Convex Risk Measures and Risk-Based Valuation

Value at Risk (VaR) has emerged as a popular measure of risk and has widely been adopted in the finance and insurance industries. Despite its popularity, VaR is flawed as observed by Artzner et al. [2], where an axiomatic approach for risk measures and the notion of coherent risk measures were proposed. They pointed out that VaR does not, in general, satisfy the sub-additive property, which is one of the four properties that should be satisfied by a coherent risk measure. The intuition of this property is that allocating assets over two risky positions should reduce risk. Another shortcoming of VaR is that it may lead to some bizarre and sub-optimal decisions if it is used as a binding constraint in portfolio selection, (see [44] for the discrete-time case and [3] for the continuous-time case).

The class of coherent risk measures can provide a remedy for some of the defects of VaR. However it was argued in [16] and [14] that in practice, the risk of a trading position might increase nonlinearly with the size of the position. This is attributed to the liquidity of a large trading position. To incorporate the nonlinear dependence of the risk of a trading position on the additional liquidity risk, Frittelli and Rosazza-Gianin in [16] and Föllmer and Schied in [14] relaxed the sub-additive and positive homogeneous properties of coherent risk measures and replaced these two properties by a convex property. They introduced independently the notion of convex risk measures. It is a generalization of the concept of coherent risk measures proposed by Artzner et al. in [2]. Here we employ a convex risk measure as a measure of risk in the risk-based valuation of contingent claims. In what follows, we first give the notion of a convex risk measure and then present the risk-based valuation problems for determining a seller's and a buyer's indifference prices of an option contract.

As in [16] and [14], a convex risk measure is defined as follows:

Definition 3.1. Let $\mathcal{V}$ be a class of H-measurable random variables that are bounded below, where H is a σ-algebra of Ω. A convex risk measure is a functional $\rho : \mathcal{V} \to \Re$ that satisfies the following three properties:

(1) If $L \in \mathcal{V}$ and $\beta \in \Re$, then
$$\rho(L + \beta) = \rho(L) - \beta \ ;$$
(2) for any $L_1, L_2 \in \mathcal{V}$, if $L_1(\omega) \leq L_2(\omega)$, for all $\omega \in \Omega$, then $\rho(L_1) \geq \rho(L_2)$;
(3) for any $L_1, L_2 \in \mathcal{V}$ and $\lambda \in (0, 1)$,
$$\rho(\lambda L_1 + (1 - \lambda)L_2) \leq \lambda\rho(L_1) + (1 - \lambda)\rho(L_2) \ .$$

Remark 3.2. $\mathcal{V}$ can be interpreted as the class of all possible financial positions whose values are known given the information set H. $\rho(L)$ can be interpreted as the risk capital for a position $L \in \mathcal{V}$; This interpretation can be seen easily from the first property, namely, the translation invariance property. The second property is intuitive and states that a financial position having a higher future net worth has lower risk. This property is called monotonicity. The last property is convexity. It replaces the two properties, namely, positive homogeneity and subadditivity, in the definition of a coherent risk measure. This property takes into account the liquidity risk of a large financial position.

Föllmer and Schied in [14] and Frittelli and Rosazza-Gianin in [16] gave an elegant representation for convex risk measures so as to provide a general characterization of the class of convex risk measures. We state the representation theorem without giving the proof.

Theorem 3.3. *A functional $\rho : \mathcal{V} \to \Re$ is a convex risk measure if and only if there is a family $\mathcal{M}_a$ of probability measures $\mathcal{Q}$ which are absolutely continuous with respect to $\mathcal{P}$, (i.e. $\mathcal{Q} << \mathcal{P}$), on H and a convex "penalty function" $\eta : \mathcal{V} \to \bar{\Re}$ with $\inf_{\mathcal{Q} \in \mathcal{L}} \eta(\mathcal{Q}) = \eta(\mathcal{P}) = 0$ such that*

$$\rho(L) = \sup_{\mathcal{Q} \in \mathcal{M}_a} \{E_{\mathcal{Q}}[-L] - \eta(\mathcal{Q})\} \ , \quad L \in \mathcal{V} \ . \tag{3.1}$$

Here $E_{\mathcal{Q}}$ is expectation under $\mathcal{Q}$.

Remark 3.4. The representation theorem 3.3 implies that one can generate different convex risk measures by suitable choices of $\mathcal{M}_a$ and η. With $\mathcal{P}$ being interpreted as the "true" underlying probability measure that generates the share price data, each $\mathcal{Q} \in \mathcal{M}_a$ may be interpreted as the probability measure of an approximating model. Consequently, the penalty function η may be interpreted as a function which penalizes a "wrong" choice of the model. When $\eta(\mathcal{Q}) < 0$ ($\eta(\mathcal{Q}) > 0$), the measure $\mathcal{Q}$ may give a more optimistic (pessimistic) estimate of risk than that predicted by the "true" measure $\mathcal{P}$ and this is adjusted accordingly in the representation of the convex risk measures. When $\eta(\mathcal{Q}) = 0$, there is no misspecification of the "true" data-generating probability measure. Then the representation for convex risk measures becomes that for coherent risk measures as defined in [2]. Frittelli and Rosazza-Gianin in [16] provided the definition and representation of convex risk measures in the space $\mathcal{L}^p$ of p-integrable random variables.

We now present the risk-based valuation problems for the seller's and buyer's indifference prices of a European-style option contract.

Consider a general European-style contingent claim on the share S and the non-traded factor Y whose payoff at the expiration time T is $g(S(T), Y(T))$, where $g : \Re_+ \times \Re \to \Re_+$ is called the payoff function. We first define the risk-based seller's indifference price of the claim g, denoted as P_s, by considering the following two "scenarios".

Scenario 1: Suppose the claim g is sold. The minimal risk the seller faces is given by:

$$\Phi_1(v + P_s) = \inf_{\pi \in \mathcal{A}} \rho(V^{\pi}_{v+P_s}(T) - g(S(T), Y(T))) \ .$$

Here, writing $v + P_s$ for initial wealth, $V^{\pi}_{v+P_s}(T)$ is the wealth of the seller at the expiration time T when the initial wealth is invested following the portfolio process $\pi \in \mathcal{A}$.

Scenario 2: Suppose the claim g is not sold. Then, the minimal risk the seller faces is:

$$\Phi_2(v) = \inf_{\pi \in \mathcal{A}} \rho(V^{\pi}_{v}(T)) \ .$$

The following gives a definition of the risk-based indifference price of the the claim g from the seller's, or writer's, perspective:

Definition 3.5. The *risk-based seller's indifference price* P_s of the claim g is defined as the solution P_s of the following equation:

$$\Phi_1(v + P_s) = \Phi_2(v) \ .$$

With the representation of convex risk measures in Theorem 3.3, we can formulate the determination of the seller's indifference price P_s as two, zero-sum, stochastic differential games, each of which involves two players, the seller and the market, (or the nature). The first stochastic differential game corresponds to Scenario 1, where the seller wishes to minimize the risk if the claim g is sold, and is presented as follows:

$$\Phi_1(v+P_s) = \inf_{\pi\in\mathcal{A}} \left(\sup_{\mathcal{Q}\in\mathcal{M}_a} \mathrm{E}_{\mathcal{Q}}[-V^{\pi}_{v+P_s}(T) + g(S(T), Y(T))] - \eta(\mathcal{Q}) \right) ,$$

Here the seller selects a trading strategy so as to minimize the risk of the trading position at the terminal time T. On the other hand, the market might respond antagonistically to the seller's action by selecting a real-world probability measure corresponding to the "worst-case" scenario where the risk is maximized.

