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Susanne C. Brenner
Louisiana State University

Joscha Gedicke
Universität Heidelberg

Li Yeng Sung
Louisiana State University

Yi Zhang
The University of Tennessee, Knoxville

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AN *A POSTERIORI* ANALYSIS OF C^0 INTERIOR PENALTY METHODS FOR THE OBSTACLE PROBLEM OF CLAMPED KIRCHHOFF PLATES

SUSANNE C. BRENNER, JOSCHA GEDICKE, LI-YENG SUNG, AND YI ZHANG

Abstract. We develop an *a posteriori* analysis of C^0 interior penalty methods for the displacement obstacle problem of clamped Kirchhoff plates. We show that a residual based error estimator originally designed for C^0 interior penalty methods for the boundary value problem of clamped Kirchhoff plates can also be used for the obstacle problem. We obtain reliability and efficiency estimates for the error estimator and introduce an adaptive algorithm based on this error estimator. Numerical results indicate that the performance of the adaptive algorithm is optimal for both quadratic and cubic C^0 interior penalty methods.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain, $f \in L_2(\Omega)$, $\psi \in C(\bar{\Omega}) \cap C^2(\Omega)$ and $\psi < 0$ on $\partial\Omega$. The displacement obstacle problem for the clamped Kirchhoff plate is to find

$$u = \underset{v \in K}{\operatorname{argmin}} \left[\frac{1}{2} a(v,v) - (f,v) \right] \tag{1.1}$$

where

$$a(w,v) = \int_\Omega D^2 w : D^2 v \, dx = \int_\Omega \sum_{i,j=1}^{2} \Big(\frac{\partial^2 w}{\partial x_i \partial x_j} \Big) \Big(\frac{\partial^2 v}{\partial x_i \partial x_j} \Big) dx, \quad (f,v) = \int_\Omega f v \, dx \tag{1.2}$$

and

$$K = \{ v \in H_0^2(\Omega) : \, v \geq \psi \quad \text{in} \quad \Omega \}. \tag{1.3}$$

The unique solution $u \in K$ of (1.1)–(1.3) is characterized by the variational inequality

$$a(u, v-u) \geq (f, v-u) \qquad \forall \, v \in K,$$

which can be written in the following equivalent complementarity form:

$$\int_\Omega (u - \psi) \, d\lambda = 0, \tag{1.4}$$

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where the Lagrange multiplier λ is the nonnegative Borel measure defined by

$$a(u,v) = (f,v) + \int_\Omega v\,d\lambda \qquad \forall\, v \in H_0^2(\Omega). \tag{1.5}$$

Remark 1.1. Since $u > \psi$ near $\partial\Omega$, the support of λ is disjoint from $\partial\Omega$ because of (1.4).

Remark 1.2. We can treat λ as a member of $H^{-2}(\Omega) = [H_0^2(\Omega)]'$ such that

$$\langle \lambda, v\rangle = \int_\Omega v\,d\lambda \qquad \forall\, v \in H_0^2(\Omega).$$

C^0 interior penalty methods [23, 13, 11, 9, 8, 26] form a natural hierarchy of discontinuous Galerkin methods that are proven to be effective for fourth order elliptic boundary value problems. The goal of this paper is to develop an *a posteriori* error analysis of C^0 interior penalty methods for the obstacle problem defined by (1.1)–(1.3). While there is a substantial literature on the *a posteriori* error analysis of finite element methods for second order obstacle problems (cf. [30, 19, 37, 34, 2, 35, 36, 7, 6, 27, 28, 18] and the references therein), as far as we know this is the first paper on the *a posteriori* error analysis for the displacement obstacle problem of Kirchhoff plates. We note that there is a fundamental difference between second order and fourth order obstacle problems, namely that the Lagrange multipliers for the fourth order discrete obstacle problems can be represented naturally as sums of Dirac point measures (cf. Section 2), which leads to a simpler *a posteriori* error analysis (cf. Section 4 and Section 5).

The rest of the paper is organized as follows. We recall the C^0 interior penalty methods in Section 2 and analyze a mesh-dependent boundary value problem in Section 3 that plays an important role in the *a posteriori* error analysis carried out in Section 4 and Section 5. An adaptive algorithm motivated by the *a posteriori* error analysis is introduced in Section 6 and we report results of several numerical experiments in Section 7. We end the paper with some concluding remarks in Section 8.

2. C^0 Interior Penalty Methods

Let $\mathcal{T}_h$ be a triangulation of Ω, $\mathcal{V}_h$ be the set of the vertices of $\mathcal{T}_h$, $\mathcal{E}_h$ be the set of the edges of $\mathcal{T}_h$, and $V_h \subset H_0^1(\Omega)$ be the P_k Lagrange finite element space ($k \geq 2$) associated with $\mathcal{T}_h$. The discrete problem for the C^0 interior penalty method [14, 15] is to find

$$u_h = \operatorname*{argmin}_{v\in K_h} \Big[\frac{1}{2}a_h(v,v) - (f,v)\Big], \tag{2.1}$$

where $K_h = \{v \in V_h : v(p) \geq \psi(p) \quad \text{for all } p \in \mathcal{V}_h\}$,

$$a_h(w,v) = \sum_{T\in\mathcal{T}_h}\int_T D^2w : D^2v\,dx + \sum_{e\in\mathcal{E}_h}\int_e \Big(\Big\{\!\!\Big\{\frac{\partial^2 w}{\partial n^2}\Big\}\!\!\Big\}\Big[\!\!\Big[\frac{\partial v}{\partial n}\Big]\!\!\Big] + \Big\{\!\!\Big\{\frac{\partial^2 v}{\partial n^2}\Big\}\!\!\Big\}\Big[\!\!\Big[\frac{\partial w}{\partial n}\Big]\!\!\Big]\Big)ds$$
$$+ \sum_{e\in\mathcal{E}_h}\frac{\sigma}{|e|}\int_e \Big[\!\!\Big[\frac{\partial w}{\partial n}\Big]\!\!\Big]\Big[\!\!\Big[\frac{\partial v}{\partial n}\Big]\!\!\Big]\,ds,$$

$\{\!\{\cdot\}\!\}$ denotes the average across an edge, $[\![\cdot]\!]$ denotes the jump across an edge, $|e|$ is the length of the edge e, and $\sigma \geq 1$ is a penalty parameter large enough so that $a_h(\cdot,\cdot)$ is positive-definite on V_h. Details for the notation and the choice of σ can be found in [13, 31].

The unique solution $u_h \in K_h$ of (2.1) is characterized by the variational inequality

$$a_h(u_h, v - u_h) \geq (f, v - u_h) \qquad \forall\, v \in K_h,$$

which can be expressed in the following equivalent complementarity form:

$$\sum_{p \in \mathcal{V}_h} \lambda_h(p)\big(u_h(p) - \psi(p)\big) = 0, \tag{2.2}$$

where the Lagrange multipliers $\lambda_h(p)$ are defined by

$$a_h(u_h, v) = (f, v) + \sum_{p \in \mathcal{V}_h} \lambda_h(p) v(p) \qquad \forall\, v \in V_h \tag{2.3}$$

and satisfy

$$\lambda_h(p) \geq 0 \qquad \forall\, p \in \mathcal{V}_h. \tag{2.4}$$

We also use λ_h to denote the measure $\sum_{p \in \mathcal{V}_h} \lambda_h(p)\delta_p$, where δ_p is the Dirac point measure at p. The equation (2.3) can therefore be written as

$$a_h(u_h, v) = (f, v) + \int_\Omega v\, d\lambda_h \qquad \forall\, v \in V_h. \tag{2.5}$$

Remark 2.1. For second order obstacle problems, the discrete Lagrange multiplier cannot be extended to $H^{-1}(\Omega)$ as a sum of Dirac point measures since such measures do not belong to $H^{-1}(\Omega)$. Consequently there are different choices for extending the discrete Lagrange multiplier to $H^{-1}(\Omega)$ [37, 34, 35]. The fact that the Lagrange multiplier for the discrete fourth order obstacle problem can be expressed naturally as a sum of Dirac point measures leads to the simple *a posteriori* error analysis in Section 4 and Section 5.