The second stochastic differential game corresponds to Scenario 2, where the claim g is not sold, and is given by:

$$\Phi_2(v) = \inf_{\pi\in\mathcal{A}} \left(\sup_{\mathcal{Q}\in\mathcal{M}_a} \mathrm{E}_{\mathcal{Q}}[-V^{\pi}_{v}(T)] - \eta(\mathcal{Q}) \right) .$$

Similarly, we can define the buyer's risk-based indifference price of the European option contract, denoted by P_b, by considering the following two scenarios:
Scenario 3: Suppose the claim g is bought. The minimal risk the buyer faces is given by:

$$\Phi_3(v-P_b) = \inf_{\pi\in\mathcal{A}} \rho(V^{\pi}_{v-P_b}(T) + g(S(T), Y(T))) .$$

Here $V^{\pi}_{v-P_b}(T)$ is the terminal wealth at time T of the buyer with initial wealth $v - P_b$ when it is invested following the portfolio process $\pi \in \mathcal{A}$.
Scenario 4: Suppose the claim g is not bought. Then, the minimal risk the buyer faces is:

$$\Phi_2(v) = \inf_{\pi\in\mathcal{A}} \rho(V^{\pi}_{v}(T)) .$$

This is the exactly the same as the minimal risk the seller faces.

Again, the stochastic differential game corresponding to Scenario 3 is:

$$\Phi_3(v-P_b) = \inf_{\pi\in\mathcal{A}} \left(\sup_{\mathcal{Q}\in\mathcal{M}_a} \{\mathrm{E}_{\mathcal{Q}}[-V^{\pi}_{v-P_b}(T) - g(S(T), Y(T))] - \eta(\mathcal{Q})\} \right) .$$

The stochastic differential game corresponding to Scenario 4 is exactly the same as that corresponds to Scenario 2.

Definition 3.6. The *risk-based buyer's indifference price* P_b of the claim g is defined as the solution P_b of the following equation:

$$\Phi_3(v+P_b) = \Phi_2(v) .$$

4. Measure Changes and Price Kernels

In this section, we specify the parametric form of price kernels, or equivalent martingale measures, by a family of products of two density processes. One of these density processes is for a measure change of the Brownian motion W_1 and another one is for a measure change of the Brownian motion W_2.

For each $i = 1, 2$, let $\theta_i := \{\theta_i(t) | t \in \mathcal{T}\}$ be a real-valued, G-progressively measurable, stochastic process on $(\Omega, \mathcal{F}, \mathcal{P})$ such that

$$\int_0^T \theta_i^2(t) dt < \infty \ , \quad \mathcal{P}\text{-a.s.}$$

Consider, for each $i = 1, 2$, a G-adapted process $\Lambda^{\theta_i} := \{\Lambda^{\theta_i}(t) | t \in \mathcal{T}\}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ defined by:

$$\Lambda^{\theta_i}(t) := \exp\left(-\int_0^t \theta_i(u) dW_i(u) - \int_0^t \theta_i^2(u) du \right) \ , \quad t \in \mathcal{T} \ . \tag{4.1}$$

Suppose, for each $i = 1, 2$, θ_i satisfies the Novikov condition:

$$\mathrm{E}\left[\exp\left(\frac{1}{2} \int_0^T \theta_i(t) dt \right) \right] < \infty \ .$$

Then Λ^{θ_i} is a $(G, \mathcal{P})$-martingale.

For each $t \in \mathcal{T}$, let $\boldsymbol{\theta}(t) := (\theta_1(t), \theta_2(t))'$ and $\boldsymbol{\theta} := \{\boldsymbol{\theta}(t) | t \in \mathcal{T}\}$. Write $\boldsymbol{\Theta}$ for the space of all such processes $\boldsymbol{\theta}$.

Define, for each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, a real-valued, G-adapted, process $\Lambda^{\boldsymbol{\theta}} := \{\Lambda^{\boldsymbol{\theta}}(t) | t \in \mathcal{T}\}$ by putting:

$$\Lambda^{\boldsymbol{\theta}}(t) := \Lambda^{\theta_1}(t) \cdot \Lambda^{\theta_2}(t) \ . \tag{4.2}$$

Consequently, $\Lambda^{\boldsymbol{\theta}}$ is a $(G, \mathcal{P})$-martingale. Hence

$$\mathrm{E}[\Lambda^{\boldsymbol{\theta}}(T)] = \Lambda^{\boldsymbol{\theta}}(0) = 1 \ , \quad \boldsymbol{\theta} \in \boldsymbol{\Theta} \ . \tag{4.3}$$

For each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, we define a new probability measure $\mathcal{P}^{\boldsymbol{\theta}}$ absolutely continuous with respect to $\mathcal{P}$ on $G(T)$ by putting:

$$\left. \frac{d\mathcal{P}^{\boldsymbol{\theta}}}{d\mathcal{P}} \right|_{G(T)} := \Lambda^{\boldsymbol{\theta}}(T) \ , \quad T < \infty \ . \tag{4.4}$$

This is the Radon-Nikodym derivative of a measure change for both the standard Brownian motions W_1 and W_2.

We assume that W_1 and W_2 are stochastically independent under $\mathcal{P}^{\boldsymbol{\theta}}$, for each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. Then we have the following lemma, which follows directly from Girsanov's theorem.

Lemma 4.1. *For each $\boldsymbol{\theta} := (\theta_1, \theta_2) \in \boldsymbol{\Theta}$, the process $W^{\theta_i} := \{W^{\theta_i}(t) | t \in \mathcal{T}\}$ defined by*

$$W^{\theta_i}(t) := W_i(t) + \int_0^t \theta_i(u) du \ , \quad t \in \mathcal{T} \ , \tag{4.5}$$

is a $(G, \mathcal{P}^{\boldsymbol{\theta}})$-standard Brownian motion.

The following lemma gives the dynamics of the share price, the non-traded factor and the wealth process under $\mathcal{P}^{\boldsymbol{\theta}}$. It follows immediately from Lemma 4.1.

Lemma 4.2. *For each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, the share price process S, the process of the non-traded factor Y and the wealth process V under $\mathcal{P}^{\boldsymbol{\theta}}$ are, respectively, governed by:*

$$\begin{aligned} dS(t) &= S(t)(\mu(t,Y(t)) - \sigma(t,Y(t))\theta_1(t))dt + S(t)\sigma(t,Y(t))dW^{\theta_1}(t) \ , \\ S(u) &= s \ . \end{aligned}$$

$$\begin{aligned} dY(t) &= (\alpha(t,Y(t)) - \rho\beta(t,Y(t))\theta_1(t) - \sqrt{1-\rho^2}\beta(t,Y(t))\theta_2(t))dt \\ &\quad +\beta(t,Y(t))(\rho dW^{\theta_1}(t) + \sqrt{1-\rho^2}dW^{\theta_2}(t)) \ , \\ Y(u) &= y \ , \end{aligned}$$

and

$$\begin{aligned} dV^{\pi}(t) &= \Big(rV^{\pi}(t) + \pi(t)(\mu(t,Y(t)) - r - \sigma(t,Y(t))\theta_1(t))\Big)dt \\ &\quad +\pi(t)\sigma(t,Y(t))dW^{\theta_1}(t) \ , \\ V^{\pi}(u) &= v \ , \quad u,t \in \mathcal{T} \ , \quad u \le t \ . \end{aligned}$$

In what follows, we define a family of equivalent martingale measures, or price kernels, denoted as $\mathcal{M}_e$. We suppose that the parametric form of price kernels is specified by a density process $\Lambda^{\boldsymbol{\theta}}$, $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. Consequently, $\mathcal{M}_e$ is a subset of $\{\mathcal{P}^{\boldsymbol{\theta}} | \boldsymbol{\theta} \in \boldsymbol{\Theta}\}$.

Harrison and Kreps in [19] and Harrison and Pliska in [20], [21] established the relationship between the absence of arbitrage opportunities and the existence of an equivalent martingale measure under which discounted asset prices are martingales. This is known as the fundamental theorem for asset pricing. A version of this theorem states that the absence of arbitrage opportunities is essentially equivalent to the existence of an equivalent martingale measure. We call the latter the martingale condition.

Let $\tilde{S} := \{\tilde{S}(t) | t \in \mathcal{T}\}$ be the discounted share price process, where $\tilde{S}(t) := e^{-rt}S(t)$, for each $t \in \mathcal{T}$. Here the martingale condition is that the discounted share price is a $(G, \mathcal{P}^{\boldsymbol{\theta}})$-martingale. That is, for each $t, u \in \mathcal{T}$ with $t \ge u$,

$$\tilde{S}(u) = \mathrm{E}^{\boldsymbol{\theta}}[\tilde{S}(t) | G(u)] \ . \tag{4.6}$$

Here $\mathrm{E}^{\boldsymbol{\theta}}$ denotes expectation under $\mathcal{P}^{\boldsymbol{\theta}}$.

Using Lemma 4.2 and applying Itô's differentiation rule to $\tilde{S}(t)$, the discounted share price process $\tilde{S}$ under $\mathcal{P}^{\boldsymbol{\theta}}$ is governed by:

$$d\tilde{S}(t) = \tilde{S}(t)(\mu(t,Y(t)) - r - \sigma(t,Y(t))\theta_1(t))dt + \tilde{S}(t)\sigma(t,Y(t))dW^{\theta_1}(t) \ . \tag{4.7}$$

By the unique decomposition of special semi-martingales, this is a $(G, \mathcal{P}^{\boldsymbol{\theta}})$-(local)-martingale if and only if the terms in the integral of "dt" sum to zero. This, if and only if,

$$\mu(t,Y(t)) - r - \sigma(t,Y(t))\theta_1(t) = 0 \ , \quad t \in \mathcal{T} \ . \tag{4.8}$$

This, if and only if,

$$\theta_1(t) = \frac{\mu(t,Y(t)) - r}{\sigma(t,Y(t))} \ , \quad t \in \mathcal{T} \ .$$

Consequently, $\theta_1(t)$ can be interpreted as the market price of risk, or the Sharpe ratio, at time t.

Consider, for each $t \in \mathcal{T}$, an operator $M_t : \boldsymbol{\Theta} \to \Re$ defined by setting

$$\begin{aligned} & M_t(\boldsymbol{\theta}) \\ := \;& \mu(t, Y(t)) - r + \sigma(t, Y(t))\theta_1(t) \ , \end{aligned} \tag{4.9}$$

Here the process $\{M_t(\boldsymbol{\theta})|t \in \mathcal{T}\}$ is G-progressively measurable.

Now we define a subset $\boldsymbol{\Theta}_e$ of $\boldsymbol{\Theta}$ as:

$$\boldsymbol{\Theta}_e := \{\boldsymbol{\theta} \in \boldsymbol{\Theta} | M_t(\boldsymbol{\theta}) = 0, \forall t \in \mathcal{T}\} \ .$$

Then the set of equivalent martingale measures $\mathcal{M}_e$ is defined as follows:

$$\mathcal{M}_e := \{\mathcal{P}^{\boldsymbol{\theta}} | \boldsymbol{\theta} \in \boldsymbol{\Theta}_e\} \ . \tag{4.10}$$

The following lemma is a direct consequence of Lemma 4.2 and gives the dynamics of the share price process, the non-traded factor process and the wealth process under $\mathcal{P}^{\boldsymbol{\theta}}$, where $\boldsymbol{\theta} \in \boldsymbol{\Theta}_e$.

Lemma 4.3. *For each $\boldsymbol{\theta} \in \boldsymbol{\Theta}_e$, the share price process, the non-traded factor process and wealth process under $\mathcal{P}^{\boldsymbol{\theta}}$ are, respectively, governed by:*

$$\begin{aligned} dS(t) &= rS(t)dt + \sigma(t, Y(t))S(t)dW^{\theta_1}(t) \ , \\ S(u) &= s \ , \end{aligned}$$

$$\begin{aligned} dY(t) &= \left[\alpha(t, Y(t)) - \rho\beta(t, Y(t))\left(\frac{\mu(t, Y(t)) - r}{\sigma(t, Y(t))}\right) - \sqrt{1-\rho^2}\beta(t, Y(t))\theta_2(t)\right]dt \\ &\quad + \beta(t, Y(t))(\rho dW^{\theta_1}(t) + \sqrt{1-\rho^2}dW^{\theta_2}(t)) \ , \\ Y(u) &= y \ , \end{aligned}$$

and

$$\begin{aligned} dV^{\pi}(t) &= rV^{\pi}(t)dt + \pi(t)\sigma(t, Y(t))dW^{\theta_1}(t) \ , \\ V^{\pi}(u) &= v \ , \quad u, t \in \mathcal{T} \ , \quad u \leq t \ . \end{aligned}$$

Consequently, it is easy to see that for each $\boldsymbol{\theta} \in \boldsymbol{\Theta}_e$, the discounted share price process and wealth process are $(G, \mathcal{P}^{\boldsymbol{\theta}})$-martingales.

Note that the process θ_2 is not determined by the martingale condition, or the martingale restriction. Consequently, the space $\mathcal{M}_e$ contains infinitely many equivalent martingale measures $\mathcal{P}^{\boldsymbol{\theta}}$.

We shall determine two equivalent martingale measures, one for the seller's indifference price and another one for buyer's indifference price, according to the saddle-point equilibrium conditions of the stochastic differential games described in the last section. We call equivalent martingale measures for the seller's and buyer's indifference prices a seller's pricing measure and a buyer's pricing measure, respectively.