Remark 2.2. We can also treat λ_h as a member of $H^{-2}(\Omega) = [H_0^2(\Omega)]'$ such that

$$\langle \lambda_h, v \rangle = \int_\Omega v\, d\lambda_h = \sum_{p \in \mathcal{V}_p} \lambda_h(p) v(p) \qquad \forall\, v \in H_0^2(\Omega).$$

Let the mesh-dependent norm $\|\cdot\|_h$ be defined by

$$\|v\|_h^2 = \sum_{T \in \mathcal{T}_h} |v|_{H^2(T)}^2 + \sum_{e \in \mathcal{E}_h} \frac{\sigma}{|e|} \|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2. \tag{2.6}$$

Note that

$$\|v\|_h = |v|_{H^2(\Omega)} \qquad \forall\, v \in H_0^2(\Omega). \tag{2.7}$$

The following *a priori* error estimate is known [15, 14]:

$$\|u - u_h\|_h \leq C h^\alpha, \tag{2.8}$$

where the index of elliptic regularity $\alpha \in (\frac{1}{2}, 1]$ is determined by the interior angles of Ω and can be taken to be 1 if Ω is convex.

Our goal is to develop *a posteriori* error estimates for $\|u - u_h\|_h$.

Two useful tools for the analysis of C^0 interior penalty methods are the nodal interpolation operator $\Pi_h : H_0^2(\Omega) \longrightarrow V_h$ and an enriching operator $E_h : V_h \longrightarrow W_h \subset H_0^2(\Omega)$, where W_h is the Hsieh-Clough-Tocher macro finite element space [20].

The operator E_h is defined by averaging (cf. [8, Section 4.1]) and hence

$$(E_h u_h)(p) = u_h(p) \quad \text{for all } p \in \mathcal{V}_h. \tag{2.9}$$

The following estimate can be found in the proof of [8, Lemma 1].

$$h_T^{-4}\|v - E_h v\|_{L_2(T)}^2 \le C \sum_{e \in \tilde{\mathcal{E}}_T} \frac{1}{|e|}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2 \qquad \forall\, T \in \mathcal{T}_h, \tag{2.10}$$

where $\tilde{\mathcal{E}}_T$ is the set of the edges of $\mathcal{T}_h$ emanating from the vertices of T, and the positive constant C depends only on k and the shape regularity of $\mathcal{T}_h$.

From (2.10) and standard inverse estimates [21, 12], we also have

$$h_T^{-2}\|v - E_h v\|_{L_\infty(T)}^2 \le C \sum_{e \in \tilde{\mathcal{E}}_T} \frac{1}{|e|}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2 \qquad \forall\, T \in \mathcal{T}_h, \tag{2.11}$$

$$\sum_{T \in \mathcal{T}_h} |v - E_h v|_{H^2(T)}^2 \le C \sum_{e \in \mathcal{E}_h} \frac{1}{|e|}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2 \qquad \forall\, v \in V_h, \tag{2.12}$$

$$\|v - E_h v\|_h^2 \le C \sum_{e \in \mathcal{E}_h} \frac{\sigma}{|e|}\|[\![\partial v/\partial n]\!]\|_{L_2(e)}^2 \qquad \forall\, v \in V_h, \tag{2.13}$$

where the positive constant C depends only on k and the shape regularity of $\mathcal{T}_h$.

3. A Mesh-Dependent Boundary Value Problem

Let $z_h \in H_0^2(\Omega)$ be defined by

$$a(z_h, v) = (f, v) + \int_\Omega v\, d\lambda_h = (f, v) + \sum_{p \in \mathcal{V}_h} \lambda_h(p) v(p) \qquad \forall\, v \in H_0^2(\Omega). \tag{3.1}$$

Then u_h is the approximate solution of (3.1) obtained by the C^0 interior penalty method.

Remark 3.1. The idea of considering such mesh-dependent boundary value problems was introduced in [5] for second order obstacle problems.

A residual based error estimator [10, 8] for u_h (as an approximate solution of (3.1)) is given by

$$\eta_h = \Big(\sum_{e \in \mathcal{E}_h} \eta_{e,1}^2 + \sum_{e \in \mathcal{E}_h^i} (\eta_{e,2}^2 + \eta_{e,3}^2) + \sum_{T \in \mathcal{T}_h} \eta_T^2\Big)^{\frac{1}{2}}, \tag{3.2}$$

where $\mathcal{E}_h^i$ is the set of the edges of $\mathcal{T}_h$ interior to Ω,

$$\eta_{e,1} = \frac{\sigma}{|e|^{\frac{1}{2}}} \|[\![\partial u_h/\partial n]\!]\|_{L_2(e)}, \tag{3.3}$$

$$\eta_{e,2} = |e|^{\frac{1}{2}} \|[\![\partial^2 u_h/\partial n^2]\!]\|_{L_2(e)}, \tag{3.4}$$

$$\eta_{e,3} = |e|^{\frac{3}{2}} \|[\![\partial^3 u_h/\partial n^3]\!]\|_{L_2(e)}, \tag{3.5}$$

$$\eta_T = h_T^2 \|f - \Delta^2 u_h\|_{L_2(T)}. \tag{3.6}$$

The following result will play an important role in the *a posteriori* error analysis of the obstacle problem. Note that its proof is made simple by the representation of the discrete Lagrange multiplier λ_h as a sum of Dirac point measures supported at the vertices of $\mathcal{T}_h$, which allows the analysis in [8] to be used here.

Lemma 3.2. *There exists a positive constant C, depending only on k and the shape regularity of $\mathcal{T}_h$, such that*

$$\|z_h - u_h\|_h \le C\eta_h. \tag{3.7}$$

Proof. We have an obvious estimate

$$\sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|} \left\| \left[\!\!\left[\frac{\partial(z_h - u_h)}{\partial n} \right]\!\!\right] \right\|_{L_2(e)}^2 = \sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|} \left\| \left[\!\!\left[\frac{\partial u_h}{\partial n} \right]\!\!\right] \right\|_{L_2(e)}^2 \le \sum_{e\in\mathcal{E}_h} \eta_{e,1}^2, \tag{3.8}$$

and it only remains to estimate $\sum_{T\in\mathcal{T}_h} |z_h - u_h|_{H^2(T)}^2$.

Let $E_h : V_h \longrightarrow H_0^2(\Omega)$ be the enriching operator. It follows from (2.12) and (3.3) that

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} |z_h - u_h|_{H^2(T)}^2 &\le 2 \sum_{T\in\mathcal{T}_h} \left[|z_h - E_h u_h|_{H^2(T)}^2 + |u_h - E_h u_h|_{H^2(T)}^2 \right] \\ &\le 2|z_h - E_h u_h|_{H^2(\Omega)}^2 + C \sum_{e\in\mathcal{E}_h} \eta_{e,1}^2,\end{aligned} \tag{3.9}$$

and, by duality,

$$|z_h - E_h u_h|_{H^2(\Omega)} = \sup_{\phi\in H_0^2(\Omega)\setminus\{0\}} \frac{a(z_h - E_h u_h, \phi)}{|\phi|_{H^2(\Omega)}}. \tag{3.10}$$

In view of (2.3) and (3.1), the numerator on the right-hand side of (3.10) becomes

$$\begin{aligned} a(z_h - E_h u_h, \phi) &= \sum_{T\in\mathcal{T}_h} \int_T D^2(z_h - E_h u_h) : D^2\phi \, dx \\ &= (f, \phi) + \sum_{p\in\mathcal{V}_h} \lambda_h(p)\phi(p) \\ &\qquad + \sum_{T\in\mathcal{T}_h} \int_T D^2(u_h - E_h u_h) : D^2\phi \, dx - \sum_{T\in\mathcal{T}_h} \int_T D^2 u_h : D^2(\phi - \Pi_h\phi) \, dx \end{aligned}$$

$$
\begin{aligned}
&\quad - \sum_{T\in\mathcal{T}_h}\int_T D^2u_h : D^2(\Pi_h\phi)\,dx\\
&= (f,\phi) + \sum_{p\in\mathcal{V}_h}\lambda_h(p)\phi(p) - (f,\Pi_h\phi) - \sum_{p\in\mathcal{V}_h}\lambda_h(p)(\Pi_h\phi)(p) + a_h(u_h,\Pi_h\phi)\\
&\quad + \sum_{T\in\mathcal{T}_h}\int_T D^2(u_h - E_hu_h) : D^2\phi\,dx - \sum_{T\in\mathcal{T}_h}\int_T D^2u_h : D^2(\phi-\Pi_h\phi)\,dx\\
&\quad - \sum_{T\in\mathcal{T}_h}\int_T D^2u_h : D^2(\Pi_h\phi)\,dx.
\end{aligned}
$$

Since ϕ and $\Pi_h\phi$ agree on the vertices of $\mathcal{T}_h$, the two terms involving λ_h cancel each other and we end up with

$$
\begin{aligned}
a(z_h - E_hu_h,\phi) &= \sum_{T\in\mathcal{T}_h}\int_T D^2(u_h - E_hu_h) : D^2\phi\,dx - \sum_{T\in\mathcal{T}_h}\int_T D^2u_h : D^2(\phi-\Pi_h\phi)\,dx\\
&\quad - \sum_{T\in\mathcal{T}_h}\int_T D^2u_h : D^2(\Pi_h\phi)\,dx + a_h(u_h,\Pi_h\phi) + (f,\phi-\Pi_h\phi),
\end{aligned}
$$

which is precisely the equation [8, (7.9)] (and which has nothing to do with either z_h or λ_h).