Consider the stochastic differential games for valuation presented in the last section. We must specify a family of probability measures $\mathcal{M}_a$ and a penalty function η for the valuation games. Here we suppose that $\mathcal{M}_a = \mathcal{M}_e$. This means that we only consider a family of equivalent martingale measures for the valuation games. This is not an unreasonable assumption from an economic point of view since only equivalent martingale measures are relevant for valuation of financial risk under the no-arbitrage principle. It remains to specify a penalty function. Indeed, we have to specify a conditional penalty function due to the dynamic nature of the valuation problems.

Let h be some convex function such that $h \in \mathcal{C}^1(\Re_+ \times \Re \times \Re)$, where $\mathcal{C}^1(\Re_+ \times \Re \times \Re)$ is the space of continuously differentiable functions on $\Re_+ \times \Re \times \Re$ and $\Re_+$ is the set of non-negative real numbers. For notational convenience, we suppress the superscript π in $V^\pi(t)$ and write $V(t)$ for $V^\pi(t)$, for each $t \in \mathcal{T}$. Then we define the conditional penalty function for a measure $\mathcal{P}^{\boldsymbol{\theta}}$, $\boldsymbol{\theta} \in \boldsymbol{\Theta}_e$, given $S(u) = s$, $Y(u) = y$ and $V(u) = v$ as follows.

$$\eta_{s,y,v}(\boldsymbol{\theta}) := \mathrm{E}^{\boldsymbol{\theta}}_{s,y,v}\left[e^{-r(T-u)} h(S(T), Y(T), V(T))\right] . \tag{4.11}$$

Here $\mathrm{E}^{\boldsymbol{\theta}}_{s,y,v}$ is the conditional expectation given $S(u) = s$, $Y(u) = y$ and $V(u) = v$ under $\mathcal{P}^{\boldsymbol{\theta}}$.

To make the conditional penalty function well-defined, we assume that

$$\mathrm{E}^{\boldsymbol{\theta}}_{s,y,v}[h(S(T), Y(T), V(T))] < \infty \ , \quad \forall \boldsymbol{\theta} \in \boldsymbol{\Theta}_e \ .$$

Now we suppose that the claim g is integrable; that is,

$$E^{\boldsymbol{\theta}}_{s,y,v}[g(S(T), Y(T))] < \infty \ , \quad \forall \boldsymbol{\theta} \in \boldsymbol{\Theta}_e \ .$$

Define, for each $(\pi, \boldsymbol{\theta}) \in \mathcal{A} \times \boldsymbol{\Theta}_e$ and each $(u, s, y, v) \in \mathcal{T} \times \Re_+ \times \Re \times \Re$, the following three objective functions:

$$\begin{aligned} & J_1^{\pi,\boldsymbol{\theta}}(u, s, y, v) \\ := \ & \mathrm{E}^{\boldsymbol{\theta}}_{s,y,v}\left[e^{-r(T-u)}\bigg(g(S(T), Y(T)) - V(T) - h(S(T), Y(T), V(T))\bigg)\right] , \end{aligned}$$

$$\begin{aligned} & J_2^{\pi,\boldsymbol{\theta}}(u, s, y, v) \\ := \ & \mathrm{E}^{\boldsymbol{\theta}}_{s,y,v}\left[e^{-r(T-u)}\bigg(- V(T) - h(S(T), Y(T), V(T))\bigg)\right] , \end{aligned}$$

and

$$\begin{aligned} & J_3^{\pi,\boldsymbol{\theta}}(u, s, y, v) \\ := \ & \mathrm{E}^{\boldsymbol{\theta}}_{s,y,v}\left[e^{-r(T-u)}\bigg(- g(S(T), Y(T)) - V(T) - h(S(T), Y(T), V(T))\bigg)\right] . \end{aligned}$$

Then the two stochastic differential games for the seller's indifference price of the claim g can be represented as:

$$\Phi_1(u, s, y, v) := \inf_{\pi \in \mathcal{A}} \left(\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}_e} J_1^{\pi,\boldsymbol{\theta}}(u, s, y, v) \right) ,$$

and

$$\Phi_2(u, s, y, v) := \inf_{\pi \in \mathcal{A}} \left(\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}_e} J_2^{\pi,\boldsymbol{\theta}}(u, s, y, v) \right) .$$

In a similar vein, the two stochastic differential games for the buyer's indifference price of the claim g can be written as:

$$\Phi_3(u,s,y,v) := \inf_{\pi\in\mathcal{A}}\left(\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_e} J_3^{\pi,\boldsymbol{\theta}}(u,s,y,v)\right) ,$$

and

$$\Phi_2(u,s,y,v) := \inf_{\pi\in\mathcal{A}}\left(\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_e} J_2^{\pi,\boldsymbol{\theta}}(u,s,y,v)\right) .$$

Consequently, to determine the seller's and buyer's indifference prices of the claim g, we must solve the three stochastic differential games with value functions Φ_1, Φ_2 and Φ_3.

5. A Dynamic Programming Approach

In this section, we adopt the HJB dynamic programming approach to find solutions to the stochastic differential games for determining the seller's and buyer's indifference prices of the claim g. The local conditions for the saddle-point solutions to the games are presented. We give the problems in a Markov framework. In other words, we require the controlled state dynamics and the control processes to be Markov. Here the controlled state dynamics are the share price process S, the non-traded factor process Y and the wealth process V. These processes are Markov with respect to the enlarged filtration G.

We shall restrict our attention to the control processes $\pi\in\mathcal{A}$ and $\boldsymbol{\theta}\in\boldsymbol{\Theta}_e$. Since the controlled state dynamics are Markov with respect to G, it is not unreasonable to assume that the control processes π and $\boldsymbol{\theta}$ are Markov with respect to G, (see, for example, [12], Chapter 16, therein). Further, under some mild conditions, Markov controls and general adapted controls have "essentially" the same performance in the classical stochastic optimal control theory, (see, for example, [32] and [33]).

Let $\mathcal{O} := (0,T)\times(0,\infty)\times(-\infty,\infty)\times(-\infty,\infty)$ so that $\mathcal{O}$ is our solvency region. Suppose K_1 be the subset of $\Re$ such that $\pi(t)\in K_1$, $\mathcal{P}$-a.s, for each $t\in\mathcal{T}$. Similarly, let K_2 be the subset of $\Re^2$ such that $\boldsymbol{\theta}(t)\in K_2$, $\mathcal{P}$-a.s., for each $t\in\mathcal{T}$. We suppose that K_1 and K_2 are neither non-empty nor singleton. Let $\bar{\pi}:\mathcal{O}\to K_1$ and $\bar{\boldsymbol{\theta}}:\mathcal{O}\to K_2$ be some given deterministic functions. Suppose, for each $(t,S(t),Y(t),V(t))\in\mathcal{O}$,

$$\pi(t)=\bar{\pi}(t,S(t),Y(t),V(t)) , \quad \boldsymbol{\theta}(t)=\bar{\boldsymbol{\theta}}(t,S(t),Y(t),V(t)) . \tag{5.1}$$

That is, π and $\boldsymbol{\theta}$ are assumed to be Markov controls.