It then follows from the estimates [8, (7.10) – (7.19)] that

$$
a(u - E_hu_h,\phi) \le C\eta_h|\phi|_{H^2(\Omega)}. \tag{3.11}
$$

The estimate (3.7) follows from (2.6) and (3.8)–(3.11). □

4. Reliability Estimates for the Obstacle Problem

We begin with a simple estimate.

Lemma 4.1. *There exists a positive constant C, depending only on k and the shape regularity of $\mathcal{T}_h$, such that*

$$
\|u-u_h\|_h + \|\lambda-\lambda_h\|_{H^{-2}(\Omega)} \le C\eta_h + \sqrt{\int_\Omega (\psi - E_hu_h)^+ d\lambda}\,. \tag{4.1}
$$

Proof. Let $E_h : V_h \longrightarrow H_0^2(\Omega)$ be the enriching operator. We can write

$$
\begin{aligned}
|u-E_hu_h|^2_{H^2(\Omega)} &= a(u-E_hu_h, u-E_hu_h)\\
&= a(u-z_h, u-E_hu_h) + a(z_h - E_hu_h, u-E_hu_h),
\end{aligned} \tag{4.2}
$$

and, in view of (2.7), (2.13), (3.3) and Lemma 3.2, the second term on the right-hand side of (4.2) is bounded by

$$
\begin{aligned}
a(z_h - E_hu, u-E_hu_h) &\le |z_h - E_hu_h|_{H^2(\Omega)}|u-E_hu_h|_{H^2(\Omega)}\\
&\le \big(\|z_h-u_h\|_h + \|u_h - E_hu_h\|_h\big)|u-E_hu_h|_{H^2(\Omega)}\\
&\le C\eta_h|u-E_hu_h|_{H^2(\Omega)}.
\end{aligned} \tag{4.3}
$$

By (1.3)–(1.5), (2.2), (2.4), (2.9) and (3.1), the first term on the right-hand side of (4.2) can be bounded as follows:

$$
\begin{aligned}
(4.4)\qquad a(u-z_h, u-E_hu_h) &= \int_\Omega (u-E_hu_h)\,d\lambda - \sum_{p\in\mathcal{V}_h} \lambda_h(p)\big(u(p)-(E_hu_h)(p)\big)\\
&= \int_\Omega (\psi-E_hu_h)\,d\lambda - \sum_{p\in\mathcal{V}_h} \lambda_h(p)\big(u(p)-\psi(p)\big) \le \int_\Omega (\psi-E_hu_h)^+\,d\lambda.
\end{aligned}
$$

It follows from (2.7) and (4.2)–(4.4) that

$$
\|u-E_hu_h\|_h \le C\eta_h + \sqrt{\int_\Omega (\psi-E_hu_h)^+ d\lambda}\;,
$$

which together with (2.13) implies

$$
\|u-u_h\|_h \le C\eta_h + \sqrt{\int_\Omega (\psi-E_hu_h)^+ d\lambda}\;. \tag{4.5}
$$

In order to estimate $\|\lambda-\lambda_h\|_{H^{-2}(\Omega)}$, we observe that (1.5), (2.7) and (3.1) imply

$$
\begin{aligned}
(4.6)\qquad \|\lambda-\lambda_h\|_{H^{-2}(\Omega)} &= \sup_{v\in H_0^2(\Omega)} \frac{\displaystyle\int_\Omega v\,d(\lambda-\lambda_h)}{|v|_{H^2(\Omega)}}\\
&= \sup_{v\in H_0^2(\Omega)} \frac{a(u-z_h,v)}{|v|_{H^2(\Omega)}} = |u-z_h|_{H^2(\Omega)} \;\le \|u-u_h\|_h + \|z_h-u_h\|_h.
\end{aligned}
$$

The estimate for $\|\lambda-\lambda_h\|_{H^{-2}(\Omega)}$ then follows from Lemma 3.2 and (4.5). □

We can also remove the inconvenient E_h in the estimate (4.1).

Theorem 4.2. *There exists a positive constant C, depending only on k and the shape regularity of $\mathcal{T}_h$, such that*

$$
\begin{aligned}
(4.7)\qquad \|u-u_h\|_h + \|\lambda-\lambda_h\|_{H^{-2}(\Omega)} &\le C\Big(\eta_h + |\lambda|^{\frac12}\sqrt{\max_{T\in\mathcal{T}_h} h_T \sum_{e\in\tilde{\mathcal{E}}_T} |e|^{-1/2}\|[\![\partial u_h/\partial n]\!]\|_{L_2(e)}}\Big)\\
&\qquad + |\lambda|^{\frac12}\|(\psi-u_h)^+\|^{\frac12}_{L_\infty(\Omega)},
\end{aligned}
$$

where $\tilde{\mathcal{E}}_T$ is the set of the edges in $\mathcal{T}_h$ that emanate from the vertices of T.

Proof. We have

$$
\int_\Omega (\psi-E_hu_h)^+\,d\lambda \le \big[\|(\psi-u_h)^+\|_{L_\infty(\Omega)} + \|u_h-E_hu_h\|_{L_\infty(\Omega)}\big]|\lambda|, \tag{4.8}
$$

and, by (2.11),

$$\|u_h - E_h u_h\|_{L_\infty(\Omega)} \le C \max_{T\in\mathcal{T}_h} h_T \sum_{e\in\tilde{\mathcal{E}}_T} |e|^{-1/2} \|[\![\partial u_h/\partial n]\!]\|_{L_2(e)}. \tag{4.9}$$

The estimate (4.7) follows from (4.1), (4.8), and (4.9). □

Remark 4.3. The estimate (4.7) is not a genuine *a posteriori* error estimate since $|\lambda|$ is not known. But it is useful for monitoring the asymptotic convergence of adaptive algorithms (cf. Lemma 6.1 and Lemma 6.2).

Remark 4.4. Under the stronger assumption $\psi \in C^2(\bar{\Omega})$ on the obstacle function, one can also obtain a genuine *a posteriori* error estimate by replacing $|\lambda|$ with a computable bound.

Indeed, for any $w \in K$, we have

$$\frac{1}{2}|u|^2_{H^2(\Omega)} \le \frac{1}{2}|w|^2_{H^2(\Omega)} - (f,w) + (f,u) \le \frac{1}{2}|w|^2_{H^2(\Omega)} - (f,w) + C\|f\|^2_{L_2(\Omega)} + \frac{1}{4}|u|^2_{H^2(\Omega)},$$

by a Poincaré-Friedrichs inequality [33] and the arithmetic-geometric means inequality, and hence

$$|u|^2_{H^2(\Omega)} \le 2|w|^2_{H^2(\Omega)} - 4(f,w) + C\|f\|^2_{L_2(\Omega)}, \tag{4.10}$$

where C is a computable positive constant. Combining (4.10) with the Sobolev embedding (cf. [1]) $H^2(\Omega) \hookrightarrow C^{0,\gamma}(\Omega)$ for any $\gamma < 1$, we see that there is a computable $\delta > 0$ such that $u(x) > \psi(x)$ if the distance from x to $\partial\Omega$ is $< \delta$. Therefore there is a computable $\phi \in C_c^\infty(\Omega)$ such that $\phi = 1$ on the support of λ.

We then have, in view of (1.5) and (4.10),

$$|\lambda| = a(u,\phi) - (f,\phi) \le |u|_{H^2(\Omega)}|\phi|_{H^2(\Omega)} + \|f\|_{L_2(\Omega)}\|\phi\|_{L_2(\Omega)} \le C,$$

where the positive constant C is computable.