Define Π and $\mathcal{K}$ as the following sets of Markov controls:

$$\Pi := \{\pi\in\mathcal{A}|\pi(t)=\bar{\pi}(t,S(t),Y(t),V(t)),\forall t\in(0,T)\} ,$$

and

$$\mathcal{K} := \{\boldsymbol{\theta}\in\boldsymbol{\Theta}_e|\boldsymbol{\theta}(t)=\bar{\boldsymbol{\theta}}(t,S(t),Y(t),V(t)),\forall t\in(0,T)\} .$$

With a slight abuse of notation, we do not distinguish between π and $\bar{\pi}$ and between $\boldsymbol{\theta}$ and $\bar{\boldsymbol{\theta}}$. Then we can identify the control processes $\pi(t)$ and $\boldsymbol{\theta}(t)$ with deterministic functions $\pi(t,s,y,v)$ and $\boldsymbol{\theta}(t,s,y,v)$, respectively, for each $(t,s,y,v)\in\mathcal{O}$. These are called feedback admissible controls.

Since both the controlled state dynamics and the control processes are Markov, the dynamic programming principle applies, (see, for example, [13]). In the sequel, we present a verification theorem for the HJB solution to each of the stochastic differential games for determining the seller's and buyer's indifference prices.

Let $\mathcal{D}$ be the space of functions $\phi : \mathcal{T} \times \Re^+ \times \Re \times \Re \to \Re$ such that

$$\phi \in \mathcal{C}^{1,2,2,2}(\mathcal{T} \times \Re_+ \times \Re \times \Re) \ .$$

Then for each pair $(\boldsymbol{\theta}, \pi) \in \mathcal{K} \times \Pi$, the generator of the controlled state process $(S(t), Y(t), V(t))$ is a partial differential operator $\mathcal{L}^{\boldsymbol{\theta},\pi}$ acting on $\mathcal{D}$ such that for each $(u, s, y, v) \in \mathcal{T} \times \Re_+ \times \Re \times \Re$,

$$\begin{aligned}
& \mathcal{L}^{\boldsymbol{\theta},\pi}[\phi(u, s, y, v)] \\
= \ & \frac{\partial \phi}{\partial u} + rs\frac{\partial \phi}{\partial s} + \left[\alpha(u, y) - \rho\beta(u, y)\left(\frac{\mu(u, y) - r}{\sigma(u, y)}\right) - \sqrt{1 - \rho^2}\beta(u, y)\theta_2\right]\frac{\partial \phi}{\partial y} \\
& + rv\frac{\partial \phi}{\partial v} + \frac{1}{2}\sigma^2(u, y)s^2\frac{\partial^2 \phi}{\partial s^2} + \frac{1}{2}\beta^2(u, y)\frac{\partial^2 \phi}{\partial y^2} + \frac{1}{2}\pi^2\sigma^2(u, y)\frac{\partial^2 \phi}{\partial v^2} \\
& + \sigma(u, y)s\beta(u, y)\rho\frac{\partial^2 \phi}{\partial s \partial y} + \beta(u, y)\rho\pi\sigma(u, y)\frac{\partial^2 \phi}{\partial y \partial v} + \sigma^2(u, y)s\pi\frac{\partial^2 \phi}{\partial s \partial v} \ . \qquad (5.2)
\end{aligned}$$

Lemma 5.1. *Suppose $\phi(t, s, y, v) \in \mathcal{C}^{1,2,2,2}(\mathcal{T} \times \Re^+ \times \Re \times \Re)$. Let τ be an (optional) stopping time such that $\tau < \infty$, $\mathcal{P}$-a.s. Assume further that for each $(\boldsymbol{\theta}, \pi) \in \mathcal{K} \times \Pi$, $|\phi(t, s, y, v)|$ and $|\mathcal{L}^{\boldsymbol{\theta},\pi}[\phi(t, s, y, v)]|$ are bounded on $(t, s, y, v) \in \mathcal{T} \times \Re_+ \times \Re \times \Re$. Then,*

$$\begin{aligned}
& E^{s,y,v}[\phi(\tau, S(\tau), Y(\tau), V(\tau))] \\
= \ & \phi(u, s, y, v) + E^{s,y,v}\left(\int_u^\tau \mathcal{L}^{\theta,\pi}[\phi(t, S(t), Y(t), V(t))]dt\right) \ . \qquad (5.3)
\end{aligned}$$

Proof. The result follows by applying differentiation rule to $\phi(t, S(t), Y(t), V(t))$ and conditioning on $(S(u), Y(u), V(u)) = (s, y, v)$ under $\mathcal{P}$. □

We first present a verification theorem for the HJB solutions to the three stochastic differential games for determining the seller's and buyer's indifference prices. This is a saddle-point result, (see, for example, [10]). We only state the main result without giving the proof. For the proof of this theorem, interested readers may refer to Theorem 3.2 in [31] and to consider the partial differential operator in Lemma 5.1 here.

Theorem 5.2. *Let $\bar{\mathcal{O}}$ be the closure of $\mathcal{O}$. Suppose, for each $i = 1, 2, 3$, there exists a function ϕ_i and a Markov control $(\boldsymbol{\theta}_i^*, \pi_i^*) \in \mathcal{K} \times \Pi$ such that $\phi_i \in \mathcal{C}^2(\mathcal{O}) \cap \mathcal{C}(\bar{\mathcal{O}})$ and the following conditions are satisfied:*

(1) $\mathcal{L}^{\boldsymbol{\theta},\pi_i^*(u,s,y,v)}[\phi_i(u, s, y, v)] \leq 0$, $\forall \boldsymbol{\theta} \in \mathcal{K}$ *and* $\forall (u, s, y, v) \in \mathcal{O}$*;*
(2) $\mathcal{L}^{\boldsymbol{\theta}_i^*(u,s,y,v),\pi}[\phi_i(u, s, y, v)] \geq 0$, $\forall \pi \in \Pi$ *and* $\forall (u, s, y, v) \in \mathcal{O}$*;*
(3) $\mathcal{L}^{\boldsymbol{\theta}_i^*(u,s,y,v),\pi_i^*(u,s,y,v)}[\phi_i(u, s, y, v)] = 0$, $\forall (u, s, y, v) \in \mathcal{O}$*;*

(4) *for each* $(\boldsymbol{\theta}, \pi) \in \mathcal{K} \times \Pi$,

$$\begin{aligned}
& \lim_{u \to T^-} \phi_i(u, S(u), Y(u), V(u)) \\
= \;& (g(S(T), Y(T)) - V(T) - h(S(T), Y(T), V(T))) I_{\{i=1\}} \\
& + (-V(T) - h(S(T), Y(T), V(T))) I_{\{i=2\}} \\
& + (-g(S(T), Y(T)) - V(T) - h(S(T), Y(T), V(T))) I_{\{i=3\}} \;;
\end{aligned}$$