5. Efficiency Estimates for the Obstacle Problem

Let the local data oscillation $\mathrm{Osc}(f;T)$ be defined by

$$\mathrm{Osc}(f;T) = h_T^2\|f - \bar{f}_T\|_{L_2(T)},$$

where $\bar{f}_T$ is the L_2 projection of f in the polynomial space $P_j(T)$ with $j = \max(k-4, 0)$. The global data oscillation is then given by

$$\mathrm{Osc}(f;\mathcal{T}_h) = \Big(\sum_{T\in\mathcal{T}_h} \mathrm{Osc}(f;T)^2\Big)^{\frac{1}{2}}.$$

Theorem 5.1. *There exists a positive constant C, depending only on the shape regularity of $\mathcal{T}_h$, such that*

$$\eta_{e,1} \le \frac{\sigma}{|e|^{\frac{1}{2}}}\|[\![\partial(u-u_h)/\partial n]\!]\|_{L_2(e)} \qquad \forall\, e \in \mathcal{E}_h,$$

$$\eta_{e,2} \le C\Big[\sum_{T\in\mathcal{T}_e}\big[|u-u_h|_{H^2(T)} + \mathrm{Osc}(f;T)\big] + \|\lambda-\lambda_h\|_{H^{-2}(\Omega_e)}\Big] \qquad \forall\, e\in\mathcal{E}_h^i,$$

$$\eta_{e,3} \le C\Big[\sum_{T\in\mathcal{T}_e}\big[|u-u_h|_{H^2(T)} + \mathrm{Osc}(f;T)\big] + \|\lambda-\lambda_h\|_{H^{-2}(\Omega_e)} + \frac{1}{|e|}\|[\![\partial(u-u_h)/\partial n]\!]\|_{L_2(e)}^2\Big] \qquad \forall\, e\in\mathcal{E}_h^i,$$

$$\eta_T \le C\big(|u-u_h|_{H^2(T)} + \mathrm{Osc}(f;T) + \|\lambda-\lambda_h\|_{H^{-2}(T)}\big) \qquad \forall\, T\in\mathcal{T}_h,$$

where $\mathcal{T}_e$ is the set of the two triangles that share the edge e and Ω_e is the interior of $\bigcup_{T\in\mathcal{T}_e}\bar{T}$.

Proof. The estimate for $\eta_{e,1}$ is obvious. The other estimates are obtained by modifying the arguments in [8, Section 5.3].

In the proof of the estimate [8, (5.17)] (with $v = u_h$), we replace the relation

$$\int_T (\bar{f}_T - \Delta^2 u_h) z\,dx = \int_T D^2(u-u_h) : D^2 z\,dx + \int_T (\bar{f}_T - f) z\,dx$$

by

$$\int_T (\bar{f}_T - \Delta^2 u_h) z\,dx = \int_T D^2(u-u_h) : D^2 z\,dx + \int_T (\bar{f}_T - f) z\,dx - \int_T z\,d(\lambda-\lambda_h) \tag{5.1}$$

to obtain the estimate

$$\int_T (\bar{f}_T - \Delta^2 u_h) z \le C\big(h_T^{-2}|u-u_h|_{H^2(T)} + \|f-\bar{f}_T\|_{L_2(T)} + h_T^{-2}\|\lambda-\lambda_h\|_{H^{-2}(T)}\big)\|z\|_{L_2(T)},$$

which then leads to the estimate for η_T. Note that (5.1) holds because the bubble function z vanishes at the vertices of $\mathcal{T}_h$.

In the proof of the estimate [8, (5.26)] (with $v = u_h$), we replace the relation

$$\sum_{T\in\mathcal{T}_e}\Big(-\int_T D^2u_h : D^2(\zeta_1\zeta_2)\,dx + \int_T (\Delta^2 u_h)(\zeta_1\zeta_2)\,dx\Big) = \sum_{T\in\mathcal{T}_e}\int_T D^2(u-u_h) : D^2(\zeta_1\zeta_2)\,dx - \sum_{T\in\mathcal{T}_e}\int_T (f-\Delta^2 u_h)(\zeta_1\zeta_2)\,dx$$

that appears in [8, (5.24)] by

$$\sum_{T\in\mathcal{T}_e}\Big(-\int_T D^2u_h : D^2(\zeta_1\zeta_2)\,dx + \int_T (\Delta^2 u_h)(\zeta_1\zeta_2)\,dx\Big) = \sum_{T\in\mathcal{T}_e}\int_T D^2(u-u_h) : D^2(\zeta_1\zeta_2)\,dx - \sum_{T\in\mathcal{T}_e}\int_T (f-\Delta^2 u_h)(\zeta_1\zeta_2)\,dx - \int_{\Omega_e}(\zeta_1\zeta_2)\,d(\lambda-\lambda_h) \tag{5.2}$$

to obtain the estimate

$$\sum_{T\in\mathcal{T}_e}\Big(-\int_T D^2u_h : D^2(\zeta_1\zeta_2)\,dx + \int_T(\Delta^2 u_h)(\zeta_1\zeta_2)\,dx\Big)$$
$$\leq C\Big[\sum_{T\in\mathcal{T}_e}\big(h_T^{-2}|u-u_h|_{H^2(T)}+\|f-\Delta^2u_h\|_{L_2(T)}\big)+h_T^{-2}\|\lambda-\lambda_h\|_{H^{-2}(\Omega_e)}\Big]\|\zeta_1\zeta_2\|_{L_2(\Omega_e)},$$

which then leads to the estimate for $\eta_{e,2}$. Note that (5.2) holds because the bubble function $\zeta_1\zeta_2$ vanishes at the vertices of $\mathcal{T}_h$.

Finally, in the proof of the estimate [8, (5.32)] (with $v = u_h$), we replace the relation

$$\sum_{T\in\mathcal{T}_e}\Big(\int_T D^2u_h : D^2(\zeta_2\zeta_3)\,dx - \int_T(\Delta^2 u_h)(\zeta_2\zeta_3)\,dx\Big)$$
$$= \sum_{T\in\mathcal{T}_e}\int_T D^2(u_h-u):D^2(\zeta_2\zeta_3)\,dx + \sum_{T\in\mathcal{T}_2}\int_T(f-\Delta^2u_h)(\zeta_2\zeta_3)\,dx$$

that appears in [8, (5.30)] by

$$\sum_{T\in\mathcal{T}_e}\Big(\int_T D^2u_h : D^2(\zeta_2\zeta_3)\,dx - \int_T(\Delta^2 u_h)(\zeta_2\zeta_3)\,dx\Big)$$
$$= \sum_{T\in\mathcal{T}_e}\int_T D^2(u_h-u):D^2(\zeta_2\zeta_3)\,dx + \sum_{T\in\mathcal{T}_2}\int_T(f-\Delta^2u_h)(\zeta_2\zeta_3)\,dx \tag{5.3}$$
$$+\int_{\Omega_e}(\zeta_2\zeta_3)\,d(\lambda-\lambda_h)$$

to obtain the estimate

$$\sum_{T\in\mathcal{T}_e}\Big(\int_T D^2u_h : D^2(\zeta_2\zeta_3)\,dx - \int_T(\Delta^2 u_h)(\zeta_2\zeta_3)\,dx\Big)$$
$$\leq C\Big[\sum_{T\in\mathcal{T}_e}\big(h_T^{-2}|u-u_h|_{H^2(T)}+\|f-\Delta^2u_h\|_{L_2(T)}\big)+h_T^{-2}\|\lambda-\lambda_h\|_{H^{-2}(\Omega_e)}\Big]\|\zeta_2\zeta_3\|_{L_2(\Omega_e)},$$

which then leads to the estimate for $\eta_{e,3}$. Again (5.3) holds because the bubble function $\zeta_2\zeta_3$ vanishes at the vertices of $\mathcal{T}_h$. □

We can also prove a global efficiency result under the following assumption:

(5.4) The triangles (resp. interior edges) of $\mathcal{T}_h$ can be divided into n disjoint groups so that the ratio of the diameters of any two triangles (resp. interior edges) in the same group is bounded above by a constant $\tau \geq 1$.

Theorem 5.2. *Under assumption* (5.4)*, there exists a positive constant C depending only on τ, k and the shape regularity of $\mathcal{T}_h$ such that*

$$\eta_h \leq C\big(\sqrt{\sigma}\|u-u_h\|_h + \sqrt{n}\|\lambda-\lambda_h\|_{H^{-2}(\Omega)} + \mathrm{Osc}(f;\mathcal{T}_h)\big). \tag{5.5}$$

Proof. We have a trivial estimate

$$\sum_{e\in\mathcal{E}_h}\eta_{e,1}^2\le C\sum_{e\in\mathcal{E}_h}\frac{\sigma^2}{|e|}\Big\|\Big[\!\!\Big[\frac{\partial(u-u_h)}{\partial n}\Big]\!\!\Big]\Big\|_{L_2(e)}^2.$$

For the estimate involving η_T, we first write $\mathcal{T}_h$ as the disjoint union $\mathcal{T}_{h,1}\cup\cdots\cup\mathcal{T}_{h,n}$ so that the ratio of the diameters of any two triangles in $\mathcal{T}_{h,j}$ is bounded by τ. For $1\le j\le n$, the subdomain Ω_j is the interior of $\cup_{T\in\mathcal{T}_{h,j}}\bar{T}$.

For any $T\in\mathcal{T}_{h,j}$, let z_T be the bubble function in [8, Section 5.3.2] associated with T and we define $z_j=\sum_{T\in\mathcal{T}_{h,j}}z_T\in H_0^2(\Omega_j)$. It follows from [8, (5.16)], (5.1) and a standard inverse estimate that

$$\begin{aligned}\|\bar{f}_T-\Delta^2u_h\|_{L_2(T)}^2&\le C\int_T(\bar{f}_T-\Delta^2u_h)z_T\,dx\\&\le C\Big(\big[h_T^{-2}|u-u_h|_{H^2(T)}+\|f-\bar{f}_T\|_{L_2(T)}\big]\|z_T\|_{L_2(T)}-\int_T z_T\,d(\lambda-\lambda_h)\Big)\end{aligned}$$

and hence

$$\begin{aligned}\sum_{T\in\mathcal{T}_{h,j}}\|\bar{f}_T-\Delta^2u_h\|_{L_2(T)}^2&\le C\Big(\sum_{T\in\mathcal{T}_{h,j}}\big[h_T^{-2}|u-u_h|_{H^2(T)}+\|f-\bar{f}_T\|_{L_2(T)}\big]\|z_T\|_{L_2(T)}\\&\qquad\qquad-\int_{\Omega_j}z_j\,d(\lambda-\lambda_h)\Big)\\&\le C\Big[\Big(\sum_{T\in\mathcal{T}_{h,j}}\big[h_T^{-4}|u-u_h|_{H^2(T)}^2+\|f-\bar{f}_T\|_{L_2(T)}^2\big]\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_{h,j}}\|z_T\|_{L_2(T)}^2\Big)^{\frac12}\\&\qquad+\|\lambda-\lambda_h\|_{H^{-2}(\Omega_j)}\Big(\sum_{T\in\mathcal{T}_{h,j}}h_T^{-4}\|z_T\|_{L_2(T)}^2\Big)^{\frac12}\Big]\end{aligned}$$

by a standard inverse estimate.

Therefore we have

$$\begin{aligned}\sum_{T\in\mathcal{T}_{h,j}}h_T^4\|\bar{f}-\Delta^2u_h\|_{L_2(T)}^2&\le C\Big(\sum_{T\in\mathcal{T}_{h,j}}\big[h_T^4\|f-\bar{f}\|_{L_2(T)}^2+|u-u_h|_{H^2(T)}^2\big]\\&\qquad+\|\lambda-\lambda_h\|_{H^{-2}(\Omega_j)}^2\Big)\end{aligned}\tag{5.6}$$

because (cf. [8, (5.16)])

$$\|z_T\|_{L_2(T)}\approx\|\bar{f}-\Delta^2u_h\|_{L_2(T)}$$

and the diameters h_T are comparable for $T\in\mathcal{T}_{h,j}$.

It follows from (5.6) that

$$\sum_{T\in\mathcal{T}_h}\eta_T^2=\sum_{T\in\mathcal{T}_h}h_T^4\|f-\Delta^2u_h\|_{L_2(T)}^2$$

$$
\begin{aligned}
&\le \sum_{j=1}^{n}\sum_{T\in\mathcal{T}_{h,j}} 2h_T^4\big[\|\bar f_T-\Delta^2 u_h\|_{L_2(T)}+\|f-\bar f_T\|_{L_2(T)}\big]^2\\
&\le C\sum_{j=1}^{n}\Big(\sum_{T\in\mathcal{T}_{h,j}}\big[h_T^4\|f-\bar f_T\|_{L_2(T)}^2+|u-u_h|_{H^2(T)}^2\big]\\
&\qquad\qquad +\|\lambda-\lambda_h\|_{H^{-2}(\Omega_j)}^2\Big)\\
&\le C\Big(\mathrm{Osc}(f;\mathcal{T}_h)^2+\sum_{T\in\mathcal{T}_h}|u-u_h|_{H^2(T)}^2+n\|\lambda-\lambda_h\|_{H^{-2}(\Omega)}^2\Big),
\end{aligned}
$$

where we have also used the trivial estimate $\|\lambda-\lambda_h\|_{H^{-2}(\Omega_j)}\le\|\lambda-\lambda_h\|_{H^{-2}(\Omega)}$.

The estimates for $\eta_{e,2}$ and $\eta_{e,3}$ can be established by using (5.2), (5.3) and results in [8, Sections 5.3.3 and 5.3.4]. Their derivations are similar to the derivation for η_T and hence are omitted. □

6. An Adaptive Algorithm

In view of the efficiency estimates in Section 5, we will use η_h from (3.2) as the error indicator in the adaptive loop

$$\mathsf{Solve}\longrightarrow\mathsf{Estimate}\longrightarrow\mathsf{Mark}\longrightarrow\mathsf{Refine}$$

to define an adaptive algorithm for the C^0 interior penalty methods for (1.1)–(1.3).

In the step Solve, we compute the solution of the discrete obstacle problem (2.1) by a primal-dual active set method [3, 29]. In the step Estimate, we compute $\eta_{e,1}$, $\eta_{e,2}$, $\eta_{e,3}$ and η_T defined in (3.3)–(3.6). In the step Mark, we use the Dörfler marking strategy [22] to mark a minimum number of triangles and edges whose contributions exceed $\theta\eta_h$ for some $\theta\in(0,1)$. In the step Refine, we refine the marked triangles and edges followed by a closure algorithm that preserves the conformity of the triangulation.

In the adaptive setting the subscript h will be replaced by the subscript ℓ, where $\ell = 0,1,\dots$ denotes the level of refinements. The adaptive algorithm generates a sequence of triangulations $\mathcal{T}_\ell$ of Ω, a sequence of solutions $u_\ell\in V_\ell$ of the discrete obstacle problems and a sequence of error indicators η_ℓ.

According to Theorem 4.2, we can use the following result to monitor the asymptotic convergence rate of the adaptive algorithm.

Lemma 6.1. *Suppose $\eta_\ell=O(N_\ell^{-\gamma})$, where N_ℓ is the number of degrees of freedom (dof) at the refinement level ℓ. Then we have*

$$\|u-u_\ell\|_\ell+\|\lambda-\lambda_\ell\|_{H^{-2}(\Omega)}=O(N_\ell^{-\gamma}) \tag{6.1}$$

provided that

$$Q_{\ell,1}=\sqrt{\max_{T\in\mathcal{T}_\ell}h_T\sum_{e\in\tilde{\mathcal{E}}_T}|e|^{-1/2}\|[\![\partial u_\ell/\partial n]\!]\|_{L_2(e)}}=O(N_\ell^{-\gamma}), \tag{6.2}$$

$$Q_{\ell,2} = \|(\psi - u_\ell)^+\|_{L_\infty(\Omega)}^{\frac{1}{2}} = O(N_\ell^{-\gamma}). \tag{6.3}$$

In particular, the estimate (6.1) *holds if* $Q_{\ell,1}$ *and* $Q_{\ell,2}$ *are dominated by* η_ℓ.

Note that $\|\lambda - \lambda_h\|_{H^{-2}(\Omega)}$ is not computable. However we can test the convergence of $\|\lambda - \lambda_h\|_{H^{-2}(\Omega)}$ indirectly as follows. Let $\phi \in C_c^\infty(\Omega)$ be equal to 1 on the supports of λ and the λ_ℓ's. Then we have

$$|\lambda| - |\lambda_\ell| = \int_\Omega \phi \, d(\lambda - \lambda_\ell) \le |\phi|_{H^2(\Omega)} \|\lambda - \lambda_\ell\|_{H^{-2}(\Omega)}, \tag{6.4}$$

which implies

$$\begin{aligned} |\lambda_\ell| - |\lambda_{\ell+1}| &= (|\lambda_\ell| - |\lambda|) + (|\lambda| - |\lambda_{\ell+1}|) \\ &\le |\phi|_{H^2(\Omega)} \big(\|\lambda - \lambda_\ell\|_{H^{-2}(\Omega)} + \|\lambda - \lambda_{\ell+1}\|_{H^{-2}(\Omega)}\big). \end{aligned} \tag{6.5}$$

Let Λ_ℓ be defined by

$$\Lambda_\ell = |(|\lambda_\ell| - |\lambda_{\ell+1}|)|. \tag{6.6}$$

The following result is an immediate consequence of Lemma 6.1 and (6.5).