(5) *let* $\mathcal{S}_T$ *be the set of stopping times* $\tau := \tau(\omega) \leq T$, *for all* $\omega \in \Omega$; *the family* $\{\phi(\tau, S(\tau), Y(\tau), V(\tau))\}_{\tau \in \mathcal{K}}$ *is uniformly integrable.*

Then,

$$\begin{aligned}
\phi_i(u, s, y, v) &= \Phi_i(u, s, y, v) \\
&= \inf_{\boldsymbol{\theta} \in \mathcal{K}} \left(\sup_{\pi \in \Pi} J_i^{\boldsymbol{\theta}, \pi}(u, s, y, v) \right) \\
&= \sup_{\pi \in \Pi} \left(\inf_{\boldsymbol{\theta} \in \mathcal{K}} J_i^{\boldsymbol{\theta}, \pi}(u, s, y, v) \right) \\
&= \sup_{\pi \in \Pi} J_i^{\boldsymbol{\theta}^*_i, \pi}(u, s, y, v) = \inf_{\boldsymbol{\theta} \in \mathcal{K}} J_i^{\boldsymbol{\theta}, \pi^*_i}(u, s, y, v) \\
&= J_i^{\boldsymbol{\theta}^*_i, \pi^*_i}(u, s, y, v) \;, \quad \forall (u, s, y, v) \in \mathcal{O} \;, \qquad (5.4)
\end{aligned}$$

and $(\boldsymbol{\theta}^*_i, \pi^*_i)$ *is an optimal Markov control.*

In what follows, we determine the local conditions for the seller's and buyer's choices of price kernels and optimal portfolios. From the saddle-point equilibrium condition of the stochastic differential games, (Condition 3 of Theorem 5.1),

$$\mathcal{L}^{\boldsymbol{\theta}^*_i(u,s,y,v), \pi^*_i(u,s,y,v)}[\phi_i(u, s, y, v)] = 0 \;, \quad \forall (u, s, y, v) \in \mathcal{O} \;, \quad i = 1, 2, 3 \;,$$

with the following terminal conditions:

$$\begin{aligned}
& \phi_i(T, S(T), Y(T), V(T)) \\
= \;& (g(S(T), Y(T)) - V(T) - h(S(T), Y(T), V(T))) I_{\{i=1\}} \\
& + (-V(T) - h(S(T), Y(T), V(T))) I_{\{i=2\}} \\
& + (-g(S(T), Y(T)) - V(T) - h(S(T), Y(T), V(T))) I_{\{i=3\}} \;.
\end{aligned}$$

To simplify the notation, write, for each $i = 1, 2, 3$,

$$\begin{aligned}
& \phi^i_u := \frac{\partial \phi_i}{\partial u} \;, \quad \phi^i_s := \frac{\partial \phi_i}{\partial s} \;, \quad \phi^i_y := \frac{\partial \phi_i}{\partial y} \;, \quad \phi^i_v := \frac{\partial \phi_i}{\partial v} \;, \\
& \phi^i_{ss} := \frac{\partial^2 \phi_i}{\partial s^2} \;, \quad \phi^i_{yy} := \frac{\partial^2 \phi_i}{\partial y^2} \;, \quad \phi^i_{vv} := \frac{\partial^2 \phi_i}{\partial v^2} \;, \\
& \phi^i_{sy} := \frac{\partial^2 \phi_i}{\partial s \partial y} \;, \quad \phi^i_{yv} := \frac{\partial^2 \phi_i}{\partial y \partial v} \;, \quad \phi^i_{sv} := \frac{\partial^2 \phi_i}{\partial s \partial v} \;.
\end{aligned}$$

For each $i = 1, 2, 3$, the first-order condition of the saddle-point equilibrium of the stochastic differential game having value function Φ_i with respect to $\pi_i \in \Pi$ is given by:

$$\pi_i^*(u,s,y,v) = \frac{-\beta(u,y)\rho\sigma(u,y)\phi^i_{yv}(u,s,y,v) - \sigma^2(u,y)s\phi^i_{sv}(u,s,y,v)}{\sigma^2(u,y)\phi^i_{vv}(u,s,y,v)} \ . \tag{5.5}$$

Of course, this solution exists if $\phi^i_{vv}(u,s,y,v) \neq 0$.

Recall that $\boldsymbol{\theta} \in K_2 \subset \Re^2$. Let $\gamma : \Re^2 \to \Re$ be a projection map such that for each $(y_1, y_2) \in \Re^2$, $\gamma(y_1, y_2) = y_2$. Write K_{22} for the projection of K_2 under γ; that is,

$$K_{22} := \gamma(K_2) \ .$$

Then there exists $\theta_2^-, \theta_2^+ \in \Re$ with $\theta_2^- < \theta_2^+$ such that

$$K_{22} = [\theta_2^-, \theta_2^+] \ .$$

For each $i = 1, 2, 3$, the optimality condition of the saddle-point equilibrium of the stochastic differential game having value function Φ_i with respect to $\boldsymbol{\theta}_i \in \mathcal{K}$ is given by:

$$\theta^*_{2i}(u,s,y,v) = \begin{cases} \theta_2^- & \text{if} \quad \phi^i_y(u,s,y,v) > 0 \\ \theta_2^+ & \text{if} \quad \phi^i_y(u,s,y,v) < 0 \ . \end{cases} \tag{5.6}$$

This solution is valid if $\phi^i_y(u,s,y,v) \neq 0$.

Consequently, the seller's optimal portfolio is given by $\pi_1^*(t,s,y,v)$ in (22) and the seller's price kernel is given by:

$$\begin{aligned}
\Lambda^{\boldsymbol{\theta}_1^*}(t) \quad = \quad & \exp\Bigg[-\int_0^t \left(\frac{\mu(u,Y(u)) - r}{\sigma(u,Y(u))}\right) dW_1(u) - \int_0^t \left(\frac{\mu(u,Y(u)) - r}{\sigma(u,Y(u))}\right)^2 du \\
& - \int_0^t \Big(\theta_2^- I_{\{\phi^1_y(u,s,y,v)>0\}} + \theta_2^+ I_{\{\phi^1_y(u,s,y,v)<0\}}\Big) dW_2(u) \\
& - \int_0^t \Big(\theta_2^- I_{\{\phi^1_y(u,s,y,v)>0\}} + \theta_2^+ I_{\{\phi^1_y(u,s,y,v)<0\}}\Big)^2 du \Bigg] \ .
\end{aligned}$$

Similarly, the buyer's optimal portfolio is given by $\pi_3^*(t,s,y,v)$ in (22) and the seller's price kernel is given by:

$$\begin{aligned}
\Lambda^{\boldsymbol{\theta}_3^*}(t) \quad = \quad & \exp\Bigg[-\int_0^t \left(\frac{\mu(u,Y(u)) - r}{\sigma(u,Y(u))}\right) dW_1(u) - \int_0^t \left(\frac{\mu(u,Y(u)) - r}{\sigma(u,Y(u))}\right)^2 du \\
& - \int_0^t \Big(\theta_2^- I_{\{\phi^3_y(u,s,y,v)>0\}} + \theta_2^+ I_{\{\phi^3_y(u,s,y,v)<0\}}\Big) dW_2(u) \\
& - \int_0^t \Big(\theta_2^- I_{\{\phi^3_y(u,s,y,v)>0\}} + \theta_2^+ I_{\{\phi^3_y(u,s,y,v)<0\}}\Big)^2 du \Bigg] \ .
\end{aligned}$$

6. The Maximum Principle

Sufficient maximum principles to the solutions of the games consisting of the Arrow condition based on the Hamiltonian and the backward stochastic partial differential equations of the adjoint processes are presented. Here we do not require the assumption of Markov controls and consider general progressively-measurable control processes.

Firstly, we consider the stochastic differential game with value function Φ_1. Define the Hamiltonian $H_1 : \mathcal{T} \times \Re_+ \times \Re \times \Re \times [\theta_2^-, \theta_2^+] \times K_1 \times \Re^3 \times \Re^3 \times \Re \to \Re$ for this game by:

$$
\begin{aligned}
& H_1(t, S(t), Y(t), V(t), \theta_2(t), \pi(t), \mathbf{p}^1(t), \mathbf{q}^1(t), r^1(t)) \\
= & \left[\alpha(t, Y(t)) - \rho\beta(t, Y(t))\left(\frac{\mu(t, Y(t)) - r}{\sigma(t, Y(t))}\right) - \sqrt{1-\rho^2}\beta(t, Y(t))\theta_2(t)\right]p_2^1(t) \\
& + rS(t)p_1^1(t) + rV(t)p_3^1(t) + \sigma(t, Y(t))S(t)q_1^1(t) + \beta(t, Y(t))\rho q_2^1(t) \\
& + \pi\sigma(t, Y(t))q_3^1(t) + \beta(t, Y(t))\sqrt{1-\rho^2}r^1(t) \ . \qquad (6.1)
\end{aligned}
$$

We suppose that H_1 is differentiable with respect to $S(t)$, $Y(t)$ and $V(t)$. The adjoint equations in the unknown G-adapted processes $\mathbf{p}^1 := \{\mathbf{p}^1(t) | t \in \mathcal{T}\}$, $\mathbf{q}^1 := \{\mathbf{q}^1(t) | t \in \mathcal{T}\}$ and $r^1 := \{r^1(t) | t \in \mathcal{T}\}$, where $\mathbf{p}^1(t) := (p_1^1(t), p_2^1(t), p_3^1(t))'$ and $\mathbf{q}^1(t) := (q_1^1(t), q_2^1(t), q_3^1(t))'$, satisfy the following backward stochastic partial differential equations (BSPDEs):

$$
\begin{aligned}
dp_1^1(t) &= (rp_1^1(t) + \sigma(t, Y(t))q_1^1(t))dt + q_1^1(t)dW^{\theta_1}(t) \ , \\
p_1^1(T) &= -\frac{\partial h}{\partial s}(S(T), Y(T), V(T)) + \frac{\partial g}{\partial s}(S(T), Y(T)) \ , \qquad (6.2)
\end{aligned}
$$

$$
\begin{aligned}
& dp_2^1(t) \\
= & \left[\frac{\partial \alpha}{\partial y}(t, Y(t)) - \rho\left(\frac{\mu(t, Y(t)) - r}{\sigma(t, Y(t))}\right)\left(\frac{\partial \beta}{\partial y}(t, Y(t)) - \frac{\beta(t, Y(t))}{\sigma(t, Y(t))}\frac{\partial \sigma}{\partial y}(t, Y(t))\right)\right. \\
& - \frac{\rho\beta(t, Y(t))}{\sigma(t, Y(t))}\frac{\partial \mu}{\partial y}(t, Y(t)) - \sqrt{1-\rho^2}\theta_2(t)\frac{\partial \beta}{\partial y}(t, Y(t)) + S(t)q_1^1(t)\frac{\partial \sigma}{\partial y}(t, Y(t)) \\
& \left. + \rho q_2^1(t)\frac{\partial \beta}{\partial y}(t, Y(t)) + \pi(t)q_3^1(t)\frac{\partial \sigma}{\partial y}(t, Y(t)) + \sqrt{1-\rho^2}r^1(t)\frac{\partial \beta}{\partial y}(t, Y(t))\right]dt \\
& + q_2^1(t)dW^{\theta_1}(t) + r^1(t)dW^{\theta_2}(t) \ ,
\end{aligned}
$$

$$
\begin{aligned}
& p_2^1(T) \\
= & -\frac{\partial h}{\partial y}(S(T), Y(T), V(T)) + \frac{\partial g}{\partial y}(S(T), Y(T)) \ , \qquad (6.3)
\end{aligned}
$$

and

$$
\begin{aligned}
dp_3^1(t) &= rp_3^1(t)dt + q_3^1(t)dW^{\theta_1}(t) \ , \\
p_3^1(T) &= -1 - \frac{\partial h}{\partial v}(S(T), Y(T), V(T)) \ . \qquad (6.4)
\end{aligned}
$$

Define the Hamiltonian $H_2 : \mathcal{T} \times \Re_+ \times \Re \times \Re \times [\theta_2^-, \theta_2^+] \times K_1 \times \Re^3 \times \Re^3 \times \Re \to \Re$ for the stochastic differential game with value function Φ_2 by:

$$
\begin{aligned}
& H_2(t, S(t), Y(t), V(t), \theta_2(t), \pi(t), \mathbf{p}^2(t), \mathbf{q}^2(t), r^2(t)) \\
= & \left[\alpha(t, Y(t)) - \rho\beta(t, Y(t))\left(\frac{\mu(t, Y(t)) - r}{\sigma(t, Y(t))}\right) - \sqrt{1-\rho^2}\beta(t, Y(t))\theta_2(t)\right]p_2^2(t) \\
& + rS(t)p_1^2(t) + rV(t)p_3^2(t) + \sigma(t, Y(t))S(t)q_1^2(t) + \beta(t, Y(t))\rho q_2^2(t) \\
& + \pi\sigma(t, Y(t))q_3^2(t) + \beta(t, Y(t))\sqrt{1-\rho^2}r^2(t) \ . \qquad (6.5)
\end{aligned}
$$

Again, we suppose that H_2 is differentiable with respect to $S(t)$, $Y(t)$ and $V(t)$. The adjoint equations in the unknown G-adapted processes $\mathbf{p}^2 := \{\mathbf{p}^2(t)|t \in \mathcal{T}\}$, $\mathbf{q}^2 := \{\mathbf{q}^2(t)|t \in \mathcal{T}\}$ and $r^2 := \{r^2(t)|t \in \mathcal{T}\}$, where $\mathbf{p}^2(t) := (p_1^2(t), p_2^2(t), p_3^2(t))'$, $\mathbf{q}^2(t) := (q_1^2(t), q_2^2(t), q_3^2(t))'$, satisfy the same BSPDEs for $\mathbf{p}^1$, $\mathbf{q}^1$ and r^1, except that the terminal conditions for $\mathbf{p}^2$ are:

$$\begin{aligned} p_1^2(T) &= -\frac{\partial h}{\partial s}(S(T), Y(T), V(T)) \ , \\ p_2^2(T) &= -\frac{\partial h}{\partial y}(S(T), Y(T), V(T)) \ , \\ p_3^2(T) &= -1 - \frac{\partial h}{\partial v}(S(T), Y(T), V(T)) \ . \end{aligned} \tag{6.6}$$