Lemma 6.2. *Suppose* $\eta_\ell = O(N_\ell^{-\gamma})$, *where* N_ℓ *is the number of dof at the refinement level* ℓ. *Then we have*

$$\Lambda_\ell = O(N_\ell^{-\gamma})$$

provided that (6.2) *and* (6.3) *are valid.*

Remark 6.3. In view of (6.4), we can also replace $|\lambda|$ by $|\lambda_\ell|$ in (4.7) to obtain a true *a posteriori* error estimate that is asymptotically reliable under the assumptions of Lemma 6.1.

7. Numerical Experiments

In this section we report numerical results that demonstrate the estimate (4.7) and illustrate the performance of the adaptive algorithm for quadratic and cubic C^0 interior penalty methods. We choose the penalty parameter σ to be 6 (resp. 18) for the quadratic (resp. cubic) C^0 interior penalty method. We also take θ to be 0.5 in the Dörfler marking strategy.

We will consider three examples. The first one concerns a problem on the unit square with known exact solution. The second one is about a problem on a L-shaped domain with a two dimensional coincidence set (where $u = \psi$) that has a fairly smooth boundary. The third example is also about a problem on a L-shaped domain but with a coincidence set that is one dimensional. For the second and third examples where the exact solution is not known, we estimate the error $\|u - u_\ell\|_\ell$ by using a reference solution computed on the mesh obtained by a uniform refinement of the last mesh generated by the refinement procedure.

In each of the experiment for the adaptive algorithm, we will present figures that display the convergence histories for $\|u - u_\ell\|_\ell$ and η_ℓ, and for the quantities $Q_{\ell,1}$ and $Q_{\ell,2}$ defined in (6.2) and (6.3). We also present tables that contain numerical results for the quantity Λ_ℓ defined in (6.6) and examples of adaptively generated meshes.

7.1. **Example 1.** In this example we consider an obstacle problem on the unit square $\Omega = (-0.5, 0.5)^2$ from [14, Example 1] with $f = 0$, $\psi = 1 - |x|^2$ and nonhomogeneous boundary conditions, whose exact solution is given by

$$u(x) = \begin{cases} C_1|x|^2\ln(|x|) + C_2|x|^2 + C_3\ln(|x|) + C_4 & r_0 < |x| \\ 1 - |x|^2 & |x| \le r_0 \end{cases},$$

where $r_0 \approx 0.18134453$, $C_1 \approx 0.52504063$, $C_2 \approx -0.62860905$, $C_3 \approx 0.017266401$ and $C_4 \approx 1.0467463$.

For this example the coincidence set is the disc centered at the origin with radius r_0 whose boundary is the free boundary, and we have $|\lambda| = 8\pi C_1 \approx 13.1957$.

Due to the nonhomogeneous boundary conditions, we modify the discrete obstacle problem (cf. [14]) to find

$$u_h = \operatorname*{argmin}_{v\in K_h}\Big[\frac{1}{2}a_h(v,v) - F(v)\Big],$$

where $K_h = \{v \in V_h : v - \Pi_h u \in H_0^1(\Omega),\ v(p) \ge \psi(p) \quad \forall\, p \in \mathcal{V}_h\}$,

$$F(v) = (f, v) + \sum_{e\in\mathcal{E}_h^b}\int_e \left(\left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\} + \frac{\sigma}{|e|}\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right]\right)\left[\!\!\left[\frac{\partial u}{\partial n}\right]\!\!\right] ds,$$

and $\mathcal{E}_h^b$ is the set of the edges of $\mathcal{T}_h$ that are on the boundary of Ω. We also modify the residual based error estimator:

$$\eta_h = \Big(\sum_{e\in\mathcal{E}_h^i}\eta_{e,1}^2 + \sum_{e\in\mathcal{E}_h^i}(\eta_{e,2}^2 + \eta_{e,3}^2) + \sum_{T\in\mathcal{T}_h}\eta_T^2 + \sum_{e\in\mathcal{E}_h^b}\sigma^2|e|^{-1}\|[\![\partial(u_h - u)/\partial n]\!]\|_{L_2(e)}^2\Big)^{\frac{1}{2}}.$$

In the first experiment we solve the discrete problem with the P_2 element on uniform meshes and compute the quantity

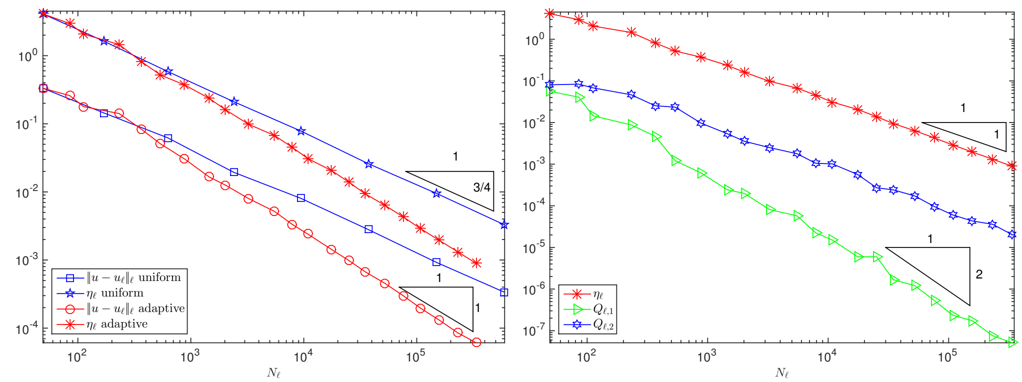
$$Q_h = C\Big(\eta_h + |\lambda|^{\frac{1}{2}}\sqrt{\max_{T\in\mathcal{T}_h} h_T \sum_{e\in\tilde{\mathcal{E}}_T}|e|^{-1/2}\|[\![\partial u_h/\partial n]\!]\|_{L_2(e)}} + |\lambda|^{\frac{1}{2}}\|(\psi - u_h)^+\|_{L_\infty(\Omega)}^{\frac{1}{2}}\Big) \tag{7.1}$$

that appears on the right-hand side of (4.7), with $C = 0.32$ and $|\lambda| = 13.196$. The results for $\|u - u_h\|_h/Q_h$ (cf. Table 7.1) clearly demonstrate the estimate (4.7).

h	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
$\|u - u_h\|_h/Q_h$	0.93	0.94	0.96	0.97	0.97	0.97	0.98	0.98	0.99	1.04

Table 7.1. Numerical results for the estimate (4.7)

In the second experiment we solve the discrete obstacle problem with the cubic element on uniform and adaptive meshes. We observe optimal (resp. suboptimal) convergence rate for adaptive (resp. uniform) meshes in Figure 7.1 (a) and also the reliability of η_ℓ. Furthermore the optimal $O(N_\ell^{-1})$ convergence rate of $\|u-u_\ell\|_\ell$ is justified by Figure 7.1 (b) and Lemma 6.1.



(a) (b)

FIGURE 7.1. Convergence histories for the cubic C^0 interior penalty method for Example 1: (a) $\|u-u_\ell\|_\ell$ and η_ℓ, (b) η_ℓ, $Q_{\ell,1}$ and $Q_{\ell,2}$

According to Lemma 6.2 and Figure 7.1 (b), the magnitude of Λ_ℓ should be $O(N_\ell^{-1})$. This is confirmed by the results in Table 7.2, where N_ℓ increases from $N_0 = 49$ to $N_{20} = 231328$.

ℓ	0	1	2	3	4	5	6	7	8	9	10
$\Lambda_\ell N_\ell$	155	250	60.5	214	164	99.6	101	75.8	21.8	31.9	17.9

ℓ	11	12	13	14	15	16	17	18	19	20	
$\Lambda_\ell N_\ell$	12.2	1.48	23.8	5.37	0.403	4.65	20.5	51.1	259	730	

TABLE 7.2. $\Lambda_\ell N_\ell$ for the adaptive cubic C^0 interior penalty method for Example 1

An adaptive mesh with roughly 3000 nodes is depicted in Figure 7.2 and strong refinement near the free boundary is observed.

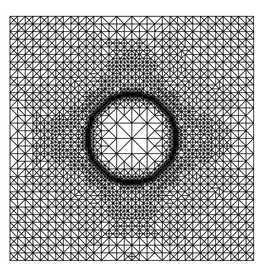
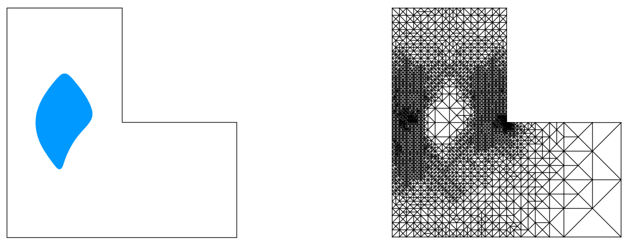


FIGURE 7.2. Adaptive mesh for the cubic C^0 interior penalty method for Example 1

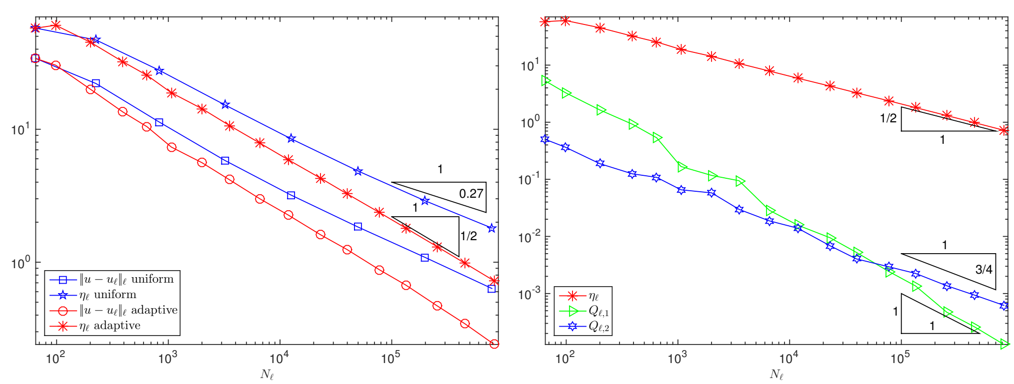
7.2. **Example 2.** In this example we consider the obstacle problem from [14, Example 4] for a clamped plate occupying the L-shaped domain $\Omega = (-0.5, 0.5)^2 \setminus [0, 0.5]^2$ with $f = 0$ and $\psi(x) = 1 - \Big[\dfrac{(x_1 + 1/4)^2}{0.2^2} + \dfrac{x_2^2}{0.35^2}\Big]$. The coincidence set for this problem is presented in Figure 7.3 (a).



(a) (b) (c)

FIGURE 7.3. L-shaped domain for Example 2: (a) Coincidence set for the obstacle problem (b) Adaptive mesh with ≈ 3000 nodes for the P_2 element (c) Adaptive mesh with ≈ 5000 nodes for the P_3 element

In the first experiment we solve the discrete obstacle problem with the P_2 element on uniform and adaptive meshes. Optimal (resp. suboptimal) convergence rate for adaptive (resp. uniform) meshes and the reliability of η_ℓ are observed in Figure 7.4 (a), and the $O(N_\ell^{-1/2})$ convergence rate of $\|u - u_\ell\|_\ell$ is justified by Figure 7.4 (b) and Lemma 6.1.



(a) (b)

FIGURE 7.4. Convergence histories for the quadratic C^0 interior penalty method for Example 2: (a) $\|u - u_\ell\|_\ell$ and η_ℓ, (b) η_ℓ, $Q_{\ell,1}$ and $Q_{\ell,2}$

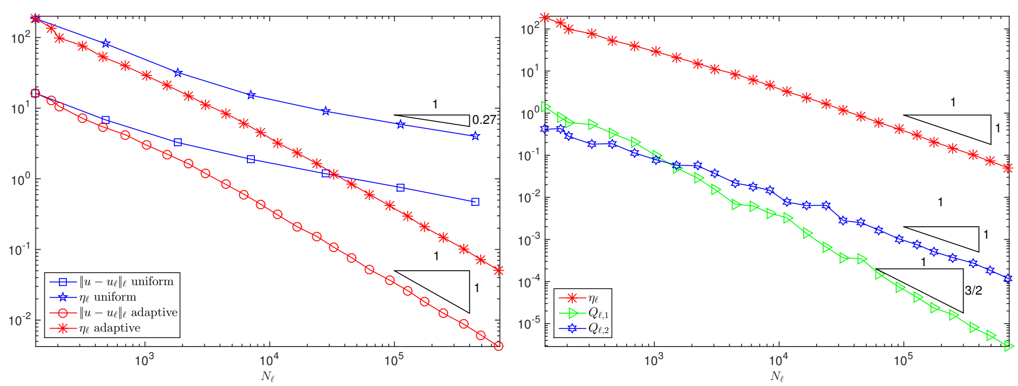
The $O(N_\ell^{-1/2})$ bound for Λ_ℓ predicted by Lemma 6.2 and Figure 7.4 (b) is observed in Table 7.3, where N_ℓ increases from 65 to 827483.

ℓ	0	1	2	3	4	5	6	7	8
$\Lambda_\ell N_\ell^{1/2}$	2715	391	637	756	1454	654	613	467	411
ℓ	9	10	11	12	13	14	15	16	
$\Lambda_\ell N_\ell^{1/2}$	360	149	255	105	144	70	72	52	

TABLE 7.3. $\Lambda_\ell N_\ell^{1/2}$ for the adaptive quadratic C^0 interior penalty method for Example 2

An adaptive mesh with roughly 3000 nodes is displayed in Figure 7.3 (b), where we observe a strong refinement near the reentrant corner. In contrast the refinement near the free boundary is mild. This is due to the fact that away from the reentrant corner the solution belongs to H^3 (cf. [24, 4]) and we are using the P_2 element.

In the second experiment we solve the obstacle problem with the P_3 element on uniform and adaptive meshes. We observe optimal (resp. suboptimal) convergence rate for adaptive (resp. uniform) meshes in Figure 7.5 (a) and that η_ℓ is reliable in both cases. Moreover the $O(N_\ell^{-1})$ convergence rate of $\|u - u_\ell\|_\ell$ is justified by Figure 7.5 (b) and Lemma 6.1.



(a) (b)

FIGURE 7.5. Convergence histories for the cubic C^0 interior penalty method for Example 2: (a) $\|u - u_\ell\|_\ell$ and η_ℓ, (b) η_ℓ, $Q_{\ell,1}$ and $Q_{\ell,2}$

The results for Λ_ℓ are reported in Table 7.4, where the $O(N_\ell^{-1})$ bound for Λ_ℓ predicted by Lemma 6.2 and Figure 7.5 (b) can be observed. Note that there are large oscillations at the beginning before the coincidence has been captured by the adaptive mesh. Here N_ℓ increases from $N_0 = 133$ to $N_{22} = 358792$.

An adaptive mesh with roughly 5000 nodes is displayed in Figure 7.3 (c), where we observe strong refinement near both the reentrant corner and the free boundary.

7.3. **Example 3.** In this example we consider the obstacle problem on the L-shaped domain $\Omega = (-0.5, 0.5)^2 \setminus [0, 0.5]^2$ with

$$\psi(x) = -[\sin(2\pi(x_1 + 0.5)(x_2 + 0.5)) \sin(4\pi(x_1 - 0.5)(x_2 - 0.5))] - 0.35$$

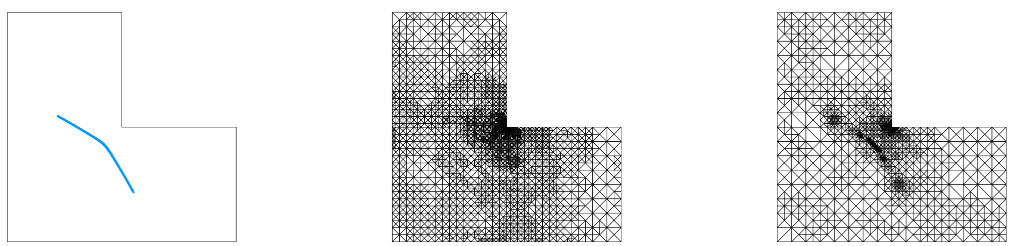
ℓ	0	1	2	3	4	5	6	7
$\Lambda_\ell N_\ell$	15398	16877	2893	1035	11806	8925	15493	5993
ℓ	8	9	10	11	12	13	14	15
$\Lambda_\ell N_\ell$	1048	5162	3544	3271	1362	119	580	778
ℓ	16	17	18	19	20	20	21	22
$\Lambda_\ell N_\ell$	77	96	68	92	147	116	754	885

TABLE 7.4. $\Lambda_\ell N_\ell$ for the adaptive cubic C^0 interior penalty method for Example 2

and

$$f(x) = \begin{cases} 10^3\left(\frac{1}{2}e^{(x_1+0.25)^2+(x_2+0.25)^2}\right) & x_1 \le 0, x_2 > 0 \\ 0 & x_1 \le 0, x_2 \le 0 \\ 10^3\left(\frac{1}{2} + [(x_1-0.25)^2 + (x_2+0.25)^2]^{3/2}\right) & x_1 \ge 0, x_2 \le 0 \end{cases}.$$

For this example, the coincidence set is one dimensional (cf. Figure 7.6(a)).