The Hamiltonian $H_3(t, S(t), Y(T), V(t), \theta_2(t), \pi(t), \mathbf{p}^3(t), \mathbf{q}^3(t), r^3(t))$ for the game with value function Φ_3 can be defined in the same vein with the following terminal conditions for the adjoint processes:

$$\begin{aligned} p_1^3(T) &= -\frac{\partial h}{\partial s}(S(T), Y(T), V(T)) - \frac{\partial g}{\partial s}(S(T), Y(T)) \ , \\ p_2^3(T) &= -\frac{\partial h}{\partial y}(S(T), Y(T), V(T)) - \frac{\partial g}{\partial y}(S(T), Y(T)) \ , \\ p_3^3(T) &= -1 - \frac{\partial h}{\partial v}(S(T), Y(T), V(T)) \ . \end{aligned} \tag{6.7}$$

The following theorem gives the sufficient maximum principles for the three stochastic differential games.

Theorem 6.1. *For each $k = 1, 2, 3$, let $(\boldsymbol{\theta}_k^*, \pi_k^*) \in \boldsymbol{\Theta}_e \times \mathcal{A}$. Suppose $(\mathbf{p}^k, \mathbf{q}^k, r^k)$ satisfy the adjoint equations with corresponding terminal conditions. Further, for each $t \in \mathcal{T}$, the following maximum principle holds:*

$$\begin{aligned} &\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}_e} H_k(t, S(t), Y(t), V(t), \theta_2(t), \pi_k^*(t), \mathbf{p}^k(t), \mathbf{q}^k(t), r^k(t)) \\ = \ &H_k(t, S(t), Y(t), V(t), \theta_{k2}^*(t), \pi_k^*(t), \mathbf{p}^k(t), \mathbf{q}^k(t), r^k(t)) \\ = \ &\sup_{\pi \in \mathcal{A}} H_k(t, S(t), Y(t), V(t), \theta_{k2}^*(t), \pi(t), \mathbf{p}^k(t), \mathbf{q}^k(t), r^k(t)) \ . \end{aligned} \tag{6.8}$$

Suppose, for each $k = 1, 2, 3$, $\boldsymbol{\theta} \to J_k^{\pi, \boldsymbol{\theta}}(t, s, y, v)$ is concave and $\pi \to J_k^{\pi, \boldsymbol{\theta}}(t, s, y, v)$ is convex. Then $(\boldsymbol{\theta}_k^*, \pi_k^*)$ is an optimal control and

$$\begin{aligned} \Phi_k(t, s, y, v) &= \inf_{\pi \in \mathcal{A}} \left(\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}_e} J_k^{\pi, \boldsymbol{\theta}}(t, s, y, v) \right) \\ &= \sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}_e} \left(\inf_{\pi \in \mathcal{A}} J_k^{\pi, \boldsymbol{\theta}}(t, s, y, v) \right) \\ &= \sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}_e} J_k^{\pi_k^*, \boldsymbol{\theta}}(t, s, y, v) \ , \\ &= \inf_{\pi \in \mathcal{A}} J^{\pi, \boldsymbol{\theta}_k^*}(t, s, y, v) \ , \\ &= J_k^{\pi_k^*, \boldsymbol{\theta}_k^*}(t, s, y, v) \ . \end{aligned} \tag{6.9}$$

The proof of Theorem is adapted to that of Theorem 1 in An et al. (2008). So we do not repeat it here.

7. The Seller's and Buyer's Indifference Prices

We now use the solutions $(\boldsymbol{\theta}^*_k, \pi^*_k, \Phi_k(t,s,y,v))$, $k = 1,2,3$, of the three stochastic differential games to determine the seller's and buyers' indifference prices. Here we demonstrate the method of finding the seller's and buyer's indifference prices by considering the case that these solutions are obtained from the maximum principle. The case that these solutions are given by the HJB dynamic programming principle can be treated by the same method. The following theorem gives the risk-based seller's indifference price.

Theorem 7.1. *The risk-based seller's indifference price is given by:*

$$P_s = P_s(t,s,y,v) = \Phi_1(t,s,y,v) - \Phi_2(t,s,y,v) \ . \tag{7.1}$$

Proof. First, by the martingale property of the discounted wealth process under $\mathcal{P}^{\boldsymbol{\theta}}$, $\boldsymbol{\theta} \in \mathcal{K}$, we get

$$J_1^{\pi,\boldsymbol{\theta}}(t,s,y,v) = \mathrm{E}\Big[e^{-r(T-t)}(g(S(T),Y(T)) - h(S(T),Y(T),V(T)))\Big] - v \ .$$

Consequently,

$$\begin{aligned}
& \Phi_1(t,s,y,v+P_s) \\
= \ & \inf_{\pi\in\mathcal{A}} \left(\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_e} J_1^{\pi,\boldsymbol{\theta}}(t,s,y,v+P_s) \right) \\
= \ & \inf_{\pi\in\mathcal{A}} \left\{ \sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_e} \mathrm{E}\Big[e^{-r(T-t)}(g(S(T),Y(T)) - h(S(T),Y(T),V(T)))\Big] - v - P_s \right\} \\
= \ & \inf_{\pi\in\mathcal{A}} \left(\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_e} J_1^{\pi,\boldsymbol{\theta}}(t,s,y,v) \right) - P_s \\
= \ & \Phi_1(t,s,y,v) - P_s \ .
\end{aligned}$$

Therefore the result follows by noting that

$$\Phi_1(t,s,y,v+P_s) = \Phi_2(t,s,y,v) \ .$$

□

Similarly, the following theorem gives the buyer's indifference price.

Theorem 7.2. *The risk-based buyer's indifference price is:*

$$P_b := P_b(t,s,y,v) = \Phi_3(t,s,y,v) - \Phi_2(t,s,y,v) \ . \tag{7.2}$$

Note that, in general, the risk-based seller's, (buyer's), indifference price depends on the current share price, the current level of non-traded factor and the current level of the seller's, (buyer's), wealth. In other words, the seller, (the buyer), determines a price they are willing to sell, (buy), the claim g by not only looking at the current share price and non-traded factor levels, but also their current financial situation.

8. Summary

We discussed the risk-based approach for determining indifference prices of a general contingent claim in a continuous-time stochastic volatility model. A risk-based valuation approach based on the notion of convex risk measures was used to determine the seller's and buyer's indifference prices of an option. With convex risk measures as measures of risk, the risk-based valuation problems were formulated as two-person, zero-sum, stochastic differential games with a market participant and the market as players. Two approaches, namely, the HJB dynamic programming approach and the stochastic maximum principle, were adopted to find solutions to the games.

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