(a) (b) (c)

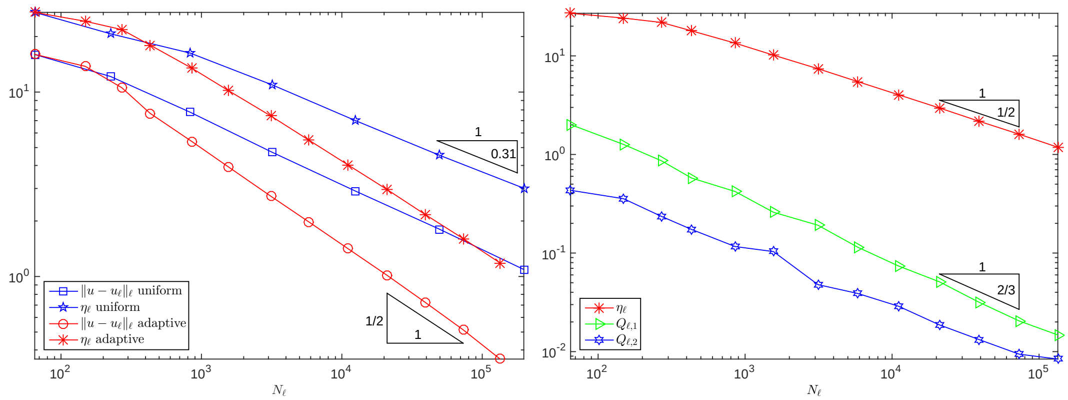
FIGURE 7.6. L-shaped domain for Example 3: (a) Coincidence set for the obstacle problem (b) Adaptive mesh with 11062 dof for the P_2 element (c) Adaptive mesh with 12841 dof for the P_3 element

In the first experiment we solve the obstacle problem with the P_2 element on uniform and adaptive meshes. We observe optimal (resp. suboptimal) convergence rate for adaptive (resp. uniform) meshes in Figure 7.7(a) and also the reliability of η_ℓ. The $O(N_\ell^{-1/2})$ convergence rate of $\|u-u_\ell\|_\ell$ is confirmed by Figure 7.7(b) and Lemma 6.1.

The results in Table 7.5 agrees with the $O(N_\ell^{-1/2})$ bound for Λ_ℓ that follows from Lemma 6.2 and Figure 7.7(b). The number of dof increases from $N_0 = 65$ to $N_{12} = 134096$.

An adaptive mesh with 11062 dof is depicted in Figure 7.6(b), where we observe that the only strong refinement is around the reentrant corner. This is again due to the fact that away from the reentrant corner the solution belongs to H^3 and we are using the P_2 element.

In the second experiment we solve the obstacle problem with the P_3 element on uniform and adaptive meshes. We observe optimal (resp. suboptimal) convergence rate for adaptive



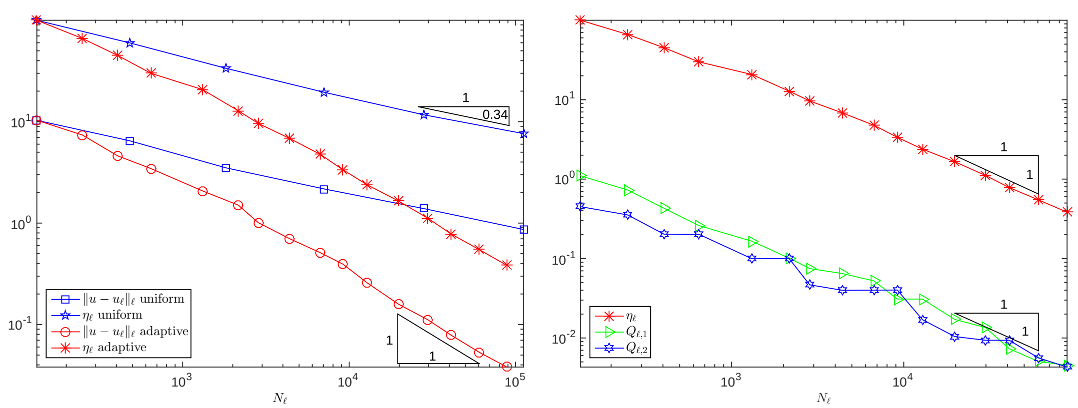
(a) (b)

FIGURE 7.7. Convergence histories for the quadratic C^0 interior penalty method for Example 3: (a) $\|u-u_\ell\|_\ell$ and η_ℓ, (b) η_ℓ, $Q_{\ell,1}$ and $Q_{\ell,2}$

ℓ	0	1	2	3	4	5	6	7	8	9	10	11	12
$\Lambda_\ell N_\ell^{1/2}$	1151	501	92	201	120	419	201	98	75	34	76	40	36

TABLE 7.5. $\Lambda_\ell N_\ell^{1/2}$ for the adaptive quadratic C^0 interior penalty method for Example 3

(resp. uniform) meshes in Figure 7.8(a) and also the reliability of η_ℓ. Furthermore the $O(N_\ell^{-1})$ convergence rate for $\|u-u_\ell\|_\ell$ is justified by Figure 7.7(b) and Lemma 6.1.



(a) (b)

FIGURE 7.8. Convergence histories for the cubic C^0 interior penalty method for Example 3: (a) $\|u-u_\ell\|_\ell$ and η_ℓ, (b) η_ℓ, $Q_{\ell,1}$ and $Q_{\ell,2}$

The results in Table 7.6 agrees with the $O(N_\ell^{-1})$ bound for Λ_ℓ predicted by Lemma 6.2 and Figure 7.8(b). Here N_ℓ increases from $N_0 = 133$ to $N_{15} = 88699$.

ℓ	0	1	2	3	4	5	6	7
$\Lambda_\ell N_\ell$	11724	1782	1842	32888	1046	6439	2974	2588
ℓ	8	9	10	11	12	13	14	15
$\Lambda_\ell N_\ell$	2781	25657	2215	3805	5177	2030	1092	2355

TABLE 7.6. $\Lambda_\ell N_\ell$ for the adaptive cubic C^0 interior penalty method for Example 3

An adaptive mesh with 12841 dof is depicted in Figure 7.6(c), where we observe strong refinement around the reentrant corner and the coincidence set.

8. CONCLUSIONS

We have developed a simple *a posteriori* error analysis of C^0 interior penalty methods for the displacement obstacle problem of clamped Kirchhoff plates by taking advantage of the fact that the Lagrange multiplier for the discrete problem can be represented naturally as the sum of Dirac point measures supported at the vertices of the triangulation. Numerical results indicate that the adaptive algorithm based on a standard *a posteriori* error estimator originally developed for boundary value problems also performs optimally for quadratic and cubic C^0 interior penalty methods for obstacle problems. However the theoretical justification of convergence and optimality for adaptive C^0 interior penalty methods remains open even in the case when the obstacle is absent.

The results in this paper can be extended to the displacement obstacle problem of the biharmonic equation with the boundary conditions of simply supported plates or the Cahn-Hilliard type. In the case where Ω is convex, such problems are related to distributed elliptic optimal control problems with pointwise state constraints [32, 25, 16, 17] and can also be considered in three dimensional domains. Adaptive finite element methods for these problems based on the approach in this paper are ongoing projects.

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Susanne C. Brenner, Department of Mathematics and Center for Computation and Technology, Louisiana State University, Baton Rouge, LA 70803
E-mail address: brenner@math.lsu.edu

Joscha Gedicke, Department of Mathematics and Center for Computation and Technology, Louisiana State University, Baton Rouge, LA 70803
E-mail address: jgedicke@math.lsu.edu

Li-yeng Sung, Department of Mathematics and Center for Computation and Technology, Louisiana State University, Baton Rouge, LA 70803
E-mail address: sung@math.lsu.edu

Yi Zhang, Department of Mathematics, University of Tennessee, Knoxville, TN 37996
E-mail address: yzhan112@utk.edu